INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR COST-BENEFIT ANALYSIS

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Abstract. Current OMB guidelines use the interest rate as a basis for the discount rate, and have nothing to say about an intergenerationally fair discount rate. A traditional utilitarian approach leads to too high values for the latter, in a wide range. We propose to apply Relative Utilitarianism to derive the discount rate, and find it should equal the growth rate of real per-capita consumption, independent of the interest rate.

1. Introduction

A central issue with regard to the successful evaluation of a public project is the appropriate choice of the discount rate. In a typical cost-benefit analysis setting both costs and benefits are spread over time and their comparison crucially depends on the way both are discounted, or translated into present (consumption) terms. In this paper we use relative utilitarianism to derive an intergenerational discount rate to evaluate benefits accruing from a public project in a general equilibrium model with overlapping generations.

The issue of discounting has been controversial in the literature since, probably, Ramsey (1928) who, developing a theory of savings, presents discounting future utility ('enjoyments') as a "practice which is ethically indefensible and arises merely from the weakness of the imagination." Not surprisingly, one can easily find practical recommendations that echo this view.[1] On the other hand, Circular A4 of the U.S. Office of Management and Budget (September 2003) mandates that all executive agencies and establishments conduct a "regulatory analysis" for any new proposal, and more specifically (pp. 33–36), a cost-benefit analysis, at the rates of both 3% and 7%.[2] This, however, is a discounting on

[1] "Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable." "*Disposal Concepts for Radioactive Waste*" by W. Wildi, D. Appel, M. Buser, F. Dermange, A. Eckhardt, P. Hufschmied, H.-R. Keusen written on behalf of the Federal Department for the Environment, Transport, Energy and Communication of Switzerland, 2000.

[2] Both rates are rationalized there as the interest rate: the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

future *consumption*, not enjoyment thereof; and Ramsey (1928) himself stressed the importance of distinguishing the two. He also conjectured that population growth and "future inventions and improvements in organisation" might have an effect on the trade-off between current and future consumption.

The OMB circular does refer to this requirement of equity vis-à-vis of future generations, and acknowledges it by requiring, for projects that might have substantial long-term impact, a further analysis at a "still lower but positive" discount rate — but more specific suggestions are hard to find. This is the question we want to address.

In their fundamental work Arrow and Kurz (1970) offer a criterion, or a social welfare function, that has been widely used to evaluate public investments in the literature since then. Denote by N_t population at time t, let c_t be per-capita consumption and β be a (subjective) discount rate, then the criterion (in its simplest formulation)[3] is

$$\tilde{W}\left((c_t)_t\right) \equiv \int_0^\infty e^{-\beta t} N_t u\left(c_t\right) dt, \tag{1}$$

where u is a concave and increasing function of per-capita consumption. To put it in their own words,

> The flow of felicity to society is the sum over individuals at a given time; the total utility from a policy is taken to be the sum over all time of the felicities of each time, discounted back to the present at a constant rate.

Criterion (1) can be presented as a true social welfare function, i.e., a function of individual (lifetime) utilities. Indeed, assume, for example, all individuals live for a fixed period of time (unity), and that an individual born at time t has a life-time utility of the form

$$U_t\left(c.\right) = \int_t^{t+1} e^{-\alpha(s-t)} u\left(c_s\right) ds,$$

where $c.$ is the time-path of consumption (as a function of age), and α is the individual time preference. Assume also that population grows exponentially at a rate ν. Then, aggregating over all individuals (integrate over t from $-\infty$ to $+\infty$) when discounting their life-time utilities at a rate β, one gets the following criterion:[4]

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[3]More generally, the utility, u, can depend directly on government investment, k_g in a given period.

[4]The discount rate $\beta > 0$ is often introduced only for the 'technical' reason of making sure the social welfare function returns a finite number for strictly positive consumption profiles, given that $u(c) > 0$ if $c > 0$. Ramsey (1928) avoided this difficulty by suggesting to use a bounded function $u(c)$ and, then to minimize the difference between u and the 'bliss', B,, or the highest attainable utility, using that as the criterion. Following Chiang (1992) we will refer to this substitution as "the Ramsey device".

$$W \equiv \int_{-\infty}^{\infty} N_t e^{-\beta t} U_t(c.)\,dt \tag{2}$$

$$= \int_{-\infty}^{\infty} N_t e^{-\beta t} \int_t^{t+1} e^{-\alpha(s-t)} u(c_s)\,ds dt \tag{3}$$

$$= N_0 \int_{-\infty}^{\infty} e^{-\alpha s} u(c_s) \int_{s-1}^{s} e^{(\alpha+\nu-\beta)t} dt \tag{4}$$

$$= M \int_{-\infty}^{\infty} e^{-\beta t} N_t u(c_t)\,dt, \text{ where } M \equiv \int_0^1 e^{(\alpha+\nu-\beta)x} dx \tag{5}$$

Observe that, under the above assumptions, the two criteria (W and $\tilde{W}$) rank the policies that affect per-capita streams of consumption only after time zero in the same fashion. The advantage of using criterion W is its generality: it encompasses the Arrow and Kurz (1970) criterion and also allows for life-time utilities that are not necessarily time-separable. Finally, re-interpreting the criterion in this way allows to separate the individual time preference, α, from the social discount rate, β, and, as we will see, it allows as well to separate completely attitudes towards risk from the time preferences (and, in particular, from the inter-temporal substitution).

To illustrate the trade-off between current and future consumption that stem from this criterion, consider, following Arrow and Kurz (1970), a constant relative risk aversion function with coefficient, $\rho > 0$, so that $u(c) = c^{1-\rho}/(1-\rho)$; and suppose the economy is on a steady growth path with per-capita consumption growing exponentially at a rate $\gamma > 0$. Consider a policy that affects population for some time in the future, i.e., it involves a variation in aggregate consumption of δC_t, and it is to be evaluated at time 0. The status-quo per-capita consumption at time t is $c_0 e^{\gamma t}$, where c_0 is the initial (time 0) per-capita consumption. Doing, for clarity, the computation in discrete time, it follows that the net benefit equals

$$\begin{aligned} & \sum e^{-\beta t} N_t \left[u\left(c_0 e^{\gamma t} + \tfrac{\delta C_t}{N_t}\right) - u\left(c_0 e^{\gamma t}\right)\right] \\ = & \sum e^{-\beta t} N_t u'\left(c_0 e^{\gamma t}\right) \cdot \tfrac{\delta C_t}{N_t} = \sum e^{-\beta t} u'\left(c_0 e^{\gamma t}\right) \cdot \delta C_t = \sum c_0^{-\rho} e^{-(\rho\gamma+\beta)t} \delta C_t \end{aligned}$$

This means that future consumption is discounted at the rate $\rho\gamma + \beta$ under this criterion. Even if we are to follow Ramsey (1928) and set $\beta = 0$ (which still leaves the criterion W meaningful, under the specification of u suggested above), the magnitude of the suggested discount rate, $\rho\gamma$, (for most part) is far above any rates applicable in practice, besides, the range of acceptable values is extremely wide, as the next subsection demonstrates.

1.1. **Get orders of magnitude.** To estimate γ one may use a measure of growth of real per-capita GDP. Based on the data from the Bureau of Economic Analysis, over the past 70 years the average in the U.S. is estimated to be around $2 - 2.5\%$ per annum (with averages over various decades since 1950 ranging from 3% to 1.8%).

The obvious interpretation of our discrete time framework is that individuals live for 1 period, so the *only* role of ρ is to determine the individuals' attitudes towards risk. And consistency with, e.g., Harsanyi's axiomatization(s) of such additive SWF's forces then to interpret u as the individual's von Neumann-Morgenstern utility function, and hence ρ as his coefficient of relative risk aversion. One of the most recent overviews compiling various (micro) estimates of the risk aversion coefficients is contained in Einav and Cohen (2005). Remarkable is both the range as well as the magnitude of the suggested values ranging from single- to three-digit values. Einav and Cohen (2005) measure relative risk aversion coefficients from individual-level data on car insurance and annual income, obtaining *two-digit* estimates. Clearly, cost-benefit analysis will then only allow for very short-sighted policies. This remains true even with more conservative estimates, like, say, derived by Drèze (1981) ($\rho \sim$ 12–15), or like those which seem accepted as corresponding to individual behaviour in financial markets — say 3, leading to $\rho\gamma \sim 7\%$, way too high.

In sum, it is impossible to view the traditional utilitarian approach described above as a correct interpretation of "treating future generations equally" — which is exactly what our S.W.F. tried to embody, by using $\beta = 0$.

1.2. **Relative utilitarianism.** Since the traditional utilitarian approach failed so badly, let us now look at Relative Utilitarianism, introduced in Dhillon and Mertens (1999).

The axiomatization consists basically of applying Arrow's axioms to preferences over lotteries, after "surgically removing" from them everything which is clearly objectionable — i.e., which anyone would expect a good S.W.F. to violate: the implications that variations in the intensity of preference of x over y don't matter.

After this removal, one can add anonymity (implying here also that individuals of different generations are treated equally) to obtain an axiomatization of a unique S.W.F., relative utilitarianism, that takes for each individual's preferences the unique von Neumann-Morgenstern representation having minimum 0 and maximum 1 over the feasible set, and sums those to obtain a representative of the corresponding social preferences.

It is stressed in that paper that this dependence on the feasible set implies that in actual use it should be applied with some universal feasible set, to quote "all alternatives that are feasible and just".

In particular, in the present situation, the feasible set should consist not only of the "baseline" and the different proposals under consideration, but of all policies and policy-changes that might be considered by any agency of the government.

In (exogenous) growth models, γ, the "technological rate of growth", is unaffected by any policy variable: policies affect only the height of the growth path — i.e. multiply c_t by some constant.

Thus, we have to normalize $u(c_t)$ between some $(1-\eta)c_0e^{\gamma t}$ and $(1+\zeta)c_0e^{\gamma t}$

$$v(c_0e^{\gamma t}+\delta c_t)=\frac{u(c_0e^{\gamma t}+\delta c_t)}{u\big((1+\zeta)c_0e^{\gamma t}\big)-u\big((1-\eta)c_0e^{\gamma t}\big)}$$

i.e., we divide by

$$\frac{e^{(1-\rho)\gamma t}}{c_0^\alpha}\left[\frac{-1}{(1+\zeta)^{(\rho-1)}}+\frac{1}{(1-\eta)^{(\rho-1)}}\right]\quad\sim e^{(1-\rho)\,\gamma\, t}$$

So the variation of our S.W.F. becomes

$$\sum e^{(\rho-1)\,\gamma\, t}\delta C_t u'\left(c_0\, e^{\gamma\, t}\right)=\sum e^{(\rho-1)\,\gamma\, t}e^{-\rho\gamma\, t}\delta C_t=\sum e^{-\gamma\, t}\delta C_t$$

I.e., the previous discount rate of $\rho\gamma$ becomes now simply γ, $2-2\frac{1}{2}$ %, right in the ball-park of "positive and $< 3\%$".

Clearly, in the above economy, there is no reason to save anything: in every period, all individuals are unanimous in wanting to disinvest as much as possible. So there is no reason for (non-negative) growth either. Further, in a real model where there is growth and savings, there is also an interest rate — and individuals would smooth the shock over their lifetime using the going interest rate: so one would expect the result to be driven back to the interest rate, to a large extent at least.

We conjecture nevertheless that the result does remain valid in the following much more general framework.

2. The model

We use a general-equilibrium model, cast in an exogenous growth framework.

2.1. **The Consumption Sector.**

2.1.1. *Population Dynamics.* Time is continuous. There are several types of individuals. An individual of type τ dies at age T_τ. We could specify fully the dynamics by non-decreasing right-continuous functions $F_{\tau,\tau'}$, defined on $[0,\tau]$ s.t. $F_{\tau,\tau'}(s)$ is the number of children of type τ' an individual of type τ has at age s, and then specifying that we are looking at the corresponding invariant distribution.[5] But as long as we are not introducing bequest motives or the like, it is only this distribution that

[5]We keep everything deterministic here, just to avoid having to discuss irrelevant insurance markets for idiosyncratic risks.

matters. It is such that, at time t, the number of individuals of type τ in the age-group $(s, s+ds)$ $(0 \le s \le T_\tau)$ is given by $N_\tau e^{\nu(t-s)} ds$.

2.1.2. *Preferences and Endowments.* At each instant of his life, an individual of type τ consumes non-negative quantities of n goods and allocates fractions of his time to l types of labour (hence the sum of those fractions is always ≤ 1).

His preferences over (say integrable) life-time consumption-streams in $\mathbb{R}^{n+l}$ are represented by a utility function U^τ (concave, differentiable, increasing in the goods, decreasing in labour).[6]

For balanced growth to be at all possible, we assume U^τ to be homogeneous, say of degree $1-\rho^\tau$, in the n first coordinates (consumption-stream of the goods).

Endowments are 0 — except for the "endowment of leisure" of 24h/day, which is represented by the constraint on the consumption set that the sum of fractions of time devoted to all possible occupations is always ≤ 1.

In what follows, perturbations of endowments will only refer to perturbations in the goods, $\omega_i = (\delta\omega)_i$ for $i = 1 \dots n$.

2.2. **Production.** There are k capital goods K^i $(i = 1 \dots k)$, each with its corresponding investment good I^i, depreciation rate δ_i, and capital-accumulation equation $\frac{dK_t^i}{dt} = I_t^i - \delta_i K_t^i$.

Instantaneous production is described by a closed convex cone $Y \subset \mathbb{R}_+^{k+l+n+k}$, describing feasible production plans transforming $k+l$ inputs (k of capital and l types of (effective) labour, z_i) into n consumption goods and k investment goods.

Individuals supply labour (time) to the firms, or, more precisely they translate time worked in a given occupation, $l_i(s)$ (which is the argument of their utility function), into effective labour — the input in Y — (which is marketed), so that the amount z_i of effective labour of type i received at time t by the firm from an individual of type τ and age s is

$$z_i(t) = e^{\gamma t} \varepsilon_i^\tau(s)\, l_i(s)$$

where $\varepsilon_i^\tau(s)$ is the life-cycle 'productivity' (in occupation i) of an individual of type τ and age s,[7] and where γ is (labour-enhancing) technological progress.

2.3. **Markets.** Individuals face a life-time budget constraint, and markets are perfectly competitive.

[6] Index consumption streams by age, in $[0, T_\tau]$, so all individuals of the same type have the same consumption set and utility function, independently of their birth-date.

[7] E.g., in the standard OLG models, ε would be 1 during the first half of life and 0 after.

3. Evaluating Future Benefits

3.1. **The Feasible Set.** To formulate the social welfare function we need the feasible set, and the simplistic formulation used in section 1.2 is no longer adequate (multiple goods, types, etc). Ideally this should be defined in the space of policies, but since one of our aims is to prove that our result is completely independent of it, we will define it as the corresponding set in the space of (final — i.e., after all equilibrium readjustments) allocations.

Let $\Phi\colon t \mapsto [\phi_1(t), \phi_2(t)]$ be a uniformly bounded set of functions of the form $\phi_1(t) \equiv \left(\left(l_j^\tau(s)\right)_{j=1}^l\right)_\tau$, $\phi_2(t) \equiv ((c_i^\tau(s))_{i=1}^n)_\tau$, thought of as the allocation at age s of an individual of type τ born at time t. Assume that Φ is translation invariant, i.e., $\phi \in \Phi \Rightarrow T^h(\phi) : t \mapsto \phi(t+h) \in \Phi$. Assume also that $\sum_{j=1}^l l_j^\tau(s)$ is uniformly bounded away from 1 (over all $\phi_1 \in \Phi$, all τ, all birth-dates t and ages s), and that similarly $\inf \phi_2$ is bounded away from 0.[8]

Define $F = \{t \mapsto (\phi_1(t), e^{\gamma t}\phi_2(t)) \mid (\phi_1, \phi_2) \in \Phi\}$ to be the feasible set.

So, the translation invariance, with the factor $e^{\gamma t}$, are there to capture the previous idea that policies affect only the height of the growth path — while leaving the geometry completely arbitrary in all other respects.

Now, for the social welfare function to be well-defined in the neighbourhood of the status-quo point (which will be a balanced growth path), remark that the '0–1'-normalisation of utilities on the feasible set is an irrelevant convention, nothing changes by adding a constant to an individual's utility function. We will choose therefore this constant such as to have utility 0 at the status-quo.

3.2. **The distribution of costs and benefits.** We associate with any policy-change a corresponding perturbation of individual endowments of consumption goods over time. We want to evaluate the corresponding variation of social welfare, after individuals trade to a new equilibrium.

Let $\delta\omega_y^\tau(t)$ be a perturbation of consumption (vector) of individual of type τ who was born at time y. It is clear that by just taking on a given day consumption away from the old and giving it to the young one could achieve artificial welfare increases: indeed, since their utilities at birth are weighted equally in the social welfare function, their own time-impatience will have for effect that the benefits of the transfer to the young is much greater than the disutility to the old.

Thus our variation of welfare will in general depend on the whole perturbation of endowments, not only on the aggregate.

One faces this problem as soon as one uses individual weights in the social welfare function for which the given equilibrium path is not optimal.

[8] Or just that $\inf_\phi U_\tau(\phi_{1,\tau}(0), \phi_{2,\tau}(0)) > U_\tau(0,0)$ for all τ.

One may want to approach this problem (problem — in as much one wants to adhere to this idea of intergenerationally fair social welfare function —) in at least two different ways.

The first would be to argue that individual preferences must be respected — by the model, by the "state" —; that if somebody goes to the casino and loses all his money (or robs a bank and gets to jail), it would break all incentives for the state to bail him out afterwards — and similarly if he exhibits such time-preferences as to spend all his money in his youth. This is roughly the point of view of the present model, and the reason for insisting that the social welfare function be formulated in the terms of the individuals' *expected utilities at birth*.

In this vein, one would want to reformulate individual utilities in the model to encompass both a "bequest motive" (e.g. in the form of a utility depending recursively on that of one's children too), and some form of altruism vis-à-vis of one's parents: both effects tend to lengthen individuals time horizon, i.e., to decrease their impatience, hence probably to reduce claims of inadequacy of this approach. And de facto, it seems that in traditional societies those 2 aspects prevented any form of gross injustice.

If the above approach is not sufficiently adequate, – or anyway, since it is not a solution in principle –, one might want to take a more paternalistic approach, and argue that, in the same way the state has to protect future generations against short-sightedness of the current generation, it also has to protect each individual against the consequences in his old age of his own short-sightedness when young. The various policy instruments used to achieve this (in the extreme, some forms of forced savings, etc.) should then be incorporated into the model, to get rid of the problem — i.e., to reduce in effect to the same model, but where individual time-preferences have been corrected to fit with the social welfare function.

It is clear that such things require much more work, and thought, and lead us astray from our subject — the discount rate for cost-benefit analysis. Hence, to be able to pursue our analysis, in a way unaffected by this problem, we will assume that somehow this problem is being taken care of by current policy, and that the aggregate perturbation $\delta\Omega(t)$ gets distributed in a fixed (i.e., time- and commodity-independent) way across age groups and types. So the variation in welfare will be a function just of the aggregate $\delta\Omega$.

Let thus $\vartheta^\tau(s)$ be some integrable function, the distribution of endowments, with $\vartheta^\tau(s) = 0$ for $s < 0$ and $s > T_\tau$, and with $\sum_\tau \int_{-\infty}^{+\infty} \vartheta^\tau(s)ds = 1$. Then, a perturbation of consumption (vector) of individual of type τ who was born at time y is related to the aggregate perturbation in the following way,

$$\delta\omega_y^\tau(t) = \vartheta^\tau(t-y)\frac{\delta\Omega\,(t)}{N^\tau e^{\nu y}}$$

4. The Main Statement

Taking a balanced growth path as status-quo point, we can now view the social welfare function W as a real-valued function of aggregate endowment perturbations $\Omega(t)$. It is true that as a result of such a perturbation several equilibria might emerge. Out of these we choose the one closest to the initial stable growth path in terms of the distance $\sum_i \int |\ln p_i(t) - \ln p_i^{\Omega}(t)|dt$, where $p(t)$ is the price vector at time t prevailing at the initial equilibrium and $p^{\Omega}(t)$ is the price vector of a perturbed economy. Provided the perturbed economy has an equilibrium, W is a well defined function of aggregate endowment perturbations. We want to compute the differential of this map at 0 (the status-quo point) for evaluating the effect of small perturbations, and to prove that whenever it exists it is of the form $\int \langle q, \Omega(t)\rangle e^{-\gamma t}dt$ for some $q \in \mathbb{R}^n$ — i.e., that the discount rate used equals γ.

To make this last statement as strong as possible, we need to use the weakest notion of differential, that of Gateaux-differential.[9] We also need to specify the space of perturbations and its topology; we will use the space K (defined below), because that way the statement implies the same statement for about any other space of perturbations, since K embeds continuously as a dense subspace in about any other space.

We follow Gelfand and Shilov (1959) in defining K and the space K^* of continuous linear functionals on K (i.e., generalized functions).

Definition 1. *K is the space of infinitely differentiable functions with compact support, and a sequence of functions $\varphi_n \in K$ converges to zero if $\exists h \in \mathbb{R}$: $|x| \geq h \Longrightarrow \varphi_n(x) = 0$ for all n, and φ_n and all its successive derivatives converge uniformly to zero.*

K^* is the space of linear functionals ψ on K s.t. $\psi(\varphi_n) \to 0$ whenever $\varphi_n \to 0$ in K.

The economic meaning of $\Omega \in K^n$ is that the endowments are perturbed only over a bounded interval of time.

Now we can formulate the statement precisely:[10]

Theorem 2. *If the map from $\Omega \in K^n$ to W is Gateaux-differentiable at $\Omega = 0$, then its differential equals $\int \langle q, \Omega(t)\rangle e^{-\gamma t}dt$ for some $q \in \mathbb{R}^n$.*

This implies that the discount rate is the growth rate of real per-capita consumption, γ.

[9] I.e., W is differentiable at $x = 0$ along every straight line $x\Omega$ ($x \in \mathbb{R}$) and those directional derivatives form a continuous linear functional in Ω — this is then the differential.

[10] A similar result could be shown in the traditional set-up, provided (the multi-dimensional analog of) risk-aversion, ρ_τ, is independent of the type τ — giving then a discount factor of $\rho\gamma$, and hence showing the robustness of our conclusions from the mini-model in the introduction.

Proof. (sketch thereof.) By definition of Gateaux differential,

$$\begin{aligned} DW(\Omega) &= \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon W(\Omega)}{\varepsilon}, \\ \delta_\varepsilon W(\Omega) &= W(\Omega + \varepsilon\delta\Omega) - W(\Omega) \end{aligned}$$

By assumption,

$$DW(0) = \langle \delta\Omega, \mu \rangle$$

where $\mu \in (K^*)^n$, i.e., the differential at $\Omega = 0$ is linear in $\delta\Omega$. It is sufficient for what follows to describe $\delta_\varepsilon W(\Omega)$, i.e., the change in the social welfare function caused by the perturbation of endowments, which amounts to subtracting a constant from each agent's utility, the utility on the baseline, thus the criterion of interest is the difference δW.

To construct δW let us first normalise life-time utilities. Recall the feasible set is $F = \{t \mapsto (\phi_1(t), e^{\gamma t}\phi_2(t)) \mid (\phi_1, \phi_2) \in \Phi\}$, where Φ is translation invariant. Consider an individual of type τ who was born at time t. His normalised utility, $U_t^{*\tau}$ (that enters the social welfare function) equals

$$\frac{U^\tau(\cdot)}{\sup_{\phi\in\Phi} U^\tau(\phi_1(t), e^{\gamma t}\phi_2(t)) - \inf_{\phi\in\Phi} U^\tau(\phi_1(t), e^{\gamma t}\phi_2(t))}.$$

By homogeneity of the utility function U^τ of degree $(1-\rho^\tau)$ and by translation-invariance, the denominator can be represented as $e^{(1-\rho^\tau)\gamma t}/w^\tau$, with

$$w^\tau \equiv \left(\sup_{\phi\in\Phi} U^\tau(\phi_1(0), \phi_2(0)) - \inf_{\phi\in\Phi} U^\tau(\phi_1(0), \phi_2(0)) \right)^{-1}$$

Therefore, we have

$$U_t^{*\tau} = e^{(\rho^\tau - 1)\gamma t} w^\tau U^\tau$$

We, therefore, can write the social welfare function in the following form

$$\delta W(\cdot) \equiv \int_{-\infty}^{\infty} \sum_\tau N_t^\tau (\delta U_t^{*\tau})\, dt \tag{6}$$

We require the consumption and time allocation plans to be consistent with an equilibrium arising on a stable growth path. Let us define $V_t^\tau : \Omega_t \mapsto \mathbb{R}$ to be a map from (consumption) endowments into the level of utility of individual of type τ at time t, the utility that is attained in an equilibrium after the endowments have been traded and the prices have reacted accordingly.

$$\begin{aligned} W(\Omega) &= \sum_\tau w^\tau W^\tau(\Omega) \\ W^\tau(\Omega) &\equiv \int_{-\infty}^{\infty} N_t^\tau e^{(\rho^\tau - 1)\gamma t} V_t^\tau(\Omega_t)\, dt \end{aligned}$$

Consider now the perturbation $\delta\tilde{\Omega}_t$, where

$$\delta\tilde{\Omega}_{t+h} = e^{(\gamma+\nu)h}\delta\Omega_t \tag{7}$$

By Lemma 4 the corresponding "response" of the system is obviously obtained from the response to $\delta\Omega_t$ by delaying everything by h, multiplying all aggregate quantities of goods by $e^{(\gamma+\nu)h}$, and all per-capita quantities by $e^{\gamma h}$, while a factor 1 is used for good zero, and correspondingly for prices.

Hence, for utilities, by their homogeneity property in goods $1,..,n$,

$$V^\tau_{t+h}\left(\delta\tilde{\Omega}\right) = e^{(1-\rho^\tau)\gamma h} V^\tau_t\left(\delta\Omega\right)$$

and, in particular, when $\delta\tilde{\Omega} = \delta\Omega = 0$,

$$V^\tau_{t+h}\left(\delta\tilde{\Omega}\right) - V^\tau_{t+h}\left(0\right) = e^{(1-\rho^\tau)\gamma h}\left(V^\tau_t\left(\delta\Omega\right) - V^\tau_t\left(0\right)\right)$$

Therefore,

$$\begin{aligned} W^\tau\left(\delta\tilde{\Omega}\right) - W^\tau\left(0\right) &= \int_{-\infty}^{+\infty} N^\tau_0 e^{\nu(t+h)} e^{(\rho^\tau-1)\gamma(t+h)}\left[V^\tau_{t+h}\left(\delta\tilde{\Omega}\right) - V^\tau_{t+h}\left(0\right)\right] d\left(t+h\right) \\ &= \int_{-\infty}^{+\infty} N^\tau_0 e^{\nu(t+h)+(1-\rho^\tau)\gamma h} e^{(\rho^\tau-1)\gamma(t+h)}\left[V^\tau_t\left(\delta\Omega\right) - V^\tau_t\left(0\right)\right] dt \\ &= e^{\nu h}\int_{-\infty}^{+\infty} N^\tau_t e^{(\rho^\tau-1)\gamma t}\left[V^\tau_t\left(\delta\Omega\right) - V^\tau_t\left(0\right)\right] dt = e^{\nu h}\left[W^\tau\left(\delta\Omega\right) - W^\tau\left(0\right)\right] \end{aligned}$$

(i.e., the factor $\left(1-\rho^\tau\right)\gamma$ drops out). As a consequence, the total change in welfare is

$$\begin{aligned} W\left(\delta\tilde{\Omega}\right) - W\left(0\right) &= \sum_\tau w^\tau\left(W^\tau\left(\delta\tilde{\Omega}\right) - W^\tau\left(0\right)\right) \\ &= e^{\nu h}\left[W\left(\delta\Omega\right) - W\left(0\right)\right] \end{aligned}$$

Therefore, applying the definition of the derivative, we get

$$\begin{aligned} \langle\delta\tilde{\Omega},\mu\rangle &= \lim_{\varepsilon\to 0}\frac{W\left(\varepsilon\delta\tilde{\Omega}\right) - W\left(0\right)}{\varepsilon} = \\ &= e^{\nu h}\lim_{\varepsilon\to 0}\frac{W\left(\varepsilon\delta\Omega\right) - W\left(0\right)}{\varepsilon} = e^{\nu h}\langle\delta\Omega,\mu\rangle \end{aligned} \tag{8}$$

Define $T_h : t \longmapsto \xi\left(t+h\right)$. By (7)

$$\delta\tilde{\Omega} = e^{(\gamma+\nu)h} T_h \delta\Omega$$

Combining with (8), we get

$$e^{(\gamma+\nu)h}\langle T_h\left(\delta\Omega\right),\mu\rangle = \langle\delta\tilde{\Omega},\mu\rangle = e^{\nu h}\langle\delta\Omega,\mu\rangle$$

and, due to arbitrariness of h and $\delta\Omega$, the following condition holds for all $h \in \mathbb{R}$ and all perturbations $\delta\Omega \in K^n$,

$$\langle\delta\Omega - e^{\gamma h} T_h\left(\delta\Omega\right),\mu\rangle = 0$$

for $\mu \in \left(K^*\right)^n$. Dividing by h and taking limit as $h \to 0$, we get

$$\langle\gamma\delta\Omega - \left(\delta\Omega\right)',\mu\rangle = 0$$

Rearranging and using the definition of a derivative of a generalized function,

$$\left\langle \mu', f \right\rangle = -\left\langle \mu, f' \right\rangle, \ f \in K, \ \mu \in K^*$$

results in

$$\left\langle \gamma\mu + \mu', \delta\Omega \right\rangle = 0, \ \forall \delta\Omega \in K^n$$

so we have to solve a differential equation $\gamma\mu + \mu' = 0$, which, by Lemma (3) gives only the solutions of the form $\mu = q \otimes e^{-\gamma t}$ for some $q \in \mathbb{R}^n$, therefore,

$$DW = e^{-\gamma t} \langle q, \delta\Omega \rangle = \int_{-\infty}^{+\infty} e^{-\gamma t} \langle q, \delta\Omega_t \rangle dt, \ \forall \delta\Omega \in K^n.$$

□

Lemma 3. *Consider a homogeneous differential equation of the form*

$$y' = \lambda y, \tag{9}$$

for a given constant λ. Then every solution of that system in the class K^* of generalized functions is of the form

$$y = Ce^{\lambda t}, \ C \in \mathbb{R}$$

i.e., is a "classical solution".

Proof. From (9) we have that for any $\varphi \in K$, $\langle y', \varphi \rangle = \lambda \langle y, \varphi \rangle$; by definition of the derivative of a generalized function this implies $\langle y, -\varphi' \rangle = \lambda \langle y, \varphi \rangle$, and so $\langle y, \lambda\varphi + \varphi' \rangle = 0$. Let $K_\lambda = \{\psi \in K | \int_{-\infty}^{\infty} e^{\lambda t} \psi(t)\, dt = 0\}$. Observe that $\forall \psi \in K_\lambda \exists \varphi \in K : \psi = \lambda\varphi + \varphi'$: take $\varphi(t) = \int_{-\infty}^{t} e^{\lambda s} \psi(s) ds$ (the converse is true as well, but we won't use it). So $y = 0$ on K_λ.

Note that any $\varphi \in K$ can be represented in the form $\varphi = \psi + c\varphi_0$, where $\psi \in K_\lambda$, c is a constant and $\varphi_0 \in K \setminus K_\lambda$ is fixed: choose $c = \frac{\int_{-\infty}^{\infty} e^{\lambda t} \varphi(t) dt}{\int_{-\infty}^{\infty} e^{\lambda t} \varphi_0(t) dt}$, then $\psi = \varphi - c\varphi_0 \in K_\lambda$.

Thus $\langle y, \varphi \rangle = c \langle y, \varphi_0 \rangle$, so, letting the constant $C = \frac{\langle y, \varphi_0 \rangle}{\int_{-\infty}^{\infty} e^{\lambda t} \varphi_0(t) dt}$, we get $\langle y, \varphi \rangle = C \int_{-\infty}^{\infty} e^{\lambda t} \varphi(t)\, dt$, $\forall \varphi \in K$, i.e., $y = Ce^{\lambda t}$. □

Lemma 4. *Postponing all variables — including endowments — by a duration of h, and multiplying all non-labour individual quantities (endowments of goods, allocations of goods) by $\exp(\gamma h)$, while multiplying all aggregate quantities of such goods (capital, investment, total output of consumer-goods) by $\exp((\gamma + \nu)h)$, and the aggregate quantities of population and labour by $\exp(\nu h)$ is an automorphism of the model:*

- *it maps feasible production plans in a 1-to-1 way onto feasible production plans.*
- *it maps the preferences of each consumer between different consumption bundles (consumption=goods+labour) to the corresponding preferences of his image, born time h later. And his initial endowment is mapped as well to the initial endowment of his image.*

Proof. The second part is obvious: for the endowments, it holds by definition of the transformation, and for the preferences, it follows because all agents of the same type have the same utility function over their consumption set, which is homogeneous in the goods: so multiplying the "goods-component" by a constant just multiplies to whole utility function by a constant, and hence doesn't change preferences.

For the first part, note that the capital-accumulation equations are not affected, since they are linear and homogeneous in the aggregate goods. Remains to check for the "instantaneous production cone" Y that it too is preserved by the transformation. Assume thus for some t a vector (y, L) in Y_t — i.e., $(y, \exp(\gamma t)L)$ in Y — before the transformation — where the coordinates of y are all aggregate consumption and investment outputs and capital inputs, and those of L are the aggregate labour input. Then, after the transformation, this vector becomes $[\exp((\gamma + \nu)h)\, y, \exp(\nu h)L]$, and we have to show that this belongs to Y_{t+h} — i.e., that $[\exp((\gamma + \nu)h)y, \exp(\gamma(t + h)) \exp(\nu h)L]$ belongs to Y. Since this vector equals $\exp((\gamma + \nu)h)\, [y, \exp(\gamma t)L]$, this follows straight from the fact that Y is a cone. □

5. Additional Questions

There first remains to improve on the statement and the proof of the conjecture: e.g., the actual proof shows much more. Also, the approach can be extended to be directly applicable to the study of the marginal welfare impact of small policy variations, without having first to translate them into an equivalent flow of consumption goods, thus, enabling analytical evaluation of policies in this class of models.

The next step is to prove that the main statement is non-vacuous, as it would be, e.g., in case of indeterminacy.

Going much further in that direction, a third point would be to address the vague conjecture that, in fact, the balanced growth paths at which the social welfare function is differentiable are exactly the stable[11] balanced growth paths (at least in case of 0 population growth). But this would require extending much of the basic general equilibrium theory to those models, and then, even more, looking at the stability issue.

5.1. **Related literature.** We do not attempt to provide an overview of the vast literature applying and analyzing the overlapping generations model, referring instead to a survey by Kotlikoff (1998), who, in particular, stresses the importance of OLG modeling in analysing tax reform and privatizing social security; and to Erosa and Gervais

[11]This does not refer to some dynamic system; we mean here that equilibria of a slightly perturbed economy converge to those of the unperturbed economy sufficiently far away in time from the perturbation.

(2001), who compare policy implications derived from life-cycle models with those based on models with an infinitely-lived representative agent, and stress the importance of the former. The by-now classical two-period life-cycle OLG model and its policy implications are analyzed in detail in Kotlikoff (2002). In a more general set-up with age-related individual productivity, Erosa and Gervais (2002) offer an analysis of tax equivalence. In most of those models, however, there is no technological growth.

Restrictions on economic fundamentals to allow for a stable growth are summarized in King, Plosser, and Rebelo (2002), some of which are to be used in the current project, e.g., homogeneity of utility functions with respect to consumption goods, and constant returns to scale in production. Some other contributions in this vein are discussed in Arrow and Kurz (1970).

Indeterminacy is known to plague some classes of OLG models (Geanakoplos and Brown (1985)); hence the need to show that this problem is avoided in our case — in particular, making a policy change meaningful, in the sense that it generates predictable (determinate) changes in the economy.

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