Some remarks on pseudotrees

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Abstract

This paper provides new results on pseudotrees. First, it is shown that pseudotrees are precisely those posets for which consistent sets, directed sets, and nonempty chains coincide. Second, we show that chain-complete pseudotrees yield complete meet-semilattices. Third, we prove that pseudotrees are precisely those posets that admit a set representation by sets of appropriate chains. This latter result generalizes results needed for applications in game theory and economics.

1 Preliminaries

A subset A of a partially ordered set (poset) P is a *chain* if it is totally ordered under the restricted order. Equivalently, A is a chain if every finite, nonempty subset F of A has an upper bound in F (hence a maximum).

There are two notable generalizations of the concept of (nonempty) chains, that differ only with respect to which set contains the upper bound from the previous sentence. A subset A of a poset P is *directed* if it is nonempty and every finite subset of A has an upper bound *in A*. Note that all chains are directed sets. A subset A of a poset P is *consistent* if it is nonempty and every finite subset of A has an upper bound *in P*. Obviously, all directed sets and hence all chains are consistent.

A poset is totally or linearly ordered if and only if it is a chain. A generalization of the concept of linearly ordered set has been recently proposed and studied in the theory of Boolean Algebras. A *pseudotree* is a poset $(T, \leq)$ such that, for each $t \in T$, the down-set (or principal order ideal, see [6, p. 185])

$$\downarrow t = \{x \in T | x \leq t\}$$

is a chain (see e.g. [8]).[1] A pseudotree is a *tree* if these down-sets are well-ordered, i.e. for all $t \in T$ every subset of $\downarrow t$ has a first element (a minimum) (see e.g. [7]).

A *dual pseudotree* is a poset $(T, \geq)$ such that, for each $t \in T$, the up-set (or principal order filter)

$$\uparrow t = \{x \in T | t \leq x\}$$

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[1] Other authors call such a poset simply a tree, e.g. [6, Exercise 1.7]

is a chain, i.e. if $(T, \leq)$ is a pseudotree. Of course, working with pseudotrees or dual pseudotrees is purely a matter of convention, as one can be converted to the other by reversing the partial order. We follow here the notation of [8].

A poset $(P, \leq)$ has a *bottom element* if there exists $x_0 \in P$ such that $x_0 \leq x \, \forall \, x \in P$. A *top element* is defined dually. A pseudotree is *rooted* if it has a bottom element; notice, though, that general pseudotrees need not be rooted.

Koppelberg and Monk [8] study the *pseudotree algebra* of a pseudotree $(T, \leq)$, i.e. the set algebra generated by $\{\uparrow x | x \in T\}$. Baur and Heindorf [4] and Baur [3] study the *initial chain algebra*, i.e. the algebra generated by $\{\downarrow x | x \in T\}$. Alós-Ferrer and Ritzberger [1, 2] study the set representation of certain classes of pseudotrees which are of special interest in sequential decision theory and game theory. In these latter papers, we use dual pseudotrees (there called simply trees) for notational convenience, due to the fact that the natural set representations are of the form $(N, \supseteq)$, with $\supseteq$ being set inclusion.

2 Directed Sets and Pseudotrees

Often a result holds for all directed sets in a poset if and only if it holds for all chains. An outstanding example is the following. A poset $(P, \leq)$ is a *CPO* (short for Complete Partially Ordered set) if it has a bottom element and every directed subset $D \subseteq P$ has a supremum (join) $\bigvee D$ (see e.g. [6]). A poset is *chain-complete* if every chain, including the empty chain, has a supremum. It is a well-known but nontrivial result (whose proof uses the machinery of ordinals, see [10]) that a poset is a CPO if and only if it is chain-complete.

Results like this one raise the question of how much does the generalization of the concept of chain to that of directed sets adds to the understanding of general posets. A complementary question could be phrased like this: for which posets are all directed sets chains? In such posets, results like the one quoted above would collapse to trivial statements.

The next (straightforward) proposition shows that any poset such that all its directed sets are chains is necessarily a pseudotree. The converse is even stronger, since in a pseudotree, all consistent sets are chains.

Proposition 1. *Let $(P, \leq)$ be a poset. The following statements are equivalent:*

(a) $(P, \leq)$ is a pseudotree.

(b) All consistent subsets of P are chains.

(c) All directed subsets of P are chains.

Proof. **(a) implies (b).** Let $(P, \leq)$ be a pseudotree. Let C be a consistent subset of T. Let $d, d' \in C$. Since C is consistent, there exists an upper bound in T for $\{d, d'\}$, i.e. $x \in P$ such that $d \leq x$ and $d' \leq x$. It follows that $d, d' \in \downarrow x$. Since $\downarrow x$ is a chain, either $d \leq d'$ or $d' \leq d$, proving that C is a chain.

(b) implies (c). Immediate.

(c) implies (a). Let $(P, \leq)$ be a poset such that all its directed subsets are chains. Let $x \in P$. Since $x \in \downarrow x$ is an upper bound for all subsets of $\downarrow x$, it follows that $\downarrow x$ is directed and hence a chain. It follows that $(P, \leq)$ is a pseudotree. ■

This result offers a characterization of pseudotrees as the class of posets where the generalization of the concept of chain to directed (or consistent) sets is vacuous. Note that pseudotrees provide an example of posets where all consistent sets are actually directed, i.e. the generalization of the concept of directed sets to consistent sets is vacuous. One may ask whether there are other examples of posets satisfying this property, i.e. posets where all consistent sets are directed but some directed sets are not chains. The next proposition shows that the answer is negative.

Proposition 2. *A poset is a pseudotree if and only if all its consistent sets are directed.*

Proof. The "only if" implication follows from Proposition 1. To see the converse, let $(P,\leq)$ be poset such that all its consistent subsets are directed.

Let $x \in P$. We have to show that $\downarrow x$ is a chain. Let $y,z \in \downarrow x$. Since $y \leq x$ and $z \leq x$, it follows that the set $\{x,y\}$ is consistent. By hypothesis, it must then be directed. Since $\{y,z\}$ is a finite subset of itself, it follows that either $y \leq z$ or $z \leq y$, showing that $\downarrow x$ is a chain. ■

Given a poset P, consider the three classes of subsets given by nonempty chains, directed sets, and consistent sets. Propositions 1 and 2 show that it is enough that any two of these three classes coincide to conclude that P is a pseudotree. Reciprocally, it follows that, for any poset which is not a pseudotree, there must exist at least a consistent subset which is not directed, and a directed subset which is not a chain.

3 Pseudotrees as (Semi)Lattices

In view of Proposition 1, we obtain the following, immediate Corollary.

Corollary 3. *A pseudotree is a CPO if and only if it is chain-complete.*

Of course, this is just a restatement of the celebrated result in [10], which holds for arbitrary posets. We merely remark here that the result is trivial for pseudotrees. This observation, though, points out that chain-complete pseudotrees have a strong algebraic structure. Indeed, we want to argue that such pseudotrees are a subclass of *semilattices*, and they turn into lattices under minimal, additional requirements.

Recall the following concepts. A poset $(P,\leq)$ is a *lattice* if any two elements $x,y \in P$ have a least upper bound (supremum), denoted $x \vee y$ (the *join* of x and y) and a greatest lower bound (infimum), denoted $x \wedge y$ (the *meet* of x and y). A lattice is *complete* if every subset $S \subseteq P$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$. Since $S = \emptyset$ is allowed, any complete lattice has both top and bottom elements.

A *meet-semilattice* is a poset such that any two elements have an infimum. A complete meet-semilattice is a CPO satisfying any of the equivalent conditions of the following, well-known Lemma (see [6, Lemma 3.20]).

Lemma 4. *Let $(P,\leq)$ be a CPO. The following conditions are equivalent:*

(SL1) *P is* consistently complete*, i.e. the join $\bigvee S$ exists for every consistent subset S of P.*

(SL2) *The join $\bigvee S$ exists whenever $S \subseteq P$ has an upper bound.*

(SL3) *The meet $\bigwedge S$ exists whenever $S \subseteq P$ is nonempty.*

(SL4) *$\downarrow x$ is a complete lattice for all $x \in P$.*

(SL5) $P \bigoplus 1$ *is a complete lattice, where* $P \bigoplus 1$ *denotes the poset formed by adding a new element (the 1) to P and extending the order on P in such a way that 1 is a top element, i.e.* $x \leq 1$ *for all* $x \in P$.

In view of this definition, it is immediate to observe that chain-complete pseudotrees (i.e. pseudotrees which are CPOs) coincide with complete meet-semilattices.

Proposition 5. *A pseudotree* $(T, \leq)$ *is chain-complete if and only if it is a complete meet-semilattice.*

Proof. Suppose $(T, \leq)$ is chain-complete. By Proposition 1(b), the fact that every chain has a supremum implies condition (SL1) in Lemma 4.

Suppose now $(T, \leq)$ is a complete meet-semilattice. Then, it is a CPO by definition, hence chain-complete by Corollary 3. ■

Hence, conditions (SL1-5) hold for all chain-complete pseudotrees. In particular, adjoining a top to a chain-complete pseudotree yields a complete lattice. We now show that a weaker property characterizes the class of rooted pseudotrees which turn into (not necessarily complete) lattices when a top element is adjoined.

Baur and Heindorf [4] introduce the concept of *well-met pseudotrees*: A pseudotree is *well-met* if any two compatible elements[2] have a greatest lower bound (infimum). They show [4, Corollary 1.3] that any pseudotree can be monomorphically embedded in a well-met pseudotree.

In a rooted pseudotree all elements are compatible. Hence, in a well-met, rooted pseudotree, any two elements have an infimum. Thus, a rooted pseudotree is well-met if and only if it is a meet-semilattice. It is well-known, though, that adjoining a top element to a general meet-semilattice does *not* necessarily yield a lattice.[3]

Proposition 6. *Let* $(T, \leq)$ *be a rooted pseudotree. Then* T *is well-met (i.e. a meet-semilattice) if and only if* $T \bigoplus 1$ *is a lattice.*

Proof. “if": Suppose that $T \bigoplus 1$ is a lattice, and let $x, y \in T$ (which are compatible because T is rooted). Since $T \bigoplus 1$ is a lattice, there exists an infimum z for x and y in $T \bigoplus 1$. If $z = 1$, it follows that $x = z \notin T$, a contradiction, and hence $z \in T$, proving the claim.

“only if": Consider any two elements $x, y \in T$. Either they are related by $\leq$ or not. If $x \leq y$, y is the supremum of x and y, and reciprocally if $y \leq x$. If x, y are unrelated by $\leq$, then there can exist no upper bound. For, if $x \leq z$ and $y \leq z$, then $x, y \in \downarrow z$, which is a chain, a contradiction. Thus, once the top element is added, it becomes the least upper bound of any pair of unrelated elements. ■

Finally, we observe that the class of well-met, rooted pseudotrees is larger than the class of chain-complete pseudotrees. This result also follows indirectly from Proposition 6 and (SL5).

Proposition 7. *Every chain-complete pseudotree is well-met.*

Proof. If $x, y \in T$ are compatible, the set $S = \downarrow x \cap \downarrow y$ is nonempty. Since both $\downarrow x$ and $\downarrow y$ are chains, S has a supremum, z. Since both x and y are upper bounds for S, it follows that $z \in S$ and hence it is a greatest lower bound for x and y. ■

[2] Two elements x, y of a poset $(P, \leq)$ are compatible if there exists some $z \in P$ such that $z \leq x$ and $z \leq y$.

[3] This is true if all chains have finite length (see [5, p.23]), but, as the next Proposition shows, this condition is not necessary.

The converse is not true, as the pseudotree $(\mathbb{Z}, \leq)$ shows.

Notice that, in the proof of the last Proposition, we only make use of the property that every *nonempty* chain in T has a supremum in T. The difference with chain-completeness is small but significative, since a supremum for the empty chain is necessarily a bottom element, i.e. a root for the pseudotree.

4 Representation of pseudotrees

Trees and pseudotrees are of particular interest to economics and game theory, where they serve as a transparent model of sequential decision making. Intuitively, nodes (elements) represent decision points belonging to some decision maker. Indeed, a key ingredient of an *extensive form game* (as defined by [9], see also [12, 11]) is the tree describing the possible decisions.

As game theory has moved beyond the confines of discreteness, trees have proved insufficient. For example, decision problems in continuous time cannot be modelled with trees. In a related paper [1], we have shown that certain classes of pseudotrees (there simply called *trees*) constitute a necessary element for a general theory of sequential decision making.

The emphasis in game theory is on *plays* and outcomes, as those provide a domain for decision-makers' preferences. Intuitively, an outcome is the final result of the chain of decisions taken in the course of a game, while a play (or maximal history) is a full record of all decisions taken. Following [12], we may think of decisions taken along the trees as sampling subsets of a universal set of outcomes. We should then be able to identify each node (element of the tree) with the set of outcomes (or plays) which have still not been discarded at the decision point represented by the node.

This intuition has been first formalized in [1]. Mathematically, it consists of representation theorems for certain classes of pseudotrees. We now review the necessary concepts and present a generalization which, in our view, sheds light on the concept of a pseudotree.

Let $(P, \leq)$ be a nonempty poset. A chain $w \subseteq P$ is maximal if there exists no $x \in P \setminus w$ such that $w \cup \{x\}$ is a chain. That is, a maximal chain is a *play* in the game-theoretic sense. Let W denote the set of maximal chains of $(P, \leq)$ and, for each $x \in P$, denote by $W(x) = \{w \in W | x \in w\}$ the set of maximal chains "passing through" x. By the Hausdorff Maximality Principle, $W(x)$ is nonempty for all $x \in P$ (see e.g. [5, Chapter VIII]). We can define the poset $(W(P), \supseteq)$, where $W(P) = \{W(x) | x \in P\}$ and $\supseteq$ is set inclusion, and consider $W(\cdot)$ as a mapping from P to $W(P)$. Then, the following result holds:

Lemma 8. *Let $(T, \leq)$ be a pseudotree. If $x, y \in T$ are such that $x \leq y$, then $W(x) \supseteq W(y)$.*

Proof. Let $w \in W(y)$. Notice that $\downarrow y \subseteq w$ by maximality of w. To see this, fix $z \in \downarrow y$, and consider any $t \in w$. Since w is a chain and $y \in w$, either $y \leq t$ or $t \leq y$. In the first case, $z \leq y \leq t$ implies $z \leq t$. In the second case, $z, t \in \downarrow y$ implies (since $\downarrow y$ is a chain that either $z \leq t$ or $t \leq z$. This proves that $w \cup \{z\}$ is a chain, which, by maximality of w, implies $z \in w$.

Now, suppose $x \leq y$, and consider any $w \in W(y)$. By the previous statement, $x \in w$, i.e. $w \in W(x)$, proving the claim. ■

A *decision pseudotree* is a pseudotree $(T, \leq)$ such that if $W(x) = W(y)$ for two elements $x, y \in T$, then necessarily $x = y$. The name comes from the fact that, in a decision pseudotree, every element has to be followed (in the sense of $\leq$) by at least two other elements. Elements of the pseudotree can be identified with “decision points" (see [1] for details). It is straightforward to show that a pseudotree $(T, \leq)$ is a decision pseudotree if and only if the mapping $W(\cdot)$ is an order isomorphism. (This fact will become a corollary of the results proved below.) The pseudotree $(W(T), \supseteq)$ is then a set representation of $(T, \leq)$.

While decision pseudotrees are or particular interest for decision theory,[4] from the theoretical point of view the question remains whether a similar characterization can be given for general pseudotrees. The question is also of interest for game theory, for certain definitions of extensive form games (notably the one in [11]) make use of trees which are not decision trees.

In this section such a characterization is provided. Of course, in a tree which is not a decision tree, there might exist distinct elements x, y such that $W(x) = W(y)$. Intuitively, there is no decision to be taken at x, and y immediately follows x. Thus, a representation theorem cannot be based on maximal chains alone. Instead, it must rely on a weaker concept.

Given a poset $(P, \leq)$, say that a chain $w \subseteq P$ is an *extensible chain* if there exists $x \in P$ such that $w = \downarrow x$ and

$$\bigcap \left\{ w' \in W \;\middle|\; w \subseteq w' \right\} \setminus w \neq \emptyset.$$

In words, a chain of the form $w = \downarrow x$ is extensible if all maximal chains that contain it have a common part outside w. In decision-theoretic terms, this common part comes forcefully “after” w. Hence, a decision at x cannot “discard” anything, but becomes a trivial “decision.” For the class of pseudotrees the following characterization of extensible chains obtains.

Lemma 9. *Given a pseudotree $(T, \leq)$, a chain $w \subseteq T$ is an extensible chain if and only if there exists $x \in T$ and $y \in \uparrow x \setminus \{x\}$ such that $w = \downarrow x$ and $W(x) = W(y)$.*

Proof. Suppose $w = \downarrow x$ is extensible, and let $y \in \bigcap \{ w' \in W \mid w \subseteq w' \}$ such that $y \notin w$. It follows that $y \in \uparrow x \setminus \{x\}$. By Lemma 8, $W(y) \subseteq W(x)$ and thus we only have to show that $W(x) \subseteq W(y)$.

Let $w' \in W(x)$, i. e. $x \in w' \in W$. Since w' is a maximal chain, $w = \downarrow x \subseteq w'$. By the choice of y, we have that $y \in w'$, thus $w' \in W(y)$. This completes the proof of the “only if” implication.

Suppose now $w = \downarrow x$ and $W(x) = W(y)$ with $y \in \uparrow x \setminus \{x\}$, $x \in T$. Let $w' \in W$ such that $w \subseteq w'$. Since $w = \downarrow x$, we have that $w' \in W(x) = W(y)$, thus $y \in w'$. Since $y \notin w$, this proves that w is extensible. ■

Denote by Ω the set of chains of $(P, \leq)$ which are either maximal or extensible and, given $x \in P$, let $\Omega(x) = \{w \in \Omega | x \in w\}$. Obviously, $W(x) \subseteq \Omega(x)$. We can consider $\Omega(\cdot)$ as a mapping from $(P, \leq)$ to the poset $(\Omega(P), \supseteq)$, where $\Omega(P) = \{\Omega(x) | x \in P\}$ and $\supseteq$ is set inclusion. Then, the next Theorem shows that this latter poset yields a set representation of $(P, \leq)$.

Example 1. *Let $P = \{x, y, z_1, z_2\}$ and $\leq$ be given by $x \leq y, z_1, z_2$ and $y \leq z_1, z_2$. $(P, \leq)$ is a pseudotree. There are only two maximal chains, $w_1 = \{x, y, z_1\}$ and $w_2 = \{x, y, z_2\}$.*

[4] The focus in [1] is on the representation of decision (pseudo)trees by sets of maximal chains.

Further, $W(x) = \{w_1, w_2\} = W(y)$, and thus $W(P)$ does not yield a set representation of P. However, there is an extensible chain $w_0 = \{x\}$, thus $\Omega(x) = \{w_0, w_1, w_2\} \supsetneq \{w_1, w_2\} = \Omega(y)$, and $\Omega(P)$ yields the desired set representation.

Theorem 10. *A poset $(P, \leq)$ is a pseudotree if and only if the mapping Ω is an order isomorphism.*

Proof. "if": Suppose that $\Omega(\cdot)$ is an order isomorphism. Let $z \in P$. We have to show that $\downarrow z$ is a chain. Let $x, y \in \downarrow z$. By order isomorphism, it follows that $\Omega(z) \subseteq \Omega(x) \bigcap \Omega(y)$. Let $w \in \Omega(z)$. Then, $x, y \in w$, and the conclusion follows from the fact that w is a chain.

"only if": Suppose $(P, \leq)$ is a pseudotree. The mapping $\Omega(\cdot)$ is surjective by construction. Since any order-embedding surjection is an order isomorphism, it is enough to show that $\Omega(\cdot)$ is order-embedding, i.e. $x \leq y$ if and only if $\Omega(x) \supseteq \Omega(y)$.

Let $x, y \in P$ such that $x \leq y$. Let $w \in \Omega(y)$. If $w \in W(y)$, then, by Lemma 8, $w \in W(x) \subseteq \Omega(x)$. If $w \notin W(y)$, then by Lemma 9 there exists $z \in P$ such that $w = \downarrow z$. Since $x \leq y \in w$, it follows that $x \leq z$ and thus $x \in \downarrow z = w$, implying that $w \in \Omega(x)$. Hence, $\Omega(x) \supseteq \Omega(y)$.

Suppose now $\Omega(x) \supseteq \Omega(y)$ for some $x, y \in P$. In particular, this implies that $W(x) \supseteq W(y)$. Let $w \in \Omega(y)$; then, $x, y \in w$ and, since w is a chain, either $x \leq y$ or $y \leq x$. Suppose $y \leq x$ and $x \neq y$. That $y \leq x$ implies $W(x) = W(y)$ by Lemma 8 and the hypothesis. Then, that $x \in \uparrow y \setminus \{y\}$ implies that $w^* = \downarrow y$ is an extensible chain. But $w^* \in \Omega(y)$ implies $\omega \in \Omega(x)$ by the hypothesis, in contradiction to $x \notin w^* = \downarrow y$. ■

As a consequence of Theorem 10, a characterization of decision pseudotrees is obtained.

Corollary 11. *A pseudotree is a decision pseudotree if and only if there are no extensible chains.*

Proof. "if": Let $(T, \leq)$ be a pseudotree. Let $x, y \in T$ such that $W(x) = W(y)$. If there are no extensible chains, $\Omega(x) = \Omega(y)$ and, by Theorem 10, $x = y$.

"only if": Let $(T, \leq)$ be a decision pseudotree, and let w be an extensible chain. Then, by Lemma 9 there exist $x, y \in T$ such that $w = \downarrow x$, $y \in \uparrow x \setminus \{x\}$, and $W(x) = W(y)$. Since $(T, \leq)$ is a decision pseudotree, we must have $x = y$, a contradiction. ■

That $(T, \leq)$ is a decision pseudotree if and only if the mapping $W(\cdot)$ is an order isomorphism is an obvious corollary of Theorem 10 and Corollary 11. This is Theorem 1 in [1].

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