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Research Article

Trees and decisions★

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Summary. The traditional model of sequential decision making, for instance, in extensive form games, is a tree. Most texts define a tree as a connected directed graph without loops and a distinguished node, called the root. But an abstract graph is not a domain for decision theory. Decision theory perceives of acts as functions from states to consequences. Sequential decisions, accordingly, get conceptualized by mappings from sets of states to sets of consequences. Thus, the question arises whether a natural definition of a tree can be given, where nodes are sets of states. We show that, indeed, trees can be defined as specific collections of sets. Without loss of generality the elements of these sets can be interpreted as representing plays. Therefore, the elements can serve as states and consequences at the same time.

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1 Introduction

Traditional decision theory under uncertainty is a static theory. The objects are lotteries over consequences (von Neumann and Morgenstern [20]), functions from states to consequences (Savage [23]), or functions from states to lotteries over consequences (Anscombe and Aumann [3]). In a sequential context possible states are restricted and probabilities are updated by Bayes' rule. Otherwise, each consecutive decision is treated like a static decision problem.

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This rules out many features that appear relevant to sequential decisions: preference for flexibility (Kreps [18]), temptation and self-control (Gül and Pesendorfer [13]), or unforeseen contingencies (Dekel, Lipman, and Rustichini [9]). Such issues can be addressed by preferences over subsets, rather than elements, of the space of possible consequences.

The best-known example is that of making a reservation at a restaurant. "Imagine that the only way that restaurants vary is in the menu of meals which they will serve. The individual is assumed to know the menus at all restaurants that he might select. Eventually, the individual will choose a meal, but his initial choice is of a restaurant/menu from which he will later choose his meal." (Kreps [18], p. 565)

There is another traditional domain for sequential decision making: the *tree* of the extensive form representation of a game (Kuhn [19]). Trees serve as a transparent graphical model of how consecutive decisions refine the selection among possible outcomes.[1] Yet, trees are usually defined as directed connected graphs without loops and with a distinguished node (the "root"). This is formally at variance with sequential decisions over (increasingly smaller) sets of outcomes. The issue arises whether arbitrary trees can be recast into collections of subsets of some underlying space, thereby making them an adequate domain for sequential decision theory.

This paper addresses that issue. Starting from the order-theoretic concept of a tree (e.g. the traditional graphical model) we show that it can be represented as a set of sets with a particular structure. To interpret elements of these sets as consequences/outcomes requires that maximal chains of sets (called *plays*) identify elements that all sets in the chain contain. In the restaurant example, a menu needs to correspond to a collection of meals, because meals are ultimately consumed.

Every tree has a set representation that meets this requirement. Characterizing this leads to a definition of set trees that is appropriate for sequential decision theory. For these set trees a node is like an event in probability theory, i.e. a set of states. Furthermore, elements of these sets/nodes correspond to plays, thereby providing a framework for sequential decision theory.

The present paper is a first step towards a general definition of an extensive form as a framework for truly sequential decision theories. This is why we start with utmost generality. For instance, in von Neumann games ([20], Chapter II) the number of predecessors of every node in an information set is required to be the same. In our set-up immediate predecessors may not even exist, and the number of predecessors may not be finite. This allows us to consider examples as exotic as decision problems in continuous time ("differential games"), long cheap-talk games à la Aumann and Hart [4], (infinitely) repeated games, bilateral bargaining games à la Rubinstein [22], and stochastic games à la Shapley [24] (see Sect. 2.2).

1.1 Overview

Every tree can be represented as a collection of (sub)sets (of some underlying set) with a particular structure (Proposition 1). But in a set-theoretic environment this

[1] As already highlighted by von Neumann and Morgenstern ([20], p. 65), trees are closely related to collections of subsets of an underlying space of consequences or outcomes.

structure can be “cleaned” without affecting the properties of the trees. We “clean” it in three steps, where each step corresponds to adding structure that enables increasingly specific interpretations of the tree.

In Section 2 characterizing set representations of trees (Proposition 2) reveals that the set-theoretic analog of the defining order-theoretic structure can be modified such that *unordered* nodes correspond to *disjoint* sets. Every tree has a set representation that satisfies this “Trivial Intersection”-property. Next, ruling out trivial structures leads to “decision trees.” Characterizing set representations of these (Proposition 3) shows that, under Trivial Intersection, the set-theoretic analog of the defining order-theoretic property of decision trees is strengthened, to “Separability”. Every decision tree has a set representation that satisfies the two strong properties (Trivial Intersection and Separability). Moreover, every decision tree has a “canonical” set representation, where the elements of the underlying set are *plays* (Theorem 1), that satisfies those properties (Corollary 1). This yields the first milestone: “set trees.”

When can the elements of the underlying set be perceived as representing plays, as suggested by Theorem 1? Section 3 shows that this requires the underlying set to be neither too large nor too small.

In a “reduced form” (Proposition 4), where redundancies in the underlying set are eliminated, the elements of the underlying set map *one-to-one* into plays (Proposition 5). In fact, Trivial Intersection is equivalent to this property. A set tree already is in reduced form if, roughly, no element of the underlying set can be dropped: if the set tree is “irreducible.” This is equivalent to the elements of the underlying set in the reduced form being the singleton sets of the originally underlying set (Proposition 6). Defining a “proper” order isomorphism as an order isomorphism that preserves the “strong” properties (Sect. 3.3) yields a characterization: a collection of sets is a set tree if and only if it is properly isomorphic to its reduced form and the latter is an irreducible set tree (Theorem 2). If Irreducibility of the reduced form implies Irreducibility of the original set tree, the two are “doubly isomorphic.” Irreducible set trees are precisely those that are doubly isomorphic to their reduced forms (Proposition 7). This clarifies when the underlying set is not too large.

It may still be too small. “Boundedness” of a set tree (Sect. 3.4) ensures that the elements of the underlying set map *onto* plays. Achieving this involves, possibly, enlarging the underlying set. Irreducible set trees are precisely those, where elements can be added, so that every play is represented by a distinct element of the underlying set (Proposition 8). Hence, a set tree is bounded if and only if the elements of the underlying set in the reduced form represent *all* plays (Proposition 9). This yields the second milestone: “game trees,” defined as bounded irreducible set trees.

Section 4 characterizes game trees as those for which there is a bijection between elements of the underlying set and plays; equivalently, they are precisely those decision trees that are their own “canonical” set representation by plays (Theorem 3). Hence, there is no loss of generality in assuming Boundedness and Irreducibility. Yet, Boundedness does not imply that the singletons from the underlying set belong to the set of nodes. But adding the singletons as nodes to the tree does not change any essential features of the tree, provided it is a game tree (Proposition 10). This yields the third and last milestone: "complete game trees." A set tree is a complete game tree if and only if it is irreducible and every play has a minimum (Proposition 11). An even simpler characterization is obtained if all plays are finite (Proposition 12).

As an application, Section 5 shows that extensive decision problems (extensive forms) can be consistently defined with game trees (Theorem 4). The familiar strategy notions translate smoothly to this general framework. Even large, exotic decision problems, like a differential game, have a representation in these terms. Interestingly, though, such a definition automatically rules out problems related to absent-mindedness (Proposition 13). Section 6 discusses directions for further research.

Proofs of major results are included in the text; proofs of selected Lemmata are relegated to the Appendix. Straightforward proofs are omitted.

2 Set representations

A *preordered set* is a pair $(N, \geq)$ consisting of a nonempty set N and a reflexive and transitive binary relation $\geq$ on N. A preordered set $(N, \geq)$ for which the relation $\geq$ is antisymmetric is a (partially) *ordered set* (or a *poset*). A V*-poset* is a poset $(M, \supseteq)$ where M is a collection of nonempty subsets of a given set V and $\supseteq$ is set inclusion. A nonempty subset $c \subseteq N$ of a preordered set $(N, \geq)$ is a *chain* if for all $x, y \in c$ either $x \geq y$ or $y \geq x$ (or both), i.e. if the induced preorder on c is complete.

Given a preordered set $(N, \geq)$ and an element $x \in N$ define the *up-set* (or *order filter*) $\uparrow x$ and the *down-set* (or *order ideal*) $\downarrow x$ by

$$\uparrow x = \{y \in N \,|y \geq x\} \text{ and } \downarrow x = \{y \in N \,|x \geq y\}. \tag{1}$$

Intuitively, in a decision-theoretic context, the up-set corresponds to the "past" and the down-set to the "future." Denote by $\downarrow N = \{\downarrow x \,|x \in N\} \subseteq 2^N$ the set of all down-sets of $(N, \geq)$. An *order isomorphism* between two preordered sets $(N_1, \geq_1)$ and $(N_2, \geq_2)$ is a bijection $\varphi : N_1 \to N_2$ such that

$$x \geq_1 y \text{ if and only if } \varphi(x) \geq_2 \varphi(y) \tag{2}$$

for all $x, y \in N_1$. This last property is referred to as "order embedding." Two order-isomorphic preordered sets can be regarded as identical for all practical purposes.

Remark 1 If $(N_1, \geq_1)$ is a *poset*, $(N_2, \geq_2)$ a preordered set, and $\varphi : N_1 \to N_2$ an order embedding function, then φ is necessarily injective (one-to-one). For, given $x, y \in N_1$ such that $\varphi(x) = \varphi(y)$, reflexivity of $\geq_2$ implies $\varphi(x) \geq_2 \varphi(y)$ and $\varphi(y) \geq_2 \varphi(x)$ and hence $x \geq_1 y$ and $y \geq_1 x$ (by the "if"-part of (2)), together implying $x = y$ (by antisymmetry for $\geq_1$). In particular, any order-embedding *surjection* (onto function) between two posets is an order isomorphism.

Say that a preordered set $(N, \geq)$ *admits a set representation* if there is an order isomorphism between $(N, \geq)$ and a V-poset $(M, \supseteq)$.

Proposition 1 *A preordered set* $(N, \geq)$ *admits a set representation if and only if it is a poset.*

Proof. "if:" Suppose $(N, \geq)$ is a poset. Then $\varphi : N \to \downarrow N$ given by $\varphi(x) = \downarrow x$ is onto by construction. Let $x, y \in N$ and $y \geq x$. Consider any $z \in \downarrow x$. By transitivity $y \geq x \geq z$ implies $z \in \downarrow y$, so $\varphi(y) \supseteq \varphi(x)$. Conversely, let $x, y \in N$ and $\varphi(y) \supseteq \varphi(x)$. Then $x \in \downarrow x = \varphi(x) \subseteq \varphi(y) = \downarrow y$ implies $y \geq x$. Thus, $y \geq x \Leftrightarrow \varphi(y) \supseteq \varphi(x)$ shows that the surjection φ is order embedding, hence an order isomorphism by Remark 1.

"only if:" Let $(N, \geq)$ be a preordered set which admits a set representation. Let $(M, \supseteq)$ be the associated poset and $\psi : N \to M$ the order isomorphism. If both $x \geq y$ and $y \geq x$ hold for some $x, y \in N$, then by (2) $\psi(x) = \psi(y) \in M$ implies $x = y$, because ψ is one-to-one. Hence, $\geq$ is antisymmetric. □

Proposition 1 identifies an order isomorphism, $\varphi(x) = \downarrow x$ for all $x \in N$, between N and $\downarrow N$. The resulting set representation is referred to as the *set representation by principal (order) ideals*. Similar results are known, for instance, for finite arbitrary ordered sets (Davey and Priestley [7], Theorem 8.19).

2.1 Trees and subtrees

Definition 1 *A* **tree** *is a poset* $(N, \geq)$ *such that* $\uparrow x$ *is a chain for all* $x \in N$. *In a tree the elements of* N *are called* **nodes**. *For nodes* $x, y \in N$ *say that* x **precedes** *(resp. follows)* y *if* $x \geq y$ *(resp.* $y \geq x$*) and* $x \neq y$. *A tree is* **rooted** *if there is a node* $x_o \in N$, *called the root, such that* $x_o \geq x$ *for all* $x \in N$.

This is the most general definition of trees in order theory. It could also be stated dually, i.e. with an element that is not followed by other nodes (a "bottom" instead of a "top") and $\downarrow x$. Here we prefer the opposite convention to be able to associate the order relation $\geq$ on an abstract tree with set inclusion $\supseteq$ on its set representation.

Remark 2 Definition 1 (stated dually) is given as an example of a poset by Davey and Priestley ([7], p. 23). However, in order theory the word "tree" is usually reserved for posets such that the sets $\uparrow x$ are (dually) well-ordered: all their subsets have a first element according to $\geq$ (see Koppelberg [16], Chapter 6). This implies that immediate successors of non-terminal nodes are well-defined. Koppelberg and Monk [17] dropped the well-ordered requirement and called the resulting concept a *pseudotree*.[2]

The tree-property is preserved by order isomorphism, i.e., if a poset is order isomorphic to a tree, then it is itself a tree. By Proposition 1, every tree $(N, \geq)$ has a set representation by principal (order) ideals, $(\downarrow N, \supseteq)$. This is called the tree's *set representation by subtrees*, as for any $x \in N$ the ordered set $(\downarrow x, \geq)$ is itself a tree. Writing the defining property of a tree more explicitly yields:

[2] Order-theoretic analysis of pseudotrees has concentrated on the various (set) Boolean algebras that they give rise to: Koppelberg and Monk [17] study the algebra of subtrees (down-sets); Baur and Heindorf [5] study the initial chain algebra (up-sets). For an order-theoretic characterization of pseudotrees, see Alós-Ferrer and Ritzberger [2].

Lemma 1 *A poset* $(N, \geq)$ *is a tree if and only if, for all* $x, y, z \in N$

$$\text{if } y \geq x \text{ and } z \geq x \text{ then } y \geq z \text{ or } z \geq y. \tag{3}$$

If this last property is translated into set-theoretic terms, two alternatives emerge: Say that a V-poset $(M, \supseteq)$ satisfies *Trivial Intersection* if, for all $a, b \in M$,

$$\text{if } a \cap b \neq \emptyset \text{ then } a \subset b \text{ or } b \subseteq a \tag{4}$$

and that it satisfies *Weak Trivial Intersection* if, for all $a, b, c \in M$,

$$\text{if } c \subseteq a \cap b \text{ then } a \subset b \text{ or } b \subseteq a. \tag{5}$$

Clearly, Trivial Intersection implies Weak Trivial Intersection. If the latter is written in terms of an abstract partial order $\geq$, property (3) is obtained. Trivial Intersection, on the other hand, cannot be translated back into arbitrary posets, since in general the intersection of two nodes might be nonempty but not contain any other node.

Proposition 2 *(a) A* V*-poset* $(M, \supseteq)$ *is a tree if and only if it satisfies Weak Trivial Intersection.*
(b) A poset $(N, \geq)$ *is a tree if and only if its set representation by principal ideals* $(\downarrow N, \supseteq)$ *satisfies Trivial Intersection.*

Proof. (a) Note that (5) is equivalent to property (3) and apply Lemma 1.
(b) "if:" If $(\downarrow N, \supseteq)$ satisfies Trivial Intersection, then it satisfies Weak Trivial Intersection and by part (a) it is a tree. By isomorphism $(N, \geq)$ is a tree.

"only if:" Let $(N, \geq)$ be a tree and let $x, y \in N$ such that $\downarrow x \cap \downarrow y \neq \emptyset$. Let $z \in \downarrow x \cap \downarrow y$. It follows that $\downarrow z \subseteq \downarrow x \cap \downarrow y$. By isomorphism and part (a), $(\downarrow N, \supseteq)$ satisfies Weak Trivial Intersection, and hence either $\downarrow x \subset \downarrow y$ or $\downarrow y \subseteq \downarrow x$. □

Any set representation of a tree is necessarily a tree (by order isomorphism), and, hence, Proposition 2(a) characterizes *all* set representations of trees. Proposition 2(b) establishes that trees can also be characterized as those posets whose set representations by principal ideals satisfy Trivial Intersection. The implication is rather natural if one observes that, for $(\downarrow N, \supseteq)$, there is no difference between an intersection of two nodes being empty and not containing any other node, i.e. Weak Trivial Intersection and Trivial Intersection coincide. Still, there may be set representations of a tree which satisfy Weak Trivial Intersection but not Trivial Intersection.

Example 1 Let $(M, \supseteq)$ be the $\{1,2,3\}$-poset with $M = \{\{1,2,3\}, \{1,2\}, \{2,3\}\}$. Trivial Intersection fails, because $\{1,2\} \cap \{2,3\} \neq \emptyset$ and neither of the nodes contains the other. But Weak Trivial Intersection holds, because its hypothesis is void, for $\{1,2\}$ and $\{2,3\}$. The set representation by subtrees is $M' = \{\{s_1, s_2, s_3\}, \{s_2\}, \{s_3\}\}$, where $s_1 = \{1,2,3\}$, $s_2 = \{1,2\}$, and $s_3 = \{2,3\}$. Now $\{s_2\} \cap \{s_3\} = \emptyset$, as nodes are elements of 2^M rather than M.

This set representation is still not satisfactory. Intuitively, one would like to remove the element 2 from $V = \{1,2,3\}$ and represent the tree as a $\{1,3\}$-poset $(M'', \supseteq)$ with $M'' = \{\{1,3\}, \{1\}, \{3\}\}$. In M' the primitives are the nodes, i.e.,

M' is formed by subsets of an underlying set (of sets). By contrast, in M'' only ultimate "outcomes" are elements of an underlying set of which the elements of M'' are subsets.

Intuitively, chains of nodes in a tree represent possible ways in which the underlying decision process may proceed. Maximal chains will then correspond to complete descriptions of all decisions taken from the beginning to the end. The classical name for such descriptions is "plays."

For a poset $(N, \geq)$ a chain $c \in 2^N$ is *maximal* if there is no $x \in N \setminus c$ such that $c \cup \{x\}$ is a chain. Every chain is contained in a maximal chain by the Hausdorff Maximality Principle (an equivalent form of the Axiom of Choice and, hence, Zorn's Lemma; see Birkhoff [6], Chapter VIII, or Hewitt and Stromberg [14], Chapter 1).

Hausdorff Maximality Principle. Let $(N, \geq)$ be a poset and $c \subseteq N$ a chain in N. Then there exists a maximal chain w in N such that $c \subseteq w$.

For a tree $(N, \geq)$ a *play* w is a maximal chain in N. Denote by W the set of all plays. Given a node $x \in N$, let $W(x) = \{w \in W \,|x \in w\}$ be the set of all plays *passing through* x.

At this point plays are *a priori* complicated objects. E.g., in a V-poset where nodes are themselves sets (subsets of V), plays are sets of subsets of V. Hence, a set of plays is a set of sets of subsets of V. Yet, there is a natural way of constructing trees in which plays, rather than being complex objects, are the primitives on which decision theory can operate.

2.2 Motivating examples

In this subsection examples are presented that illustrate the generality of the approach, and can be referred to as we proceed.

2.2.1 Bilateral bargaining

The first example corresponds to the tree of an infinite bilateral bargaining game à la Rubinstein [22]. Let $V = (\cup_{t=1}^{\infty} V_t) \cup V_\infty$, where $V_t = \{(a_\tau)_{\tau=1}^{t} | a_\tau \in [0, \delta^{\tau-1}], \forall \tau = 1, ..., t\}$ and $V_\infty = \{(a_\tau)_{\tau=1}^{\infty} | a_\tau \in [0, \delta^{\tau-1}], \forall \tau \geq 1\}$, for some fixed $\delta \in (0, 1)$, and let N be given by V together with $(\{v\})_{v \in V_\infty}$ and

$$(\{x \subseteq (\cup_{\tau=t}^{\infty} V_\tau) \cup V_\infty \,|a_\tau = b_\tau, \forall \tau = 1, ..., t, \forall a, b \in x\}, (\{v\})_{v \in V_t})_{t=1}^{\infty}.$$

Then, $(N, \supseteq)$ is a rooted tree which has infinitely many plays of infinite length (corresponding to the elements of V_∞) and infinitely many plays of finite length (the elements of V_t). While the latter end in "terminal nodes," the former do not. Payoffs can still be defined on the set of plays which is naturally one-to-one with the underlying set V.

2.2.2 Repeated games

In a repeated game a constituent one-shot game is repeated infinitely often. Representing this in extensive form gives rise to a large tree. Let I be a player set and A_i an action set for player $i \in I$, actions chosen in discrete time, $t = 0, 1, 2...$ A set representation for the tree is as follows. Let $A = \times_{i \in I} A_i$ and $V = A^\infty = \{a : \mathbb{N} \to A\}$ be the set of all possible sequences of decisions (plays) from "time" $t = 0$ to infinity. Define $N_t = \{x \subseteq V \,|a(\tau) = b(\tau), \forall \tau = 0, 1, ..., t, \forall a, b \in x\}$ for all $t \geq 1$. With $N = \{V\} \cup (\cup_{t=1}^{\infty} N_t)$ the A^∞-poset $(N, \supseteq)$ is the rooted tree of the repeated game.[3] To show that it is a tree, we verify Trivial Intersection, (4), and apply Proposition 2. Let $x, y \in N$ be such that $x \cap y \neq \emptyset$. If either x or y are the root V, the conclusion follows. If not, choose $a \in x \cap y$ and let t, τ be such that $x \in N_t$ and $y \in N_\tau$. If $t = \tau$ then $b(s) = a(s)$ for all $s = 0, 1, ..., t$ for all $b \in x \cup y$, and $x = y$ follows from the definition. If $t \neq \tau$, assume without loss of generality that $t < \tau$. Then, $b(s) = a(s)$ for all $s = 0, 1, ..., \tau$ for all $b \in y$, and $a \in x$ implies $y \subseteq x$.

The interpretation of this tree is straightforward. A node $x \in N_t \subseteq N$ is a specification of all the actions taken up to "time" t. No future actions have yet been discarded. At this node, all players simultaneously choose actions and this results in a new node $y \in N_{t+1}$ being reached. Note that in repeated games there are no "terminal nodes" on which payoff can be defined. There is still a natural one-to-one correspondence between the elements of the underlying set $V = A^\infty$ and the set of plays, so that payoffs can be defined on these.

2.2.3 Stochastic games

The trees for stochastic games (see [24]) are also covered by the present framework. Following Friedman [11], consider a stochastic game to be a situation in which an n-player normal-form game γ from a given set Γ is selected in every period $t = 0, 1, 2, ...$ according to a transition function $q(\gamma' \,|\gamma, s\,)$, where $q(\gamma' \,|\gamma, s\,)$ denotes the probability that next period's game is γ', given that in the current period game γ is played and players have chosen the strategy combination $s \in S(\gamma)$ in γ. (The set $S(\gamma)$ is the set of strategy combinations in the game γ.)

Of course, the transition function does not affect the underlying tree. The latter describes *possibilities*. In an extensive form representation of the stochastic game the transition function becomes merely a (fixed) strategy for a distinguished player, called "nature." The tree can be constructed in analogy to repeated games with the underlying set $V = \{a : \mathbb{N} \to \Gamma \times \cup_{\gamma \in \Gamma} S(\gamma) \,| a_t = (\gamma_t, s_t) \text{ with } s_t \in S(\gamma_t)\,\}$. Chance will decide at nodes where all past decisions (including the selected games) up to time t are given. Personal players decide at nodes following those, where the next game γ_{t+1} has been decided, but the strategy profile is still open.

2.2.4 Long cheap talk

In the repeated and stochastic games the plays contain at most countably many nodes. Some games, though, involve larger trees. One example is the following

[3] In this specification players, who choose simultaneously, are active at the same node; see Section 5 for details.

"long cheap-talk" game by Aumann and Hart [4]. First, in the information phase, a finite two-player normal-form game is selected at random from a given set Γ. All games in Γ have the same action sets, A_1 and A_2. Second, a talk phase goes on for infinitely many periods $1, 2, ...$ In every period each player sends a message (from a set M) to the other. Third, *after* these infinitely many periods of cheap-talk, the action phase takes place, where players simultaneously choose actions in the normal-form game $\gamma \in \Gamma$.

In the tree for this "transfinite" game the "length" of the plays is $\omega + 1$, where ω is the first infinite ordinal. It fits the present framework, though. Let $A = A_1 \times A_2$ and $(M^2)^\infty$ be the set of all sequences of elements from $M^2 = M \times M$. Denote a typical sequence $m \in (M^2)^\infty$ by $m = (m_1, ..., m_t, ...)$, where $m_t = (m_t^1, m_t^2)$. The underlying set V is given by $V = \Gamma \times (M^2)^\infty \times A$. The set of nodes, ordered by set inclusion, is the union $N = N_I \cup N_T \cup N_A$. Nodes in N_I model the information phase: $N_I = \{V\} \cup \{x(\gamma) \,|\gamma \in \Gamma\}$, where V is the root and $x(\gamma) = \{(\gamma', m, a) \in V \,|\gamma' = \gamma\}$ is the move reached after the normal-form game $\gamma \in \Gamma$ has been selected. The first messages are sent at the nodes $x(\gamma)$. The set N_T contains all the intermediate moves, i.e. nodes of the talk phase,

$$
\begin{aligned}
N_T = \{x_t(\gamma, m)|\gamma{\in}\Gamma, m{\in}(M^2)^t, t{=}1, 2, ...\} \cup \{x_\infty(\gamma, m)|\gamma{\in}\Gamma, m{\in}(M^2)^\infty\} \\
\text{where } x_t(\gamma, m){=}\{(\gamma', m', a){\in}V|\gamma'{=}\gamma \text{ and } m'_\tau{=}m_\tau \text{ for all } \tau{=}1, ..., t\} \\
\text{and } x_\infty(\gamma, m) = \{(\gamma', m', a) \in V|\gamma' = \gamma \text{ and } m' = m\}.
\end{aligned}
$$

The nodes $x_t(\gamma, m)$ are those, where t pairs of messages have already been sent and the $(t+1)$-pair has to be chosen. At the nodes $x_\infty(\gamma, m)$ infinitely many messages have already been sent, and the actions have to be taken. Finally, the set N_A contains the "terminal nodes" selected in the action phase, $N_A = \{\{v\} \,|v \in V\}$. Obviously, in this well-behaved example there is a three-sided, one-to-one correspondence between the terminal nodes in N_A, the elements of the underlying set V, and the set of plays. To complete the game, payoffs can be defined either on plays, on the "terminal nodes" in N_A, or on the underlying set V.

2.2.5 Differential games

Up to this point trees had a discrete structure. The purpose of the following is to illustrate that the present concepts go further. They include the trees of so-called "differential games" (decision problems in continuous time). The reader is referred to Dockner et al. [10] for an introduction to the topic.

Let V be the set of functions $f : \mathbb{R}_+ \to A$, where A is some given set of "actions," containing at least two elements, and let $N = \{x_t(g) \,|g \in V,\ t \in \mathbb{R}_+\}$, where $x_t(g) = \{f \in V \,|f(\tau) = g(\tau), \forall \tau \in [0, t)\}$ for any $g \in V$ and $t \in \mathbb{R}_+$. Intuitively, at each point in time $t \in \mathbb{R}_+$ a decision $a_t \in A$ is taken. The "history" of all decisions taken in the past (up to, but exclusive of, time t) is a function $f : [0, t) \to A$, i.e. $f(\tau) = a_\tau$ for all $\tau \in [0, t)$. A node at "time" t is the set of all functions which coincide with f on $[0, t)$, all possibilities still open for their values thereafter.

We claim that $(N, \supseteq)$ is a V-poset satisfying Trivial Intersection, and hence a tree by Proposition 2(a). To verify this claim, let $x_t(g)$ and $x_\tau(h)$ be two arbitrary nodes, with $g, h \in V$ and $t, \tau \in \mathbb{R}_+$. If $x_t(g) \cap x_\tau(h) \neq \emptyset$, then there is some $f \in V$ such that $f(s) = g(s)$ for all $s \in [0, t)$ and $f(s) = h(s)$ for all $s \in [0, \tau)$. If, say, $\tau \leq t$, then $g(s) = f(s) = h(s)$ for all $s \in [0, \tau)$, implying that $x_t(g) \subseteq x_\tau(h)$ as required.

There is no "point in time" where the decision between two distinct nodes $x_t(g)$ and $x_t(h)$ for which $g(\tau) = h(\tau)$ for all $\tau \in [0, t)$, but $g(t) \neq h(t)$, is actually "taken."[4] Still, the definition is operational: in each node $x_t(g)$ the decision that an agent has to take is his action, and the history up to that point is fully specified. Ultimately, a function $f \in V$ becomes a complete description of all decisions taken from the beginning to the end, i.e. it represents exactly one play. Though this equivalence is transparent in the example, it is not clear in general.

The example puts differential games into a clear game-theoretic context. The question of whether differential games, and continuous-time decision processes in general, can actually be seen as "proper games" has permeated this branch of the literature. Recall, for instance, the "chattering problem" arising when one attempts to define strategies in continuous time as limits of discrete-time strategies as the time steps shrink to zero; see e.g. Davidson and Harris [8] and Fudenberg and Levine [12].

The examples show that the framework is sufficiently general to capture even very large decision problems. In all cases the problem has a representation such that there is a one-to-one correspondence between plays and elements of the underlying set. This is not a coincidence, as the subsequent analysis shows.

2.3 Decision trees

The next result identifies key properties of the mapping $W : N \to W$ that associates to each node $x \in N$ the plays passing through x. Its proof is straightforward and omitted (notice, e.g., that part (a) follows directly from the Hausdorff Maximality Principle).

Lemma 2 *For any tree $(N, \geq)$ and all nodes $x, y \in N$:*
(a) The set $W(x)$ of plays passing through x is nonempty,
(b) if $x \geq y$ then $W(x) \supseteq W(y)$.

Any chain is a tree. But in a chain nodes that follow a given node do not represent alternatives, because there is only one play for the whole tree. To model decisions, a given node should be followed by several others that are not related by $\geq$. The idea is that $\geq$ expresses "history," while nodes not related by $\geq$ model decisions among *alternative* "histories." If every node represents a decision, the following is obtained.

Definition 2 *A* **decision tree** *is a tree $(N, \geq)$ such that for all $x, y \in N$*

$$\text{if } W(x) = W(y) \text{ then } x = y. \tag{6}$$

[4] Suppose the convention in the specification of nodes would be changed such that for two functions to belong to the same node they would have to agree on the closed interval $[0, t]$. Then there would be no "point in time" when the decision actually "becomes effective."

A decision tree is a tree without irrelevant nodes, where a node is irrelevant if it is followed only by one other node. The presence of irrelevant nodes would make it impossible to recover nodes as sets of plays, since the plays passing through two different nodes may be identical. Since irrelevant nodes serve no purpose for decision theory, Definition 2 rules them out.

Lemma 3 *A tree $(N, \geq)$ is a decision tree if and only if for all $x, y \in N$*

$$\begin{array}{c}\text{if } x \geq y \text{ and } y \not\geq x \text{ then there is } z \in N \\ \text{such that } x \geq z,\ y \not\geq z, \text{ and } z \not\geq y.\end{array} \tag{7}$$

Since property (7) is given purely in terms of the partial order $\geq$, the property of being a decision tree is preserved by order isomorphism. That is, if a poset is order isomorphic to a decision tree, it must itself be a decision tree. The translation of (7) into set-theoretic terms gives rise to the following definitions: A V-poset $(M, \supseteq)$ satisfies *Separability* if, for all $a, b \in M$,

$$\text{if } b \subset a \text{ then there is } c \in M \text{ such that } c \subseteq a \text{ and } b \cap c = \emptyset \tag{8}$$

and it satisfies *Weak Separability* if, for all $a, b \in M$,

$$\text{if } b \subset a \text{ then there is } c \in M \text{ such that } c \subseteq a \text{ but } c \setminus b \neq \emptyset \text{ and } b \setminus c \neq \emptyset. \tag{9}$$

Clearly, Separability implies Weak Separability. If the latter is written in terms of an abstract partial order $\geq$, property (7) is obtained. Separability, like Trivial Intersection, cannot be translated back into arbitrary posets. The difference exists, however, only in the absence of Trivial Intersection.

Lemma 4 *Let $(M, \supseteq)$ be a V-poset satisfying Trivial Intersection, and let $b, c \in M$. Then, $b \setminus c \neq \emptyset$ and $c \setminus b \neq \emptyset$ if and only if $b \cap c = \emptyset$. In particular, under Trivial Intersection, Weak Separability holds if and only if Separability holds.*

It follows that Separability and Weak Separability are equivalent for the set representation by subtrees, but not necessarily for arbitrary set representations of trees.

Proposition 3 *(a) A V-poset $(M, \supseteq)$ is a decision tree if and only if it satisfies Weak Trivial Intersection and Weak Separability.*
(b) A poset $(N, \geq)$ is a decision tree if and only if its set representation by subtrees $(\downarrow N, \supseteq)$ satisfies Trivial Intersection and Separability.

Proof. (a) follows from Lemma 3 and Proposition 2(a), as (9) is equivalent to (7). (b) "if:" If $(\downarrow N, \supseteq)$ satisfies Trivial Intersection and Separability, then it satisfies Weak Trivial Intersection and Weak Separability and by part (a) it is a decision tree. By isomorphism $(N, \geq)$ is a decision tree.

"only if:" Let $(N, \geq)$ be a decision tree. By Proposition 2(a), $(\downarrow N, \supseteq)$ satisfies Trivial Intersection. But, by isomorphism, $(\downarrow N, \supseteq)$ is a decision tree and by part (a) satisfies Weak Separability. By Lemma 4, we have that $(\downarrow N, \supseteq)$ satisfies also Separability. □

This result is the analogue to Proposition 2 for decision trees. Any set representation of a decision tree is a decision tree, and, hence, Proposition 3(a) characterizes *all* set representations of decision trees. By Proposition 3(b) decision trees are those posets whose set representations by principal ideals satisfy Trivial Intersection and Separability. There may be set representations of a decision tree for which Separability fails, but then, by Lemma 4, Trivial Intersection must also fail.

Example 2 Reconsider Example 1. Separability (8) does not hold, because $\{1,2\} \subset \{1,2,3\}$ and yet the only other node contained in $\{1,2,3\}$, that is, $\{2,3\}$, has a nonempty intersection with $\{1,2\}$. However, Weak Separability, (9), holds. Neither of the nodes $\{1,2\}$ and $\{2,3\}$ contains the other.

Example 3 For the differential game tree (Sect. 2.2.5), on the other hand, Separability holds. For, let $x_t(g)$ and $x_\tau(h)$ be two nodes, with $g, h \in V$ and $t, \tau \in \mathbb{R}_+$, such that $x_t(g) \subset x_\tau(h)$. Then, $\tau < t$. Choose any $f \in V$ such that $f(s) = h(s)$ for all $s \in [0, \tau)$ and $f(\tau) \neq g(\tau)$. Then, for any s with $\tau < s < t$, we have that $x_s(f) \subseteq x_\tau(h)$ but $x_s(f) \cap x_t(g) = \emptyset$, verifying Separability.

2.4 Representation by plays

The arbitrariness of the V-poset representing a tree makes it hard to interpret the elements of V. Yet, every decision tree $(N, \geq)$ admits a set representation $(M, \supseteq)$ where $M \subseteq 2^W$ is a collection of nonempty sets of plays, i.e., every decision tree can be represented by a W-poset.

Intuitively, one should be able to take plays and nodes alternatively as the primitives. If nodes are the primitives, plays are derived as maximal chains. If plays are the primitives, nodes are recovered as sets of plays sharing a common history. Formally, for a tree $(N, \geq)$ its *image in plays* is the tree $(W(N), \supseteq)$, where

$$W(N) = \{W(x)\}_{x \in N} = \{a \in 2^W \mid \exists\, x \in N : a = W(x)\}.$$

It is easy to see that a tree's image in plays satisfies Trivial Intersection, and, hence, is itself a tree (by Proposition 2(a)).

Lemma 5 *If $(N, \geq)$ is a tree, its image in plays satisfies Trivial Intersection.*

Say that a tree $(N, \geq)$ can be *represented by plays* if the mapping[5] $W : N \to W(N)$ is an order isomorphism between $(N, \geq)$ and its image in plays $(W(N), \supseteq)$. The latter is then called the tree's (set) *representation by plays*. The image in plays is the natural candidate for a "canonical" set representation. An arbitrary tree, though, need not be order isomorphic to its image in plays.

Theorem 1 *A tree can be represented by plays if and only if it is a decision tree.*

[5] No confusion should arise between the mapping $W(\cdot)$ assigning to each node x the set of plays passing through x and the set W of all plays.

Proof. "if:" Let W be the set of plays. The set $W(N)$ and its elements are nonempty by Lemma 2(a). The mapping $W: N \to W(N)$ is one-to-one by (6) and onto by construction. Next, it is verified that the bijection W is order embedding.

Let $x, y \in N$. If $y \geq x$, then by Lemma 2(b) $W(x) \subseteq W(y)$. Conversely, suppose $W(x) \subseteq W(y)$. Choose $w \in W(x) \subseteq W(y)$. Since $x, y \in w$, either $x \geq y$ or $y \geq x$. In the first case, the previous argument would imply $W(x) = W(y)$ and, therefore, $x = y$, because W is one-to-one. Since $\geq$ is reflexive, in both cases $y \geq x$. Hence, $y \geq x \Leftrightarrow W(x) \subseteq W(y)$ for all $x, y \in N$, i.e. $W(\cdot)$ is an order isomorphism.

"only if:" If $W(x) = W(y)$ for $x, y \in N$, then $x = y$ as the mapping W is one-to-one. □

The set representation by plays of a decision tree is itself a decision tree which satisfies Trivial Intersection by Lemma 5 and Weak Separability by Proposition 3(a). Hence, it also satisfies Separability (by Lemma 4).

Corollary 1 *If $(N, \geq)$ is a decision tree then its image in plays $(W(N), \supseteq)$ satisfies Trivial Intersection and Separability.*

Hence, the set representation by plays of a decision tree is order-isomorphic to the decision tree and satisfies Trivial Intersection and Separability. These results can also be understood as follows. The properties that characterize set representations of decision trees, Weak Trivial Intersection and Weak Separability, have order-theoretic analogues, that are preserved by order isomorphisms. Trivial Intersection and Separability, on the other hand, make sense only for V-posets and, hence, are not preserved by order isomorphisms. However, both the set representation by subtrees and the set representation by plays of a decision tree satisfy Trivial Intersection and Separability and can be taken as "canonical." The former gives a particularly "bulky" representation, while the latter conforms to our intuition, according to which we should be able to take either nodes or plays as primitives.

Example 4 Let $(M, \supseteq)$ be a decision tree with $M = \{\{1,2,3\},\{1,2\},\{3\}\}$. This tree satisfies Trivial Intersection and Separability, but a decision between 1 and 2 is never taken, i.e., there is a redundant element in the underlying set. Its set representation by subtrees is given by $M' = \{\{s_1, s_2, s_3\},\{s_2\},\{s_3\}\}$, where $s_1 = \{1,2,3\}$, $s_2 = \{1,2\}$, and $s_3 = \{3\}$. In this representation there is also an irrelevant element in the underlying set, because no decision is ever taken to select s_1. The representation by plays of $(M, \supseteq)$ is given by $M' = \{\{w_1, w_2\},\{w_1\},\{w_2\}\}$, where $w_1 = \{\{1,2,3\},\{1,2\}\}$ and $w_2 = \{\{1,2,3\},\{3\}\}$. In a sense, the redundant element 2 has disappeared.

This example shows that the representation by plays "reduces" the underlying set, eliminating irrelevant elements.

Example 5 Since the differential game tree example of Section 2.2.5 is a decision tree, it can be represented by plays. Given a node $x_t(g)$, the set of plays passing through it is given by $W(x_t(g)) = \{\{x_\tau(f)\}_{\tau \in [0,\infty)} | f \in V \text{ with } f(\tau) = g(\tau) \, \forall \, \tau \in [0,t)\}$.

Remark 3 Focusing on decision trees entails the position that nodes followed by exactly one other node represent no decision and, hence, do not belong to the description of a decision problem. This is not a necessary position, though. If trivial structures are allowed for, a result analogous to Theorem 1 obtains: General trees can be represented by *elementary chains* just like in Theorem 1. A chain $w \subseteq N$ is *elementary* if, whenever there exists $z \in N \setminus w$ such that $w \cup \{z\}$ is a chain, there exist $x, y \in N$ with $x \geq y$ and $x \neq y$ such that $w = \uparrow x$ and $W(x) = W(y)$; see Alós-Ferrer and Ritzberger [2] for details.

3 Set trees

Definition 3 *A V-poset $(M, \supseteq)$ is a V-***set tree** *if it satisfies Trivial Intersection, (4), and Separability, (8). A V-set tree is rooted if $V \in M$.*

All V-set trees are decision trees, but not all V-posets, that are decision trees, are also V-set trees (see Proposition 3(a)). Given a decision tree $(N, \geq)$, though, we can find two alternative set representations which turn out to be V-set trees. The first is the set representation by subtrees, which is a $\downarrow N$-set tree (where $\downarrow N$ is the set of subtrees of $(N, \geq)$) by Proposition 3(b). The second (by Theorem 1) is the image in plays, which is a W-set tree (where W is the set of plays of $(N, \geq)$) by Corollary 1.

3.1 Reduced-form posets

Intuitively, Separability for a V-set tree $(M, \supseteq)$ ensures that there are *no redundant nodes* in M. Yet, there may still be *redundant elements* in V. Roughly, an element $v \in V$ is redundant, if it can be deleted without affecting the structure of the tree. But there are two meanings for when an element of V is redundant.

Example 6 Let $V = \{1, 2, 3, 4, 5\}$ and $M = \{\{1, 2, 3, 4\}, \{1, 2\}, \{3\}\}$. Then $(M, \supseteq)$ satisfies Trivial Intersection (4) and Separability (8). For, if $b \subset a$ then $a = \{1, 2, 3, 4\}$, so that there always is $c \in M \setminus \{a, b\}$ such that $c \subset a$ and $b \cap c = \emptyset$. On the other hand, V contains redundant elements for two reasons.

First, $4 \notin \{1, 2\} \cup \{3\}$, but $4 \in a \in M$ implies $a = \{1, 2, 3, 4\}$ so that $\{1, 2\} \cup \{3\} \subset a$; hence, there is no $b \in M$ with $v \in b \setminus a$ for $v = 1, 2, 3$. Intuitively, element $4 \in V$ is *not separable*. Similarly, since there is no $a \in M$ with $5 \in a$, there are no $a, b \in M$ such that $5 \in a \setminus b$ and $v \in b \setminus a$ for $v = 1, 2, 3, 4$. Second, $1 \neq 2$, but $1 \in c \in M$ if and only if $2 \in c \in M$. Intuitively, elements $1, 2 \in V$ are *duplicates*.

In this example we attribute the first redundancy to the two elements $4, 5 \in V$ not being *separable*. The structure of the tree $(M, \supseteq)$ would not be affected by eliminating elements 4 and 5 from V. The second redundancy we attribute to elements $1, 2 \in V$ being *duplicates*. If one of them were eliminated (or they would be identified), the structure of the tree $(M, \supseteq)$ would not be affected.

Let $(M, \supseteq)$ be a V-poset, $v \in V$, and define $\uparrow \{v\} = \{a \in M \mid v \in a\}$. If $\{v\} \in M$, this coincides with the previously defined up-set. With this convention, the aforementioned redundancies can be tackled. Starting with duplicates, define the equivalence relation $\sim$ on V by

$$v \sim v' \text{ if } \uparrow\{v\} = \uparrow\{v'\}, \tag{10}$$

that is, if, for all $a \in M$, $v \in a \Leftrightarrow v' \in a$. Note that $v \in a \Leftrightarrow [v] \subseteq a$ for all $a \in M$ and all $v \in V$, where $[v]$ denotes the equivalence class (with respect to $\sim$) to which v belongs. In Example 6 we have $1 \sim 2$, so $[1] = [2] = \{1, 2\}$, but $[v] = \{v\}$ for $v = 3, 4, 5$. By definition, it is now justified to write $\uparrow [v] = \uparrow \{v\}$.

Obviously, any V-poset $(M, \supseteq)$ can be identified with a $(V/\sim)$-poset, where $V/\sim$ is the quotient set, and this representation will contain no duplicate elements.

Turning to separable elements, consider the subset $S(V)$ of the quotient space $V/\sim$ defined by

$$S(V) = \{[v] \in V/\sim \,\big|\, \cap_{a \in \uparrow[v]} a = [v]\} \tag{11}$$

which will be referred to as the set of *separable* equivalence classes. In Example 6 we have $\uparrow [4] = \{\{1, 2, 3, 4\}\}$ and $\uparrow [5] = \emptyset$, so $[4], [5] \notin S(V)$, while $\cap_{a \in \uparrow[v]} a = [v]$ for $v = 1, 2, 3$. The following justifies the name chosen for these classes, by characterizing them as those equivalence classes that can be "separated" from other classes by nodes.

Lemma 6 *Let $(M, \supseteq)$ be a V-poset. The equivalence class $[v] \in V/\sim$ is separable, i.e. $[v] \in S(V)$, if and only if for all $v' \in V \setminus [v]$ there is $a \in M$ such that $[v] \subseteq a$ and $v' \notin a$, i.e. $V \setminus [v] = V \setminus \cap_{a \in \uparrow[v]} a = \cup_{a \in \uparrow[v]} (V \setminus a)$*

The intersection of two (not necessarily distinct) elements from a V-poset contains at least one separable equivalence class.

Lemma 7 *Let $(M, \supseteq)$ be a V-poset. If $a, b \in M$ are such that $a \cap b \neq \emptyset$ (not necessarily $a \neq b$) then there is $[v] \in S(V)$ such that $[v] \subseteq a \cap b$.*

The *reduced form* of a V-poset $(M, \supseteq)$ is the $S(V)$-poset $(M^*, \supseteq)$, where

$$M^* = \{a^* \subseteq S(V) \mid \exists\, a \in M : [v] \in a^* \Leftrightarrow [v] \subseteq a\}. \tag{12}$$

For instance, $S(V) = \{[1], [3]\}$ and $M^* = \{\{[1], [3]\}, \{[1]\}, \{[3]\}\}$ in Example 6.

Proposition 4 *If the V-poset $(M, \supseteq)$ is a V-set tree, then it is order isomorphic to its reduced form with order isomorphism $\varphi : M \to M^*$ given by $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\}$.*

Proof. We first show that the mapping φ is onto. Let $a^* \in M^*$. Then there is $a \in M$ such that $[v] \in a^*$ if and only if $[v] \subseteq a$, i.e. $a^* = \varphi(a)$ and φ is onto.

Let $a, b \in M$ be such that $a \subseteq b$. Then $a \supseteq [v] \in S(V)$ implies $b \supseteq [v] \in S(V)$, so $\varphi(a) \subseteq \varphi(b)$. Conversely, if $a, b \in M$ are such that $\varphi(a) \subseteq \varphi(b)$, then $[v] \in \varphi(a)$ implies $[v] \in \varphi(b)$, so $a \supseteq [v] \in S(V)$ implies $b \supseteq [v]$, hence, $a \cap b \neq \emptyset$. By Trivial Intersection, either $a \subseteq b$ or $b \subset a$. If $b \subset a$, then by Separability there is

$c \in M$ such that $c \subseteq a$ and $b \cap c = \emptyset$. By Lemma 7, we can choose $[v'] \in S(V)$ such that $[v'] \subseteq c$. Then $\varphi(c) \subseteq \varphi(a) \subseteq \varphi(b)$ implies $[v'] \subseteq b$, in contradiction to $b \cap c = \emptyset$. Hence, $a \subseteq b$ must hold and φ is order embedding. By Remark 1, the statement is verified. □

That the hypothesis of a V-set tree (rather than a V-poset) is *necessary* for Proposition 4 is illustrated by the following example.

Example 7 Reconsider Example 1. There,

$$\begin{aligned}\uparrow\{1\} &= \{\{1,2,3\},\{1,2\}\},\\ \uparrow\{2\} &= \{\{1,2,3\},\{1,2\},\{2,3\}\} = M,\\ \uparrow\{3\} &= \{\{1,2,3\},\{2,3\}\},\end{aligned}$$

so all equivalence classes with respect to $\sim$ are singletons, but only $2 \in V$ is separable, i.e. $S(V) = \{[2]\}$, and $[2] \subseteq a$ for all $a \in M$. Therefore, $M^* = \{\{2\}\}$ cannot be order isomorphic to $(M, \supseteq)$. This is due to a violation of Trivial Intersection.

Yet, this example does not mean that Proposition 4 can be strengthened to a characterization. The next example shows that there are V-posets (in fact, trees) that are order isomorphic to their reduced form, but are not V-set trees. The crucial point is the step from Weak Trivial Intersection to Trivial Intersection.

Example 8 Let $V = \{1,2,3\}$ and $M = \{\{1,2,3\},\{1,2\},\{2,3\},\{1,3\}\}$. Then,

$$\begin{aligned}\Uparrow\{1\} &= \{\{1,2,3\},\{1,2\},\{1,3\}\} \text{ and } \cap_{a\in\uparrow\{1\}}\, a = \{1\} = [1],\\ \Uparrow\{2\} &= \{\{1,2,3\},\{1,2\},\{2,3\}\} \text{ and } \cap_{a\in\uparrow\{2\}}\, a = \{2\} = [2],\\ \Uparrow\{3\} &= \{\{1,2,3\},\{2,3\},\{1,3\}\} \text{ and } \cap_{a\in\uparrow\{3\}}\, a = \{3\} = [3],\end{aligned}$$

so all equivalence classes are singletons and all elements of V are separable, i.e., $S(V) = \{[1],[2],[3]\}$. Therefore, $M^* = \{\{[1],[2],[3]\},\{[1],[2]\},\{[2],[3]\},\{[1],[3]\}\}$, so $(M, \supseteq)$ and $(M^*, \supseteq)$ are order isomorphic by $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\}$.

This also shows that without Trivial Intersection (but with Separability) there may be no subset $V' \subseteq V$ such that Trivial Intersection holds for $(M', \supseteq)$, with $M' = \{a' \subseteq V' \,|\, \exists\, a \in M : a' = a \cap V'\}$, and $(M', \supseteq)$ is order isomorphic to $(M, \supseteq)$. For, if $1 \in V'$ then $1 \in \{1,2\} \cap V'$ and $1 \in \{1,3\} \cap V'$. So, if $(M', \supseteq)$ satisfies Trivial Intersection, then $1 \notin V'$. Thus, if $\{1,2\} \cap V' \neq \emptyset$, it follows that $2 \in V'$ and $2 \in \{1,2\} \cap V'$. But then $2 \in \{2,3\} \cap V'$ contradicts Trivial Intersection on $(M', \supseteq)$. Hence, $(M', \supseteq)$ cannot satisfy Trivial Intersection and be order isomorphic to $(M, \supseteq)$.

If for a V-poset $(M, \supseteq)$ and $v \in V$ there exists some $a \in M$ such that $a \subseteq [v]$, then $[v] \in S(V)$. For, $a \subseteq [v]$ implies $a = [v]$ (because $v' \in [v]$ implies $v' \sim v$ and, therefore, $v' \in a$ and $[v] \subseteq a$) so that $a \in \uparrow [v]$ and $a = [v] \subseteq b$ for all $b \in \uparrow [v]$ imply that $a = \cap_{b\in\Uparrow[v]} b = [v]$, as required. That there is $a \in M$ such that $a \subseteq [v]$ is, therefore, sufficient for $[v] \in S(V)$. But it is not necessary, as the next example shows.

Example 9 Let $V = [0,1]$ and $M = \{(\{v\})_{v\in(0,1]}, (x_t)_{t=1}^{\infty}\}$, where $x_t = [0, 1/t]$ for all $t = 1, 2, ...$ Then $\cap_{a\in\uparrow[0]} a = \{0\} = [0]$, but there is no $a \in M$ such that $a = [0]$.

Hence, there are more separable equivalence classes than those that coincide with a node without a successor.[6]

Proposition 5 *Let $(M, \supseteq)$ be a V-poset.*

(a) If $\uparrow[v] = \uparrow[v']$ then $[v] = [v']$, for all $[v], [v'] \in S(V)$.

(b) $(M, \supseteq)$ satisfies Trivial Intersection if and only if $\uparrow[v] \in W$ for all $[v] \in S(V)$, where W denotes the set of plays (maximal chains) for $(M, \supseteq)$.

Proof. (a) If $\uparrow[v] = \uparrow[v']$ then $[v] = \cap_{a\in\uparrow[v]} a = \cap_{a\in\uparrow[v']} = [v']$.
(b) "if:" Let W be the set of maximal chains for $(M, \supseteq)$ and assume that $\uparrow[v] \in W$ for all $[v] \in S(V)$. If $a, b \in M$ are such that $a \cap b \neq \emptyset$ then by Lemma 7 there is $[v] \in S(V)$ such that $[v] \subseteq a \cap b$, i.e., $a, b \in \uparrow[v]$. But then $\uparrow[v] \in W$ implies either $a \subset b$ or $b \subseteq a$, verifying Trivial Intersection.

"only if:" By Trivial Intersection $\uparrow[v]$ is a chain for all $[v] \in S(V)$. Suppose there is $a \in M \setminus \uparrow[v]$ such that $\uparrow[v] \cup \{a\}$ is a chain. If there would be some $b \in \uparrow[v]$ such that $b \subseteq a$, then $[v] \subseteq a$ in contradiction to $a \notin \uparrow[v]$. Thus, if $\uparrow[v] \cup \{a\}$ is a chain, then $a \subset b$ for all $b \in \uparrow[v]$. Then $a \subseteq \cap_{b\in\uparrow[v]} b = [v]$, i.e. $a = [v]$, again in contradiction to $a \notin \uparrow[v]$. □

Proposition 5(a) shows that, even for arbitrary V-posets, on the separable equivalence classes $S(V)$ the mapping $[v] \mapsto \uparrow[v]$ is one-to-one (injective). Moreover, part (b) says that Trivial Intersection is equivalent to $\{\uparrow[v] \,|\, [v] \in S(V)\} \subseteq W$. The reverse inclusion, though, is not true, as a slight modification of the last example shows.

Example 10 Let $V = (0,1]$ and $M = \{(\{v\})_{v\in(0,1]}, (x_t)_{t=1}^{\infty}\}$, where $x_t = (0, 1/t]$ for all $t = 1, 2, ...$ Then $w = \{x_t\}_{t=1}^{\infty}$ is a play that corresponds to no separable class.

The problem in this example is that V itself is not large enough, since intuitively a play fails to lead to an ultimate outcome (even in the limit).

3.2 Irreducible set trees

Proposition 5 suggests that separable equivalence classes in $S(V)$ can be used to represent plays for a V-set tree. If the elements of $S(V)$ would correspond to singletons in V, this would yield an interpretation of the elements of V as representatives of plays. What is still missing is a further separation property which guarantees that, given two elements of the underlying set, there is always a decision to distinguish between them.

[6] A Cantor-set construction would allow for an example with infinitely many such classes.

Call a V-poset $(M, \supseteq)$ *irreducible* if, for all $v, v' \in V$,

$$\text{if } v \neq v' \text{ then } \exists\, a, b \in M \text{ such that } v \in a \setminus b \text{ and } v' \in b \setminus a. \tag{13}$$

By Lemma 4 it follows that, if a V-poset $(M, \supseteq)$ satisfies Trivial Intersection, then Irreducibility holds if and only if *Strong Irreducibility* holds:[7] for all $v, v' \in V$

$$\text{if } v \neq v' \text{ then } \exists\, a, b \in M : v \in a,\ v' \in b, \text{ and } a \cap b = \emptyset. \tag{14}$$

Lemma 8 *If a V-poset $(M, \supseteq)$ satisfies Trivial Intersection and Irreducibility, then it satisfies Separability.*

The converse of Lemma 8 is not true. The trivial V-set tree $(\{V\}, \supseteq)$ satisfies Separability, because the hypothesis is void, but it fails Irreducibility, whenever V is not a singleton set.

The set representation by plays of a decision tree satisfies Trivial Intersection by Lemma 5. It is easy to show that it also satisfies Irreducibility and, therefore, is an irreducible set tree.[8] Hence, every decision tree $(N, \geq)$ is order isomorphic to an irreducible W-set tree. But the hypothesis of a *decision* tree is only required to make the image in plays a set representation.

Lemma 9 *If $(N, \geq)$ is a tree, its image in plays $(W(N), \supseteq)$ is an irreducible tree.*

Recall that, by Lemma 6, separable equivalence classes can be "separated" from other classes by choosing appropriate elements of M. This implies:

Lemma 10 *Let $(M, \supseteq)$ be a V-poset. Its reduced form $(M^*, \supseteq)$ is irreducible.*

Proposition 6 *A V-poset is irreducible if and only if $S(V) = \{\{v\}\}_{v \in V}$.*

Proof. "if:" If $S(V) = \{\{v\}\}_{v \in V}$ then by Lemma 10 Irreducibility holds for all $v, v' \in V$ (the set tree and its reduced form must then be identical).

"only if:" Let $(M, \supseteq)$ be an irreducible V-poset and consider any $v \in V$. By Irreducibility for any $v' \in V \setminus \{v\}$ there are $a, b \in M$ such that $v \in a \setminus b$ and $v' \in b \setminus a$, implying that $[v] \neq [v']$ and, therefore, $[v] = \{v\}$ for all $v \in V$. That is, all equivalence classes are singletons. It remains to show that they are separable. Consider any class $[v] = \{v\}$. If $[v] \subset \cap_{a \in \uparrow [v]} a$ then there is $v' \neq v$ such that $v' \in a$ for all $a \in \uparrow \{v\}$. This contradicts Irreducibility, as the latter requires the existence of some $a \in \uparrow \{v\}$ and $b \in M$ such that $v' \in b \setminus a$. Hence, $[v] = \cap_{a \in \uparrow [v]} a$, i.e. $[v] = \{v\} \in S(V)$. This implies $\{\{v\}\}_{v \in V} \subseteq S(V)$. Since all equivalence classes are singletons, the reverse inclusion follows. □

For irreducible V-set trees the set $S(V)$ simply consists of the collection of all singleton subsets of V. Thus, for an irreducible V-set tree the mapping $v \mapsto \uparrow \{v\}$ on V is a one-to-one function into the set W of plays (by Proposition 5).

[7] Note the formal analogy with the definition of a Hausdorff space in topology.

[8] The set representation by subtrees of a decision tree *cannot* satisfy Irreducibility. For, if $x \neq y$ and $x \geq y$ then $\downarrow y \subseteq \downarrow x$, so that $x \in \downarrow z$ and $y \in \downarrow z'$ for $z, z' \in N$ implies $\downarrow x \subseteq \downarrow z$ and $\downarrow y \subseteq \downarrow z'$ and, therefore, $y \notin \downarrow z' \setminus \downarrow z$.

Example 11 The differential game tree of Section 2.2.5 satisfies (Strong) Irreducibility. For, if $g, f \in V$ and $g \neq f$, then there is some $t \in \mathbb{R}_+$ such that $f(t) \neq g(t)$. For any τ with $t < \tau$ it then follows that $x_\tau(f) \cap x_\tau(g) = \emptyset$ (because elements of V are functions), verifying (14). Moreover, every $g \in V$ uniquely induces the play $\uparrow\{g\} = \{x_t(g)\}_{t\in[0,\infty)}$.

3.3 Proper order isomorphism

When are the "strong" properties, Trivial Intersection and Separability, preserved by order isomorphisms? Let $(M, \supseteq)$ be a V-poset, $(M', \supseteq)$ a V'-poset, and $\varphi : M \to M'$ an order isomorphism between the two. The order isomorphism φ is *proper* if

$$\varphi(a) \cap \varphi(b) = \emptyset \text{ implies } a \cap b = \emptyset \text{ for all } a, b \in M. \tag{15}$$

Properness need not be symmetric, i.e., that $(M, \supseteq)$ is properly isomorphic to $(M', \supseteq)$ does not necessarily imply that $(M', \supseteq)$ is properly isomorphic to $(M, \supseteq)$. Yet, properness is necessary and sufficient to preserve the strong properties:

Lemma 11 *Let $(M, \supseteq)$ be a V-poset, $(M', \supseteq)$ a V'-poset, and $\varphi : M \to M'$ an order isomorphism between the two.*
(a) If $(M, \supseteq)$ satisfies Trivial Intersection, then φ is proper.
(b) If $(M', \supseteq)$ satisfies Trivial Intersection, then $(M, \supseteq)$ satisfies Trivial Intersection if and only if φ is proper.
(c) If $(M', \supseteq)$ is a V'-set tree, then $(M, \supseteq)$ is a V-set tree if and only if φ is proper.

Consider a decision tree $(M, \supseteq)$. Its image in plays, $(W(M), \supseteq)$, is order isomorphic to $(M, \supseteq)$ by Theorem 1, but the order isomorphism is not necessarily proper. To see why, recall that decision trees are characterized only by Weak Trivial Intersection and Weak Separability (Proposition 3(a)), while $(W(M), \supseteq)$ is a W-set tree by Corollary 1. Hence, the concept of proper order isomorphism adds further structure.

On the other hand, if both $(M, \supseteq)$ and $(M', \supseteq)$ satisfy Trivial Intersection, then all order isomorphism between them *and their inverses* are trivially proper. Thus, the following characterization says that set trees are precisely those posets (of sets) that are properly order isomorphic to their reduced forms, provided the latter are irreducible set trees.

Theorem 2 *A V-poset $(M, \supseteq)$ is a V-set tree if and only if its reduced form $(M^*, \supseteq)$ is an irreducible $S(V)$-set tree and $(M, \supseteq)$ is properly (order) isomorphic to $(M^*, \supseteq)$.*

Proof. The "if"-part is immediate from Lemma 11. For the "only if"-part, note that Irreducibility of $(M^*, \supseteq)$ follows from Lemma 10. That $(M, \supseteq)$ is order isomorphic to its reduced form follows from Proposition 4. That the order isomorphism

$$\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\}$$

from Proposition 4 is proper follows from Lemma 7. For, if $a \cap b \neq \emptyset$, for some $a, b \in M$, then by Lemma 7 there is $[v] \in S(V)$ such that $[v] \subseteq a \cap b$, hence $[v] \in \varphi(a) \cap \varphi(b)$ by the construction of φ. □

Intuitively, Theorem 2 shows that any V-set tree is properly order-isomorphic to an irreducible set tree, obtained by appropriately shrinking the underlying set V. Moreover, by Proposition 5 for any V-set tree the mapping $[v] \mapsto \uparrow [v]$ on $S(V)$ is an injection into the set W of plays. If this mapping were onto, W and $S(V)$ could be identified. (Recall, though, Example 10.) In particular, if this mapping were onto for an *irreducible* V-set tree, then there would be no distinction between W and V (rather than $S(V)$), due to Proposition 6.

This motivates the following definitions: Let $(M, \supseteq)$ be a V-poset and $(M', \supseteq)$ a V'-poset; an order isomorphism $\varphi : M \to M'$ is an *isomorphic embedding* if there is an injection $f : V \to V'$ such that

$$f(a) = \{v' \in V' \,|\, \exists\, v \in a : v' = f(v)\} \subseteq \varphi(a) \text{ for all } a \in M. \tag{16}$$

If $(M, \supseteq)$ is isomorphically embedded in $(M', \supseteq)$ and, moreover, f is onto and satisfies $f(a) = \varphi(a)$ for all $a \in M$, then $(M, \supseteq)$ and $(M', \supseteq)$ are *doubly* (order) *isomorphic*.

Every isomorphic embedding is a proper order isomorphism. For, if $(M, \supseteq)$ is isomorphically embedded in $(M', \supseteq)$ and $v \in a \cap b$ with $a, b \in M$, then $f(v) \in \varphi(a) \cap \varphi(b)$ implies $\varphi(a) \cap \varphi(b) \neq \emptyset$. The converse is false: By Theorem 2 a V-set tree is properly order isomorphic to its reduced form, the latter is irreducible, but the former may not be. Still, only irreducible V-posets can be isomorphically embedded in irreducible V'-posets.

Lemma 12 *Let $(M, \supseteq)$ be a V-poset which is isomorphically embedded in a V'-poset $(M', \supseteq)$. If $(M', \supseteq)$ satisfies Irreducibility, then so does $(M, \supseteq)$.*

In other words, like other "strong" properties, Irreducibility is inherited by V-posets isomorphically embedded in irreducible V'-posets.

Proposition 7 *A V-set-tree is irreducible if and only if it is doubly isomorphic to its reduced form.*

Proof. The "if"-part follows from Lemmata 10 and 12. For the "only if"-part, let $(M, \supseteq)$ be an irreducible V-set tree. By Theorem 2 it is properly order isomorphic to its reduced form. By Proposition 6, $S(V) = \{\{v\}\}_{v \in V}$ and the mapping f, given by $f(v) = [v]$, is a bijection such that $\varphi(a) = \{[v] \in S(V) \,|\, [v] \subseteq a\} = f(a)$ for all $a \in M$. □

This clarifies the status of Irreducibility. It is equivalent to the property that a set tree is not only properly order isomorphic to its reduced form, but the underlying sets – V for the set tree and $S(V)$ for its reduced form – also "look alike."

3.4 Bounded set trees

Proposition 5 says that certain plays for a V-set tree $(M, \supseteq)$ can be represented by elements of $S(V)$. According to Proposition 6 equivalence classes in $S(V)$ have to be used to represent plays, because V may be "too large." If the V-set tree were irreducible, V could be used directly. But even Irreducibility does not ensure that *all* plays for $(M, \supseteq)$ can be represented by elements of V. The problem is that for some play $w \in W$ the set $\{v \in V \,|v \in a, \forall a \in w\}$ may be empty, so that not every play is represented by some $v \in V$, i.e., the given set V may be "too small." This was the case, for instance, in Example 10.

This suggests that for an irreducible V-set tree the underlying set V could be used to represent all plays, provided V is "large enough." The following definition makes precise what "large enough" V means: Call a V-poset $(M, \supseteq)$ *bounded* (from below) if every chain in M has a lower bound in V, i.e., if

$$\text{for all chains } c \in 2^M \text{ there is } v \in V \text{ such that } v \in a \text{ for all } a \in c. \tag{17}$$

The image in plays of a tree $(N, \geq)$ is bounded. For, if $c \subseteq W(N)$ is a chain, then there is a chain $c' \subseteq N$ such that $x \in c'$ if and only if $W(x) \in c$. By the Hausdorff Maximality Principle there is a play $w \in W$ for $(N, \geq)$ such that $c' \subseteq w$. Therefore, $w \in W(x)$ for all $W(x) \in c$. It follows from Theorem 1 that the set representation by plays of any decision tree is bounded (and irreducible by Lemma 9).

Example 12 The differential game tree (Sect. 2.2.5) is bounded. For, consider any chain $c \in 2^N$ and let $x_t(g), x_\tau(h) \in c$. If $\tau \leq t$, then (since c is a chain) $x_t(g) \subseteq x_\tau(h)$, and it follows that $g(s) = h(s)$ for all $s \in [0, \tau)$. Hence, the mapping $f_c : \mathbb{R}_+ \to A$ defined in (18) below is a well defined function.

$$f_c(t) = \begin{cases} f(t) & \text{if there exists } x_\tau(f) \in c \text{ with } \tau > t \\ a_o \in A & \text{otherwise.} \end{cases} \tag{18}$$

By construction, $f_c \in x_\tau(f)$ for all $x_\tau(f) \in c$, which proves the claim.

Lemma 13 *A V-poset $(M, \supseteq)$ satisfies Trivial Intersection and is bounded (from below) if and only if*

$$c \in 2^M \textit{ is a chain if and only if } \exists\, v \in V : v \in a,\ \forall a \in c. \tag{19}$$

Condition (19) implies that $\cap_{a \in c} a \neq \emptyset$ for any chain c, thus preventing the situation in Example 10. It will now be shown that there is no loss of generality in assuming that an irreducible V-set tree is bounded. To do this, the underlying set V gets enlarged to an appropriately constructed superset V_B such that a new V_B-set tree is obtained, that is bounded and properly order isomorphic to the original tree. The idea of enlarging is captured by an isomorphic embedding, where $f : V \to V_B$ is simply the identity.

Proposition 8 *A V-poset $(M, \supseteq)$ is an irreducible V-set tree if and only if it is isomorphically embedded in some bounded irreducible V_B-set tree $(M_B, \supseteq)$.*

Proof. The "if"-part follows from Lemmata 11 and 12. To show the "only if"-part, let $(M, \supseteq)$ be an irreducible V-set tree. By Proposition 6, $S(V) = \{\{v\}\}_{v\in V}$ and, by Proposition 5, the mapping $v \mapsto \uparrow\{v\}$ on V is an injection into the set W of plays. Let $W^* = \{\uparrow\{v\}\,|v \in V\}$ and define the superset V_B of V by $V_B = V \cup (W \backslash W^*)$. By construction, this union is disjoint.

For any $a \in M$ let $\phi(a) = a \cup (W(a) \setminus W^*) \subseteq V_B$. Hence $\phi(a) \cap V = a$. Let

$$M_B = \phi(M) = \{a' \subseteq V_B\,|\exists\, a \in M : a' = \phi(a)\}.$$

It follows that $\phi : M \to M_B$ is onto and also one-to-one, because if $a, b \in M$ are such that $\phi(a) = \phi(b)$, then $a = \phi(a) \cap V = \phi(b) \cap V = b$. Moreover, $\phi(a) \subseteq \phi(b)$ implies $a = \phi(a) \cap V \subseteq \phi(b) \cap V = b$ for all $\phi(a), \phi(b) \in M_B$. Conversely, if $a, b \in M$ are such that $a \subseteq b$ then by Lemma 2(b) $W(a) \subseteq W(b)$ which implies (by the construction of ϕ) that $\phi(a) \subseteq \phi(b)$. Hence, ϕ is an order isomorphism. Condition (16) holds with $f : V \to V_B$ given by $f(v) = v$ for all $v \in V$, so that ϕ is an isomorphic embedding.

To establish that $(M_B, \supseteq)$ is a bounded irreducible V_B-set tree, the following is needed.

Claim. If $a \cap b = \emptyset$ then $\phi(a) \cap \phi(b) = \emptyset$ for all $a, b \in M$.
To see this, recall that $V_B = V \cup (W \backslash W^*)$ and the union is disjoint. If $\phi(a) \cap \phi(b) \neq \emptyset$ and $a \cap b = \emptyset$, then there exists $w \in W(a) \cap W(b)$, in contradiction to $a \cap b = \emptyset$.

Clearly, ϕ has inverse $\phi^{-1} : M_B \to M$ given by $\phi^{-1}(a') = a' \cap V$ for all $a' \in M_B$. The Claim implies that ϕ^{-1} is a proper order isomorphism and hence Lemma 11(b) implies that $(M_B, \supseteq)$ satisfies Trivial Intersection. Next, we verify Irreducibility on V_B. If $v', w' \in V_B$ are such that $v' \neq w'$, there are three possibilities.

If $v', w' \in V$, Irreducibility for $(M, \supseteq)$ implies that there are $a, b \in M$ such that $v' \in a \setminus b$ and $w' \in b \setminus a$. It follows that $v' \in \phi(a) \setminus \phi(b)$ and $w' \in \phi(b) \setminus \phi(a)$.

If $v' \in V$ and $w' \in V_B \backslash V$ (and analogously for the reciprocal case), then w' is a play for $(M, \supseteq)$. If $v' \in a$ for all $a \in w'$, then $w' = \uparrow\{v'\} \in \cup_{v\in V} \uparrow\{v\}$, a contradiction. Hence, there exists $a \in w'$ such that $v' \notin a$. Therefore, $w' \neq \uparrow\{v'\}$, i.e., they are two different plays in $(M, \supseteq)$. Let $a, b \in M$ such that $a \in \uparrow\{v'\} \backslash w'$ and $b \in w' \backslash \uparrow\{v'\}$. Then $v' \in a$, because $a \in \uparrow\{v'\}$. And $v' \notin b$, because $b \in w' \backslash \uparrow\{v'\}$, i.e., $v' \in a \backslash b = a \backslash \phi(b) \subset \phi(a) \setminus \phi(b)$. On the other hand, because $b \in w' \in V_B \backslash V$, it follows that $w' \in W(b) \setminus (\cup_{v\in V} \uparrow\{v\}) \subset \phi(b)$. Since $a \notin w'$, we have that $w' \notin \phi(a)$ and hence $w' \in \phi(b) \setminus \phi(a)$.

Finally, if $v', w' \in V_B \backslash V = W \backslash (\cup_{v\in V} \uparrow\{v\})$, then v' and w' are two different plays in $(M, \supseteq)$. Let $a, b \in M$ such that $a \in v' \backslash w'$ and $b \in w' \backslash v'$. It follows that $v' \in \phi(a) \setminus \phi(b)$ and $w' \in \phi(b) \setminus \phi(a)$.

This shows that Irreducibility holds for all $v', w' \in V_B$. By Lemma 8, it follows that the V_B-poset $(M_B, \supseteq)$ is an irreducible V_B-set tree. To establish that $(M_B, \supseteq)$ is also bounded from below, consider any chain c' in M_B. By Irreducibility and the Claim above,

$$c = \phi^{-1}(c') \equiv \{a \in M\,|\exists\, a' \in c' : a' = \phi(a)\}$$

is a chain in M. By the Hausdorff Maximality Principle, there exists a play w in M such that $c \subseteq w$. If $w \in \cup_{v \in V} \uparrow\{v\}$, let $v \in V \subset V_B$ be such that $w = \uparrow\{v\}$. Then v is a lower bound for c and, therefore, for c'. If $w \in W \setminus \cup_{v \in V} \uparrow\{v\} \subset V_B$, then $w \in W(a) \setminus (\cup_{v \in V} \uparrow\{v\}) = \phi(a)$ for all $a \in c$ and, hence, is a lower bound for c'. □

Combining this with Theorem 2 it follows that any V-set tree is properly order isomorphic to an irreducible set tree which itself is isomorphically embedded in a bounded irreducible set tree. In this sense, there is no loss of generality in assuming Irreducibility and Boundedness when working with set trees.

Proposition 9 *A V-set tree $(M, \supseteq)$ is bounded if and only if the map $[v] \mapsto \uparrow[v]$ on $S(V)$ is onto W, where W is the set of plays for $(M, \supseteq)$.*

Proof. "if:" Suppose the mapping $[v] \mapsto \uparrow[v]$ on $S(V)$ is onto W. Let $c \in 2^M$ be a chain in M. By the Hausdorff Maximality Principle there is a play $w \in W$ such that $c \subseteq w$. By hypothesis there is $[v] \in S(V)$ such that $w = \uparrow[v]$. Therefore, $v \in a$ for all $a \in c \subseteq w$.

"only if:" Let $(M, \supseteq)$ be a bounded V-set tree and $w \in W$ a play. By hypothesis there is $v \in V$ such that $v \in a$ for all $a \in w$. By Trivial Intersection and the fact that w is a maximal chain, $\uparrow[v] = w$ and $[v] = \cap\{a \,|\, a \in w\}$, and, hence, $[v] \in S(V)$. □

Combining the last result with Proposition 6 it follows that for a bounded irreducible V-set tree the mapping $v \mapsto \uparrow\{v\}$ is a bijection from V onto the set W of plays. This result is particularly transparent in the differential game (Sect. 2.2.5).

4 Game trees

Theorem 2 and Proposition 8 show that every set tree can be modified into a bounded irreducible set tree by appropriately shrinking and enlarging the underlying set. Then, the underlying set V and the set W of plays can be identified (by Propositions 5 and 9), and the mapping W from nodes to (sets of) plays passing through them becomes the identity.

Definition 4 *A* **game tree** *is a W-poset $(N, \supseteq)$ that satisfies*

$$c \in 2^N \textit{ is a chain if and only if } \exists\, w \in W : w \in x, \forall x \in c \qquad (20)$$

and for all $w, w' \in W$

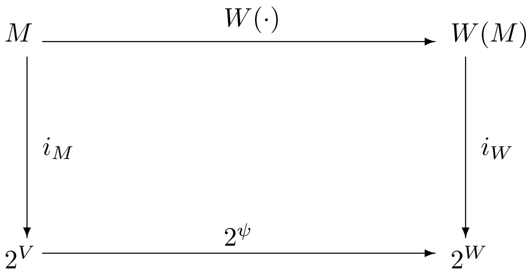
$$\textit{if } w \neq w' \textit{ then } \exists\, x, y \in N \textit{ such that } w \in x \setminus y \textit{ and } w' \in y \setminus x. \qquad (21)$$

By Lemma 13 condition (20) holds if and only if Trivial Intersection holds and every chain in N has a lower bound in W. Since condition (21) is simply Irreducibility, a W-poset $(N, \supseteq)$ is a game tree if and only if it is a bounded irreducible W-set tree.

It will now be shown that a game tree is *its own set representation by plays*. Yet, when precisely is a set tree its own set representation by plays? The idea is that, not only the set of plays W and the underlying set V are bijective, but, additionally, this bijection can be used to reconstruct the plays passing through a node from the elements which the node contains. This is formalized as a double isomorphism.

Definition 5 *A V-poset $(M, \supseteq)$ is* **its own set representation by plays** *if there exists a bijection $\psi : V \to W$ such that $\psi(a) \equiv \{\psi(v)\}_{v\in a} = W(a)$ for all $a \in M$, where $W : M \to W(M)$ is the mapping assigning plays passing through a node, given by $W(a) = \{w \in W \,|a \in w\}$.*

Remark 4 Let $2^{\psi} : 2^V \to 2^W$ be the trivial extension of ψ to the power set given by $2^{\psi}(a) = \psi(a)$ for all $a \subseteq V$. Let $i_M : M \to 2^V$ be the immersion of M into the power set of V (given by $i_M(x) = x$ for all $x \in M$) and $i_W : W(M) \to 2^W$ the analogous immersion of $W(M)$ into the set of sets of plays. Then, the previous definition amounts to the following diagram being commutative:



Lemma 14 *A tree $(M, \supseteq)$ that is its own representation by plays is a decision tree.*

Combining Theorem 1 with Lemma 14 shows that *a tree is its own representation by plays if and only if it is doubly isomorphic to its image in plays via the natural order isomorphism.*

Example 10 shows that Trivial Intersection and Irreducibility are *not* sufficient for $(M, \supseteq)$ to be its own set representation by plays. This is purely due to the fact that the underlying set V is given, and is unrelated to properties of the decision tree. That the underlying set V is large enough to contain a lower bound for every chain in M is expressed by adding the converse to the implication in Trivial Intersection, as in condition (20). The "only if"-part of condition (20) in Definition 4 is purely a convention on how the set representation is chosen. That is, by contrast to its "if"-part (viz. Trivial Intersection) and Irreducibility, (21), it has no impact on the corresponding decision tree $(N, \geq)$, by Proposition 8.

Theorem 3 *For any V-poset $(M, \supseteq)$ the following statements are equivalent:*
(a) $(M, \supseteq)$ is a game tree.
(b) $(M, \supseteq)$ is a tree and its own representation by plays.
(c) $\psi(v) = \uparrow\{v\}$ defines a bijection from V to the set W of maximal chains in M.

Proof. "(a) implies (b):" By Proposition 6, $S(V) = \{\{v\}\}_{v\in V}$ and, by Proposition 5(b), for every $v \in V$, the set $\uparrow \{v\} \equiv \{a \in M \,|v \in a\} \subseteq M$ is a play (maximal chain).

Define $\psi : V \to W$ by $\psi(v) = \uparrow\{v\}$ for all $v \in V$. By Proposition 5(b) this function is one-to-one and by Proposition 9 it is onto. Hence, it is a bijection. Moreover, it follows that $\psi^{-1}(w)$ is given by the only element in $\cap_{a \in w} a = \{v \in V \,|v \in a, \forall a \in w\}$.

Next, it is verified that $\psi(a) \equiv \{\psi(v)\}_{v \in a} = W(a)$ for all $a \in M$. For $a \in M$ the plays passing through are $W(a) = \{w \in W \,|a \in w\}$, as usual. Let $w' \in W(a)$ and $v' = \psi^{-1}(w')$. Since $a \in w' = \psi(v') = \{a' \in M \,|v' \in a'\}$, it follows that $v' \in a$. Hence $w' \in \psi(a)$. Conversely, let $w' \in \psi(a)$. Then there is $v' \in a$ such that $\psi(v') = w'$. Since $w' = \psi(v') = \{a' \in M \,|v' \in a'\}$, it follows that $a \in w'$. Hence, $w' \in W(a)$. In summary, $\psi(a) = W(a)$ for all $a \in M$, as required by Definition 5.

"(b) implies (c):" Suppose that $(M, \supseteq)$ is a decision tree and that there exists some bijection $\widetilde{\psi} : V \to W$ such that $\widetilde{\psi}(a) \equiv \{\widetilde{\psi}(v)\}_{v \in a} = W(a)$ for all $a \in M$. Since $(W(M), \supseteq)$ is a decision tree by Lemma 9, it follows from Lemmata 11 and 12 that $(M, \supseteq)$ is an irreducible V-set tree.

Consequently, by Propositions 5 and 6, $\psi(v) = \uparrow\{v\}$ is a play for all $v \in V$. It suffices to show that $\psi(v) = \widetilde{\psi}(v)$ for all $v \in V$. Let $v \in V$ and consider any $a \in \psi(v) = \uparrow\{v\}$. Then $v \in a$ implies $\widetilde{\psi}(v) \in \widetilde{\psi}(a) = W(a)$ and, hence, $a \in \widetilde{\psi}(v)$. It follows that $\psi(v) \subseteq \widetilde{\psi}(v)$ and, since both $\psi(v)$ and $\widetilde{\psi}(v)$ are plays, $\psi(v) = \widetilde{\psi}(v)$ by maximality.

"(c) implies (a):" Suppose there is $v \in V$ such that $v \in a$ for all $a \in c$ for some $c \in 2^M$. Then $c \subseteq \psi(v) = \uparrow\{v\} \in W$ implies that c is a chain. This verifies the "if"-part of (20).

If $c \in 2^M$ is chain, by the Hausdorff Maximality Principle there is a maximal chain $c' \in W$ such that $c \subseteq c'$. Since ψ is onto, there is $v \in V$ such that $\psi(v) = c' \in W$. Then $v \in b$ for all $b \in c'$ implies $v \in a$ for all $a \in c$. This verifies the "only if"-part of (20).

Let $v, v' \in V$ be such that $v \neq v'$. Then $\psi(v)$ and $\psi(v')$ are distinct plays, because ψ is one-to-one. If for every $a \in \psi(v)$ also $v' \in a$ would hold, then $\psi(v) \subseteq \psi(v')$ would imply the contradiction $\psi(v) = \psi(v')$ by maximality. Hence, there is $a \in \psi(v)$ such that $v' \notin a$. A symmetric argument shows that there is $b \in \psi(v')$ such that $v \notin b$. Thus $v \in a \setminus b$ and $w \in b \setminus a$ verifies (21). □

Along the way it has been shown that it is justified to call the mapping $\psi : V \to W$ defined by $\psi(v) = \uparrow\{v\}$ *the canonical mapping*. For, the proof of "(b) implies (c)" shows that the bijection from the underlying set to the set of plays is *unique*.

Corollary 2 *For any game tree $(N, \supseteq)$ the bijection in Definition 5 from V to the set W of plays is unique and given by $\psi(v) = \uparrow\{v\}$ for all $v \in V$.*

In a game tree, by Theorem 3, the sets V and W can be identified. The bijection $\psi : V \to W$ becomes the identity on all of $V = W$, and then $\psi(a) = W(a)$ for all $a \in M$, i.e., a node can be identified with the set of plays that pass through it. Game trees are those trees for which it is inconsequential whether nodes or plays are taken to be the primitives.

4.1 Complete game trees

Even for a game tree some plays may not end at nodes. Yet, it will be shown that terminal nodes (singletons) can be added without altering the structure of the tree. Hence, having all plays ending at nodes or not becomes a modelling decision.

Definition 6 *A game tree $(N, \supseteq)$ is* **complete** *if $\{w\} \in N$ for all $w \in W$ (where W is the underlying set). A complete game tree is* rooted *if $W \in N$.*

A W-poset $(N, \supseteq)$ is a complete game tree if and only if (20) holds and $\{w\} \in N$ for all $w \in W$, because then Irreducibility, (21), holds trivially for all $w, w' \in W$. Theorem 3 implies that every game tree can be *completed* by adding all singletons $\{w\}$ for $w \in W$ to N without affecting the set of plays:

Proposition 10 *If the V-poset $(N, \supseteq)$ is a game tree and $Z = \{\{v\} \mid v \in V\}$ is the collection of singleton sets, then $(N \cup Z, \supseteq)$ is a complete game tree. Moreover, $\phi(w) = w \cup \{\{\psi^{-1}(w)\} \cap Z\}$ defines a bijection between the set of plays for $(N, \supseteq)$ and the set of plays for $(N \cup Z, \supseteq)$.*

Proof. Let W be the set of plays for the game tree $(N, \supseteq)$ and W' the set of plays for $(N', \supseteq)$ where $N' = N \cup Z$. Let $\psi : V \to W$ be the canonical mapping, defined by $\psi(v) = \{x \in N \,|v \in x\} =\uparrow \{v\}$. Define $\phi : W \to W'$ by $\phi(w) = w \cup \{\{\psi^{-1}(w)\} \cap Z\}$. We show that ϕ is a bijection.

Let $w' \in W'$ and $w = w' \cap N$. Then $w \in W$, $\psi^{-1}(w) \in V$, and $\phi(w) = w' \in W'$. Hence, ϕ is onto (surjective). If $\phi(w) = \phi(\hat{w})$ for $w, \hat{w} \in W$ then

$$w = \phi(w) \cap N = \phi(\hat{w}) \cap N = \hat{w}.$$

Thus, ϕ is one-to-one (injective). It follows that W and W' are set-isomorphic (bijective). By Theorem 3 (applied to $(N, \supseteq)$), ψ is bijective and hence $\psi' \equiv \phi \circ \psi : V \to W'$ is also bijective. But

$$\begin{aligned}\psi'(v) &= \{x \in N | v \in x\} \cup \{\{v' \in V | v' \in x, \forall x \in \psi(v)\} \cap Z\} \\ &= \{x' \in N' | v \in x'\},\end{aligned}$$

because $\{v' \in V \,|v' \in x, \forall x \in \psi(v)\} = \{\psi^{-1}(\psi(v))\} = \{v\}$ by Theorem 3. Therefore, Theorem 3 now implies that $(N', \supseteq)$ is a complete game tree. □

In Proposition 10, all singletons are added to N. But some of the singletons may already have to belong to N, by Irreducibility, (21). The next example shows this.

Example 13 (Twins) Let $V = [0, 1]$ and $N = \{(\{v\})_{v \in V}, (x_t)_{t=1}^{\infty}\}$, where

$$x_t = \left[0, \frac{1}{t+1}\right] \cup \left[\frac{t}{t+1}, 1\right] \text{ for all } t = 1, 2, ...$$

Because $\{v\} \in N$ for all $v \in V$, the poset $(N, \supseteq)$ is a complete game tree. The set $c_\infty = \{x_t \in N \,|t = 1, 2, ...\}$ is not a play (because $c_\infty \cup \{0\}$ is a chain), nor is any set $c_t = \{x_\tau \in N \,|\tau = 1, ..., t\}$ (for the same reason). The set of plays is given by

$$W = \left\{ \left((c_t \cup \{v\})_{v \in (\frac{1}{t+2}, \frac{1}{t+1}]}, (c_t \cup \{v\})_{v \in [\frac{t}{t+1}, \frac{t+1}{t+2})} \right)_{t=1}^{\infty}, \right.$$
$$\left. c_\infty \cup \{0\}, c_\infty \cup \{1\} \right\}.$$

Hence, V and W are naturally isomorphic by the bijection $\psi : V \to W$, where $\psi(v) = c_{t(v)} \cup \{v\}$ and $t(v)$ is the largest integer such that $t(v) \leq (1 - v)/v$ for all $v \in (0, 1/2]$, $\psi(v) = c_{t(v)} \cup \{v\}$ and $t(v)$ is the largest integer such that $t(v) \leq v/(1 - v)$ for all $v \in (1/2, 1)$, $\psi(0) = c_\infty \cup \{0\}$, and $\psi(1) = c_\infty \cup \{1\}$.

The singletons $\{0\}$ and $\{1\}$ could *not* have been added, as in Proposition 10, to a game tree. For, if originally, say $\{0\}$ would not be a node, then the original tree fails Irreducibility, (21). This is, because without $\{0\}$ there would not be any node that separates $0 \in V$ from $1 \in V$ (i.e. that $0 \in V$ belongs to a node would imply that $1 \in V$ belongs to this node). Only if the underlying set is modified to become $V' = (0, 1]$, the resulting ordered set $(N \setminus \{\{0\}\}, \supseteq)$ would be a game tree; but then $\{0\}$ could not be added. Hence, the class of singletons that can truly be added (i.e. without already being there), as in Proposition 10, forms a particular subset of the set of all singletons. To unveil what special subset that is, however, takes extra concepts that are deferred to a separate paper (Alós-Ferrer and Ritzberger [1]).

By Lemma 5 the representation by plays of a decision tree is an irreducible set tree. By Proposition 8 every irreducible set tree can be modified – by adding elements to the underlying set V – to become a bounded irreducible tree, i.e. a game tree. (Hence, every decision tree is order isomorphic to a game tree.) By Proposition 10 every game tree can be modified – by adding nodes – to become a complete game tree. Neither of these modifications changes any essential features of the tree.

Proposition 11 *Let $(N, \supseteq)$ be a V-set tree. Then, $(N, \supseteq)$ is a complete game tree if and only if it is irreducible and every play has a minimum.*

Proof. "if:" Let $(N, \supseteq)$ be an irreducible V-set tree for which every play has a minimum. Then $(N, \supseteq)$ is a game tree by Theorem 3. For any $v \in V$ let $\psi(v) = \uparrow \{v\} = w \in W$ be the play associated to v by the canonical mapping. By hypothesis w has a lower bound $z \in w$ and, by the definition of w, we have that $v \in z$.

If there is $v' \in z$ with $v' \neq v$, then by Irreducibility there are $x, x' \in N$ such that $v \in x \setminus x'$ and $v' \in x' \setminus x$. Since $v \in x$, we have $x \in w$. Since $v' \in z$ and $z \subseteq x$ (because $z \subseteq y$ for all $y \in w$), it follows that $v' \in x$, a contradiction. Therefore, $z = \{v\} \in N$.

"only if:" If $(N, \supseteq)$ is a complete game tree, then it is irreducible, because $\{v\} \in N$ for all $v \in V$. If $w \in W$ then by (20) there is $v \in V$ such that $v \in x$

for all $x \in w$. By definition $x \supseteq \{v\} \in N$ for all $x \in w$, so $w \subseteq \uparrow\{v\}$. Since $\uparrow\{v\} \in W$ by Theorem 3(c), maximality implies $w = \uparrow\{v\}$. Therefore, if $z \in N$ is such that $z \subseteq x$ for all $x \in w$, then, in particular, $z \subseteq \{v\}$ implies $z = \{v\}$. Since, moreover, $\{v\} \in w$ the play $w = \uparrow\{v\}$ has the minimum $\{v\} \in N$. □

In one important special case a complete game tree is characterized by a much simpler condition than the combination of (20) and (21) from Definition 4.

Proposition 12 *If $(N, \supseteq)$ is a V-poset such that all chains in N are finite, then $(N, \supseteq)$ is a complete game tree if and only if it satisfies Trivial Intersection and $\{v\} \in N$ for all $v \in V$.*

Proof. "if:" If $v, v' \in V$ are such that $v \neq v'$ then, because $\{\hat{v}\} \in N$ for all $\hat{v} \in V$, $v \in \{v\} \setminus \{v'\}$ and $v' \in \{v'\} \setminus \{v\}$ verifies (21). If $c \in 2^N$ is a chain, then by hypothesis it is finite, so that $\cap_{x \in c} x \neq \emptyset$ implies that there is $v \in \cap_{x \in c} x \subseteq V$. Since the chain c is arbitrary, the tree is bounded. By Lemma 13, (20) holds, and $(N, \supseteq)$ is a game tree.

"only if:" If $(N, \supseteq)$ is a complete game tree, then Trivial Intersection follows from (20) and Lemma 13. For any $v \in V$ the chain $\uparrow\{v\}$ is finite by hypothesis and, therefore, contains a smallest node $x^* \in N$ such that $v \in x^*$ (i.e. $v \in y \in N \Rightarrow x^* \subseteq y$). If there is $v' \in V \setminus \{v\}$ such that $v' \in x^*$, then by (21) there are $x, y \in N$ such that $v' \in x \setminus y$ and $v \in y \setminus x$. But $v \in y$ implies $v \in x^* \subseteq y$ in contradiction to $v' \notin y$. Hence, there is no $v' \in V \setminus \{v\}$ such that $v' \in x^*$ and, therefore, $\{v\} = x^* \in N$. □

All game trees with underlying *finite* set V are necessarily complete game trees. Without finiteness, however, a game tree as in Definition 6 is quite general. It includes the decision trees for all examples from Section 2.2.5.

Example 14 The differential game tree (Sect. 2.2.5) is irreducible and bounded, and hence a game tree. To turn it into a complete game tree, it is enough to add the singletons (applying Proposition 10), e.g. simply allowing t to be infinite in the definition of $x_t(g)$ and N.

A chain in this tree need not have a minimum, but only an infimum. For, let c be a chain. If, for all t, there exists $x_\tau(f) \in c$ with $\tau > t$, then $\{f_c\}$, where f_c is defined in (18), is an infimum for c. If there exists t such that c contains no node $x_\tau(f)$ with $\tau > t$, then let t^* be the supremum over all such t. The node $x_{t^*}(f_c)$ is an infimum for the chain. To see this, take any node $x_\tau(f) \in c$; necessarily, $\tau \leq t^*$ and $x_{t^*}(f_c) \subseteq x_\tau(f)$. However, $x_{t^*}(f_c) \notin c$ in general. Intuitively, this says that no node other than the root can be reached in a finite number of "steps" from the root.

The last observation raises the issue whether or not a game tree has enough structure to serve as the "objective" description of what may happen in the course of an extensive form game. The next section tackles this issue.

5 Extensive decision problems

A necessary condition for a game to have complete rules is that it can be represented in extensive form. To verify the latter requires an appropriate specification of a tree.

Here it will be shown that game trees lend themselves to such an exercise. Since the aim is only to provide an extensive form *representation* – rather than a vehicle for solving games – the proposed definition has to be viewed as preliminary, though.

An extensive form requires a specification of the players' choices on top of the specification of the tree. This is also true for the traditional definition of an extensive form: choices of players (and, thereby, the information structures) have to be specified separately. In general, however, a given choice may not always have an associated "information set" (a set of nodes, where this choice is available).

Let $T = (N, \supseteq)$ be a game tree with set of plays W and let the set of *moves* be $X = \{x \in N \,|\, \exists\, z \in N : z \subset x\}$. For a set $a \subseteq W$ of plays let $\downarrow a = \{x \in N \,|\, x \subseteq a\}$ be its *down-set* and define the set of (immediate) *predecessors* of a as

$$P(a) = \{x \in N \,|\, \exists\, y \in \downarrow a : \uparrow x = \uparrow y \setminus \downarrow a\}. \tag{22}$$

Since every node $x \in N$ is a set of plays, nodes too may, but need not, have immediate predecessors. Say that a set a of plays is *available at* the move $x \in X$ if $x \in P(a)$.

Definition 7 *An* **extensive decision problem** *with player set I is a pair (T, C), where $T = (N, \supseteq)$ is a game tree with set of plays W and $C = (C_i)_{i \in I}$ is a system of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of plays) for all $i \in I$, such that*

- *(i) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$ then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ and all $i \in I$;*
- *(ii) $x \cap [\cap_{i \in J(x)} c_i] \neq \emptyset$ for all $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ and all $x \in X$;*
- *(iii) if $y, y' \in N$ satisfy $y \cap y' = \emptyset$ then there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$;*
- *(iv) if $x \supset y \in N$ then for every $i \in J(x)$ there is $c \in A_i(x)$ such that $y \subseteq c$, for all $x \in X$;*

where $A_i(x) = \{c \in C_i \,|\, x \in P(c)\}$ are the choices available to i at x for all $i \in I$ and the set of decision makers at x, $J(x) = \{i \in I \,|\, A_i(x) \neq \emptyset\}$, is required to be nonempty for all $x \in X$.

What is added to the tree to obtain an extensive decision problem are collections of "choices" $c \in C_i$ (i.e. collections of sets of plays) for all $i \in I$. The set I may contain a distinguished player $i = 0$, called "chance," that models events not under the control of personal players. The behavior of "chance" can then be determined by fixing a behavior strategy (see below) for $i = 0$. The sets of choices C_i have to satisfy four constraints.

First, (i) stands in for information sets. If two distinct choices are available at a common move, then their immediate predecessors are identical and the choices are disjoint. Thus, the player cannot infer from the available menu of choices at which move (in the common set of predecessors, i.e. information set) she chooses. And, two choices that are simultaneously available cannot overlap.[9]

[9] Property (i) is not necessary for the behavior of the distinguished player "chance."

Second, by (ii), if a combination of choices (one for each decision maker) is available at a common move, then the combination has a nonempty intersection. Note that here, when several players take decisions simultaneously, this may (but need not) be represented by a single move, in contrast to the usual approach of specifying several nodes and using information sets to preserve informational simultaneity.[10]

Third, (iii) deals with the selection of outcomes. If two nodes are disjoint, then some player (possibly chance) has choices that separates them. This will be shown to imply that the intersection of all choices that contain a particular play yields precisely this play. Hence, "in the end" whatever is not decided by personal players will be decided by chance.

Fourth, (iv) implies the traditional exclusion of absent-mindedness (Kuhn [19]). In the absence of such a condition a play may cross an information set more than once (Piccione and Rubinstein [21]) or, in the present formalism, the same choice may be available more than once along the same play. It will be shown (Proposition 13) that (iv) implies "no-absent-mindedness."

It is not difficult to construct examples showing that the properties (i)-(iv) are independent. Therefore, we here only show that (iv) may fail even if (i)-(iii) hold.

Example 15 (Two-sided absent-minded driver paradox) Let $W = \{w_1, ..., w_4\}$, $N = \{W, \{w_3, w_4\}, (\{w\})_{w \in W}\}$, $I = \{0, 1\}$, $C_0 = \{\{w_1, w_2\}, \{w_3, w_4\}\}$, and, finally, $C_1 = \{\{w_1, w_3\}, \{w_2, w_4\}\}$. That $P(\{w_1, w_3\}) = \{W, \{w_3, w_4\}\} = P(\{w_2, w_4\})$ verifies (i). Because $J(W) = \{0, 1\}$ and $J(\{w_3, w_4\}) = \{1\}$ and $c_0 \cap c_1 \neq \emptyset$ for all $(c_0, c_1) \in C_0 \times C_1$, property (ii) also holds. Since for two nodes to be disjoint requires at least one of them not to be a move and none of them to be the root, the hypothesis of (iii) applies only if either $y = \{w_3, w_4\}$ and $y' = \{w\}$ for some $w \in \{w_1, w_2\}$ or both y and y' are singletons. For pairs of singletons $y, y' \in N$ there clearly is always a disjoint pair of choices of the same decision maker that separates them. Furthermore, the singletons $\{w_1\}$ and $\{w_2\}$ are separated from $\{w_3, w_4\}$ by the two choices of chance. Hence, (iii) also holds true. But (iv) fails, as $1 \in J(W)$ and $x = W \supset y = \{w_3, w_4\}$, but there is no $c \in A_1(W) = C_1$ such that $\{w_3, w_4\} \subseteq c$.

Definition 7 captures the following quasi-operational specification of an extensive decision problem. At every move $x \in X$ each player $i \in I$ is told (by an "umpire") which choices $c \in C_i$ she has available (in the sense that $x \in P(c)$, $c \in C_i$) and asked to select one of those. No other information is released to players. Given the decisions by all personal players and the possible chance moves, taking the intersection gives a nonempty set of plays, which shows how the game will continue.

[10] Friedman [11, p.109] has coined the term "semi-extensive form" for such a representation of true simultaneity.

5.1 Implications

The next result shows that every play is realized when all the choices compatible with the play are actually taken. That is, the players' decisions ultimately do lead to the realization of a play, respectively an outcome.

Theorem 4 *Let (T, C) be an extensive decision problem with player set I. Then,*

$$\cap\{c \in \cup_{i \in I} C_i \,|w \in c\} = \{w\} \text{ for all plays } w \in W.$$

Proof. If $N = \{W\}$, i.e. the tree is trivial, there is nothing to prove. In nontrivial cases all plays pass through at least two nodes.

Let $C(w) = \{c \in \cup_{i \in I} C_i \,|w \in c\}$ and note that this set is nonempty by (iii). For, let $x, y \in N$ be such that $w \in x \cap y$ and $x \supset y$. Since a game tree satisfies Separability, there is a third node $z \in N$ such that $x \supset z$ and $y \cap z = \emptyset$. By (iii) there is $i \in I$ and disjoint choices $c, c' \in C_i$ such that $y \subseteq c$ and $z \subseteq c'$. Since $w \in y \subseteq c$, $c \in C(w)$ verifies that $C(w)$ is nonempty.

That $w \in \cap_{c \in C(w)} c$ follows from the definition. Suppose there is $w' \in W \setminus \{w\}$ such that $w' \in \cap_{c \in C(w)} c$. Because T is a game tree, there are $x, x' \in N$ such that $w \in x$, $w' \in x'$, and $x \cap x' = \emptyset$ (by Irreducibility). By (iii) there are $i \in I$ and $c, c' \in C_i$ such that $x \subseteq c$, $x' \subseteq c'$, and $c \cap c' = \emptyset$. Since $w \in x \subseteq c$, the choice c belongs to $C(w)$, so that by hypothesis $w' \in c$. But $w' \in x' \subseteq c'$ contradicts $c \cap c' = \emptyset$. □

This shows that plays "build up" from consecutive decisions by players (and/or chance) on sets of plays. Hence, the framework achieves what it was designed for. Theorem 4 does *not* say, however, that the players' decisions *as described by strategies* (defined in the next subsection below) will always induce outcomes. In this sense, representability as an extensive decision problem is only a weak consistency check.

An important feature of the present definition of an extensive decision problem is that information sets need not exist. If they do, they are given by the set of immediate predecessors of available choices. But at this level of generality nothing ensures that all choices have immediate predecessors. Still, in the following version of the differential game example information sets do exist.

Example 16 Turn the tree of the differential game (Sect. 2.2.5) into an extensive decision problem, say, with two personal players, as follows. Let the set of actions A be a product set $A = A_1 \times A_2$. Given a function $f \in V$, denote $f = (f_1, f_2)$. The interpretation is as follows. At any point in time the two players $i = 1, 2$ simultaneously decide on an action $a_i \in A_i$ for $i = 1, 2$. Up to that moment, they know the entire history, but cannot anticipate the decision taken by the other player at t. Choices are of the form $c = c_{it}(f) = \{g \in V \,|g \in x_t(f), g_i(t) = f_i(t)\}$ for some $f \in V$, some $t \in \mathbb{R}_+$, and $i = 1, 2$. (If player i were not to observe previous decisions by the other player, but recalls her own, choices would be defined only by the property that $g_i(\tau) = f_i(\tau)$ for all $\tau \in [0, t]$, rather than $g \in x_t(f)$.) When two players decide their choices at t by picking, say, $c_{it}(f^i) \in C_i$ for $i = 1, 2$, their intersection, $c_{1t}(f^1) \cap c_{2t}(f^2) = \{g \in V \,\big|(g_1(\tau), g_2(\tau)) = (f_1^1(\tau), f_2^2(\tau)), \forall \tau \in [0, t]\}$, keeps track of both decisions while leaving all possibilities open for the future. That choices are unions of nodes follows from $c_{it}(f) = \cup_{\tau > t} \cup_{g \in c_{it}(f)} x_\tau(g)$.

The current definition of an extensive decision problem excludes cases of absent-mindedness (see Piccione and Rubinstein [21]), where a play crosses an information set more than once. Because in the present framework players choose among sets of ultimate outcomes, they cannot "choose not to choose," that is, pick a choice that will become available once more, later on. Owing to (iv), if a choice is available at two distinct moves, then these moves cannot be ordered:

Proposition 13 *Let (T, C) be an extensive decision problem with player set I as in Definition 7. Then, for all $x, y \in X$,*

$$\textit{if } A_i(x) \cap A_i(y) \neq \emptyset \textit{ and } y \subseteq x \textit{ then } y = x, \textit{ for all } i \in I. \tag{23}$$

Proof. Suppose for some $i \in I$ there are $c \in C_i$ and $x, y \in N$ such that $x, y \in P(c)$, i.e. $c \in A_i(x) \cap A_i(y)$, and $y \subseteq x$. By $y \in P(c)$ there is $y' \in \downarrow c$ such that $\uparrow y = \uparrow y' \setminus \downarrow c$. Hence, $y' \subset y \subseteq x$ and $y \setminus c \neq \emptyset$. If $y \subset x$ would hold, then by (iv) there would be $c' \in A_i(x)$ such that $y \subseteq c'$, implying that $c \neq c'$ from $y \setminus c \neq \emptyset$. But then $x \in P(c) \cap P(c')$ would imply that $c \cap c' = \emptyset$ by (i), in contradiction to $y' \subseteq c \cap c'$. Hence, $y \subseteq x$ must imply $y = x$, as desired. □

Given (i), condition (iv) implies that for all $x \in X$ the collection $\{x \cap c \,|\, c \in A_i(x)\}$ is a partition of x, for all $i \in J(x)$. (For, if two choices are available at x, then by (i) they must be disjoint. But, for any $w \in x \in X$, there is a node $y \in N$ such that $w \in y \subset x$. By (iv) y must be contained in a choice $c \in A_i(x)$ available at x.) If instead of (iv) only this partitional property were required, though, some examples of absent-mindedness become feasible, as Example 15 shows. There, conditions (i)-(iii) hold, while (iv) fails. But the partitional property holds.

5.2 Strategies

Whether or not strategies induce outcomes is a more demanding question than whether strategies can merely be defined for a given extensive decision problem. The latter can be answered affirmatively in the present framework. To see this, let $X_i = \{x \in N \,|\, A_i(x) \neq \emptyset\}$ denote player i's decision points and define a *pure strategy* for player $i \in I$ as a function $s_i : X_i \to C_i$ such that, for all $c \in s_i(X_i) = \cup_{x \in X_i} s_i(x)$,

$$s_i(x) = c \text{ if and only if } x \in P(c). \tag{24}$$

By the "if"-part of (24), if a choice $c \in C_i$ is selected at all, then wherever c is available, s_i picks this choice c. The "only if"-part of (24) says that, if $c \in C_i$ is chosen by s_i, then it is chosen only where it is available.

Similarly, a *behavior strategy* for player $i \in I$ is a function ρ_i from X_i to the set of probability distributions on (a σ-algebra containing) C_i such that, for all $b \in \rho_i(X_i) = \cup_{x \in X_i} \rho_i(x)$,

$$\rho_i(x) = b \text{ if and only if } x \in \cap_{c \in \text{supp}(b)} P(c). \tag{25}$$

By the "if"-part of (25), the same probability distribution b is selected by ρ_i at all moves, where choices in the support of b are available. For, if b is a distribution for which two choices c and c' in its support do not have the same predecessor set, then by (i) their predecessor sets are disjoint, i.e. there is no $x \in P(c) \cap P(c')$. Thus, ρ_i can only assign a distribution for which all choices in its support have the same predecessor sets. If b is such a distribution, $x, x' \in \cap_{c \in \text{supp}(b)} P(c)$, $\rho_i(x) = b$, and $\rho_i(x') = b'$, then the "if"-part of (25) implies $b = b'$. Likewise, by the "only if"-part of (25), if $\rho_i(x) = b$, then $x \in P(c)$ for all choices $c \in C_i$ in the support of b, i.e., ρ_i assigns to $x \in X_i$ only distributions supported on choices available at x.

These specifications illustrate that the familiar strategy notions can be defined naturally for the present concept of an extensive decision problem.

6 Discussion

This paper studies how arbitrary trees can be represented by a collection of sets. The purpose of such a representation is to provide a domain for sequential decision theory. To do this requires two things: First, a node should be an event in the sense of probability theory, i.e. a set of states. Second, the elements of the nodes/sets should have meaning as representatives of ultimate outcomes. We show that both desiderata can be met without any substantial loss of generality by the current definition of a game tree: a collection of subsets of an underlying set (of plays) such that (i) a family of those subsets is a chain if and only if all its elements (nodes) contain a common element (play), and (ii) for any two distinct elements (plays) there are two sets (nodes) such that the first set (node) contains the first element (play), but not the second, and the second set (node) contains the second element (play), but not the first.

This definition is essentially equivalent to the notion of a decision tree, i.e., all requirements on top of being a set representation of an (order-theoretic) decision tree are purely modelling conventions. At the same time it captures the intuition that nodes and plays could both serve as the primitives for the model. Intuitively, Theorem 3 says that game trees are characterized by

$$\text{“} x \in w \Leftrightarrow w \in x \text{”}$$

for nodes $x \in N$ and plays $w \in W$ (abusing notation, of course).

As an application we show that game trees are sufficient to define extensive decision problems by adding sets of choices for all players. The traditional strategy notions can then be translated into this general framework.

Some problems remain open for further research, though. For instance, with the present generality nodes need not have immediate predecessors (even if choices do), as in the differential game example (Sect. 2.2.5). This poses a problem with alternating moves, as they appear in games of perfect information. In the differential game example with two players, one could let A_2 be the set of functions from A_1 to an ultimate action space for player 2, modelling that player 2's decision conditions on what player 1 has chosen. But, in the tree, the two players' decision would formally be taken simultaneously. Ideally, one would like to let player 1 move at the immediate predecessor of player 2's decision points. In continuous time, however, immediate predecessors do not exist. On the other hand, alternating moves could be viewed as a property of the situation that is to be modelled by the tree, so that the tree inherits a discrete structure from what it models.

The results presented in this paper refer to representability of the strategic situation. They say nothing about whether or not an extensive form game can be "solved" in terms of strategies, let alone equilibria. An open issue is under which conditions (pure or behavior) strategy combinations induce unique outcomes. Even if pure strategies do, there also remains, of course, a measurability issue whether behavior strategy combinations induce well defined probability distributions on plays.[11] These questions are left for future research.

A Appendix

Proof of Lemma 3. "if:" Let $(N, \geq)$ be a tree and $x, y \in N$ such that $W(x) = W(y)$. Then, for any $w \in W(x) = W(y)$, that $x, y \in w$ implies either $x \geq y$ or $y \geq x$ (or both), because $w \in W$ is a chain. Assume, without loss of generality, that $x \geq y$. Suppose $y \not\geq x$. Then, by (7) there exists $z \in \downarrow x$ such that $z \not\geq y$ and $y \not\geq z$. Since $W(z) \subseteq W(x)$ by Lemma 2(b), $y \notin w$ for all $w \in W(z)$. Hence, $W(y) \subseteq W(x) \setminus W(z)$ contradicts $W(x) = W(y)$. Thus, also $y \geq x$ must hold, so that $x = y$ (by antisymmetry) verifies (6).

"only if:" Let $(N, \geq)$ be a decision tree, and let $x, y \in N$ such that $x \geq y$ and $y \not\geq x$. By Lemma 2(b), $W(x) \supseteq W(y)$. By (6), $W(x) \supset W(y)$, i.e. there exists $w \in W(x) \setminus W(y)$. For any $z \in w$, either $z \geq x$ or $x \geq z$. If $z \geq x$, transitivity implies $z \geq y$. Hence, there must be some $z \in w$ such that $x \geq z$ and both $z \not\geq y$ and $y \not\geq z$ hold. For, otherwise for all $z \in w$ either $z \geq y$ or $y \geq z$, which implies that $w \cup \{y\}$ is a chain. By maximality of $w \in W$, it follows that $y \in w$ and $w \in W(y)$, a contradiction. □

Proof of Lemma 6. First, let $[v] \in V/\sim$ be such that $\uparrow[v] = \emptyset$. Then, $\cap_{a \in \uparrow[v]} a = \emptyset$, i.e. $[v]$ is not separable. Since there is no $a \in M$ such that $[v] \subseteq a$, the property is false, verifying the equivalence in this case. Let now $[v] \in V/\sim$ be such that $\uparrow[v] \neq \emptyset$. Note that $[v] \subseteq \cap_{a \in \uparrow[v]} a$ whenever $\uparrow[v] \neq \emptyset$. Then, $[v]$ is separable if and only if $[v] = \cap_{a \in \uparrow[v]} a$, or, equivalently, $V \setminus [v] = V \setminus \cap_{a \in \uparrow[v]} a = \cup_{a \in \uparrow[v]} (V \setminus a)$, which proves the claim. □

Proof of Lemma 7. For all $v \in a \cap b$ we have $[v] \subseteq \cap_{c \in \uparrow[v]} c \neq \emptyset$. If there is $v \in a \cap b$ such that $\cap_{c \in \uparrow[v]} c \subseteq [v]$ the statement is verified. Hence, suppose that $[v] \subset \cap_{c \in \uparrow[v]} c$ for all $v \in a \cap b$. But then $a \cap b = \cup_{v \in a \cap b} [v] \subset \cup_{v \in a \cap b} (\cap_{c \in \uparrow[v]} c) \subseteq a \cap b$ yields a contradiction. □

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[11] For instance, consider the well-known problem with the law of large numbers arising if a player tosses a coin repeatedly in continuous time (e.g. Judd [15]). For the purpose of completing an extensive decision problem to a full-fledged game, these problems can be bypassed by defining payoff functions directly on the space of strategy combinations.

Proof of Lemma 11. (a) Assume that $(M, \supseteq)$ satisfies Trivial Intersection and let $a, b \in M$ be such that $a \cap b \neq \emptyset$. Then either $a \subseteq b$ or $b \subset a$; hence, by order isomorphism, (2), either $\varphi(a) \subseteq \varphi(b)$ or $\varphi(b) \subset \varphi(a)$, i.e. $\varphi(a) \cap \varphi(b) \neq \emptyset$.
(b) Suppose $(M', \supseteq)$ satisfies Trivial Intersection. If is φ proper and $a \cap b \neq \emptyset$, for $a, b \in M$, then by (15) $\varphi(a) \cap \varphi(b) \neq \emptyset$. Therefore, either $\varphi(a) \subseteq \varphi(b)$ or $\varphi(b) \subset \varphi(a)$; by order isomorphism, (2), either $a \subseteq b$ or $b \subset a$. The "only if"-part follows from (a).
(c) By (b) and Lemma 4 it is enough to establish Weak Separability, under the hypothesis that $(M', \supseteq)$ is a V'-set tree. But Weak Separability is preserved by order isomorphism. Hence, this is immediate. □

Proof of Lemma 12. Assume that $(M, \supseteq)$ is isomorphically embedded in $(M', \supseteq)$. Let $v, w \in V$ such that $v \neq w$. Since the mapping f is one-to-one, $f(v) \neq f(w)$. By Irreducibility for $(M', \supseteq)$, there are $a', b' \in M'$ such that $f(v) \in a' \backslash b'$ and $f(w) \in b' \backslash a'$. Let $a, b \in M$ such that $a' = \varphi(a)$ and $b' = \varphi(b)$, where φ is the order isomorphism. Since $f(v) \in a'$ and f is one-to-one, it follows that $v \in a$. Since $f(v) \notin b'$, it follows that $v \notin b$. Hence, $v \in a \backslash b$ and, analogously, $w \in b \backslash a$. □

Proof of Lemma 13. "if:" Suppose that $(M, \supseteq)$ satisfies that $c \in 2^M$ is a chain if and only if there is $v \in V$ such that $v \in a$ for all $a \in c$. Then, for any chain $c \in 2^M$ the "only if" part implies that there is $v \in V$ which forms a lower bound on c. Furthermore, if $a, a' \in M$ are such that $a \cap a' \neq \emptyset$, then that there is $v \in a \cap a'$ implies from the "if" part that $\{a, a'\} \in 2^M$ is a chain, i.e., either $a' \subset a$ or $a \subseteq a'$, verifying Trivial Intersection.

"only if:" Suppose $(M, \supseteq)$ satisfies Trivial Intersection and every chain in M has a lower bound in V. Then if $c \in 2^M$ is a chain, there is $v \in V$ such that $v \in a$ for all $a \in c$. On the other hand, if $v \in a$ for all $a \in c$ for some $c \in 2^M$, then if $a, a' \in c$ that $v \in a \cap a'$ implies from Trivial Intersection either $a' \subset a$ or $a \subseteq a'$, i.e., $c \in 2^M$ is a chain. □

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