Beliefs Revealed in Bayesian-Nash Equilibrium

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Abstract

Standard belief hierarchies are an insufficient description of uncertainty for the Bayesian-Nash Equilibrium (BNE) solution concept. Two states with the same belief hierarchy profile may have different sets of BNE action profiles, for many games. We construct new hierarchy profiles with explicit beliefs about payoff irrelevant signals. We show that BNE can be described within those hierarchy profiles and is characterized by Bayesian rationality conditions. We also show that those hierarchy profiles are minimal, revealed by the BNE play, in the following sense: if any two states differ on them, then there is a game for which they have different sets of BNE action profiles.

1 Introduction

Harsanyi, in a series of papers (cf. [17]), introduced the Bayesian games framework for modelling and solving games with incomplete information. Half of the novelty consisted in replacing decision theoretically well motivated but unwieldy hierarchy of beliefs with a simple device, a type space, as a representation of incomplete information. The other half was the introduction of Bayesian-Nash equilibrium (BNE) as the solution concept. The resulting framework - tandem of BNE and type spaces - paved the way for the contemporary literature on auctions, bargaining, insurance, moral hazard, principal-agent, rational expectations, repeated games, reputation, signalling etc.

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Even though the Bayesian games framework is unquestionably *the* framework used for modelling and solving strategic situations with incomplete information, its decision theoretic foundation is flawed, in the following sense. Type spaces and belief hierarchies are not interchangeable. The former are genuinely more detailed description of incomplete information environment. Moreover, the added level of detail is crucial for the BNE prediction: two profiles of types corresponding to the same profile of hierarchies can give rise to two different sets of action tuples that can be played in BNE (see Bergmann and Morris [6], Battigalli and Siniscalchi [5]). Accordingly, it becomes unclear what a type space really represents. Furthermore, the program of characterizing the possible BNE play in terms of rationality conditions on belief hierarchies is not well defined. We cannot characterize BNE solution concept in terms of Bayesian rationality.

Let us illustrate the problems with the following example, based on Ely and Peski [13].

Example 1 *(Distributed Information) Consider the environment with agents* 1 *and* 2 *and two payoff relevant states of nature,* $\Omega = \{H, L\}$*. The incomplete information, as described by a belief hierarchy, is the following: Each agent deems states* H *and* L *equally likely. Those are first order beliefs. Furthermore, each agent puts probability* 1 *to each agent having such beliefs, each agent puts probability* 1 *to each agent putting probability* 1 *... - those are second, third ... order beliefs. In other words, there is common belief that the agents find* H *and* L *to be equally likely.*

Now consider the following two type spaces modelling this environment. In the type space $S := \{S_i, \mu_i^S\}_{i=1,2}$ *each player has only one possible type,* $S_i = \{s_i\}$*,* $i \in \{1, 2\}$ *and the beliefs are described by* $\mu_i^S : S_i \to \Delta(\Omega \times_{i=1,2} S_i)$ *such that* $\mu_i^S(s_i)(\{(H, s_1, s_2)\}) = \mu_i^S(s_i)(\{(L, s_1, s_2)\}) = 1/2$*. In the type space* $T := \{T_i, \mu_i^T\}_{i=1,2}$ *each player has two possible types,* $T_i = \{t_i, t'_i\}$*. The beliefs are described by* $\mu_i^T : T_i \to \Delta(\Omega \times_{i=1,2} T_i)$ *with* $\mu_i^T(t_i)(\{(H, t_i, t_{-i})\}) = \mu_i^T(t_i)(\{(L, t_i, t'_{-i})\}) = 1/2$, $\mu_i^T(t'_i)(\{(H, t'_i, t'_{-i})\}) = \mu_i^T(t'_i)(\{(L, t'_i, t_{-i})\}) = 1/2$.[1]

[1] In Figures 1 and 2 probabilistic beliefs are obtained by conditioning the uniform distribution over the elements in $\Omega \times_{i=1,2} T_i$ and $\Omega \times_{i=1,2} S_i$ on the information partition of each player.

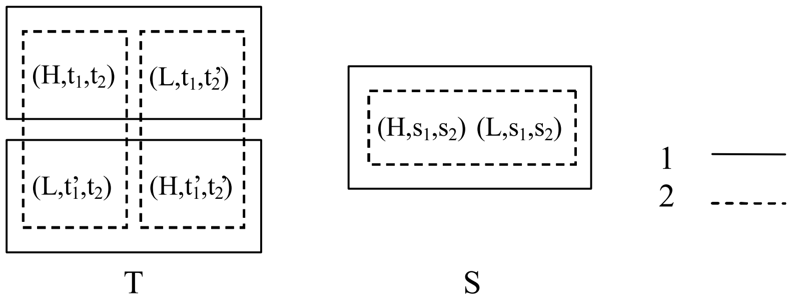

Figure 1: Type spaces T and S.

Consider the game Γ with payoffs at H and L given by:

$H:$	l	r	s
u	$3,3$	$0,0$	$0,2$
d	$0,0$	$3,3$	$0,2$
s	$2,0$	$2,0$	$2,2$

$L:$	l	r	s
u	$0,0$	$3,3$	$0,2$
d	$3,3$	$0,0$	$0,2$
s	$2,0$	$2,0$	$2,2$

There is a BNE for Γ over the type space T such that agents always coordinate on the action profile optimal for H or L, and so $(3,3)$ is the payoff at every state: t_1 *plays* u , t_1' *plays* d, t_2 *plays* l *and* t_2' *plays* r. *However, it is easy to see that the unique BNE for Γ over the type space S has both agents play the "safe" actions (s,s). In particular, types (t_1,t_2) in T and (s_1,s_2) in S model the same pair of belief hierarchies, but have different action profiles played in BNE for Γ.*

In the example above, intuitively, even though all pairs of types in T and S are described by the same pair of belief hierarchies, only in T the agents have distributed information: if they were to pool their information, they would identify whether the state of nature is H or L. It enables the agents to appropriately correlate their actions in equilibrium, but is not reflected in the belief hierarchies. This case of distributed information suggests that some relevant aspects of incomplete information are left out from the hierarchy description. Moreover, for the hierarchy description and the game as in the example, the choice of e.g. action u by player 1 sometimes appears to be rational, and sometimes not, depending on the type space.

This gap between type spaces and BNE on the one hand, and belief hierarchies and Bayesian rationality on the other leads to the following series of questions. First, what is the implicit feature of a type space that is crucial for BNE? Since a type space is often considered to be just a convenient, thought model for representing beliefs of the agents (see e.g. Gul [16]), does an explicit description that extends standard belief hierarchies exist? (In particular, such description has to distinguish the profile of types (t_1, t_2) in T from (s_1, s_2) in S in Example 1.) Secondly, belief hierarchies should not only provide a sufficient description of a state, i.e. a profile of types, but also have no spurious, redundant information. Is there any principled way of choosing the right definition of belief hierarchies for the BNE? Lastly, given the new belief hierarchies, can we characterize the BNE solution concept in terms of Bayesian rationality?

The purpose of our article is to answer those questions. We define new hierarchy profiles, called "$X-$belief hierarchy profiles", which provide richer, yet readily interpretable description of a state. Our main contribution is twofold. First, by employing a type of revelation argument in the multiple agent, equilibrium setting, we show that the $X-$belief hierarchy profile of a state in a type space is characterized by the local BNE behavior at this state, for all games. Locality here means that we look at BNE over the generated sub-type space (see Mertens and Zamir [22]). It follows that the $X-$belief hierarchy profile description of an incomplete information state is rich enough, yet has no redundant information, as measured by the yardstick of local BNE solution concept. Second, the fact that for some game a certain action tuple can be played in a local BNE by a profile of types can be expressed *within* the $X-$belief hierarchy profile, by the rationality and common belief of rationality conditions on beliefs. In other words, we provide an epistemic justification for local BNE.

In the following we provide more details about the results and review the related literature.

1.1 $X-$belief hierarchy profiles

A profile of types in a type space is supposed to represent a complete, comprehensive epistemic state of affairs. In particular, it should pin down beliefs of the players about *all* facts, not only about the payoff relevant facts. Our $X-$belief hierarchy profiles are exactly geared to describe such beliefs in a consistent way. For any fixed sets of players $\{1, .., I\}$ and payoff relevant states of nature Ω, standard belief hierarchy profile corresponds to, roughly, sequences of agents' beliefs about Ω, agents' beliefs about agents' beliefs about Ω, etc. For a space X of variables, symbols, $X-$belief hierarchy profile is the set of all sequences of agents' beliefs about Ω and X, beliefs about beliefs about Ω and X etc, for all ways of assigning the types of players to the variables.

The added variables are commonly used in logic or mathematics for naming, referring to objects. "There exists an x such that x is the square root of 2" is an example of a statement that makes use of a variable. In our case we should think of variables as ranging over signals: each signal is a payoff irrelevant fact, such that some agent is always convinced whether it holds or not. For example, an $X-$belief hierarchy profile might express that there is common belief of the following: player 1 is sure whether x_1 or x_1' holds, player 2 is sure whether x_2 or x_2' holds, x_1 and x_2 together imply H, x_1 and x_2' together imply L (see Example 1).

On the one hand, an $X-$belief hierarchy profile affords a much finer description of a state than the standard hierarchy profile. In particular, it specifies the beliefs about sheer sunspots as well as the (co)relations between the beliefs of the agents over payoff relevant states of nature (see Examples 3, 4). On the other, such signals are essential for the equilibrium behavior. Signals are directly payoff irrelevant, but they may be crucial for the choice of own action as others may condition actions on beliefs about them.

The following (somewhat degenerate) example is based on the celebrated example by Aumann [2]. Here the set of payoff relevant states of nature is trivial, and so there is no uncertainty about the agents' payoffs.

Example 2 *There are two agents* 1 *and* 2 *and the single state of nature,* $\Omega = \{\omega\}$. *The description of incomplete information in terms of a belief hierarchy profile is trivial: there is a common belief that* ω. *However, consider the following two type spaces representing this environment. First, in the "Nash Equilibrium" type space* $S := \{S_i, \mu_i^S\}_{i=1,2}$ *each player has only one type,* $S_i = \{s_i\}$, $i \in \{1, 2\}$, *and the trivial beliefs. In the "Correlated Equilibrium" type space* $T := \{T_i, \mu_i^T\}_{i=1,2}$ *each player has two possible types,* $T_i = \{t_i, t_i'\}$, $i \in \{1, 2\}$, *and the beliefs are represented below.*

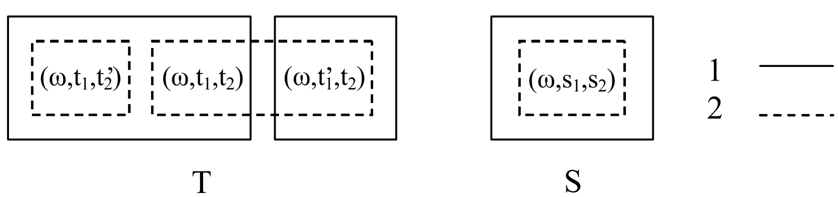

Figure 2: Type spaces T *and* S.

Consider the game Γ *represented by the following payoffs matrix:*

$\omega:$	l	r
u	$6,6$	$2,7$
d	$7,2$	$0,0$

The action profiles that can be played in BE by types (s_1, s_2) *in the type space* S *are exactly the Nash Equilibria of* $\Gamma : (u,r)$, (d,l) *and the mixed* $(\{\frac{2}{3}u, \frac{1}{3}d\}, \{\frac{2}{3}l, \frac{1}{3}r\})$. *In the type space* T, *if the agents appropriately correlate their actions, an additional BE has* (u,l) *played by the types* (t_1, t_2). *In sum,* (t_1, t_2) *in* T *and* (s_1, s_2) *in* S *have the same (standard) hierarchy profile but different action profiles played in BE for* Γ *(see also Example 5).*

1.2 Characterization

$X-$belief hierarchy profile provides an intuitive description of a state, which encodes beliefs about payoff relevant facts and payoff irrelevant signals. The first main result of the paper is the following characterization result. We show that an $X-$belief hierarchy profile specifies exactly the details of an incomplete information state that are relevant for local BNE.

More specifically, for fixed $\{1,..,I\}$ and Ω as above, a game is a function from action profiles and Ω to $\mathbb{R}^I$. Each pair - a game Γ and a profile of types $\bar{t} = \{t_1,..,t_I\}$ in some type space T - gives rise to a *BNE prediction*: the set of action profiles that can be played by $\bar{t}$ in some BNE for Γ over T. Letting BNE to be just over a generated sub-type space S of T we arrive at a *local* BNE prediction. We prove that if two profiles of types in some type spaces give rise to the same $X-$belief hierarchy profile then they give rise to exactly the same local BNE predictions, for every game. On the other hand, if they agree on (local) BNE prediction for every finite game, the appropriate topological closures of the hierarchy profiles agree. Moreover, there exists a single infinite canonical game such that the set of (local) BNE predictions for this game exactly characterize the $X-$belief hierarchy profile of a type tuple.

To prove the latter direction we construct a countable collection of finite test games, each with a distinguished subset of action profiles. For every game and $\bar{t}$ in T an action profile from the distinguished subset is played in local BNE at $\bar{t}$ if and only if $X-$belief hierarchy profile of $\bar{t}$ belongs to a certain set of hierarchy profiles. Those sets are sufficiently fine so that the BNE predictions for every test game characterizes uniquely the closure of an $X-$belief hierarchy profile.

Our results do not contradict the previous well known results on the equivalence between standard belief hierarchies and type spaces (Mertens and Zamir [22], Brandenburger and Dekel [9]).[2] It is worth emphasizing how our methods and aims differ. There the behavior component - games, action choices and solution concepts - is absent from the framework. It is further assumed that the standard belief hierarchy profiles are independently motivated as descriptions of states. In fact, the analogous results can be proven for many other kinds of belief hierarchy profiles.[3] In contrast, we start with the actions chosen in local BNE as the criterion of equivalence. Given the local BNE behavior for some type space as a "black box" input we derive the appropriate belief hierarchy description. We believe that this method makes a step towards extending the mainstream subjectivist, or behavioral paradigm to the multiple person environments (see also section 6).

1.3 Epistemic Justification

One direction in the above characterization implies that the local BNE prediction for any game can be defined directly over an $X-$belief hierarchy profile. The second contribution of this paper is the strenghtening of this result: We show that the local BNE prediction for any game can be defined explicitly within an $X-$belief hierarchy profile, via conditions on agents' rationality. For any game and a profile of types $\overline{t}$ in T an action profile can be played by $\overline{t}$ in local BNE if and only if a specific sequence is part of the $X-$belief hierarchy profile description of $\overline{t}$. This sequence expresses the fact that for some acting of agents on their beliefs about signals there is a common belief of rationality. This provides an epistemic justification for local BNE.

1.4 Related Literature

To address the discrepancy between BNE and standard belief hierarchy profiles we identify richer hierarchy profiles that let us characterize local BNE via Bayesian rationality. Alternatively, one can identify a weaker solution concept characterized by Bayesian rationality in standard hierarchy profiles. This approach was taken by Dekel, Fudenberg and Morris [11], who pin down in this way Interim Correlated Rationalizability. See also Bergmann and

[2] They establish, roughly, the following: any state in a type space gives rise to a unique hierarchy profile; there is a unique Universal Type Space, which for any hierarchy profile (satisfying certain coherence conditions) has a tuple of types with this hierarchy profile.

[3] Think for example about the belief hierarchies with beliefs only upto 3rd order.

Morris [6], Battigalli and Siniscalchi [5] for the relation between rationality conditions on standard belief hierarchy profiles and game theoretic solution concepts.

Our search for the appropriate belief hierarchy profiles is motivated by providing epistemic justifications of the BNE solution concept. The pioneering work in this field for the complete information case and Correlated Equilibrium (Ω singleton) was done by Aumann [3] as well as Brandenburger and Dekel [8] (see also Tan and Werlang [28]).

A typical instance of such results is the following: The *union* of action profiles that can be played at a state in *any* Correlated Equilibrium - a probability space and action assignment - is the same as the *union* of action profiles played at a state in *any* information system - a type space with an action assignment - under assumption of Bayesian rationality, which is the set of correlated rationalizability action profiles. The equivalence thus holds only between the unions of Correlated Equilibrium and Bayesian rationality action profiles over the respective models of uncertainty. The set of action profiles that can be played in equilibrium at a state in a *fixed* type space, however, is not characterized. Our $X-$belief hierarchy profiles allow us to prove the exact equivalence between equilibrium and Bayesian rationality play, which holds for a fixed specification of uncertainty (see Figure 3). Furthermore, our approach affords the characterization of BNE that preserves the dependent/independent variables distinction (see Harsanyi [17], Aumann [3]). BNE can be rationalized as a prediction from two distinct components: $X-$belief hierarchy profile, specifying uncertainty at a pre-game stage and so not including actions, and a game, specifying availible actions and payoffs.

In the incomplete information case Forges [15], and more recently Liu [20] investigated the sets of type spaces corresponding to the same standard hierarchy, but differing in the descriptions of actions taken or the descriptions of additional states of nature (see also Myerson [23], [24], Forges [14], Einy, Peleg [12], Samuelson and Zhang [27]). They also investigated the unions over such type spaces of BNE action profiles. As in the complete information case, the papers do not characterize the belief hierarchies that are revealed by BNE action profiles for a fixed type space, or characterize in terms of Bayesian rationality the BNE action profiles for a fixed type space.

Finally, our method of defining $X-$belief hierarchy profiles in fit with the BNE solution concept resembles that of the recent paper by Ely and Peski [13]. They use the solution concept of Interim Independent Rationalizability and show, in the same manner as we do here for local BNE, that for the two player case the solution concept is in fit with their "$\Delta-$hierarchies" capturing conditional beliefs about payoff relevant states. Their

$\Delta-$hierarchies do not represent beliefs about sheer behavior correlating sunspots, and are not expressive enough to allow characterization of the local BNE. On the other hand, beliefs about signals in our $X-$belief hierarchy profiles subsume beliefs about conditional beliefs.

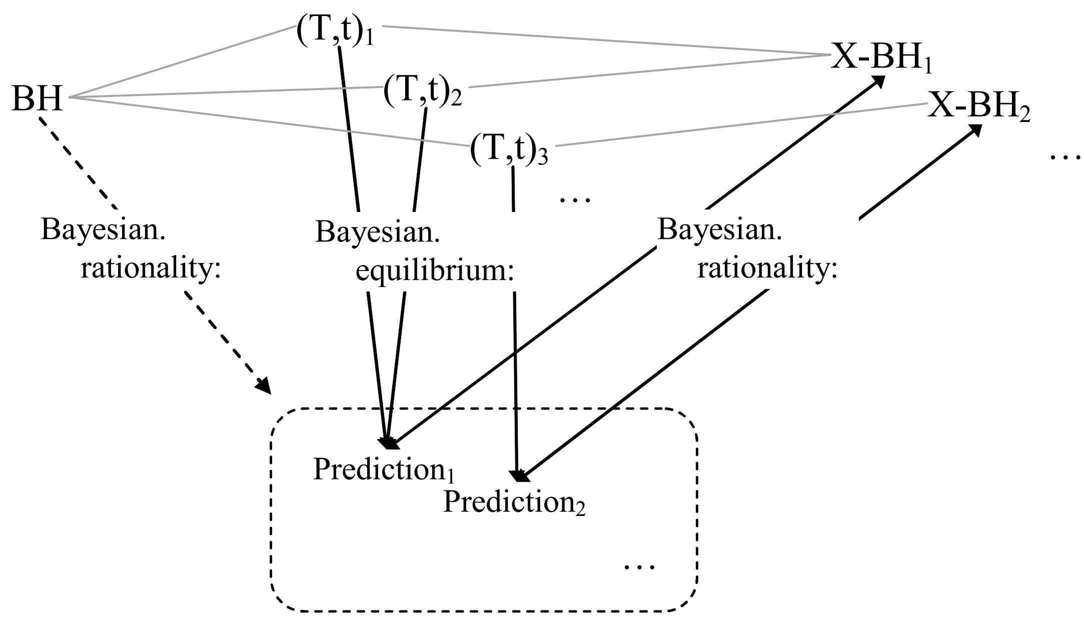

Figure 3: Relationships between the representations of incomplete information, solution concepts and predictions.

Figure 3 shows schematically relationships between the representations of incomplete information: standard belief hierarchy profiles, BH, type spaces with distinguished tuples of types, $(\mathrm{T},\bar{\mathrm{t}})_{\mathrm{n}}$, $X-$belief hierarchy profiles, $\mathrm{X\text{-}BH_n}$, as well as solution concepts and predictions. Each prediction $\mathrm{Pred_n}$ is a function from the specification of payoffs (games) to sets of action profiles. Even though $(\mathrm{T},\bar{\mathrm{t}})_1$ and $(\mathrm{T},\bar{\mathrm{t}})_3$ correspond to the same standard belief hierarchy profile BH, they give rise to different BNE predictions. Therefore e.g. $\mathrm{Prediction_1}$ cannot be characterized by the rationality conditions on BH.[4] The Bayesian rationality prediction for BH agrees with the union of BNE predictions[5] for the type spaces with distinguished tuples

[4] If we replaced in the Figure "Bayesian equilibrium" with "Interim Correlated Rationalizability" solution concept, then its predictions would agree with Bayesian rationality predictions for a BH (Dekel et al. [11]).

[5] more precisely: the function from games to sets of action profiles such that for every game its value equals the union of the respective values of $\mathrm{Pred_i}$;

of types that correspond to BH (Aumann [3], Brandenburger and Dekel [8], Forges [15], Liu [20])[6]. However, the BNE prediction for any $(\mathrm{T},\bar{\mathrm{t}})_{\mathrm{n}}$ is the same as the Bayesian rationality prediction of the corresponding X-BH_{n}. Moreover, the predictions uniquely characterize X-BH_{n}.[7]

The rest of the paper is organized as follows. The building blocks of the formalism, including the $X-$belief hierarchy profiles, are defined in Section 2. In section 3 we define BNE, local BNE prediction and the notion of Bayesian rationality prediction, as well as establish the equivalence between the last two. In the following two Sections 4 and 5 we show that X-belief hierarchy profiles are sufficient and minimal descriptions of a state for local BNE, respectively, and we conclude in Section 6.

2 Preliminaries and Framework

For a measurable space Y, the set ΔY of probability measures on Y will be identified with the measurable space with σ-algebra generated by sets of the form $\{\delta \in \Delta Y | \delta(e) \geq q\}$, for measurable e, $q \in [0,1]$ (see Heifetz, Samet [18]). For a topological space Z the set ΔZ of probability measures on Z will be identified with the topological space with the weak* topology. If Z is Polish[8] then so is ΔZ. If Y is the Borel measurable space for a Polish space Z then ΔY coincides with the Borel measurable space for the topological space ΔZ (see Battigalli Siniscalchi [4] footnote 5, Dekel *et al* [11]). For any measure $\delta \in \Delta Y$ and measurable function $f : Y \to \mathbb{R}$ the expectation of f with respect to δ is denoted $\delta[f]$.

Fix once and for all a finite set $\{1,..,I\}$ of agents and a Polish space Ω of payoff relevant states of nature, facts. Fix also Polish spaces of variables $X_i = [0,1]^{\mathbb{N}},\ i \leq I$.

Games. A *game* describes a payoff structure. It specifies the payoffs for each player, given the actions of all players and a payoff relevant state of nature. It is defined as a jointly measurable, bounded function $\Gamma : \Omega \times_{i \leq I} A_i \to \mathbb{R}^I$, such that A_i as well as $\{\Gamma_i(\cdot, a_i, \cdot)\}_{a_i \in A_i}$ with the sup metric are Polish and the functions $\Phi_i : A_i \to \{\Gamma_i(\cdot, a_i, \cdot)\}_{a_i \in A_i}$, $\Phi_i(a_i) = \Gamma_i(\cdot, a_i, \cdot)$, are measurable. We call a game *finite* if the sets A_i are finite and $\Gamma_i(\cdot, a_1, .., a_I)$ are simple functions, $a_i \in A_i, i \leq I$.[9]

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[6] Brandenburger and Dekel [8] and Forges [15] talk about equality of sets of achievable payoffs;

[7] On the figure we can also picture $\Delta-$hierarchies and the double sided arrows between them and the Interim Independent Rationalizability predictions, which are "finer" than Interim Correlated Rationalizability predictions, but "cruder" than BNE predictions (Ely, Peski [13]).

[8] i.e. a completely metrizable, separable topological space;

[9] A function $f : \Theta \to \mathbb{R}$ is simple if $f = \sum_{n=1}^{N} \alpha_n * \chi_{e_n}$, where χ_{e_n} is an indicator function for e_n, e_n

Type Spaces. A type space is one of the ways to represent incomplete information. A *type space* $T := \{T_i, \mu_i\}_{i \leq I}$ *for* I *and* Ω consists of measurable spaces T_i of types and measurable functions $\mu_i : T_i \to \Delta(\Omega \times_{i \leq I} T_i)$ that satisfy

$$t_i \notin Proj_{T_i}(e) \text{ impies } \mu_i(t_i)(e) = 0. \tag{1}$$

Condition (1) assures that agents have no doubts about own type. A *pointed type space* $T, \bar{t}$ is a type space T and a tuple of distinguished, actual types $\bar{t} \in \times_{i \leq I} T_i$. A *finite (pointed) type space* is one with all T_i finite. Finally, for two type spaces $T := \{T_i, \mu_i^T\}_{i \leq I}$ and $S := \{S_i, \mu_i^S\}_{i \leq I}$ we say that $S, \bar{s}$ is a *generated sub-type space* of $T, \bar{t}$ ($S, \bar{s} \preceq T, \bar{t}$) when $\bar{s} = \bar{t}$, S_i is a sub-space of T_i, $\mu_i^T(s_i)(\Omega \times_{i \leq I} S_i) = 1$ and $\mu_i^S(s_i) = \mu_i^T(s_i)|_{\Omega \times_{i \leq I} S_i}$, for $s_i \in S_i$ (see also Mertens, Zamir [22]).

Variable Assignments. For a type space T a *variable assignment* V is a measurable function $V := V_1 \times .. \times V_I$ with $V_i : T_i \to X_i$. Each variable assignment assigns a unique "signal" to each type. The idea is that each tuple of types, or a state, is a complete description of the state of the world. However, we do not write explicitly in the type space all the facts over which the agents reason, have beliefs. We assume, though, that two different tuples of types, or two different types (due to the product structure of type spaces) must differ on beliefs about some facts. Variable assignment provides a way of "tagging", naming those facts, so that we can refer to them in a belief hierarchy profile (see also section 6). Each type space T for Ω and I together with a variable assignment V can be identified with a type space for I and $\Omega \times_{i \leq I} X_i$, which we will denote T^V.

Belief Hierarchy profiles. Consider the spaces $Y^0 := \Omega \times_{i \leq I} X_i$, $Y^k := Y^{k-1} \times (\Delta(Y^{k-1}))^I$, $k > 0$. For a fixed variable assignment V each pointed type space $T, \bar{t}$ can be associated with a unique element $\beta(T^V, \bar{t}) \in (\times_{k=0}^{\infty} \Delta(Y^k))^I$ (see e.g. Mertens, Zamir [22], Brandenburger, Dekel [9]). For any type space $T, \bar{t}$ define its X-*belief hierarchy profile*, $\beta^X(T, \bar{t})$, $\beta^X(T, \bar{t}) \subseteq (\times_{k=0}^{\infty} \Delta(Y^k))^I$, as:

$$\beta^X(T, \bar{t}) := \bigcup_{V \text{ var. ass.}} \beta(T^V, \bar{t}). \tag{2}$$

The intended meaning of $\beta(T^V, \bar{t}) \in \beta^X(T, \bar{t})$ is that for some signals the beliefs of the agents over Ω and those signals are $\beta(T^V, \bar{t})$.

measurable, $\alpha_n \in \mathbb{R}$, $N \in \mathbb{N}$.

Each $X-$belief hierarchy profile provides a more detailed description of beliefs than standard belief hierarchy (cf. [22], [9]). The following two examples illustrate that - in contrast to the standard belief hierarchy profiles - the description is fine enough to distinguish the tuples of types from Examples 1 and 2, which differed on their BNE behavior.

Example 3 *Look at the type spaces T and S in the Example 1. For a space of states of nature $\Omega = \{H, L\}$ and some $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$ consider a subset $e \subseteq \Omega \times_{i \leq I} X_i$,*

$$e := (H, x_1, x_2) \cup (L, x_1, x_2') \cup (H, x_1', x_2') \cup (L, x_1', x_2), \tag{3}$$

and let $cb(e) \subseteq (\times_{k=0}^{\infty} \Delta(Y^k))^I$ consist of all those elements for which there is common belief that e.[10] *We have $\beta^X(S, (s_1, s_2)) \cap cb(e) = \emptyset$ but $\beta^X(T, (t_1, t_2)) \cap cb(e) \neq \emptyset$. To verify that $\beta^X(T, (t_1, t_2)) \cap cb(e) \neq \emptyset$ consider a variable assignment that assigns x_i to t_i and x_i' to t_i', $i \leq 2$. It follows that $\beta^X(S, (s_1, s_2)) \neq \beta^X(T, (t_1, t_2))$.*

Example 4 *For the type spaces T and S in Example 2, for e.g. a set $e_2 \subseteq \times_{k=0}^{\infty} \Delta(Y^k)$,*

$$e_2 := \{(\delta_2^k)_{k=1}^{\infty} \in \times_{k=0}^{\infty} \Delta(Y^k) | \delta_2^1(\{\omega, x_1\} \times X_2) = \delta_2^1(\{\omega, x_1'\} \times X_2) = 1/2\}, \tag{4}$$

with $x_1, x_1' \in X_1$, we have $\beta^X(S, (s_1, s_2)) \cap ((\times_{k=0}^{\infty} \Delta(Y^k)) \times e_2) = \emptyset$ and $\beta^X(T, (t_1, t_2)) \cap ((\times_{k=0}^{\infty} \Delta(Y^k)) \times e_2) \neq \emptyset$. Accordingly $\beta^X(S, (s_1, s_2)) \neq \beta^X(T, (t_1, t_2))$.

3 Bayesian-Nash Equilibrium and Bayesian Rationality

Fix a game Γ. In what follows we will always assume without loss of generality that A_i is a measurable subspace of X_i. (see e.g. Kechris [19] Theorem 4.14). For a type space T *Bayesian-Nash Equilibrium* (BNE) for Γ over T is defined as a variable assignment $V = V_1 \times .. \times V_I$ such that for every $i \leq I$ and type $t_i \in T_i$ we have $V_i(t_i) \in A_i$ and

$$\mu_i(t_i)[\Gamma_i \circ (id \times V)] \geq \mu_i(t_i)[\Gamma_i \circ \sigma_{x_i^*} \circ (id \times V)], \ \forall x_i^* \in A_i \tag{5}$$

[10] Formally, define the sets $b_i^k(e) \in \Delta(Y^{k-1})$, $k > 0$, $i \leq 2$, and $cb(e)$ recursively as

$$\begin{aligned} b_i^1(e) \ &: \ = \{\delta_i^1 \in \Delta(Y^0) | \delta_i^1(e) = 1\}, \\ b_i^k(e) \ &: \ = \{\delta_i^k \in \Delta(Y^k) | \delta_i^k(e \times_{l=1}^{k-1} (\times_{j \leq I} b_j^l(e))) = 1\}, \ k > 1 \\ cb(e) \ &: \ = \times_{i \leq I} (\times_{k=1}^{\infty} b_i^k(e)). \end{aligned}$$

where $\sigma_{x_i^*}: \Omega \times_{i\leq I} X_i \to \Omega \times_{i\leq I} X_i$, $\sigma_{x_i^*}(\omega, x_1, .., x_I) = (\omega, x_1, .., x_{i-1}, x_i^*, x_{i+1}, .., x_I)$. The set of action profiles $(a_1, .., a_I)$ that are played in some Bayesian-Nash Equilibrium at $T, \bar{t}$ is called the *Bayesian-Nash Equilibrium prediction at* Γ and $T, \bar{t}$, $BNE_\Gamma(T, \bar{t})$:

$$BNE_\Gamma(T, \bar{t}) := \{V(\bar{t}) \in \times_{i\leq I} A_i | V \text{ is a BNE for } T\}. \tag{6}$$

Define also a *Local Bayesian-Nash Equilibrium prediction at* Γ and $T, \bar{t}$, $LBNE_\Gamma(T, \bar{t})$, as

$$LBNE_\Gamma(T, \bar{t}) := \bigcup_{S, s \preceq T, \bar{t}} BNE_\Gamma(T, \bar{t}). \tag{7}$$

Recall that in the incomplete information case types and type spaces are convenient but artificial constructs, used to model agents' beliefs. Consequently, so is the solution concept of BNE. We now define the epistemic counterpart of Bayesian-Nash Equilibrium prediction, which is not defined over type spaces and is phrased fully in terms of rationality conditions on beliefs. First, for any game Γ define the sets $p_\Gamma, rat_{\Gamma,i} \subseteq \Delta(\Omega \times_{i\leq I} X_i)$ as

$$\begin{aligned} p_\Gamma &:= \{\delta \in \Delta(\Omega \times_{i\leq I} X_i) | \delta(\Omega \times_{i\leq I} A_i) = 1\}, \\ rat_{\Gamma,i} &:= \{\delta \in p_\Gamma | \delta[\Gamma_i] \geq \delta[\Gamma_i \circ \sigma_{x_i^*}] \ \forall x_i^* \in A_i\}. \end{aligned}$$

Let $cb(\cap_{i\leq I} rat_{\Gamma,i}) \subseteq (\times_{k=0}^\infty \Delta(Y^k))^I$ be the set of exactly those elements of $(\times_{k=0}^\infty \Delta(Y^k))^I$ in which there is common belief that $\cap_{i\leq I} rat_{\Gamma,i}$.[11]

The set of action profiles $(a_1, .., a_I)$ that are consistent with the common belief of Bayesian rationality at Γ and $\beta^X(T, \bar{t})$, $CBR_\Gamma(\beta^X(T, \bar{t}))$, is:

$$\begin{aligned} CBR_\Gamma(\beta^X(T, \bar{t})) \quad &: \quad = \{(a_1, .., a_I) \in \times_{i\leq I} A_i | \\ \{((\delta_i^k)_{k=1}^\infty)_{i\leq I} | \delta_i^1(\Omega \times \{a_i\} \times_{j\neq i} A_j) \quad &= \quad 1\} \cap cb(\cap_{i\leq I} rat_{\Gamma,i}) \cap \beta^X(T, \bar{t}) \neq 0\}. \end{aligned} \tag{8}$$

Theorem 1 *For any pointed type space* $T, \bar{t}$ *and a game* Γ

$$LBNE_\Gamma(T, \bar{t}) = CBR_\Gamma(\beta^X(T, \bar{t})). \tag{9}$$

Theorem 1 explains in what sense the choice of local BNE action profiles is rational. The

[11] see the previous footnote;

result can be given two interpretations. In the strictly Bayesian view advocated by Aumann [3] each type in a type space should specify beliefs about the full state of affairs, including the actions chosen for each particular game. It is only the limited set theoretic formalism - set Ω including only payoff relevant states of nature and not actions chosen in any game - that makes this fact implicit in a type space. Under this interpretation, the theorem shows that for a given type space (with an incomplete formal description) an action profile can be played in local BNE if and only if there is a way to extend the description of beliefs to include actions that results in the common belief of rationality.

On the other hand, we can treat type spaces as not tied to any game or action sets: they are descriptions of the environment with uncertainty at the pre-game, (a fortiori: pre-play) stage.[12] The $X-$belief hierarchy profiles allow us to interpret the correlations in type spaces epistemically and with no recourse to beliefs about actions. The theorem can be interpreted in this setting as follows: an action profile can be played in local BNE if and only if there is a way of acting of agents on their beliefs about signals and payoff relevant facts, beliefs about beliefs... such that everybody acts optimally given the acting of others, which is optimal ... etc (see also section 6).

4 Same X-Belief Hierarchies imply same LBNE

BNE, and therefore also (local) BNE prediction, is defined over type spaces as a game-theoretic solution concept. The prediction from the consistency with the common belief of Bayesian rationality defined in (8)[13] is phrased directly in terms of agents' belief hierarchy profiles over facts in Ω and signals, and does not depend on a particular type space. Accordingly, the immediate upshot of Theorem 1 is that the local BNE prediction can be defined directly over the X-belief hierarchy profiles.

Corollary 1 *If pointed type spaces $T, \bar{t}$ and $S, \bar{s}$ have the same X-belief hierarchy profile then for every game Γ*

$$LBNE_{\Gamma}(T, \bar{t}) = LBNE_{\Gamma}(S, \bar{s}). \tag{10}$$

[12] The game might actually be constructed by a shrewd mechanism designer based on the structure of the beliefs.

[13] This notion of consistency coincides with the standard, technical one: there is a model (a type space with types including the beliefs about actions chosen) such that i) it gives rise to a given $X-$belief hierarchy profile, ii) given action profile is played and iii) there is common belief that everybody acts optimally. The type space extended with the BNE assignment of actions is precisely the semantic structure verifying the above notion of consistency.

5 Same (L)BNE imply same X-Belief Hierarchy profiles

The following two theorems show that the X-belief hierarchy profiles not only provide a description of a state that is sufficiently rich to give a well defined BNE prediction for every game, but that this description is weakest possible. Any two pointed type spaces with different X-belief hierarchy profiles give rise to different BNE predictions for some game. Each X-belief hierarchy is uniquely pinned down by the acting under the BNE solution concept. In this sense all the information carried by an $X-$belief hierarchy profile is relevant, and any strictly richer description would contain information that is redundant.

The proof of Theorem 2 relies on three lemmas. Lemma 1 shows that the space $\bigcup_{T^V,\bar{t}} \beta(T^V, \bar{t}) \subseteq (\times_{k=0}^{\infty} \Delta(Y^k))^I$ is homeomorphic to its "projection" on $\times_{i \leq I}(X_i \times_{k=0}^{\infty} \Delta(U^k))$, whose elements ignore beliefs of the agents about own variables and own beliefs, but instead specify the actual variables directly. Lemma 2 allows us to construct a countable system of sets $\mathcal{C}$ in $\times_{i \leq I}(X_i \times_{k=0}^{\infty} \Delta(U_i^k))$, which is finer then the topology of $\times_{i \leq I}(X_i \times_{k=0}^{\infty} \Delta(U^k))$. The system is built up recursively from the open sets in X_i and Ω using only finite intersections and unions as well as weak probabilistic inequalities. Given that, the key Lemma 3 constructs recursively for every set $C \in \mathcal{C}$ a finite test game, whose BNE predictions at $T, \bar{t}$ determine whether the $X-$belief hierarchy $\beta^X(T, \bar{t})$ "projected" on $\times_{i \leq I}(X_i \times_{k=0}^{\infty} \Delta(U^k))$ intersects C or not.

Theorem 2 *If for type spaces $T, \bar{t}$ and $S, \bar{s}$ we have $BNE_\Gamma(T, \bar{t}) = BNE_\Gamma(S, \bar{s})$ for all finite games Γ, then*

$$cl(\beta^X(T, \bar{t})) = cl(\beta^X(S, \bar{s})), \tag{11}$$

where the closure is in the product topology over weak topologies.*

The following example illustrates the games constructed in the proof of Lemma 3.

Example 5 *Consider the type spaces T and S from Example 2 as well as the set $(\times_{k=0}^{\infty} \Delta(Y^k)) \times e_2$, for*

$$e_2 = \{(\delta_2^k)_{k=1}^{\infty} \in \times_{k=0}^{\infty} \Delta(Y^k) | \delta_2^1(\{\omega, x_1\} \times X_2) \geq 1/2,\ \delta_2^1(\{\omega, x_1'\} \times X_2) \geq 1/2\}.$$

We know that $\beta^X(S, (s_1, s_2)) \cap ((\times_{k=0}^{\infty} \Delta(Y^k)) \times e_2) = \emptyset$ and $\beta^X(T, (t_1, t_2)) \cap ((\times_{k=0}^{\infty} \Delta(Y^k)) \times e_2) \neq \emptyset$, and so there are games for which BNE predictions differ across $(S, (s_1, s_2))$ and

$(T,(t_1,t_2))$. *One instance is the game* Γ^C *constructed in the proof of Lemma 3 for the set* $C \subseteq \times_{i\leq I}(X_i \times_{k=0}^{\infty} \Delta(U^k))$,

$$C = (X_1 \times_{k=0}^{\infty} \Delta(U^k))\times$$
$$\times\{X_2 \times \{(\delta_2^k)_{k=1}^{\infty} \in \times_{k=0}^{\infty}\Delta(U^k)|\delta_2^1(\{\omega, x_1\}) \geq 1/2, \delta_2^1(\{\omega, x_1'\}) \geq 1/2\}\}$$

which corresponds to the set $((\times_{k=0}^{\infty}\Delta(Y^k)) \times e_2) \subseteq (\times_{k=0}^{\infty}\Delta(Y^k))^I$. *The game* Γ^C *is defined by the following matrix: (up to renaming of the actions):*

ω	$(a_2^{x_1+}, a_2^{x_1'+})$	$(a_2^{x_1+}, a_2^{x_1'-})$	$(a_2^{x_1-}, a_2^{x_1'+})$	$(a_2^{x_1-}, a_2^{x_1'-})$
x_1	$1,1$	$1,1.5$	$1,0.5$	$1,1$
x_1'	$1,1$	$1,0.5$	$1,1.5$	$1,1$

,

and we have

$$A_{*,1}^C = A_1^C, \quad A_{*,2}^C = \{(a_2^{x_1+}, a_2^{x_1'-})\}.$$

We can verify that e.g. $(x_1,(a_2^{x_1+}, a_2^{x_1'+})) \notin BNE_{\Gamma^C}(S,(s_1,s_2))$ *and* $(x_1,(a_2^{x_1+}, a_2^{x_1'+})) \in BNE_{\Gamma^C}(T,(t_1,t_2))$.

If we lift the restriction to finite games we can define a single canonical game $\Gamma^\times$ that exactly charachterizes the $X-$belief hierarchy profile of any state.

Theorem 3 *There exists a game* $\Gamma^\times$ *such that if for type spaces* $T, \bar{t}$ *and* $S, \bar{s}$ *we have* $BNE_{\Gamma^\times}(T,\bar{t}) = BNE_{\Gamma^\times}(S,\bar{s})$, *then*

$$\beta^X(T,\bar{t}) = \beta^X(S,\bar{s}). \tag{12}$$

Remark 1 *All the claims in this section are true if we uniformly replace* BNE_Γ *with* $LBNE_\Gamma$.

6 Summary and further issues

In the paper we defined $X-$belief hierarchy profiles, which extend standard belief hierarchies with beliefs about signals. With their help we could give the epistemic justification of the Bayesian-Nash equilibrium. We also showed that they give a description of an epistemic state that is in exact "fit" with the BNE solution concept.

6.1 Revelation argument

The crucial notion of a "fit" between $X-$belief hierarchy profiles and BNE is defined in Corollary 1 and Theorems 1, 2, 3. The analysis is supposed to be in a partial analogy with the original introduction of subjective probabilities by Ramsey [26] and De Finetti [10]. They considered the case of a *single agent* confronting monetary[14] *bets,* who acts according to some criterion of *single agent rationality.* The behavior exactly characterizes subjective probabilities and Bayesian rationality. We analyze the case of *group of agents* facing *games*, where the local BNE for some type space takes up the role of the *group rationality* criterion. The behavior in this case is exactly characterized by $X-$belief hierarchy profiles and consistency with common belief in Bayesian rationality.

The difference lies in the fact that in the first case a set of choices is introduced via axioms, and in the second case, motivated by the common practice, via a type space and local BNE solution concept. We believe that the games constructed in the proof of Theorem 2 would be helpful in solving the independent, parallel question to ours: characterizing type spaces and local BNE not by $X-$belief hierarchy profiles and Bayesian rationality but axioms on choices of action profiles in games.

6.2 Space of signals X

Each $X-$belief hierarchy profile specifies beliefs of each agent about the fundamentals Ω and the payoff irrelevant signals for each agent. To refer to the signals we make use of the variables in X_i, $i \leq I$. It is important to realize that, as usual with variables, the beliefs are about the objects themselves (payoff irrelevant signals) and not the variables, which are just convenient labels. Clearly the names of labels that we use play no role; what *does* play role, however, is the structure of the spaces X_i. We would have a different description of a state if we restricted the sets of variables to have e.g. five elements.

Due to the full characterization, the structure of X_i spaces in our hierarchies (homeomorphic to $[0,1]^{\aleph}$, the universal Polish space) is not arbitrary. It is pinned down by the structure of the games that we consider, which can be taken as primitive in the framework.

We can partially strenghten this characterization result. Consider a class of games $\mathcal{G}^n$ with at most n actions for each player, and sets of variables X_i^n for each player consisting of

[14] Only de Finetti defined the outcomes as purely monetary and entering the utility function in a straightforward, "quasilinear" fashion. Ramsey defined them as resulting in utilities; however, the utilities could be elicited independently via an exogenous objective ("ethically neutral") coin toss.

n elements. We define X^n-belief hierarchy profiles in an obvious way. In this setup it is still true that if two profiles of types have the same X^n-belief hierarchy profile then they have the same local Bayesian-Nash equilibrium predictions for games in $\mathcal{G}^n$. However, if two profiles of types have different X^n-belief hierarchy profiles, we in general need games with much larger action spaces for their local Bayesian-Nash equilibrium predictions to differ.

6.3 Profiles of hierarchies and equilibrium

We take the whole profile of hierarchies, as opposed to a single hierarchy for one agent, as a primitive description of beliefs. This is, again, dictated by the characterization of BNE behavior, as the following example suggests.

Consider a complete information case (Ω singleton) with two players and the following two type spaces. In type space S each player i has a single type s_i and trivial beliefs. In type space T for some pair of types (t_1, t_2) each player i thinks mistakenly that there is only one pair of types (t_i, t'_{-i}) and players have trivial beliefs (see figure 4).

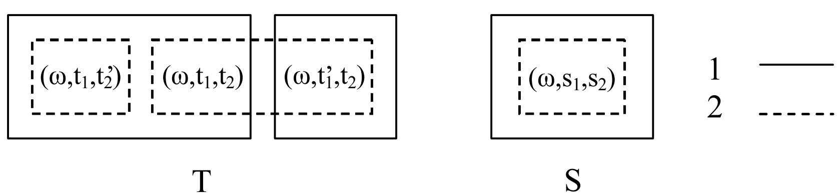

Figure 4: Type spaces T and S. In T beliefs are obtained by conditioning the prior that assigns probability one half to (ω, t_1, t'_2) and (ω, t'_1, t_2).

One can verify that the types t_i and s_i have the same X-belief hierarchy, for each player i.[15] However, for any two action coordination game, such as the Battle of the Sexes, types (s_1, s_2) must play Nash equilibrium in any Bayesian-Nash equilibrium, whereas (t_1, t_2) can miscoordinate.

The problem has to do with the fact that the set of action profiles that can be played in BNE by a type tuple is typically not a product set. This contrasts with the sets of rationalizable strategy profiles, for any known notions of rationalizability.

In regard to that notice that in interpreting an element of an X-belief hierarchy profile on

[15] where the X-belief hierarchy of i is the projection of $(\times_{k=0}^{\infty}\Delta(Y^k))^I$ on the ith coordinate;

a type space we first assign variables to signals, and then evaluate beliefs of each agent over $\Omega \times_{i \leq I} X_i$, beliefs about beliefs... In case of BNE for some game, when elements of X_i can be treated as actions, this means that the mapping from beliefs about signals to actions is common for all players. Alternatively, we would arrive at a different definition if, roughly, we used each time a new assignment for evaluating beliefs of each new type, and so players might disagree on how they act on their beliefs. Those hierarchy profiles would correspond to a new rationalizability solution concept. We leave those issues for a separate detailed analysis.

6.4 Independent definition of X-belief hierarchy profiles

Given our interest in the meaning of types, we defined an X-belief hierarchy profile as derived from some underlying type space and a profile of types. One might be interested in a definition of such belief profiles, which would not refer to a type space. This would paralell an explicit definition of a standard belief hierarchy via so called coherence conditions in [22] or [9].[16] Below we sketch such a definition.

First, each sequence of beliefs in an X-belief hierarchy profile (for a fixed variable assignment) satisfies:

C1) common belief[17] that each agent i puts probability one to one of the variables in X_i.[18]

This assumption warrants our interpretation of X_i as ranging over the signals for agent i. On the other hand, for any sequence in $(\times_{k=0}^{\infty} \Delta(Y^k))^I$ that satisfies common belief of coherency and C1 we can find a profile of types and a variable assignment that give rise to it: This is the profile of types in the generated sub-type space of the universal type space over $\Omega \times_{i \leq I} X_i$ ([22], [9]), with variable assignment that to each type assigns the variable about which she is convinced.[19]

Second, even though we characterized a single element of an X-belief hierarchy profile, for a single variable assignment, we are up for characterizing the whole set of those elements, for all variable assignments. Not any set of such elements corresponds to some X-belief hierarchy profile. For example, it is easy to notice that a version of *substitution of variables* holds for

[16] The difference is analogous to the one in logic between defining a complete theory via models and the definition of truth, and via axiom system and definition of consistency.

[17] see footnote 10;

[18] Since X_i is Polish, a subset $\{\delta \in \Delta(X_i) | \delta(x_i) = 1 \text{ for some } x_i \in X_i\}$ of $\Delta(X_i)$ is measurable.

[19] We leave to the reader an easy proof that this assignment is measurable, and that the profile of types together with this assignment gives rise to a desired element in $(\times_{k=0}^{\infty} \Delta(Y^k))^I$.

our belief profiles: If δ is an element of an X-belief hierarchy profile, then for any measurable function $\phi = \times_{i\leq I}\phi_i$, $\phi_i : X_i \rightarrow X_i$, the "composition" of δ and ϕ is also an element of it.[20]

There are more properties that link seperate elements of an X-belief hierarchy profile. To give an example, for any two elements δ_1, δ_2 there is a third element δ_3 in it and $\phi_{3,1}, \phi_{3,2} : \times_{i\leq I}X_i \rightarrow \times_{i\leq I}X_i$ such that δ_3 "composed" with $\phi_{3,1}$ gives rise to δ_1, and similarly for $\phi_{3,2}$ and δ_2. Suppose that we put the following condition on X-belief hierarchy profiles:

C2) for some injective measurable functions $\pi_i : \times_{k=0}^{\infty}\Delta(Y^k) \rightarrow X_i$, $i \leq I$:

i) there is some $\overline{\delta}$ in the X-belief hierarchy profile that satisfies the common belief that each i has beliefs $\delta_i \in \times_{k=0}^{\infty}\Delta(Y^k)$ if and only if i puts probability one to $\pi_i(\delta_i)$;

ii) δ is in the X-belief hierarchy profile if and only if there is a measurable function $\phi_\delta = \times_{i\leq I}\phi_{\delta,i}$, $\phi_{\delta,i} : X_i \rightarrow X_i$ such that δ is the "composition" of $\overline{\delta}$ and ϕ.

The C2 hold for the $X-$belief hierarchy profiles that stem from the type spaces with Polish spaces of types. On the other hand, for any subset of $(\times_{k=0}^{\infty}\Delta(Y^k))^I$ that satisfies C2 and whose elements satisfy C1 we can find a type space and a profile of types that gives rise to such an X-belief hierarchy profile. This is, again, a generated sub-type space of the universal type space over $\Omega \times_{i\leq I} X_i$, and in this case each variable assignment $\times_{i\leq I}V_i$ corresponds to a measurable function $\phi = \times_{i\leq I}\phi_i$, $\phi_i : X_i \rightarrow X_i$, such that $\phi_i = V_i \circ \pi_i^{-1}$.

7 Proofs

Proof of Theorem 1

The inclusion from left to right is clear: the assignment that constitutes a BE over a sub-type space and which assigns an action tuple $(a_1, .., a_I)$ to m also warrants that $(a_1, .., a_I) \in CBR_\Gamma(\beta^X(T, \overline{t}))$. In the opposite direction, consider a variable assignment V such that $\beta(T^V, \overline{t}) \in \{((\delta_i^k)_{k=1}^{\infty})_{i\leq I} | \delta_i^1(\Omega \times \{a_i\}) = 1\} \cap cb(\cap_{i\leq I} rat_{\Gamma,i})$. We easily verify that $T, \overline{t}$ restricted to the spaces of types $S_i := \{t_i \in T_i | \beta(T^V, (t_i, t'_{-i})) \in cb(\cap_{i\leq I} rat_{\Gamma,i})$, for some $t'_{-i} \in \times_{j\neq i}T_j\}$ is a generated sub-type space and V restricted to it is a BE for Γ.

Proof of Theorem 2

For the proof we need the following three lemmas.

[20] By the "composition" of $\delta = (\delta_i^1, \delta_i^2, ...)_{i\leq I}$ and $\phi = \times_{i\leq I}\phi_i$, $\phi_i : X_i \rightarrow X_i$, we mean the $c_\phi(\delta) = (c_\phi^1(\delta_i^1), c_\phi^2(\delta_i^2), ...)_{i\leq I}$ such that

$c_\phi^1(\delta_i^1) = \delta_i^1 \circ (id \times_{i\leq I} \phi_i^{-1})$,

$c_\phi^2(\delta_i^2) = \delta_i^2 \circ (id \times_{i\leq I} \phi_i^{-1} \times_{i\leq I} (c_\phi^1)^{-1}$

etc.

Consider the following spaces: $U_i^0 := \Omega \times_{j\neq i} X_j$, $U_i^k := U_i^{k-1} \times_{j\neq i} \Delta(U_j^{k-1})$, $i \leq I, k > 0$. Each type space $T, \bar{t}$ with a fixed variable assignment V gives rise to $\beta_U(T^V, \bar{t}) \in \times_{i\leq I}(X_i \times_{k=0}^{\infty} \Delta(U_i^k))$, which ignores the beliefs of the agents about own variables and beliefs, but instead specifies the true variables directly, in the analogous way as it gives rise to $\beta(T^V, \bar{t})$ (see Mertens, Zamir [22], Brandenburger, Dekel [9]).

Lemma 1 *For any pointed type spaces $T, \bar{t}$ and $S, \bar{s}$ as well as signal assignments V and V' we have*

$$\beta(T^V, \bar{t}) = \beta(S^{V'}, \bar{s}) \text{ iff } \beta_U(T^V, \bar{t}) = \beta_U(S^{V'}, \bar{s}). \tag{13}$$

Moreover, the mapping $\pi : \bigcup_{T^V, \bar{t}} \beta(T^V, \bar{t}) \to \bigcup_{T^V, \bar{t}} \beta_U(T^V, \bar{t})$, $\pi(\beta(T^V, \bar{t})) := \beta_U(T^V, \bar{t})$ is a homeomorphism.

Proof. (Lemma 1) In the proof, for any $\beta(T^V, \bar{t}) \in (\times_{k=0}^{\infty} \Delta(Y^k))^I$ and $\beta_U(T^V, \bar{t}) \in \times_{i\leq I}(X_i \times_{k=0}^{\infty} \Delta(U_i^k))$ the $\beta_i^k(T^V, \bar{t}) \in \Delta(Y^{k-1})$ and $\beta_{U,i}^k(T^V, \bar{t}) \in \Delta(U_i^{k-1})$ will denote the obvious projections.

π is continuous and open because for any $\beta(T^V, \bar{t})$ we have

$$\pi(\beta(T^V, \bar{t}))_i = (x_i, \pi_i'(\beta(T^V, \bar{t})) \text{ for } \beta_i^1(T^V, \bar{t})(\Omega \times \{x\}_i \times_{j\neq i} X_j) > 0, \quad (i \leq I) \tag{14}$$

where π_i' is the product of continuous and open projections $\pi_i'^l : (\times_{k=0}^{\infty} \Delta(Y^k))^I \to \Delta(U_i^l)$, $l \in \mathbb{N}$.[21] Here we prove that π is injective. This is estalished by the iterated application of the fact that each agent "knows" own variable and own beliefs: for any $T^V, \bar{t}$, a measurable subset b_i of X_i and a measurable subset e of $\Delta(\Omega \times_{i\leq I} T_i)$, $i \leq I$, $\alpha \in (0, 1]$ we have

$$\begin{aligned} \mu_i(t_i)(\{(\omega, t_1', .., t_I') &\in \Omega \times_{i\leq I} T_i | V_i(t_i') \in b_i\}) \geq \alpha \text{ iff } V_i(t_i) \in b_i, \\ \mu_i(t_i)(\{(\omega, t_1', .., t_I') &\in \Omega \times_{i\leq I} T_i | \mu_i(t_i') \in e\}) \geq \alpha \text{ iff } \mu_i(t_i) \in e. \end{aligned} \tag{15}$$

The fact follows from (1) as well as measurability of $\{(\omega, t_1', .., t_I') \in \Omega \times_{i\leq I} T_i | V_i(t_i') \in b_i\}$ and $\{(\omega, t_1', .., t_I') \in \Omega \times_{i\leq I} T_i | \mu_i(t_i') \in e\}$.

For any $K \geq 1$ we inductively define the classes E^K of sets in $\Delta(Y^{K-1})$ as follows. Each E^K is the class of finite intersections of sets of the form $\{\beta^1 \in \Delta(Y^0) | \beta^0(b_\Omega \times_{i\leq I} b_i) \geq \alpha\}$, for $K = 1$, or $\{\beta^K \in \Delta(Y^{K-1}) | \beta^K(\bigcup_{l=1}^{L}(b_\Omega^l \times_{i\leq I} b_i^l \times_{k=1}^{K-1} (\times_{i\leq I} e_i^{l,k}))) \geq \alpha\}$ for $K > 1$, where

[21] $\Theta \times_{i\in I} X_i$ projects on $\Theta \times_{j\neq i} X_j$ trivially; for a measurable projection $Proj : Y \to Z$, Z (homeomorphic to) a subspace of Y, the projection $Proj_\Delta : \Delta Y \to \Delta Z$ is defined as $(Proj_\Delta(\delta))(e) = \delta(Proj^{-1}(e))$.

$\alpha \in (0,1] \cap \mathbb{Q}$, $L \geq 1$, b_Ω, b_Ω^l and b_i, b_i^l are any measurable sets in Ω, X_i, respectively, and $e_i^{l,k} \in E^k$, $k < K$, $l \leq L$. We show that for any $i \leq I$ and $K \geq 1$ there is a function $f_i^K : E^K \to Pow(X_i \times \Delta(U_i^0) \times .. \times \Delta(U_i^{K-1}))$ such that for any $e^K \in E^K$ and $T^V, \bar{t}$ we have $\beta_i^K(T^V, \bar{t}) \in e^K$ iff $(V_i(t_i), \beta_{U,i}^1(T^V, \bar{t}), .., \beta_{U,i}^K(T^V, \bar{t})) \in f_i(e^K)$. Given that, the application of Lemma 2 yields the result.

For $K = 1$ we have for any e^1 of the form $\{\beta^1 \in \Delta(Y^0) | \beta^0(b_\Omega \times_{j \leq I} b_j) \geq \alpha\}$ that

$$\beta_i^1(T^V, \bar{t}) \in e^1 \text{ iff } \beta_{U,i}^1(T^V, \bar{t})(b_\Omega \times_{j \neq i} b_j) \geq \alpha \text{ and } V_i(t_i) \in b_i. \tag{16}$$

It follows that for any set e^1, $\beta_i^1(T^V, \bar{t}) \in e^1$ iff $(V_i(t_i), \beta_{U,i}^1(T^V, \bar{t})) \in f_i^1(e^1)$ for appropriate $f_i^1(e^1) \subseteq X_i \times \Delta(U_i^0)$.

For the induction step $K \to K+1$, for a set e^{K+1} of the form $\{\beta^{K+1} \in \Delta(Y^K) | \beta^{K+1}(\bigcup_{l=1}^L (b_\Omega^l \times_{i \leq I} b_i^l \times_{k=1}^K (\times_{i \leq I} e_i^{l,k}))) \geq \alpha\}$ we have $\beta_i^{K+1}(T^V, \bar{t}) \in e^{K+1}$ iff

$$\begin{aligned}
&\beta_i^{K+1}(T^V, \bar{t})(\textstyle\bigcup_{l=1}^L (b_\Omega^l \times_{i \leq I} b_i^l \times_{k=1}^K (\times_{i \leq I} e_i^{l,k}))) \geq \alpha \text{ iff} \\
&\beta_i^{K+1}(T^V, \bar{t})(\textstyle\bigcup_{l=1}^L (b_\Omega^l \times_{j \neq i} b_j^l \times X_i \times_{k=1}^K (\times_{j \neq i} e_j^{l,k} \times \Delta(Y^{k-1})))) \geq \alpha, \\
&V_i(t_i) \in b_i^l \text{ and } \beta_i^k(T^V, \bar{t}) \in e_i^{l,k},\ k \leq K, \text{ for some } l \leq L \text{ iff} \\
&\beta_{U,i}^{K+1}(T^V, \bar{t})(\textstyle\bigcup_{l=1}^L \{(\omega, (x_j)_{j \neq i}, (\delta_j^1)_{j \neq i}, .., (\delta_j^K)_{j \neq i})| \\
&\qquad \omega \in b_\Omega^l,\ x_j \in b_j^l,\ (x_j, \delta_j^1, .., \delta_j^k) \in f_j^k(e_j^{l,k}),\ k \leq K,\ j \neq i\}) \geq \alpha, \\
&V_i(t_i) \in b_i^l \text{ and } \beta_i^k(T^V, \bar{t}) \in e_i^{l,k},\ k \leq K, \text{ for some } l \leq L \text{ iff} \\
&\beta_{U,i}^{K+1}(T^V, \bar{t})(\textstyle\bigcup_{l=1}^L \{(\omega, (x_j)_{j \neq i}, (\delta_j^1)_{j \neq i}, .., (\delta_j^K)_{j \neq i})| \\
&\qquad \omega \in b_\Omega^l,\ x_j \in b_j^l,\ (x_j, \delta_j^1, .., \delta_j^k) \in f_j^k(e_j^{l,k}),\ k \leq K,\ j \neq i\}) \geq \alpha, \\
&V_i(t_i) \in b_i^l \text{ and } (V_i(t_i), \beta_{U,i}^1(T^V, \bar{t}), .., \beta_{U,i}^k(T^V, \bar{t})) \in f_i^k(e_i^{l,k}),\ k \leq K, \text{ for some } l \leq L \text{ iff} \\
&(V_i(t_i), \beta_{U,i}^1(T^V, \bar{t}), .., \beta_{U,i}^{K+1}(T^V, \bar{t})) \in f^{K+1}(e^{K+1}),
\end{aligned} \tag{17}$$

(the first equivalence follows from (15) and the next two follow from the induction assumption), where

$$\begin{aligned}
&f^{K+1}(e^{K+1}) := \{(x_i, \delta^1, .., \delta^{K+1}) \in X_i \times \Delta(U_i^0) \times .. \times \Delta(U_i^K)| \\
&|x_i \in b_i^l \text{ and } (x_i, \delta^1, .., \delta^k) \in f_i^k(e_i^{l,k}),\ k \leq K, \text{ for some } l \leq L, \text{ and} \\
&\delta^{K+1}(\textstyle\bigcup_{l=1}^L \{(\omega, (x_j)_{j \neq i}, (\delta_j^1)_{j \neq i}, .., (\delta_j^K)_{j \neq i})| \\
&\qquad \omega \in b_\Omega^l,\ x_j \in b_j^l,\ (x_j, \delta_j^1, .., \delta_j^k) \in f_j^k(e_j^{l,k}),\ k \leq K,\ j \neq i\}) \geq \alpha\}.
\end{aligned} \tag{18}$$

Extention to general e^{K+1} is straightforward. ■

For two systems of subsets $\mathcal{C}$ and $\mathcal{D}$ in some space Y we say that $\mathcal{C}$ *generates* $\mathcal{D}$ if for every $d \in \mathcal{D}$ we have $d = \bigcup_{c \in \mathcal{C}, c \subseteq d} c$.

Lemma 2 *Let Y be a Polish space and $\mathcal{C}$ a countable system of subsets that generates the topology of Y and is closed under finite intersections and unions.*[22] *Then:*

i) the countable system $\mathcal{C}_\Delta$, which is the closure under finite intersections and unions of the sets

$$\{\delta \in \Delta Y | \delta(c) \geq \alpha\}, \tag{19}$$

for $c \in \mathcal{C}$ and $\alpha \in [0,1] \cap \mathbb{Q}$, generates the weak topology of $\Delta(Y)$;*

ii) if topology of Y generates $\mathcal{C}$, then the weak topology of $\Delta(Y)$ generates $\mathcal{C}_\Delta$.*

Proof. (Lemma 2) We prove only part i). Fix a countable subbasis $\mathcal{B}$ of ΔY with weak* topology that consists of the sets

$$\{\delta \in \Delta Y | \delta[f] > \alpha\} \tag{20}$$

for f in some countable set F of continuous functions $f : Y \to (0,1)$ and $\alpha \in [0,1] \cap \mathbb{Q}$ (see e.g. Aliprantis and Border [1], Theorem 12.11). Fix any $B = \{\delta \in \Delta Y | \delta[f] > \alpha\} \in \mathcal{B}$ and $\delta^* \in B$. We must find $c_\Delta \in \mathcal{C}_\Delta$ with $\delta^* \in c_\Delta \subseteq B$.

Suppose that $\delta^*[f] > \alpha + 2\varepsilon$ and consider the open sets

$$o_n := \{y \in Y | f(y) \in (n/N, 1)\}, \tag{21}$$

$n = 1, .., N-1,\ 1/N \leq \varepsilon$. The function $\underline{f} := 1/N \sum_{n=1}^{N-1} \chi_{o_n}$, where χ_{o_n} is the indicator function for set o_n, satisfies $f - \varepsilon \leq f - 1/N \leq \underline{f} \leq f$. For $n = 1, .., N$ let $o_n = \bigcup_{k=1}^{\infty} c_{n,k}$ with $c_{n,k} \in \mathcal{C}$ and $c_{n,k} \subseteq c_{n,k'}$ for $k \leq k'$. Due to countable additivity of measures, we can fix $K \in \mathbb{N}$ such that for any $n = 1, .., N$ $\delta^*(o_n \backslash c_{n,K}) < \varepsilon$. For any n let $\alpha_n \in [0,1] \cap \mathbb{Q}$ be such that $\delta^*(c_{n,K}) \geq \alpha_n$ and $\alpha_n \geq \delta^*(o_n) - \varepsilon$. We have:

$$\begin{aligned} &\delta^* \in \textstyle\bigcap_{n=1}^{N} \{\delta | \delta(c_{n,K}) \geq \alpha_n\} \subseteq \bigcap_{n=1}^{N} \{\delta | \delta(o_n) \geq \alpha_n\} \subseteq \\ &\subseteq \{\delta | \delta[\underline{f}] \geq 1/N \textstyle\sum_{n=1}^{N-1} \alpha_n\} \subseteq \{\delta | \delta[f] \geq 1/N \sum_{n=1}^{N-1} \alpha_n\}, \end{aligned} \tag{22}$$

[22] See also Lemma 4.5 in Heifetz, Samet [18].

where the set inclusions follow from $\bigcup_{k=1}^{K} c_{n,k} \subseteq o_n$, the fact that for any $\delta \in \Delta Y$ $\delta[\underline{f}] = 1/N \sum_{n=1}^{N-1} \delta(o_n)$ and $\underline{f} \leq f$, consecutively. From the definition of α_n, K and $f - \varepsilon \leq \underline{f}$ we also have

$$1/N \sum_{n=1}^{N-1} \alpha_n \geq 1/N \sum_{n=1}^{N-1} (\delta^*(o_n) - \varepsilon) > \delta^*[\underline{f}] - \varepsilon \geq \delta^*[f] - 2\varepsilon > \alpha, \tag{23}$$

and so $\{\delta | \delta[f] \geq 1/N \sum_{n=1}^{N-1} \alpha_n\} \subseteq \{\delta | \delta[f] \geq \alpha\}$. It follows that $c := \bigcap_{n=1}^{N} \{\delta | \delta(c_{n,K}) \geq \alpha_n\}$ is the desired set in $\mathcal{C}_\Delta$. ■

Fix some countable bases $\mathcal{B}_\Omega, \mathcal{B}_i$ of $\Omega, X_i, i \leq I$, respectively. Consider the system of sets $\mathcal{C}$ in $\times_{i \leq I} (X_i \times_{k=0}^{\infty} \Delta(U_i^k))$ which is the minimal system with the following properties:

1. $(b_i \times_{k=0}^{\infty} \Delta(U_i^k)) \times_{j \neq i} (X_j \times_{k=0}^{\infty} \Delta(U_j^k)) \in \mathcal{C}$ for $b_i \in \mathcal{B}_i,\ i \leq I$.
2. If $C, C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$.
3. If $C^l = (X_i \times_{k=0}^{\infty} \Delta(U_i^k)) \times_{j \neq i} (b_j^l \times_{k=1}^{K} C_j^{l,k} \times_{k=K}^{\infty} \Delta(U_j^k)) \in \mathcal{C}$ for $l \leq L$, where $L \geq 1,\ i \leq I,\ b_j^l \in \mathcal{B}_j,\ C_j^{l,k} \subseteq \Delta(U_j^{k-1})$, then for $b_\Omega^l \in \mathcal{B}_\Omega,\ l \leq L,\ \alpha \in [0,1] \cap \mathbb{Q}$

   $$P_i^{\geq \alpha}(\cup_{l=1}^{L} (b_\Omega^l, C^l)) \times_{j \neq i} (X_j \times_{k=0}^{\infty} \Delta(U_j^k)) \in \mathcal{C},$$

   where

   $$P_i^{\geq \alpha}(\cup_{l=1}^{L} (b_\Omega^l, C^l)) := \{(x_i, \delta^1, \delta^2, ..) | \delta^{K+1} (\bigcup_{l=1}^{L} (b_\Omega^l \times_{j \neq i} b_j^l \times_{k=1}^{K} (\times_{j \neq i} C_j^{l,k})) \geq \alpha\}. \tag{24}$$

For a fixed type space T let $\mathcal{V}_T$ be the set of variable assignments over T, and $\mathcal{E}_{\Gamma,T}$ be the set of all BNE for Γ over T.

Lemma 3 *For every $C \in \mathcal{C},\ C = \times_{i \leq I} C_i$, there is a game $\Gamma^C : \Omega \times_{i \leq I} A_i^C \to \mathbb{R}^I$ and sets of actions $A_{*,i}^C \subseteq A_i^C,\ i \leq I$, such that for every T there are functions $\iota^C : \mathcal{V}_T \to \mathcal{E}_{\Gamma^C,T}$ and $\eta^C : \mathcal{E}_{\Gamma^C,T} \to \mathcal{V}_T$ such that:*
i) for any $V \in \mathcal{V}_T$ if $(\beta_U(T^V, \bar{t}))_i \in C_i$ then $(\iota^C(V)(\bar{t}))_i \in A_{*,i}^C$;
ii) for any $V^{BNE} \in \mathcal{E}_{\Gamma^C,T}$ if $(V^{BNE}(\bar{t}))_i \in A_{*,i}^C$ then $(\beta_U(T^{\eta^C(V^{BNE})}, \bar{t}))_i \in C_i$.

Proof. (Lemma 3) We construct the appropriate games Γ^C and sets of actions $A_{*,i}^C,\ i \leq I$, in three steps corresponding to steps 1.-3. in the definition of $\mathcal{C}$. In each game $\Gamma^C : \Omega \times_{i \leq I} A_i^C \to \mathbb{R}^I$ we have $A_i^C = P_i^C \times A_i^{C,0} \times .. \times A_i^{C,Z(C)},\ Z(C) \in \mathbb{N}$, where P_i^C is a partition of X_i, and for any $z \leq Z(C)$ $A_i^{C,z}$ is a partition of $\times_{k=0}^{\infty}\Delta(U_i^k)$. Moreover, we have $\bigcup_{a_i^C \in A_{*,i}^C} Proj_{P_i^C} a_i^C \times (\bigcap_{z \leq Z(C)} Proj_{A_i^{C,z}} a_i^C) = C_i$ (where $Proj_{P_i^C}$ and $Proj_{A_i^{C,z}}$ denote the projections on P_i^C and $A_i^{C,z}$).

In each step below we verify (a) that the payoffs for any action do not depend on its first coordinate, i.e.

$$\Gamma_i^C(\cdot, (p_i^C, a_i^{C,0}, .., a_i^{C,Z(C)}), \cdot) = \Gamma_i^C(\cdot, (p_i'^C, a_i^{C,0}, .., a_i^{C,Z(C)}), \cdot), \tag{25}$$

for $p_i^C, p_i'^C \in P_i^C$, $a_i^{C,z} \in A_i^{C,z}$, $z \leq Z(C) \leq I$; (b) that for any variable assignment V if $V^C = \times_{i \leq I} V_i^C$, $V_i^C : T_i \to A_i^C$ satisfies

$$(\beta_U(T^V, (t_1, .., t_I)))_i \in Proj_{P_i^C} V_i^C(t_i) \times (\bigcap_{z \leq Z(C)} Proj_{A_i^{C,z}}(V_i^C(t_i))), \tag{26}$$

for $(t_1, .., t_I) \in \times_{i \leq I} T_i$, $i \leq I$, then V^C is a BE for Γ^C; (c) that for any BE V^C for Γ^C, if variable assignment V satisfies

$$V_i(t_i) \in Proj_{P_i^C} V_i^C(t_i) \quad (t_i \in T_i,\ i \leq I) \tag{27}$$

then (26) holds.

Note that any function $\iota^C : \mathcal{V}_T \to \mathcal{E}_{\Gamma,T}$ such that for any $V \in \mathcal{V}_T$ $\iota^C(V)$ satisfies (26) fulfills i) in the Lemma. Moreover, any function $\eta^C : \mathcal{E}_{\Gamma^C,T} \to \mathcal{V}_T$ such that for any $V^{BE} \in \mathcal{E}_{\Gamma^C,T}$ $\eta^C(V^{BE})$ satisfies (27) fulfill ii) in the Lemma.

1. Let $C = (b_i^C \times_{k=0}^{\infty} \Delta(U_i^k)) \times_{j \neq i} (X_j \times_{k=0}^{\infty} \Delta(U_j^k))$ for $b_i^C \in \mathcal{B}_i$, $i \leq I$. Let $Z(C) = 0$, and define $P_i^C := \{b_i^C, X_i \backslash b_i^C\}$, $P_j^C := \{X_j\}$ for $j \neq i$ and $A_j^{C,0} := \{\times_{k=0}^{\infty}\Delta(U_j^k)\}$ for $j \leq I$, as well as $\Gamma_j^C \equiv 1$, $j \leq I$, and $A_{*,i}^C = \{b_i^C \times_{k=0}^{\infty} \Delta(U_i^k)\}$, $A_{*,j}^C = A_j^C$ for $j \neq i$. Verification of all the conditions is immediate.

2. Suppose that we have already constructed $\Gamma^{C''}, A_{*,i}^{C''}, \Gamma^{C'}, A_{*,i}^{C'}$, $i \leq I$. For $C = C' \cap C''$ and any $i \leq I$ define A_i^C, Γ_i^C and $A_{*,i}^C$ (where $P_i^{C''} \Cap P_i^{C'}$ denotes the coarsest meet of partitions

$P_i^{C''}$ and $P_i^{C'}$)

$$\begin{aligned}
&A_i^C := (P_i^{C''} \Cap P_i^{C'}) \times A_i^{C'',0} \times .. \times A_i^{C'',Z(C'')} \times A_i^{C',0} \times .. \times A_i^{C',Z(C')}, \\
&\Gamma_i^C(\omega, (p_j^{C''} \cap p_j^{C'}, (a_j^{C'',0}, .., a_j^{C'',Z(C'')}, a_j^{C',0}, .., a_j^{C',Z(C')})_{j\le I}) := \\
&\qquad \Gamma_i^{C''}(\omega, (p_j^{C''}, (a_j^{C'',0}, .., a_j^{C'',Z(C'')})_{j\le I}) + \Gamma_i^{C'}(\omega, (p_j^{C'}, (a_j^{C',0}, .., a_j^{C',Z(C')})_{j\le I}), \qquad (28) \\
&A_{*,i}^C := \{(p_i^{C''} \cap p_i^{C'}, a_i^{C'',0}, .., a_i^{C'',Z(C'')}, a_i^{C',0}, .., a_i^{C',Z(C')})| \\
&\qquad (p_i^{C''}, a_i^{C'',0}, .., a_i^{C'',Z(C'')}) \in A_{*,i}^{C''} \text{ and } (p_i^{C'}, a_i^{C',0}, .., a_i^{C',Z(C')}) \in A_{*,i}^{C'}\}.
\end{aligned}$$

From Induction Assumpion follows that condition (25) is satisfied. For any variable assignment V, consider V^C satisfying (26). The assignments $V^{C'} = \times_{i\le I} V_i^{C'}$,

$$\begin{aligned}
V_i^{C'}(t_i) \quad : \quad &= (p_i^{C'}, a_i^{C',0}, .., a_i^{C',Z(C')}), \qquad (29) \\
&\text{for } V_i^C(t_i) = (p_i^{C''} \cap p_i^{C'}, a_i^{C'',0}, .., a_i^{C'',Z(C'')}, a_i^{C',0}, .., a_i^{C',Z(C')}),
\end{aligned}$$

$t_i \in T_i,\ i \le I$, and $V^{C''}$ defined analogously satisfy (26), and so, by Induction Assumpion, are BE for $\Gamma^{C'}$ and $\Gamma^{C''}$, respectively. It follows that V^C is a BE for Γ^C. Similarly, if V^C is a BE for Γ^C, then $V^{C'}$ defined as in (29) and an analogous $V^{C''}$ are BE for $\Gamma^{C'}$ and $\Gamma^{C''}$, respectively (here we use the fact that the payoffs do not depend on the first coordinate). Therefore any V satisfying condition (27) satisfies respective (27) for $V^{C'}$ as in (29) and $V^{C''}$, and so fulfills by Induction Assumption the respective (26), which implies (26) for V^C.

3. Suppose that for some $L \ge 1$, $i \le I$ and the sets $C^l = (X_i \times_{k=0}^{\infty} \Delta(U_i^k)) \times_{j\ne i} (b_j^l \times_{k=1}^{K} C_j^{l,k} \times_{k=K}^{\infty} \Delta(U_j^k)),\ l \le L$, we have constructed the games Γ^{C^l} and $A_{*,j}^{C^{l\prime}}$. Consider $C = \{(x_i, \delta^1, \delta^2, ..)|\delta^{K+1}(\bigcup_{l=1}^{L}(b_\Omega^l \times_{j\ne i} b_j^l \times_{k=1}^{K}(\times_{j\ne i} C_j^{\prime l,k})) \ge \alpha\} \times_{j\ne i}(X_j \times_{k=0}^{\infty} \Delta(U_j^k))$ for some $b_\Omega^l \in \mathcal{B}_\Omega,\ \alpha \in [0,1] \cap \mathbb{Q}$, and let $a_i^+ := \{(\delta^1, \delta^2, ..)|\delta^{K+1}(\bigcup_{l=1}^{L}(b_\Omega^l \times_{j\ne i} b_j^l \times_{k=1}^{K}(\times_{j\ne i} C_j^{\prime l,k})) \ge \alpha\}$, $a_i^- := \{(\delta^1, \delta^2, ..)|\delta^{K+1}(\bigcup_{l=1}^{L}(b_\Omega^l \times_{j\ne i} b_j^l \times_{k=1}^{K}(\times_{j\ne i} C_j^{\prime l,k})) < \alpha\}$ and $a_j' := \times_{k=0}^{\infty} \Delta(U_j^k),\ j \ne i$.

Define the game Γ^C and the sets $A^C_{*,i}$, $A^C_{*,j}$:

$$
\begin{aligned}
&A_i^C := (P_i^{C^1} \Cap .. \Cap P_i^{C^L}) \times A_i^{C^1,0} \times .. \times A_i^{C^1,Z(C^1)} \times .. \times A_i^{C^L,0} \times .. \times A_i^{C^L,Z(C^L)} \times \{a_i^+, a_i^-\},\\
&A_j^C := (P_j^{C^1} \Cap .. \Cap P_j^{C^L}) \times A_j^{C^1,0} \times .. \times A_j^{C^1,Z(C^1)} \times .. \times A_j^{C^L,0} \times .. \times A_j^{C^L,Z(C^L)} \times \{a_j'\},\\
&\Gamma_j^C(\omega, (p_h^{C^1} \cap .. \cap p_h^{C^L}, a_h^{C^1,0}, .., a_h^{C^1,Z(C^1)}, .., a_h^{C^L,0}, .., a_h^{C^L,Z(C^L)}, a_h)_{h\le I}) :=\\
&\quad \textstyle\sum_{l=1}^L \Gamma_j^{C^L}(\omega, (p_h^{C^l}, a_h^{C^l,0}, .., a_h^{C^l,Z(C^l)})_{h\le I}), \qquad (30)\\
&\Gamma_i^C(\omega, (p_h^{C^1} \cap .. \cap p_h^{C^L}, a_h^{C^1,0}, .., a_h^{C^1,Z(C^1)}, .., a_h^{C^L,0}, .., a_h^{C^L,Z(C^L)}, a_h)_{h\le I}) :=\\
&\quad \textstyle\sum_{l=1}^L \Gamma_j^{C^L}(\omega, (p_h^{C^l}, a_h^{C^l,0}, .., a_h^{C^l,Z(C^l)})_{h\le I}) + \alpha \text{ if } a_i = a_i^-,\\
&\Gamma_i^C(\omega, (p_h^{C^1} \cap .. \cap p_h^{C^L}, a_h^{C^1,0}, .., a_h^{C^1,Z(C^1)}, .., a_h^{C^L,0}, .., a_h^{C^L,Z(C^L)}, a_h)_{h\le I}) :=\\
&\quad \textstyle\sum_{l=1}^L \Gamma_j^{C^L}(\omega, (p_h^{C^l}, a_h^{C^l,0}, .., a_h^{C^l,Z(C^l)})_{h\le I}) + \left\{\begin{smallmatrix} 1 \text{ if } \omega\in b_\Omega^l \text{ and } a_j^{C'}\in A_{*,j}^{C^l},\ j\neq i, \text{ for some } l\le L \\ 0 \text{ otherwise}\end{smallmatrix}\right. \text{ if } a_i = a_i^+,\\
&A_{*,i}^C := \{(p_i^{C^1} \cap .. \cap p_i^{C^L}, a_i^{C^1,0}, .., a_i^{C^1,Z(C^1)}, .., a_i^{C^L,0}, .., a_i^{C^L,Z(C^L)}, a_i) | a_i = a_i^+\},\\
&A_{*,j}^C := A_j^C,
\end{aligned}
$$

for $j \neq i$. Condition (25) holds, given Induction Assumption. Consider any variable assignment V and an assignment V^C satisfying (26). For any $l \le L$ the assignment $V^{C^l} = \times_{i\le I} V_i^{C^l}$ defined as

$$
\begin{aligned}
V_i^{C^l}(t_i) \ :\ &= (p_i^{C^l}, a_i^{C^l,0}, .., a_i^{C^l,Z(C^l)}), \qquad (31)\\
&\text{for } V_i^C(t_i) = (p_h^{C^1} \cap .. \cap p_h^{C^L}, a_h^{C^1,0}, .., a_h^{C^1,Z(C^1)}, .., a_h^{C^L,0}, .., a_h^{C^L,Z(C^L)}, a_h),
\end{aligned}
$$

for $t_i \in T_i, i \le I$, is by Induction Assumption a BE for Γ^{C^l}. By (26) V^C assigns to i action with last coordinate a_i^+ exactly when i believes with probability at least α that $\omega \in b_\Omega^l$ and $a_j^{C^l} \in A_{*,j}^{C^l},\ j \neq i$, for some $l \le L$. It follows that V^C is a BE for Γ^C.

Similarly, for any BE V^C for Γ^C, for any $l \le L$ the $V^{C'}$ defined as in (31) is a BE for Γ^{C^l}. Moreover, any V satisfying (27) satisfies respective (27) for V^{C^l} as in (31) and so fulfills by Induction Assumption the respective (26). Furthermore, if V^C assigns to i action with the last coordinate a_i^+ then i believes with probability at least α that $\omega \in b_\Omega^l$ and $a_j^{C^l} \in A_{*,j}^{C^l},\ j \neq i$, for some $l \le L$, which together with the Induction Assumption implies (26). This finishes the proof of the Lemma. ■

Given the lemmas 2, 1 and 3, in order to prove $cl(\beta^X(T,\bar{t})) \subseteq cl(\beta^X(S,\bar{s}))$ and finish the proof of the Theorem we proceed as follows. Choose any point $h \in cl(\beta^X(T,\bar{t}))$ and let

$\{D^k\}_{k=1}^{\infty}$ be its countable basis in $(\times_{k=0}^{\infty}\Delta(Y^k))^I$, and so $\{\pi(D^k)\}_{k=1}^{\infty}$ is a countable basis of $\pi(h)$ in $\times_{i\leq I}(X_i \times_{k=0}^{\infty} \Delta(U_i^k))$. Fix $k \in \mathbb{N}$. Let $C^k \in \mathcal{C}$ be such that $\pi(h) \in C^k \subseteq \pi(D^k)$ and for a variable assignment V $\beta_U(T^V, \bar{t}) \in C^k$ (see Lemma 2). From i) in Lemma 3 we have that $BE_{C^k}(T, \bar{t}) \cap (\times_{i\leq I} A_{*,i}^{C^k}) \neq \emptyset$, and so, by assumption, also $BE_{C^k}(S, \bar{s}) \cap (\times_{i\leq I} A_{*,i}^{C^k}) \neq \emptyset$. From ii) in Lemma 3, there is a variable assignment V' such that $\beta_U(S^{V'}, \bar{s}) \in C^k \subseteq \pi(D^k)$. It follows that $\pi(h) \in cl(\pi(\beta^X(S, \bar{s})))$, and so $h \in cl(\beta^X(S, \bar{s}))$.

Proof of Theorem 3

Consider the countable system $\mathcal{C} = \{C^1, C^2, ..\}$ of sets in $\times_{i\leq I}(X_i \times_{k=0}^{\infty} \Delta(U_i^k))$ from the proof of Theorem 2, together with the games $\{\Gamma^{C^n} : \Omega \times_{i\leq I} (P_i^{C^n} \times A_i^{C^n,0} \times .. \times A_i^{C^n,Z(C^n)}) \to \mathbb{R}^I\}_{n\in\mathbb{N}}$ constructed in Lemma 3, where we can clearly assume that $\Gamma^{C^n}(\Omega \times_{i\leq I} (P_i^{C^n} \times A_i^{C^n,0} \times .. \times A_i^{C^n,Z(C^n)})) \subseteq (0,1)^I$. The game $\Gamma^{\times} : \Omega \times_{i\leq I} A_i^{\times} \to \mathbb{R}^I$ will have

$$\begin{aligned}
&A_i^{\times} := X_i \times_{n\in\mathbb{N}} (A_i^{C^n,0} \times .. \times A_i^{C^n,Z(C^n)}),\\
&\Gamma_i^{\times}(\omega, (x_1, (a_1^{C^n,0}, ..a_1^{C^n,Z(C^n)})_{n\in\mathbb{N}}), .., (x_I, (a_I^{C^n,0}, ..a_I^{C^n,Z(C^n)})_{n\in\mathbb{N}})) =\\
&\quad = \textstyle\sum_{n=1}^{\infty} 1/3^n * \Gamma_i^{C^n}(\omega, (p_1^{C^n}, a_1^{C^n,0}, .., a_1^{C^n,Z(C^n)}), .., (p_I^{C^n}, a_I^{C^n,0}, .., a_I^{C^n,Z(C^n)}))
\end{aligned}$$

with $p_i^{C^n}$ such that $x_i \in p_i^{C^n}$, $i \leq I$, $n \in \mathbb{N}$. The action spaces $A_i^{\times}$ are Polish in the product topology, and the subspaces $\{\Gamma_i^{\times}(\cdot, a_i^{\times}, \cdot)\}_{a_i^{\times}\in A_i^{\times}}$ are Polish in the sup metric (with the countable dense subset consisting of all finite sums of the form $\sum_{n=1}^{N} 1/3^n * \Gamma_i^{C^n}(\cdot, a_i^{C^n}, \cdot)$ with $a_i^{C^n} \in A_i^{C^n}$; completeness is clear). It is easily verified that $\Phi_i : A_i^{\times} \to \{\Gamma_i^{\times}(\cdot, a_i^{\times}, \cdot)\}_{a_i^{\times}\in A_i^{\times}}$ are continuous, and therefore measurable, $i \leq I$. Fix $\beta_U^* \in \times_{i\leq I}(X_i \times_{k=0}^{\infty} \Delta(U_i^k))$ and let $\beta_U^* = \bigcap_{n=1}^{\infty} C^{k_n}$ for $k_n \in \mathbb{N}$ (see Lemma 2). It follows from Lemma 3 that for any pointed type space $T, \bar{t}$ $\beta_U^* \in \bigcup_{V \text{ var. ass.}} \beta_U(T^V, \bar{t})$ if and only if for some $((x_1, (a_1^{C^n,0}, ..a_1^{C^n,Z(C^n)})_{n\in\mathbb{N}}), .., (x_I, (a_I^{C^n,0}, .., a_I^{C^n,Z(C^n)})_{n\in\mathbb{N}})) \in \times_{i\leq I} A_i^{\times}$ with $((p_1^{C^n}, a_1^{C^n,0}, .., a_1^{C^n,Z(C^n)}), .., (p_I^{C^n}, a_I^{C^n,0}, .., a_I^{C^n,Z(C^n)})) \in A_*^{C^{k_n}}$, for $x_i \in p_i^{C^{k_n}}$, $i \leq I$, $n \in \mathbb{N}$, we have $((x_1, (a_1^{C^n,0}, ..a_1^{C^n,Z(C^n)})_{n\in\mathbb{N}}), .., (x_I, (a_I^{C^n,0}, .., a_I^{C^n,Z(C^n)})_{n\in\mathbb{N}})) \in BE_{\Gamma^{\times}}(T, \bar{t})$.

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