Optimal Pricing Mechanisms with Unknown Demand

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August 15, 2002

Abstract

The standard profit-maximizing multi-unit auction intersects the submitted demand curve with a preset reservation supply curve, which is determined using the distribution from which the buyers' valuations are drawn. However, when this distribution is *unknown*, a preset supply curve cannot maximize monopoly profits. The optimal pricing mechanism in this situation sets a price to each buyer on the basis of the demand distribution inferred statistically from other buyers' bids. The resulting profit converges to the optimal monopoly profit with known demand as the number of buyers goes to infinity, and convergence can be substantially faster than with sequential price experimentation.

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Recent advances in information technology, most notably the Internet, enabled the use of economic allocation mechanisms that had been impractical before. Many goods traditionally sold at posted prices are now sold in auction-like mechanisms, in which buyers express their preferences by submitting bids. Some Internet websites, such as *eBay.com*, use traditional auction mechanisms, such as the English auction. Other websites have developed new mechanisms. For example, so-called "demand aggregation" sites, such as *Mercata.com*, *LetsBuyIt.com*, and *eWinWin.com*, obtain the price by intersecting the demand curve formed by the buyers' bids with a downward-sloping "price curve."

What is the profit-maximizing pricing mechanism, and does it improve upon posted pricing?[1] The present paper examines this question in the context of selling multiple homogeneous units to buyers with unit demands. First the paper makes the standard assumption of auction theory that the seller knows the distribution from which the buyers' valuations are drawn. Under this and other standard assumptions, the optimal auction can be represented by intersecting a supply curve submitted by the seller with the demand curve revealed by the buyers' bids, and selling to those buyers whose bids are above the intersection. The seller's profit-maximizing supply curve depends on her cost function as well as on the distribution of buyers' valuations. Furthermore, in two important special cases the seller cannot improve upon a posted price. One such case is when the seller's marginal cost is constant, and so her optimal supply curve is perfectly elastic. The other case is when the number of buyers is large, and by the Law of Large Numbers the seller can predict the aggregate demand curve and the price at which it intersects the optimal supply curve.

The problem ignored by this standard analysis is that in reality, the seller may not know the distribution from which buyers' valuations are drawn, and thus may be unable

[1]While this paper focuses on the profit objective, the analysis is also applicable to a socially-minded seller who needs to cover a fixed cost of production.

to calculate the optimal reservation supply curve. A typical example is the sale of tickets or subscriptions to a one-of-a-kind concert or sporting event. Even though there are many identical units for sale, such units have not been sold before and so the seller does not know the potential demand. As emphasized in microeconomic textbooks, in this situation "a monopolistic market does not have a supply curve" (Robert S. Pindyck and Daniel L. Rubinfeld (1995)), because the profit-maximizing price depends on the overall shape, and in particular elasticity, of the demand curve.[2]

This paper proposes a new pricing mechanism that maximizes the seller's profit without requiring prior knowledge of demand. The mechanism is based on the idea that buyers' bids reveal information about the distribution of their valuations. While standard auctions ignore this information, the optimal mechanism uses it for pricing. When the number of buyers is large, the seller learns the distribution precisely, and can price optimally given the revealed distribution.

To ensure that a buyer cannot obtain a better price by misreporting his valuation, he should face a price that depends only on other buyers' bids, and not on his own. Formally, such mechanisms are the only ones satisfying dominant-strategy incentive compatibility and ex post individual rationality. This paper characterizes the expected profit-maximizing mechanism satisfying these requirements. In the simple case where the seller's marginal cost is constant, the optimal mechanism offers each buyer the optimal monopoly price against the demand curve inferred from other buyers' bids.

The proposed mechanism improves substantially upon posted pricing, but is qualita-

[2]A similar motivation underlies the analysis of Andrew V. Goldberg et al. (2001). A key difference is that they assume complete ignorance of the buyers' valuations, while in this paper these valuations are drawn from the same, although unknown, distribution. Also, Goldberg et al. (2001) maximize the *worst-case* revenue (relative to that from the optimal posted price), for which purpose randomized mechanisms strictly dominate deterministic ones.

tively different from standard auctions. The key difference is that each buyer's bid has an *informational role*: it affects other buyers' allocations even when it does not affect his own. In particular, such a mechanism cannot be represented with a supply curve.[3]

With a small number of buyers, the seller's Bayesian prior affects her posterior beliefs about the distribution of valuations, and thereby optimal pricing. The optimal mechanism is thus still not completely "detail-free" in the sense of Robert B. Wilson (1987) — the dependence on the seller's priors is simply pushed to a higher level. However, as the number n of buyers grows, the information revealed by buyers' bids overwhelms the seller's priors. The paper shows that for any consistent estimation of demand and its elasticity, as $n \to \infty$, the seller's expected profit converges to the maximum profit achievable with the knowledge of the true demand distribution. In particular, this holds for Bayesian estimation provided that the prior's support includes the true distribution. This also holds for classical statistical estimation, both parametric and non-parametric. For example, the seller can use the reported empirical distribution of the valuations of all buyers other than i as an estimate of the distribution of buyer i's valuation, and offer buyer i the optimal monopoly price against this distribution.[4]

With a large number of buyers, there are many alternative ways to learn demand and attain the optimal monopoly profit asymptotically. For example, the seller can survey a small proportion of buyers and use their reported valuations to set the optimal price to the remaining buyers. Alternatively, the seller can experiment by pricing to different buyers sequentially and updating the price using purchase history (see, e.g., Philippe Aghion et al. (1991), Leonard J. Mirman et al. (1993), Godfrey Keller and Sven Rady (1999), and

[3]Multi-unit auctions that cannot be represented with a supply curve have also been considered by Yvan Lengwiler (1998) and David McAdams (2000), though with a different motivation.

[4]This mechanism is also suggested by Sandeep Baliga and Rakesh Vohra (2002), in independent and contemporaneous work.

Yongmin Chen and Ruqu Wang (1999)). However, both these strategies set a price to each buyer utilizing less information than the optimal mechanism derived in this paper. In particular, the price offered to a buyer depends only on the information received from the preceding buyers, but not from the subsequent buyers. This "informational inefficiency" may slow down convergence to the optimal monopoly profit, sometimes quite dramatically.

It should be noted that relaxing the "ex post" constraints of dominant-strategy incentive compatibility and ex post individual rationality to the corresponding "interim" constraints of Bayesian incentive-compatibility and interim individual rationality would allow the seller to extract buyer surplus using mechanisms suggested by Jacques Cremer and Richard P. McLean (1985, 1988). However, such mechanisms are not "detail-free," since they are sensitive to the buyers' knowledge about the distribution and each other's valuations, and a seller who is ignorant of the extent of such knowledge may not want to use them.

The paper is organized as follows. Section I describes the model and the class of mechanisms being considered. Section II characterizes the optimal auction with a known demand distribution, and shows that it can normally be represented as the Vickrey-Groves-Clarke mechanism in which the seller manipulates her supply curve in a way that depends on the demand distribution. The section also examines circumstances in which the seller can do just as well with a posted price. Section III derives the optimal pricing mechanism when the seller does not know the distribution of demand but has a Bayesian prior over it, and so the buyers' valuations are correlated from her viewpoint. Section IV illustrates the optimal mechanism with a parametric example in which the valuations are drawn from an exponential distribution with an unknown hazard rate. Section V shows that the seller's expected profit converges to the maximum monopoly profit achievable with known demand as the number of buyers goes to infinity. Section VI examines the rate of convergence and compares it to that achieved by sequential experimentation mechanisms. Section VII discusses and motivates the restriction to ex post mechanisms. Section VIII concludes and discusses several potential extensions.

I. Setup

A monopolistic seller faces n buyers, each of whom has unit demand.[5] Each buyer $i = 1, \ldots, n$ privately observes his valuation v_i; in particular, the valuations are not observed by the seller. Buyers' valuations are independently drawn from a distribution F on $[0, \overline{v})$ (where $\overline{v} = \infty$ is allowed), with a positive continuous density function $f(v) = F'(v)$ and a finite expectation $E[v]$. Section II will consider the standard case in which the distribution F is common knowledge, while subsequent sections will suppose that F is not known (though the seller may have a Bayesian prior over possible distributions).

An outcome is described by the allocation of the good and the buyers' payments to the seller. An allocation of the good is a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathfrak{X}^n$, where $\mathfrak{X} = \{0, 1\}$ is the set of a buyer's possible purchases from the seller. The buyers' payments to the seller constitute a vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$. All buyers' utilities as well as the seller's profit are quasilinear in the payments. The seller's cost of producing quantity X is $C(X)$.

By the Revelation Principle, the seller can restrict attention to direct revelation mechanisms, which ask each buyer to bid his valuation, and ensure that all buyers participate and bid truthfully in equilibrium. For the sake of generality, the mechanism will be allowed to specify outcomes contingent on a public randomization $\omega \in \Omega$, where Ω is a probability space. Thus, a mechanism is described by an allocation rule $\mathbf{x} : [0, \overline{v})^n \times \Omega \to \mathfrak{X}^n$ and a payment rule $\mathbf{t} : [0, \overline{v})^n \times \Omega \to \mathbb{R}^n$. A *deterministic* mechanism is one that does not use randomization, and whose allocation and payment rules are simply $\mathbf{x} : [0, \overline{v})^n \to \mathfrak{X}^n$ and $\mathbf{t} : [0, \overline{v})^n \to \mathbb{R}^n$.

It is common to impose the Bayesian Incentive-Compatibility (BIC) and Interim Indi-

[5] With obvious alterations the analysis could be applied to the problem of procuring from n sellers, each of whom has unit supply.

vidual Rationality (IIR) constraints on the mechanism. Formally, for any buyer i and any $v_i, \hat{v}_i \in [0, \overline{v})$,

$$\text{(BIC)} \quad E_{\omega, \mathbf{v}_{-i}|v_i}\left[v_i x_i(\mathbf{v}, \omega) - t_i(\mathbf{v}, \omega)\right] \geq E_{\omega, \mathbf{v}_{-i}|v_i}\left[v_i x_i(\hat{v}_i, \mathbf{v}_{-i}, \omega) - t_i(\hat{v}_i, \mathbf{v}_{-i}, \omega)\right],$$

$$\text{(IIR)} \quad E_{\omega, \mathbf{v}_{-i}|v_i}\left[v_i x_i(\mathbf{v}, \omega) - t_i(\mathbf{v}, \omega)\right] \geq 0.$$

A mechanism satisfying (BIC) and (IIR) will be called an "interim" mechanism.

Note that randomized payments can be replaced with deterministic payments $\widetilde{t}_i(\mathbf{v}) = E_\omega t_i(\mathbf{v}, \omega)$ without affecting the constraints or the seller's expected revenue. On the other hand, a randomized allocation rule $\mathbf{x}(\cdot)$ may be useful (when the seller's cost function is nonlinear), though it will prove not to be in the cases considered below.

This paper will focus on mechanisms satisfying the stronger requirements of *Dominant-strategy* Incentive Compatibility (DIC) and *Ex post* Individual Rationality (EIR). Formally, for any buyer i, any valuation profile $\mathbf{v} \in [0, \overline{v})^n$, and any $\hat{v}_i \in [0, \overline{v})$,

$$\text{(DIC)} \quad E_\omega\left[v_i x_i(\mathbf{v}, \omega) - t_i(\mathbf{v}, \omega)\right] \geq E_\omega\left[v_i x_i(\hat{v}_i, \mathbf{v}_{-i}, \omega) - t_i(\hat{v}_i, \mathbf{v}_{-i}, \omega)\right],$$

$$\text{(EIR)} \quad E_\omega\left[v_i x_i(\mathbf{v}, \omega) - t_i(\mathbf{v}, \omega)\right] \geq 0.$$

These constraints require that each buyer's incentives to participate and bid truthfully are satisfied *ex post* (for any possible realization of other buyers' valuations), rather than just *interim* (on expectation over these valuations).[6] A mechanism satisfying (DIC) and (EIR) will be called an "ex post" mechanism. Such mechanisms are also studied by Kim-Sau Chung and Jeffrey C. Ely (2001), in the more general case of interdependent valuations.

In the standard auction setup in which buyers' valuations are independently drawn from a known distribution F, the restriction to ex post mechanisms does not reduce the seller's expected profit under standard assumptions, as explained in Section II below. However,

[6] One can formulate even stronger constraints requiring incentive-compatibility and individual rationality to hold for every realization of ω. We do not do this because randomization does not prove useful in our cases of interest anyway.

when the distribution of F is unknown, and so the valuations are correlated from the seller's viewpoint, the restriction does reduce the expected profit. Nevertheless, this restriction will be motivated in Section VII with the requirement that the mechanism be robust to the buyers' knowledge of the distribution and each other's valuations.

Deterministic ex post mechanisms have a particularly simple characterization:

LEMMA 1: *A deterministic mechanism $\langle \mathbf{x}(\cdot), \mathbf{t}(\cdot) \rangle$ is an ex post mechanism if and only if for each buyer i there exist functions $p_i, s_i : [0, \overline{v})^{n-1} \to \mathbb{R}_+$ such that for every valuation profile $\mathbf{v} \in [0, \overline{v})^n$,*[7]

$$x_i(\mathbf{v}) = \begin{cases} 1 & \text{if } v_i > p_i(\mathbf{v}_{-i}) \\ 0 & \text{if } v_i < p_i(\mathbf{v}_{-i}), \end{cases} \quad \text{and } t_i(\mathbf{v}) = p_i(\mathbf{v}_{-i}) x_i(\mathbf{v}) - s_i(\mathbf{v}_{-i}).$$

PROOF:

The "if" part is easy to verify. The "only if" part follows from the Taxation Principle (see, e.g., Bernard Salanie (1997)), which under (DIC) allows to represent the mechanism faced by buyer i for any given profile $\mathbf{v}_{-i}$ of other buyers' reports as a nondecreasing tariff $T_i(\cdot, \mathbf{v}_{-i}) : \mathfrak{X} \to \mathbb{R}$. Take $s_i(\mathbf{v}_{-i}) = -T_i(0, \mathbf{v}_{-i})$ and $p_i(\mathbf{v}_{-i}) = T_i(1, \mathbf{v}_{-i}) - T_i(0, \mathbf{v}_{-i})$. (EIR) for $v_i = 0$ requires that $s_i(\mathbf{v}_{-i}) \geq 0$.

The mechanism described in Lemma 1 offers each buyer i a lump-sum subsidy $s_i(\mathbf{v}_{-i}) \geq 0$ and a price $p_i(\mathbf{v}_{-i}) \geq 0$ that depend on other buyers' reports. Buyer i receives a unit at this price if and only if the price is below his reported valuation. Such mechanisms will be called *pricing mechanisms*, and the functions $p_i(\cdot)$ and $s_i(\cdot)$ will be called the *pricing* and *subsidy functions* respectively.

Lemma 1 implies that any deterministic allocation rule that is implementable in an ex post mechanism is *monotonic*, i.e., each buyer i's consumption $x_i(v_i, \mathbf{v}_{-i})$ is nondecreasing

[7]The consumption $x_i(\mathbf{v})$ for $v_i = p_i(\mathbf{v}_{-i})$ is left indeterminate, which is not important because the probability of this occurring is zero.

in his valuation v_i for any profile $\mathbf{v}_{-i}$ of others' valuations. In the unique pricing function implementing such an allocation rule, the price to each buyer equals the minimum bid procuring him a unit:

$$p_i\left(\mathbf{v}_{-i}\right) = \inf\left\{v_i \in [0, \overline{v}) : x_i\left(v_i, \mathbf{v}_{-i}\right) = 1\right\}. \tag{1}$$

As for the subsidy functions, a profit-maximizing seller will set them identically to zero.

One example of a pricing mechanism is the Vickrey-Groves-Clarke mechanism, in which $p_i(\mathbf{v}_{-i})$ equals the externality imposed by buyer i on the others. Another example is a *posted-price mechanism*, in which $p_i(\mathbf{v}_{-i}) \equiv p^*$ for all i, i.e., buyers face a single price that does not depend on any reports.

Observe that any pricing mechanism could in principle be implemented with a two-stage procedure, in which (1) buyers report their valuations $(v_1, \ldots, v_n)$, and then (2) each buyer i decides whether to purchase at price $p_i(\mathbf{v}_{-i})$. Since a buyer's stage 1 report has no effect on the price he faces in stage 2, truthtelling is a weak equilibrium at stage 1. However, there are several concerns with such implementation. One concern is that a buyer who has an (arbitrarily small) cost of learning his valuation would not expend the cost at stage 1, expecting to avoid it when offered a very high or very low price at stage 2. Another concern is that arbitrarily small bribes could induce collusion at stage 1. For these (unmodeled) reasons, it is preferable to eliminate the buyers' discretion at stage 2, instead determining their purchases on the basis of their reported valuations. Truthtelling will then be *uniquely* optimal for each buyer given sufficient uncertainty about others' reports.

II. The Optimal Mechanism with a Known Distribution

This section describes the optimal mechanism when the distribution F is known by the seller. This problem was first analyzed by Roger B. Myerson (1981) for the case of a single unit, and the analysis was extended to the multi-unit case by Jeremy I. Bulow and John

Roberts (1989). This section offers new characterizations of the optimal mechanism for important special cases, and provides a useful benchmark for the subsequent analysis of the case in which F is unknown.

By the Revenue Equivalence Theorem, the allocation rule $\mathbf{x}(\cdot)$ fully determines the information rents of buyers in any Bayesian incentive-compatible mechanism in which the participation constraints of zero-valuation buyers bind. The seller's expected profit can be expressed as the difference between the expected social surplus and the sum of buyers' expected information rents. Upon integration by parts, this difference can be written as the expectation of the *virtual surplus*

$$J(\mathbf{x}, \mathbf{v}) = \sum_i m(v_i)x_i - C(\sum_i x_i), \text{ where } m(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)}.$$

The difference between $m(v_i)$, called buyer i's *virtual valuation*, and his true valuation v_i accounts for the buyer's information rent that is not captured by the seller. The function $m(\cdot)$ is often called the *marginal revenue function*.[8]

If the seller is restricted to ex post mechanisms, her problem can be written as that of choosing an allocation rule to maximize the expected virtual surplus subject to a monotonicity constraint associated with dominant-strategy incentive compatibility:[9]

$$(2) \qquad \max_{\mathbf{x}:[0,\overline{v})^n \times \Omega \to \mathfrak{X}^n} E_{\mathbf{v},\omega} J(\mathbf{x}(\mathbf{v}, \omega), \mathbf{v}) \text{ s.t.}$$

$$\text{(M)} \quad E_\omega x_i(\mathbf{v}, \omega) \text{ is nondecreasing in } v_i \text{ for all } i, \text{ all } \mathbf{v}_{-i} \in [0, \overline{v})^n.$$

How would the seller's problem change if she could use interim rather than ex post mechanisms? The replacement of dominant-strategy with Bayesian incentive-compatibility

[8] This name, suggested by Bulow and Roberts (1989), comes from the following parallel to the monopoly pricing problem. If $D(p) = 1 - F(p)$ is the expected demand curve for a given buyer, $P(X) = D^{-1}(X)$ is the inverse demand curve, and $R(X) = P(X)X$ is the revenue function, then $m(v) = R'(D(v))$, i.e., the marginal revenue expressed as a function of price.

[9] For deterministic mechanisms, (M) follows from Lemma 1. For a general characterization of dominant-strategy incentive-compatible mechanisms, see Dilip Mookherjee and Stephan Reichelstein (1992).

weakens the monotonicity constraint (M) to the requirement that $E_{\mathbf{v}_{-i},\omega} x_i(\mathbf{v},\omega)$ be non-decreasing in v_i for all i. At the same time, as shown by Mookherjee and Reichelstein (1992), the expected transfers generated by any given implementable allocation rule do not change. Also, transfers satisfying interim participation constraints can be modified to satisfy ex post participation constraints while preserving all interim expected transfers and all dominant-strategy incentive constraints. Therefore, the restriction to ex post mechanisms can only hurt the seller by strengthening the monotonicity constraint. In the cases considered below, this strengthening does not reduce the seller's expected profit.

A. *When a reservation supply curve is optimal*

We begin by considering the case in which the marginal revenue function $m(\cdot)$ is increasing[10] and the cost function $C(\cdot)$ is convex. In this case constraint (M) does not bind, which we prove by discarding it and showing that it is satisfied by the solution to the resulting relaxed problem. The relaxed problem is solved by maximizing the virtual surplus independently in all states $(\mathbf{v},\omega)$, which is achieved by a deterministic allocation rule $\mathbf{x}(\mathbf{v})$. The optimal rule allocates units to buyers in the descending order of their virtual valuations $m(v_i)$, proceeding while these virtual valuations exceed the incremental cost $C(X)-C(X-1)$. Since the marginal revenue function $m(\cdot)$ is increasing, the buyers receive units in the descending order of their true valuations while the valuations exceed $m^{-1}(C(X)-C(X-1)) \equiv S(X)$. The function $S(X)$, obtained by transforming the seller's incremental cost curve $C(X)-C(X-1)$ upward with the inverse marginal revenue function $m^{-1}(\cdot)$, can be interpreted as the seller's "inverse reservation supply curve." Intersecting this curve with the inverse demand curve reported by the buyers yields the optimal quantity X^*. Formally, letting $v^{(X)}$ denote the Xth highest order statistic of the reported valuations, X^*

[10] A sufficient condition for this is the monotonicity of the hazard rate $f(v)/[1-F(v)]$, which holds for many important distributions (see Mark Bagnoli and Ted Bergstrom (1989)).

is described by

$$(3) \qquad v^{(X^*)} \geq S(X^*) \text{ if } X^* > 0 \text{ and } v^{(X^*+1)} \leq S(X^*+1) \text{ if } X^* < n.$$

Note that a buyer is more likely to receive a unit when he has a higher valuation, therefore the described allocation rule indeed satisfies (M).

By Lemma 1, the ex post mechanism implementing the described allocation rule is a pricing mechanism, whose pricing rule is uniquely determined by (1).[11] In particular, each buyer receiving a unit in equilibrium pays the price equal to his highest bid that would entail either not producing his unit or giving it to the first runner-up, buyer X^*+1:

$$(4) \qquad p = \max\left\{S(X^*), v^{(X^*+1)}\right\}.$$

These conclusions are summarized as follows:

PROPOSITION 1: *Suppose that the distribution F is known, the marginal revenue function $m(\cdot)$ is increasing, and the cost function $C(\cdot)$ is convex. Then the optimal mechanism (under either ex post or interim constraints) allocates units to buyers in the descending order of their valuations while the valuations exceed $S(X) \equiv m^{-1}(C(X) - C(X-1))$. The optimal quantity X^* is thus described by (3). In the optimal ex post mechanism, losers do not pay, and all winners pay the price given by (4).*

The optimal mechanism is thus equivalent to the Vickrey-Groves-Clarke mechanism in which each buyer pays the externality he imposes on others, except that the seller misrepresents his incremental cost to be $S(X) > C(X) - C(X-1)$. The mechanism is depicted in Figure 1.

Many features of this characterization extend to the case in which the cost function $C(\cdot)$ is not convex. Namely, the allocation rule maximizing virtual surplus still satisfies

[11] The same allocation rule can be implemented with many interim mechanisms.

(M),[12] and it still allocates units to buyers in the decreasing order of their valuations. The optimal quantity X^* must still satisfy the seller's "discrete first-order conditions" (3), for otherwise she would prefer to sell either one more or one fewer unit. Thus, the optimal quantity still lies at an intersection of the reservation supply curve $S(\cdot)$ and the demand curve revealed by the buyers. Observe that when the seller's cost function is not convex, her optimal supply curve is not upward-sloping. Auctions with downward-sloping supply curves have been implemented by "demand aggregation" websites such as *Mercata.com*, *LetsBuyIt.com*, and *eWinWin.com*, presumably reflecting the sellers' economies of scale.

Complications arise when the revealed demand curve crosses the reservation supply curve several times, in which case the reservation supply curve alone cannot determine the optimal quantity X^*.[13] Furthermore, in this case some buyers may be "pivotal," meaning that without them it would be optimal to drop some other buyers so as to switch to a lower intersection point (for example, to shut down to save a fixed cost). In the pricing mechanism implementing the optimal allocation rule, such pivotal buyers should face prices that are different from (4). See Francesca Cornelli (1996) for a characterization of the optimal mechanism in the setting with a fixed cost and constant marginal costs.

B. *When a posted price is optimal*

When the seller's marginal cost is a constant c, her inverse reservation supply curve $S(X) = m^{-1}(c)$ is horizontal. Then the optimal mechanism derived in Proposition 1 reduces to a posted price $p^* = m^{-1}(c)$, which maximizes the expected per capita profit:

$$p^* \in \operatorname*{Arg\,max}_{p \in [0,\overline{v})} \pi(p),\ \pi^* = \pi(p^*), \tag{5}$$

[12] This can be seen using the Monotone Selection Theorem of Paul R. Milgrom and Chris Shannon (1994).

[13] These complications do not arise on demand aggregation websites due to their dynamic bidding procedure, which stops once the revealed demand curve first intersects the supply curve.

$$\text{where } \pi(p) = (p-c)(1-F(p)) = \int_p^{\overline{v}} (m(v)-c) f(v)dv.$$

While the above argument relies on the assumption of increasing marginal revenue, the optimality of posted pricing is more general:

PROPOSITION 2: *If the distribution F is known and $C(X) = cX$, the posted price p^* is an optimal mechanism (under either ex post or interim constraints).*

PROOF:

Note first that with a constant marginal cost, both the seller's expected profit and the constraints (BIC), (IIR) are linear in the outcome $\langle \mathbf{x}, \mathbf{t} \rangle$, and therefore she can restrict attention to deterministic mechanisms. Second, since the seller's program (2) is additively separable across buyers, she can restrict attention to mechanisms in which each buyer's allocation depends only on his own valuation. In particular, in this case the interim constraints are equivalent to the ex post constraints. Then, by Lemma 1, the seller should use a pricing mechanism, with a price p_i to each buyer that does not depend on others' announcements. Finally, by (5), p^* is an optimal price to offer to each buyer.

Furthermore, even if the marginal cost is not constant, a posted price becomes optimal asymptotically as the number n of buyers goes to infinity. For normalization across n, consider the asymptotic setting in which each buyer's set of possible purchases is $\mathfrak{X} = \{0, 1/n\}$. This ensures that the expected demand at any posted price p is $1 - F(p)$ for any n. As $n \to \infty$, by the Strong Law of Large numbers the empirical demand at a posted price p converges to $1 - F(p)$ almost surely, and the resulting profit converges almost surely to $p(1-F(p)) - C(1-F(p))$ (provided that the cost function $C(\cdot)$ is continuous). The asymptotically optimal posted price can then be defined as

$$(6) \qquad p^* \in \operatorname*{Arg\,max}_{p \in [0, \overline{v})} \pi(p),\ \pi^* = \pi(p^*),$$
$$\text{where } \pi(p) = p(1-F(p)) - C(1-F(p)).$$

The asymptotic optimality of posted pricing is again the easiest to see in the case of increasing marginal revenue and nondecreasing marginal cost, the optimal mechanism for which is described in Proposition 1. Intuitively, as $n \to \infty$, the reported demand curve converges to $1 - F(p)$ and the reservation supply curve converges to $m^{-1}(C'(X))$, therefore the price at which they intersect converges to p^* (see Figure 2). Thus, the optimal mechanism asymptotically reduces to posting price p^*. This conclusion carries over to a more general setting:

PROPOSITION 3: *Suppose that the distribution F is known, and let $D(p) = 1 - F(p)$, $P(\cdot) = D^{-1}(\cdot)$, and $R(X) = P(X)X$ (the revenue function). Suppose that the cost function $C : [0,1] \to \mathbb{R}$ is continuous, and that for $X^* = D(p^*)$, there exists $\gamma \in \mathbb{R}_+$ such that*[14]

$$X^* \in \operatorname*{Arg\,max}_{X \in [0,1]} R(X) - \gamma X, \qquad (7)$$
$$X^* \in \operatorname*{Arg\,max}_{X \in [0,1]} \gamma X - C(X).$$

Then the seller's expected profit in any mechanism (either ex post or interim) with n buyers and $\mathfrak{X} = \{0, 1/n\}$ cannot exceed π^*, while her profit from posting price p^* converges to π^* almost surely as $n \to \infty$.

PROOF:

Divide the seller's expected profit into two terms, one being as though her marginal cost were constant and equal γ, and the other being $E[\gamma X - C(X)]$ (where X is the quantity sold by the mechanism). By Proposition 2, the first term is maximized by a posted-price mechanism, and the first line in (7) implies that it is maximized by posting price p^*, which yields the maximum value $(p^* - \gamma) X^*$. As for the second term, by the second line in (7) it cannot exceed $\gamma X^* - C(X^*)$. Adding up, we see that the seller's expected profits cannot

[14]Note that when $X^* \in (0,1)$, condition (7) implies that $\gamma = R'(X^*) = C'(X^*)$ (provided that the latter derivative exists).

exceed $p^*X^* - C(X^*) = \pi^*$. On the other hand, as noted above, by the Strong Law of Large Numbers the profit from posting price p^* converges to $\pi(p^*) = \pi^*$ almost surely as $n \to \infty$.

Condition (7) says that the graphs of $R(X) - R(X^*)$ and $C(X) - C(X^*)$ can be separated with a straight line passing through the point $(X^*, 0)$, as illustrated in see Figure 3.[15] By the Separating Hyperplane Theorem, this condition is weaker than the concavity of the revenue function $R(\cdot)$ and the convexity of the cost function $C(\cdot)$, which are assumed in Proposition 1 (note that $R'(X) = m(P(X))$).

The asymptotic setting considered in Proposition 3, in which the aggregate expected demand is held fixed, should be distinguished from the setting in Dov Monderer and Moshe Tennenholtz (2001) and Zvika Neeman (2001), in which demand grows proportionally to n (e.g., $\mathfrak{X} = \{0, 1\}$ for any n). In the latter setting, the expected demand curve in the limit becomes perfectly elastic at price $\overline{v}$. By posting a price just slightly below $\overline{v}$ and optimally rationing demand at the price, the seller can extract nearly all buyer surplus, while realizing almost all available total surplus as the number of buyers goes to infinity.[16] Monderer and Tennenholtz (2001) and Neeman (2001) instead propose using the Vickrey-Groves-Clarke mechanism, which achieves the same profit asymptotically without requiring the seller to know $\overline{v}$. The present model offers a better approximation of real-life situations with many buyers in which the aggregate demand is well known and is downward-sloping, while the individual buyers' valuations are not observed by the seller. In such situations,

[15] The first line in (7) can also be interpreted as saying that the "ironed-out" marginal revenue curve coincides with $R'(X)$ at X^* (since ironing corresponds to the convexification of $R(\cdot)$). If this does not hold, then profit maximization requires convexification, as discussed in Bulow and Roberts (1989). With a large n, this convexification can be achieved by posting two different prices to different groups of buyers, and so the seller again need not resort to bidding mechanisms.

[16] If $\overline{v} = \infty$, the seller's profits would be unbounded.

the Vickrey-Groves-Clarke mechanism asymptotically reduces to posting the competitive equilibrium price, which is clearly suboptimal when demand is not perfectly elastic.

III. The Bayes Optimal Mechanism with Unknown Distribution

Now we turn to the mechanism design problem when the distribution F is unknown. This section considers the case in which the seller is endowed with a Bayesian prior over possible distributions. For example, the seller may know that the distribution belongs to a parametric family $\{F(\cdot|\theta)\}_{\theta\in\Theta}$, and have a prior over the parameter θ. Note that the buyers' valuations $(v_1, \ldots, v_n)$, while independent conditional on F, are correlated from the seller's viewpoint. The analysis of this section in fact allows the valuations to have an arbitrary symmetric joint distribution, without regard to the source of their correlation.

We again restrict attention to ex post mechanisms, though this is no longer without loss to the seller (the restriction is motivated in Section VII below). Just as in the independent value case, dominant-strategy incentive compatibility and the binding ex post participation constraints of zero-valuation buyers pin down the information rents of each buyer i. Upon integration by parts, the seller's expected profit can be expressed as the expectation of the virtual surplus

$$\begin{aligned} J(\mathbf{x}, \mathbf{v}) &= \sum_i m(v_i|\mathbf{v}_{-i})x_i - C(\sum_i x_i), \\ \text{where } m(v_i|\mathbf{v}_{-i}) &= v_i - \frac{1-\hat{F}(v_i|\mathbf{v}_{-i})}{\hat{f}(v_i|\mathbf{v}_{-i})}. \end{aligned} \tag{8}$$

The only difference from the independent value case is that buyer i's virtual valuation $m(v_i|\mathbf{v}_{-i})$ is calculated using the conditional distribution and density functions $\hat{F}(\cdot|\mathbf{v}_{-i})$ and $\hat{f}(\cdot|\mathbf{v}_{-i})$ respectively. The seller's problem again takes the form (2) of maximizing the expected virtual surplus subject to the monotonicity constraint (M).

When the marginal cost is constant, the seller's program is additively separable across buyers. Appealing to Proposition 2, the optimal mechanism is characterized as follows:

PROPOSITION 4: *If $C(X) = cX$, then the Bayes optimal ex post mechanism is a pricing mechanism with*

$$p_i(\mathbf{v}_{-i}) \in \operatorname*{Arg\,max}_{p\in[0,\overline{v})} (p - c) \cdot \left[1 - \hat{F}(p|\mathbf{v}_{-i})\right]. \tag{9}$$

In words, the mechanism sets the optimal price to each buyer using the information revealed by other buyers' bids.

This mechanism should be contrasted with "standard" auctions, in which a buyer's bid v_i affects other buyers' allocations $\mathbf{x}_{-i}(\mathbf{v})$ only through his own allocation $x_i(\mathbf{v})$. In contrast, in the optimal mechanism obtained here, a buyer's bid has an *informational* effect on other buyers' allocations even when it does not affect his own allocation. In particular, this mechanism cannot be represented with a preset reservation supply curve.

A similar informational effect arises in *efficient* ex post mechanisms in the case of interdependent buyers' valuations, which is studied by Partha Dasgupta and Eric Maskin (2000). For example, the efficient allocation of a single object between buyers i and j may depend on the valuation of a losing buyer k, which cannot be implemented with a "standard" auction. Though the present model has purely private values, a buyer's valuation does convey information about other buyers' valuations *to the seller*, which gives rise to the interdependence of buyers' *virtual* valuations $m(v_i|\mathbf{v}_{-i})$, thus creating an informational role for messages in the profit-maximizing mechanism.

For more general cost functions, we identify conditions under which the monotonicity constraint (M) does not bind and so the optimal allocation rule is obtained by maximizing the virtual surplus in each state:[17]

PROPOSITION 5: *Suppose that (i) $m(v_i|\mathbf{v}_{-i})$ is increasing in v_i, (ii) $m(v_i|\mathbf{v}_{-i}) > m(v_j|\mathbf{v}_{-j})$ whenever $v_i > v_j$, and (iii) $C(\cdot)$ is convex. Then the Bayes optimal ex post mechanism*

[17] Giuseppe Lopomo (2001) offers a related characterization of the optimal ex post mechanism for selling a single object, which also allows for interdependent valuations.

allocates units to buyers in the descending order of their valuations while their virtual valuations $m(v_i|\mathbf{v}_{-i})$ *exceed the incremental cost* $C(X)-C(X-1)$*. Thus, the optimal quantity* X^* *is described by*

$$m\left(v^{(X^*)}|\mathbf{v}^{-(X^*)}\right) \geq C(X^*) - C\left(X^*-1\right) \text{ if } X^* > 0,$$
$$m\left(v^{(X^*+1)}|\mathbf{v}^{-(X^*+1)}\right) \leq C(X^*+1) - C\left(X^*\right) \text{ if } X^* < n,$$

where $\mathbf{v}^{-(k)}$ *denotes the profile of all valuations other than the* k*th highest. The losers in the mechanism do not pay, and the price* p_i *paid by a winner* i *satisfies*

$$p_i = \max\left\{m^{-1}\left(C(\hat{X}_i) - C(\hat{X}_i - 1)|\mathbf{v}_{-i}\right), \mathbf{v}_{-i}^{(\hat{X}_i)}\right\}, \tag{10}$$

where $\hat{X}_i$ *is an optimal quantity for the valuation profile* $(p_i, \mathbf{v}_{-i})$*.*

Proof:

Condition (ii) ensures that the virtual surplus is maximized by allocating units to buyers in the descending order of their valuations, and under condition (iii) they should be allocated while the virtual valuations exceed the incremental cost. The resulting allocation rule satisfies (M), because increasing a buyer's valuation raises both his virtual valuation by (i) and its rank among all virtual valuations by (ii), thus making him more likely to receive a unit. According to (1), each winner i in the mechanism pays the price p_i equal to his lowest bid that would procure him a unit. When buyer i bids exactly p_i, the seller is indifferent between serving him and either giving his unit to the first runner-up or not producing it at all. This is described in (10), with $\hat{X}_i$ representing the optimal quantity sold in this situation.

Proposition 5 can be viewed as a generalization of Proposition 1 to the case of correlated valuations. Indeed, with independent valuations, the marginal revenue function $m\left(\cdot|\mathbf{v}_{-i}\right)$ does not depend on other buyers' valuations $\mathbf{v}_{-i}$, and so the new condition (ii) is implied by the strict monotonicity of the marginal revenue function (condition (i)). The characterization of the optimal quantity X^* in this case is equivalent to (3). As for the pricing formula (10), note that in the case of independent valuations, a reduction in buyer i's bid from v_i to p_i does not affect the optimal allocation to other buyers as long as buyer i is still served, thus we can take $\hat{X}_i = X^*$, yielding price (4). In the general correlated case, however, such reduction affects other buyers' virtual valuations and thus the quantity sold, hence identifying the price to buyer i requires solving a system of two equations with two unknowns, $\hat{X}_i$ and p_i.

The new condition (ii) of Proposition 5 holds when the buyers' valuations are *affiliated*, as defined by Milgrom and Robert J. Weber (1982). Indeed, in this case the conditional distribution $\hat{F}(v_i|\mathbf{v}_{-i})$ is nondecreasing in $\mathbf{v}_{-i}$ in the Monotone Likelihood Ratio order, in which case the distribution's hazard rate $\hat{f}(v_i|\mathbf{v}_{-i})/\left(1-\hat{F}(v_i|\mathbf{v}_{-i})\right)$ is nonincreasing in $\mathbf{v}_{-i}$ (see Louis Eeckhoudt and Christian Gollier (1995, Lemmas 1,2)), hence so is the conditional virtual valuation $m(v_i|\mathbf{v}_{-i})$. Together with the Proposition's condition (i), this implies condition (ii). Observe that valuations that are drawn independently from an unknown distribution are affiliated when the family $\{F(\cdot|\theta)\}_{\theta\in\Theta}$ of possible distributions is ordered with the Monotone Likelihood Ratio order.

In the case of affiliated valuations, we can also say more about the pricing rule in the optimal mechanism described in Proposition 5. As argued above, in this case an increase in $\mathbf{v}_{-i}$ reduces buyer i's virtual valuation, while raising the other buyers' virtual valuations by the Proposition's condition (i). This makes it less likely that buyer i is allocated a unit, which by (1) implies that the price $p_i(\mathbf{v}_{-i})$ he faces is increased. Therefore, $p_i(\mathbf{v}_{-i})$ is nondecreasing in $\mathbf{v}_{-i}$. Since the pricing rule $p_i(\cdot)$ is the same for all buyers, this also implies that for any given valuation profile $\mathbf{v}$ and any two buyers with valuations $v_i > v_j$, buyer i faces a lower price because $\mathbf{v}_{-i} = (v_j, v_{-i-j})$ is lower than $\mathbf{v}_{-j} = (v_i, v_{-i-j})$. When the valuations are *strictly* affiliated, a buyer's virtual valuation is *strictly* decreasing in the others' valuations, and the same chain of arguments implies that higher-valuation buyers pay *strictly* lower prices. This is in contrast to the case of independent valuations examined in Proposition 1, in which all winners paid the same price.

IV. A parametric example

Let the buyers' valuations be drawn from an exponential distribution:[18] $F(v|\theta) = 1 - e^{-\theta v}$. To simplify analysis, suppose that the seller's prior over the hazard parameter θ lies in the conjugate family to exponential distributions, which, according to Morris H. DeGroot (1970, p.166), consists of gamma distributions. A gamma-distribution of θ is defined by a density function of the form[19]

$$\mu(\theta|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \text{ with parameters } \alpha, \beta > 0.$$

More precisely, if the prior distribution of θ is a gamma-distribution with parameters (α_0, β_0), then its posterior conditional on a vector $\mathbf{v}_{-i}$ of $n-1$ independent draws from $F(\cdot|\theta)$ is also a gamma-distribution, with parameters $(\alpha, \beta) = \left(\alpha_0 + n - 1, \beta_0 + \sum_{j \neq i} v_j\right)$. The posterior distribution of $v_i|\mathbf{v}_{-i}$ can then be calculated as

$$\hat{F}(v_i|\mathbf{v}_{-i}) = \int F(v|\theta)\mu(\theta|\mathbf{v}_{-i})d\theta = 1 - \frac{\beta^{\alpha}}{(\beta + v_i)^{\alpha}}.$$

It is easy to verify that the marginal revenue of this distribution is increasing in v_i, thus condition (i) of Proposition 5 is satisfied. Its condition (ii) is also satisfied because the family of exponential distributions is ordered in the Monotone Likelihood Ratio order. Thus, by Proposition 5, the optimal allocation rule with a convex cost function allocates units to buyers in the descending order of their valuations while their virtual valuations remain above the incremental cost.

[18] This is one of the demand formulations considered by Jeffrey M. Perloff and Steven C. Salop (1985).

[19] Where $\Gamma(\alpha) = \int_0^{\infty} z^{\alpha-1}e^{-z}dz$.

When the marginal cost is a constant c, the optimal price to each buyer i solves (9), which yields

$$p_i(\mathbf{v}_{-i}) = \frac{\alpha c + \beta}{\alpha - 1} = \left(\frac{\alpha_0 + n - 1}{\alpha_0 + n - 2}\right) c + \frac{\beta_0 + \sum_{j \neq i} v_j}{\alpha_0 + n - 2}.$$

In particular, if the seller lacks any prior information about the demand parameter θ, she could use the improper uniform prior on $\mathbb{R}_+$ given by parameters $(\alpha_0, \beta_0) = (1, 0)$, which yields the pricing rule $p_i(\mathbf{v}_{-i}) = \frac{n}{n-1} c + \frac{1}{n-1} \sum_{j \neq i} v_j$.

Observe that a similar pricing rule obtains if, instead of using a Bayes prior, the seller estimates parameter θ using maximum likelihood estimation. Indeed, the log-likelihood of a vector $\mathbf{v}_{-i}$ is $(n-1) \log \theta - \theta \sum_{j \neq i} v_j$, which is maximized by $\hat{\theta}(\mathbf{v}_{-i}) = \left(\frac{1}{n-1} \sum_{j \neq i} v_j\right)^{-1}$. If the seller assumes that buyer i's valuation is distributed according to the estimated parameter, i.e., takes $\hat{F}(\cdot|\mathbf{v}_{-i}) = F(\cdot|\hat{\theta}(\mathbf{v}_{-i}))$, program (9) yields the pricing rule $p_i(\mathbf{v}_{-i}) = c + 1/\hat{\theta} = c + \frac{1}{n-1} \sum_{j \neq i} v_j$. That is, each buyer is offered the price equal to the marginal cost plus the average of other buyers' bids.

Note that as $n \to \infty$, under both Bayesian and maximum likelihood estimation the prices conditional on a "true" parameter value θ_0 converge to the optimal monopoly price for this parameter value. Indeed, by the Strong Law of Large Numbers, $\frac{1}{n-1} \sum_{j \neq i} v_j$ converges almost surely to $E[v|\theta_0] = 1/\theta_0$, and therefore $p_i(\mathbf{v}_{-i})$ converges almost surely to $c + 1/\theta_0$, which is the price solving the profit-maximization program (5) for the true distribution $F(\cdot|\theta_0)$. This implies that the seller's expected profit converges to the maximum profit from monopoly pricing with known demand as $n \to \infty$.

V. Convergence

The optimal mechanism derived in Section III depends on the seller's priors. However, as the example in Section IV illustrates, for a large n the priors are overwhelmed by the information obtained from the buyers' bids. As $n \to \infty$, the seller learns the distribution

F, the prices converge to the optimal posted price for F, and the resulting profit converges to the optimal monopoly profit given F. General formulations of the convergence result are given in this section.

We adopt the "frequentist" approach of classical statistics, assuming the existence of a "true" distribution to be estimated, and examining convergence conditional on this distribution.[20] This approach allows to dispense with priors altogether, letting $\hat{F}(\cdot|\mathbf{v}_{-i})$ be *any* consistent estimator of the true distribution F, and not necessarily the Bayes posterior distribution. The simplest convergence result obtains for the case of constant marginal cost:[21]

PROPOSITION 6: *Suppose that $C(X) = cX$, that for each $v_i \in [0, \overline{v})$, $\hat{F}(v_i|\mathbf{v}_{-i}) \xrightarrow{p} F(v_i)$ as $n \to \infty$, and that $v_i \left(1 - \hat{F}(v_i|\mathbf{v}_{-i})\right) \xrightarrow{p} 0$ as $v_i, n \to \infty$.[22] Then as $n \to \infty$, the expected per capita profit in the pricing mechanism described in Proposition 4 converges to the maximum expected per capita profit π^* achievable with F known, given by (5).*

For more general cost functions, a similar convergence result can be established for the asymptotic setting in which each unit contains quantity $1/n$, under the assumptions of Proposition 5 ensuring that the optimal mechanism maximizes the virtual surplus state-by-state:

PROPOSITION 7: *Suppose that for each $v_i \in [0, \overline{v})$, $\hat{F}(v_i|\mathbf{v}_{-i}) \xrightarrow{p} F(v_i)$ and $\hat{f}(v_i|\mathbf{v}_{-i}) \xrightarrow{p} f(v_i)$ as $n \to \infty$, and $m(v_i|\mathbf{v}_{-i}) = v_i - \left(1 - \hat{F}(v_i|\mathbf{v}_{-i})\right) / \hat{f}(v_i|\mathbf{v}_{-i})$ is asymptotically uni-*

[20] If convergence is uniform across possible distributions, then it also implies the convergence of the unconditional expectation of profit given any Bayesian prior over possible distributions.

[21] The statistical concepts and results used below can be found in A. W. van der Vaart (1998). The proofs of this section's results are given in the Appendix.

[22] The last assumption is vacuous when $\overline{v} < \infty$.

formly integrable as $n \to \infty$.[23] *Suppose also that conditions (i)-(iii) of Proposition 5 hold and $C : [0,1] \to \mathbb{R}$ is continuous. Then as $n \to \infty$, the expected profit in the mechanism described in Proposition 5 with n agents and $\mathfrak{X} = \{0, 1/n\}$ converges to the maximum expected profit π^* achievable asymptotically with F known, given by (6).*

When the estimate $\hat{F}(\cdot|\mathbf{v}_{-i})$ is a posterior distribution obtained by Bayes updating of a distribution parameter θ whose prior distribution is μ, the consistency assumptions of Propositions 6 and 7 are verified for μ-almost all parameter values using Doob's Consistency Theorem. The Theorem states that the Bayes posterior distribution $\theta|\mathbf{v}_{-i}$ converges to the true parameter value θ_0 weakly, in probability, as $n \to \infty$. This in turn implies that the posterior distribution and density functions, $\hat{F}(v_i|\mathbf{v}_{-i}) = E_{\theta|\mathbf{v}_{-i}} F(v_i|\theta)$ and $\hat{f}(v_i|\mathbf{v}_{-i}) = E_{\theta|\mathbf{v}_{-i}} f(v_i|\theta)$, are consistent estimators of the true distribution and density functions, respectively.

Propositions 6 and 7 are also applicable to non-Bayesian estimation. For example, the Maximum Likelihood estimator of the parameter,

$$\hat{\theta}(\mathbf{v}_{-i}) \in \operatorname*{Arg\,max}_{\theta \in \Theta} \Pi_{j \neq i} f(v_j|\theta), \tag{11}$$

is consistent under standard assumptions, leading to the consistent distribution and density estimators $\hat{F}(\cdot|\mathbf{v}_{-i}) = F\left(\cdot|\hat{\theta}(\mathbf{v}_{-i})\right)$ and $\hat{f}(\cdot|\mathbf{v}_{-i}) = f\left(\cdot|\hat{\theta}(\mathbf{v}_{-i})\right)$ respectively. Alternatively, the seller can use non-parametric estimation, the simplest example of which is given by the empirical distribution of $\mathbf{v}_{-i}$:

$$\hat{F}(v|\mathbf{v}_{-i}) = \frac{1}{n-1} \left|\{j \neq i : v_j < v\}\right|. \tag{12}$$

Consistency of this estimator is established by the Glivenko-Cantelli Theorem.

[23] In particular, the last assumption holds when for each v_i, $E_{\mathbf{v}_{-i}|F} \left(m(v_i|\mathbf{v}_{-i})\right)^2$ is uniformly bounded across n. This can be easily verified in the parametric example in Section IV.

Application of Proposition 7 to nonparametric estimation may be problematic for two reasons. One is that the mechanism described in Proposition 5 requires an estimate of the density function, in addition to that of the distribution function. The other is that the virtual valuation estimates $m(v_j|\mathbf{v}_{-j})$ obtained through nonparametric estimation of demand may fail assumptions (i) and (ii) of Proposition 5, in which case the proposed allocation rule may fail (M). For example, an increase in v_i can raise the hazard rate of the distribution estimate $\hat{F}(v|\mathbf{v}_{-j})$, thus raising buyer j's virtual valuation $m(v_j|\mathbf{v}_{-j})$ to such extent that it becomes optimal to reallocate buyer i's unit to buyer j, violating (M). Both problems can be avoided using instead the following mechanism, inspired by Goldberg et al. (2001) and Baliga and Vohra (2002): partition buyers into two equal-sized subsets S_1, S_2, and offer each subset $S = S_1, S_2$ an optimal price against the distribution estimate using the bids from the other subset:

$$p_S\left(\mathbf{v}_{N\backslash S}\right) \in \underset{p\in[0,\overline{v})}{\operatorname{Arg\,max}}\, p\left(1-\hat{F}(p|\mathbf{v}_{N\backslash S})\right) - C\left(1-\hat{F}(p|\mathbf{v}_{N\backslash S})\right).$$

Provided that $\hat{F}(\cdot|\mathbf{v}_{N\backslash S})$ is a consistent estimator of the true distribution F, and the profit-maximizing price p^* defined in (6) is unique, by the Theorem of the Maximum the prices $p_S\left(\mathbf{v}_{N\backslash S}\right)$ to both groups $S = S_1, S_2$ converge to p^* in probability, and therefore the expected profit converges to π^*. However, this pricing mechanism is not informationally efficient, for in setting each price it ignores the information received from half of the buyers.

VI. Rates of Convergence

Convergence to the optimal per capita profit π^* is not the only useful asymptotic criterion. In fact, approximating π^* with a large number n of buyers is not at all hard. For example, the seller could experiment on some buyers by offering them different prices, as in Aghion et al. (1991) and Keller and Rady (1999). Alternatively, she could ask some buyers to report their valuations, refraining from selling to them to ensure truthful reporting. Either experimentation on or surveying of a sufficiently large "test group" of buyers would reveal the demand curve and enable the seller to set an approximately optimal price to the remaining buyers. At the same time, when n is large, the size of the "test group" can be small relative to n, ensuring that the per capita profit approaches π^*. This section compares the asymptotic performance of mechanisms such as surveying and experimentation to that of the optimal mechanism, using as the criterion the rate of convergence to the optimal monopoly profit π^* as $n \to \infty$.[24]

Intuitively, surveying or experimentation may not attain the best convergence rate because they both ignore useful information in setting prices. For example, both mechanisms are *sequential pricing mechanisms*, which set the price to a given buyer using only the information obtained from the preceding buyers, rather than from all the other buyers. In addition, while the optimal sequential mechanism would offer each buyer i the optimal price $p_i(v_1, \ldots, v_{i-1})$ given the preceding buyers' reported valuations,[25] both experimentation and surveying sacrifice profits on the first buyers (by setting a suboptimal price to them or not selling to them at all) in order to acquire information about demand.

We examine this intuition in the simple case in which the marginal cost is a constant c, and so the seller's maximum expected profit π^* is given by (5). The expected loss on a given buyer i when his price $p(\mathbf{v}_{-i})$ is determined from $n-1$ other buyers' bids is

$$L_n = \pi^* - E_{\mathbf{v}_{-i}|F}\left[\pi\left(p\left(\mathbf{v}_{-i}\right)\right)\right].$$

[24]The same asymptotic criterion for mechanism design with many agents is adopted by Thomas A. Gresik and Mark A. Satterthwaite (1989) and Tymon Tatur (2001), but their objective is maximizing the total surplus rather than the designer's profits.

[25]For example, each buyer i could be asked to report his valuation after deciding whether to buy at the quoted price $p_i(v_1, \ldots, v_{i-1})$. Recall, however, from the discussion at the end of Section 2 that for buyer i to have a *strict* incentive to report truthfully, the price $p_i(v_1, \ldots, v_{i-1})$ should not be revealed to him until after his report, and he should receive the good at the revealed price if and only if his reported valuation exceeds the price.

By Proposition 6, $L_n \to 0$ as $n \to \infty$. We examine the rate of this convergence using the following terminology: two sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ *have the same convergence rate* if there exist two numbers $\underline{a}, \overline{a} \in (0, +\infty)$ such that $\alpha_n/\beta_n \in [\underline{a}, \overline{a}] < \infty$ for n large enough. We will say that the two sequences satisfy the stronger property of being *asymptotically proportional*, written as $\alpha_n \propto \beta_n$, if $\alpha_n/\beta_n \to a \in (0, +\infty)$ as $n \to \infty$.

The convergence rate of L_n will depend on how large the family $\{F(\cdot|\theta)\}_{\theta\in\Theta}$ of possible demand distributions is. We consider three cases in turn: (1) *hypothesis testing*, in which Θ is a finite set of parameters ("simple hypotheses"), (2) *parametric estimation*, in which Θ is a Euclidean (finite-dimensional) parameter space, and (3) *nonparametric estimation*, in which Θ is an infinite-dimensional space (for example, including all distribution functions of sufficient smoothness). Suppose without loss of generality that all distributions in the family are distinct, and let θ_0 denote the true parameter value, so that the true distribution is $F(\cdot|\theta_0) = F(\cdot)$.

A. *Hypothesis Testing*

In this case, the optimal mechanism achieves exponential convergence to the optimal monopoly profit as $n \to \infty$. For example, the maximum likelihood estimator given by (11) selects a false hypothesis $\hat{\theta}(\mathbf{v}_{-i}) \neq \theta_0$ with an exponentially small probability (this follows from Chernoff's Theorem — see, e.g., Robert J. Serfling (1980, Section 10.3)). Therefore, offering buyer i the optimal price $p(\mathbf{v}_{-i}) = p^*\left(\hat{\theta}(\mathbf{v}_{-i})\right)$ against this estimator, where

$$p^*(\theta) \in \operatorname*{Arg\,max}_{p\in[0,\overline{v})} (p-c)(1-F(p|\theta)), \tag{13}$$

yields exponentially small expected loss. Since this pricing mechanism is also available to a Bayesian decision maker, the expected loss in the Bayes optimal mechanism must converge to zero at least as fast conditional on each positive-probability parameter value θ. In fact, the expected loss in the Bayes optimal mechanism is exponentially small because for any full-support prior, the expected posterior probabilities of false hypotheses shrink exponentially (see, e.g., Erik N. Torgersen (1991, Section 1.4)).[26] Thus, under both Baysian and maximum likelihood estimation, the expected per capita loss L_n satisfies

$$\log L_n \propto -n. \tag{14}$$

On the other hand, the expected per capita loss in any sequential mechanism is at least of the order $1/n$, because the mechanism sets a price to buyer 1 without the benefit of any information. Thus, sequential mechanisms converge exponentially slower than the optimal mechanism. The optimal sequential mechanism in fact achieves convergence rate n^{-1}, because setting price $p(v_1, \ldots, v_{i-1}) = p^*\left(\hat{\theta}(\mathbf{v}_{-i})\right)$ to each buyer i yields expected loss L_i on this buyer, hence the *total* expected loss is bounded above by $\sum_{i=1}^{\infty} L_i < \infty$, due to (14).

Experimentation can only perform worse than the optimal sequential mechanism because is uses only past buyers' purchases rather than their reported valuations. Here, however, optimal experimentation achieves the same convergence rate as the optimal sequential mechanism, under the generic condition $F(p^*(\theta')|\theta_0) \neq F(p^*(\theta')|\theta)$ for all $\theta, \theta' \in \Theta$. To see this, note that the seller can offer each buyer the optimal price $p^*(\theta)$ given the maximum likelihood estimate of θ that uses only the past purchase observations at the most frequently set price. Since there are only $|\Theta|$ possible prices, buyer i's price will be based on at least $\lfloor (i-1)/|\Theta| \rfloor$ i.i.d. purchase observations. A Chernoff's Theorem argument then again implies that the expected loss L_i on buyer i is exponentially small in i, and therefore $\sum_{i=1}^{\infty} L_i < \infty$, hence the per capita expected loss $\frac{1}{n}\sum_{i=1}^{n} L_i$ converges to zero at rate n^{-1}.

[26] Under the second-order Taylor expansion (15) below, the minimization of Bayesian expected loss yields a price error $p(\mathbf{v}_{-i}) - p^*(\theta_0)$ that is asymptotically proportional to the seller's posteriors on false hypotheses, hence the expected loss is proportional to the square of these posteriors.

B. *Parametric Estimation*

Assume that $p^* = p^*(\theta_0) > 0$ is a unique solution to the expected profit-maximization program (5). The first-order condition for the program can be written as $m(p^*) = c$, and the second-order condition as $m'(p^*) \geq 0$. Assume that in fact $m'(p^*) > 0$, in which case the second-order Taylor expansion of π around p^* yields

$$
\begin{aligned}
(15) \qquad \pi^* - \pi(p) &= A(p - p^*)^2 + o\left((p - p^*)^2\right), \\
\text{where } A &= -\frac{1}{2}\pi''(p^*) = m'(p^*) f(p^*) > 0.
\end{aligned}
$$

This implies that the expected loss on buyer i is asymptotically proportional to the squared price error, $E(p(\mathbf{v}_{-i}) - p^*)^2$.

Suppose that the seller offers buyer i the optimal price (13) against the maximum likelihood estimator (11) of the parameter: $p(\mathbf{v}_{-i}) = p^*\left(\hat{\theta}(\mathbf{v}_{-i})\right)$. Suppose also that the function $p^*(\theta)$ is uniquely defined in a neighborhood of $\theta = \theta_0$, with the gradient $p^*_\theta(\theta_0) \neq 0$.[27] It is well known that under standard regularity conditions, $\sqrt{n}\left(\hat{\theta}(\mathbf{v}_{-i}) - \theta_0\right)$ is asymptotically normal with zero mean and a nondegenerate covariance matrix (see van der Vaart (1998), Section 5.5). By the "delta method" based on the first-order Taylor expansion of $p^*(\theta)$ around θ_0 (see van der Vaart (1998), Theorem 3.1), $\sqrt{n}(p(\mathbf{v}_{-i}) - p^*)$ is also asymptotically normal with zero mean and a positive variance. By the Bernstein-von Mises Theorem, the same asymptotic normality holds for the Bayes optimal price $p(\mathbf{v}_{-i})$, which can be viewed as the Bayes point estimate of the optimal price p^* with the loss function (15). Therefore, in both cases, $E(p(\mathbf{v}_{-i}) - p^*)^2$ is asymptotically proportional to $1/n$, hence by (15), the per capita expected loss is

$$L_n \propto n^{-1}.$$

[27] By the Implicit Function Theorem, both assumptions hold when $m_\theta(p^*|\theta_0) \neq 0$, where $m(v|\theta) = v - (1 - F(v|\theta))/f(v|\theta)$.

The optimal sequential mechanism has a slower convergence rate. Indeed, since the expected loss on buyer i in this mechanism is L_i, the per capita expected loss is

$$\frac{1}{n}\sum_{i=1}^{n} L_i \propto \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} \propto \frac{1}{n}\int_1^n \frac{di}{i} \propto \frac{\log n}{n}.$$

(The first proportionality is by Cesàro's Theorem and the second by the Integral Test for series — see Thomas J. Bromwich (1931).) Thus, here sequentiality slows down convergence by the factor $\log n$. Optimal experimentation may in fact achieve this convergence rate. Intuitively, even if the seller sets the myopically optimal price to each buyer on the basis of past purchase observations, the price will eventually arrive in a neighborhood of the optimal price p^* in which the partial derivative $F_\theta(p|\theta)$ is bounded away from zero, and so the amount of information about θ received from a purchase observation is bounded below. Then the expected loss on buyer i is asymptotically proportional to $1/i$, yielding again the expected per capita loss of the order of $n^{-1}\log n$.

C. *Nonparametric estimation*

The simplest nonparametric distribution estimator $\hat{F}(v|\mathbf{v}_{-i})$ is the empirical distribution of the other buyers' valuations, given by (12). The price $p(\mathbf{v}_{-i})$ solving program (9) against this distribution is an "M-estimator" of the correct price p^* (see van der Vaart (1998)). Kislaya Prasad (2001) shows that the distribution of $n^{1/3}(p(\mathbf{v}_{-i}) - p^*)$ converges to a distribution with a finite positive variance. Under the assumptions of the previous subsection, (15) implies that the expected per capita loss $L_n \propto n^{-2/3}$.

Faster convergence rates can be achieved using kernel estimation of the density function f, provided that f is smooth. For example, Charles J. Stone (1983) shows that if f is known to be r times continuously differentiable, the optimal uniform probabilistic convergence rate of the kernel density estimator $\hat{f}(\cdot|\mathbf{v}_{-i})$ to the true density f is $(n^{-1}\log n)^{r/(2r+1)}$. This implies that the optimal price against the estimated distribution converges in probability to p^* at least as fast, and therefore by (15) the expected per capita loss satisfies

$$L_n = O\left(\left(\frac{n}{\log n}\right)^{-\alpha}\right), \text{ where } \alpha = \frac{2r}{2r+1} < 1.$$

Optimal sequential mechanisms may in fact achieve the same convergence rate. For example, suppose that $L_n \propto (n/\log n)^{-\alpha}$ or $L_n \propto n^{-\alpha}$, with $\alpha \in (0,1)$ (recall that empirical distribution estimation yields the latter with $\alpha = 2/3$). In both cases, Cesàro's Theorem implies that

$$\lim_{n\to\infty} \frac{nL_n}{\sum_{i=1}^n L_i} = \lim_{n\to\infty} \frac{(n+1)\,L_{n+1} - nL_n}{L_{n+1}} = 1 - \alpha > 0.$$

Thus, $\frac{1}{n}\sum_{i=1}^n L_i \propto L_n$, i.e., the expected per capital loss in the optimal sequential mechanism converges at the same rate in as in the fully optimal mechanism. The optimal experimentation mechanism would be very difficult to characterize in this setting. Intuitively, it appears that its convergence rate may be slower, because the early purchases at prices that are far from p^* will prove useless for fine-tuning the price around p^*.

VII. Justifying Ex Post Mechanisms

If the ex post constraints (DIC) and (EIR) are relaxed to the corresponding interim constraints (BIC) and (IIR), the seller is typically able to extract all buyer surplus, while implementing the surplus-maximizing allocation. Cremer and McLean (1988) show how this can be done, even using a mechanism that satisfies (DIC) (but not (EIR)). Namely, the seller can employ the Vickrey-Groves-Clarke mechanism, in addition charging each buyer i a participation fee $\phi_i(\mathbf{v}_{-i})$ that depends on other buyers' reports. For a generic joint distribution of valuations, the fee function $\phi_i(\cdot)$ can be chosen so that the expected interim payoff of buyer i is zero no matter what valuation v_i he has.[28]

[28]For example, consider the parametric setting of Section IV, with $C(X) \equiv 0$. The Vickrey-Groves-Clarke mechanism then gives the good for free to all buyers. In addition, each buyer i can be charged a participation fee $\phi_i(v_j)$ that depends on the report of another buyer $j \neq i$. It can be calculated that

Neeman (2002) notes that the surplus extraction mechanisms of Cremer and McLean (1998) exploit a one-to-one correspondence between a buyer's own valuation and his belief about the others' types. In a more general information structure, two different types of buyer i with different valuations may share the same beliefs about the others' types, in which case it is impossible to fully extract the information rents of both types of buyer i. In the extreme case in which a buyer's valuation does not constrain his beliefs about others, any mechanism that is robust to the buyers' beliefs (as Wilson (1987) calls it, *detail-free*) must be an ex post mechanism, which is formally shown by John Ledyard (1979) and Dirk Bergemann and Stephen Morris (2001).

To be sure, if buyers' beliefs stem from their information about each other's valuations, the "second-best" optimal mechanism, rather than being detail-free, will elicit these beliefs. For example, if buyer i knows the distribution F from which other buyers' valuations are drawn, the mechanism can ask this buyer to set the optimal price to the other buyers. However, the seller might be wary of using this mechanism if she is not sure how well-informed buyer i is about F. For the same reason, she might also be wary of using the Cremer-McLean mechanism described above. More generally, a seller who is "ignorant" about the buyers' knowledge of each other's valuations (while being confident that they are drawn independently from an unknown distribution) might be concerned with the mechanism's worst-case performance over all information structures. I conjecture that such worst-case performance is maximized by an ex post mechanism that elicits only the buyers' valuations and not their beliefs.

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when the valuations are distributed exponentially with a gamma-prior over the hazard parameter θ with parameters (α_0, β_0), letting $\phi_i(v_j) = \alpha_0 v_j - \beta_0$ ensures that buyer i's expected surplus in the mechanism is $v_i - E[\phi_i(v_j)|v_i] = 0$ for all v_i.

VIII. Conclusion

This paper has examined the profitability of bidding mechanisms relative to posted pricing. The advantage of bidding mechanisms is that they create *interdependence* among buyers, whereby one buyer's bid v_i affects other buyers' allocations $\mathbf{x}_{-i}$. In the standard auction theory setting with a known distribution of buyers' valuations, interdependence is desirable to the extent that the seller's cost is non-separable across buyers (in the extreme case, the seller has a capacity constraint). Indeed, a buyer's bid v_i affects his allocation x_i, which due to interactions in the seller's cost function affects the other buyers' optimal allocations $\mathbf{x}_{-i}$. However, this reasoning does not apply when the seller's marginal cost is either constant or little affected by a single buyer (e.g., when there are many small buyers). In these practically important cases, interdependence is not useful, hence optimal auctions do not improve upon posted pricing in the standard setting.

Interdependence becomes useful, however, when the seller does not know the distribution from which buyers' valuations are drawn. In this case, one buyer's bid v_i conveys information about other buyers' valuations $\mathbf{v}_{-i}$, and therefore should affect their allocations $\mathbf{x}_{-i}$ even when it does not affect the buyer's own allocation x_i. In this respect, the profit-maximizing mechanism derived in this paper resembles the efficient mechanism suggested by Dasgupta and Maskin (2000) for the case of interdependent values. The mechanism is qualitatively different from standard auctions, and in particular it cannot be represented with a supply curve.

The mechanisms suggested in this paper satisfy Wilson's (1987) desideratum of being "detail-free," i.e., robust to buyers' beliefs about each other's valuations. This is ensured by imposing the "ex post" constraints of dominant-strategy incentive-compatibility and ex post individual rationality, which rule out the surplus extraction schemes proposed by Cremer and McLean (1988).

Another dimension of "detail-freedom" is robustness to the *seller's* beliefs about buyers' valuations. The rationale for this kind of robustness is not as strong: if the seller has some prior information about the distribution of buyers' valuation, for example from a history of selling similar products, there is no reason not to utilize it in designing the mechanism. At the same time, it is useful to have mechanisms that can be used even when the seller has "no idea" of the distribution. Both kinds of mechanisms are suggested in the present paper. While the Bayes optimal mechanism utilizes the seller's priors, these priors become irrelevant with a large number of buyers, and asymptotically the seller does just as well using classical statistical estimation of demand. Non-Bayesian knowledge about the distribution, such as that of its smoothness or functional form, can also be used to accelerate convergence to optimal monopoly profits. Thus, the paper provides a flexible framework allowing to utilize different kinds of prior knowledge, Bayesian or non-Bayesian, in designing the optimal mechanism.

The present analysis can be extended in several directions. One such extension is to allow buyers to demand more than one unit. The optimal mechanism would in general involve second-degree price discrimination, charging each buyer different prices for different units, as in the model of Eric Maskin and John Riley (1984). When the seller's marginal cost is constant, her problem is again additively separable across buyers, and she should offer each buyer the optimal nonlinear tariff using the information inferred from other buyers' reported preferences. It should be kept in mind, however, that unless the buyers' preferences are seriously restricted a priori (say, to a one-dimensional domain with a single-crossing property), the computation of such optimal tariff may be quite difficult.

Another possible extension is the addition of value interdependence (common-value component) among buyers, which can be analyzed using Chung and Ely's (2001) concept of ex post implementation. The buyers in this setting may need to submit more complex bids. For example, with unit demands, they could report functions describing how their valuations depend on those of others, as in the mechanism of Dasgupta and Maskin (2000).

Internet pricing mechanisms mentioned in the Introduction usually allow buyers to submit and raise their bids over time, while observing the current price they face. The mechanisms proposed in this paper can be realized in the same dynamic fashion. An interesting feature of such dynamic realization is that as the revealed demand grows, the price facing a buyer can go either up or down. Compare this to the standard auctions, in which price can only go up as demand grows, and to the "demand aggregation" mechanisms, in which price can only go down.[29]

Finally, note that the mechanisms proposed in this paper typically charge different buyers different prices for identical units. This happens because the price to each buyer is calculated excluding this buyer's bid. However, I conjecture that a uniform-pricing mechanism in which the price is calculated using *all* buyers' bids would work just as well when the number of buyers is large. Indeed, each individual buyer will realize that his bid's effect on the price is very small, and therefore will bid close to his true valuation.

[29] It should be noted that such dynamic realization could facilitate tacit collusion among buyers (which is also true of other dynamic mechanisms, such as the English auction). For example, at a bid profile at which each buyer receives a unit, no buyer has a strict incentive to raise his bid, even if it is below his valuation. He may even strictly prefer not to raise his bid to avoid retaliation by other bidders. The rules may have to be modified to reward buyers for breaking such collusive equilibria (see, e.g., the suggestions in McAdams (2000)). Note that collusive equilibria are unlikely in the one-shot mechanism, since a buyer with sufficient uncertainty about others' behavior will find it strictly optimal to bid truthfully.

Appendix: Proofs of Propositions 6 and 7

PROOF OF PROPOSITION 6: Let $\pi_i(p|\mathbf{v}_{-i})$ denote the objective function in (9). If the seller uses the pricing rule $p(\mathbf{v}_{-i})$ solving (9), her expected loss relative to π^* on buyer i given $\mathbf{v}_{-i}$ is bounded above as follows:

$$\text{(A1)} \quad \begin{aligned} \pi^* - \pi_i(p(\mathbf{v}_{-i})) &= [\pi(p^*) - \pi_i(p^*|\mathbf{v}_{-i})] + [\pi_i(p^*|\mathbf{v}_{-i}) - \pi_i(p(\mathbf{v}_{-i})|\mathbf{v}_{-i})] \\ &\quad + [\pi_i(p(\mathbf{v}_{-i})|\mathbf{v}_{-i}) - \pi(p(\mathbf{v}_{-i}))] \\ &\leq 2 \sup_{p \in [0,\overline{v})} |\pi(p) - \pi_i(p|\mathbf{v}_{-i})| . \end{aligned}$$

In words, the loss is bounded above by twice the supremum absolute difference between the objective functions in (6) and (9). This supremum absolute difference can in turn be bounded above as follows:

$$\sup_{p\in[0,\overline{v})} |\pi(p) - \pi_i(p|\mathbf{v}_{-i})| \leq M \sup_{p\in[0,\overline{v})} \left|\hat{F}(p|\mathbf{v}_{-i}) - F(p)\right| + \sup_{p\geq M} \left[p\left(1 - \hat{F}(p|\mathbf{v}_{-i})\right)\right] + \sup_{p\geq M} [p(1 - F(p))]$$

for any $M > 0$. A simple extension of Lemma 2.11 in van der Vaart (1998) shows that by the consistency of $\hat{F}(v_i|\mathbf{v}_{-i})$, the first term above goes to zero in probability as $n \to \infty$ for any fixed M. The other assumption on $\hat{F}(v_i|\mathbf{v}_{-i})$ implies that the second term goes to zero in probability as $M, n \to \infty$. Finally, the third term goes to zero as $M \to \infty$ due to the assumption that $E[v|F] < \infty$. Putting together, we see that for all $\varepsilon, \delta > 0$ we can find $M = M(\varepsilon, \delta)$ and a corresponding $\hat{n}(\varepsilon, \delta)$ such that for all $n \geq \hat{n}(\varepsilon, \delta)$, each term is less than $\varepsilon/3$ with probability at least $1 - \delta/3$. This implies that $\Pr\left\{\sup_{p\in[0,\overline{v})} |\pi(p) - \pi_i(p|\mathbf{v}_{-i})| < \varepsilon\right\} \geq 1 - \delta$ for $n \geq \hat{n}(\varepsilon, \delta)$, which by (A1) implies that $\pi_i(p(\mathbf{v}_{-i})) \xrightarrow{p} \pi^*$ as $n \to \infty$. Since $\pi_i(p(\mathbf{v}_{-i}))$ is bounded, it follows that the expected profit $E_{\mathbf{v}_{-i}|F}[\pi_i(p(\mathbf{v}_{-i}))] \to \pi^*$ as $n \to \infty$.

PROOF OF PROPOSITION 7: Note that the Proposition's assumptions verify those of Proposition 3, which implies that the maximum expected profit with F known converges to π^* as $n \to \infty$. Therefore, it suffices to show that the expected loss from not knowing F goes to zero as $n \to \infty$.

The allocation rule described in Proposition 5 maximizes the virtual surplus (8) in each state, and therefore maximizes its expectation $E_{\mathbf{v}|F} J(\mathbf{x}(\mathbf{v}), \mathbf{v})$. The seller's expected profit under the true distribution F is instead

$$E_{\mathbf{v}|F}\left[\sum_i m(v_i)x_i(\mathbf{v}) - C(\sum_i x_i(\mathbf{v}))\right].$$

By an argument similar to that in the beginning of the proof of Proposition 6, the seller's expected loss from not knowing F is bounded above by twice the supremum absolute difference between the two expectations over all allocation rules $\mathbf{x}(\cdot)$. Using symmetry, this supremum absolute difference is in turn bounded above as follows:

$$\sup_{\mathbf{x}:[0,\overline{v})^n \to \{0,1/n\}^n} \left| E_{\mathbf{v}|F}\left[\sum_i \left[m(v_i|\mathbf{v}_{-i}) - m(v_i)\right] x_i(\mathbf{v})\right]\right| \leq E_{\mathbf{v}|F}\left|m(v_i|\mathbf{v}_{-i}) - m(v_i)\right|.$$

The consistency of $\hat{F}(v_i|\mathbf{v}_{-i})$ and $\hat{f}(v_i|\mathbf{v}_{-i})$ imply that for each v_i, $m(v_i|\mathbf{v}_{-i}) - m(v_i) \xrightarrow{p} 0$ as $n \to \infty$. Since the absolute value of the difference is asymptotically uniformly integrable by assumption, Theorem 2.20 in van der Vaart (1998) implies that $E_{\mathbf{v}_{-i}|F}\left|m(v_i|\mathbf{v}_{-i}) - m(v_i)\right| \to 0$ as $n \to \infty$. Take expectation $E_{v_i|F}$ using the Lebesgue dominated convergence theorem to see that the right-hand side of the above inequality goes to zero.

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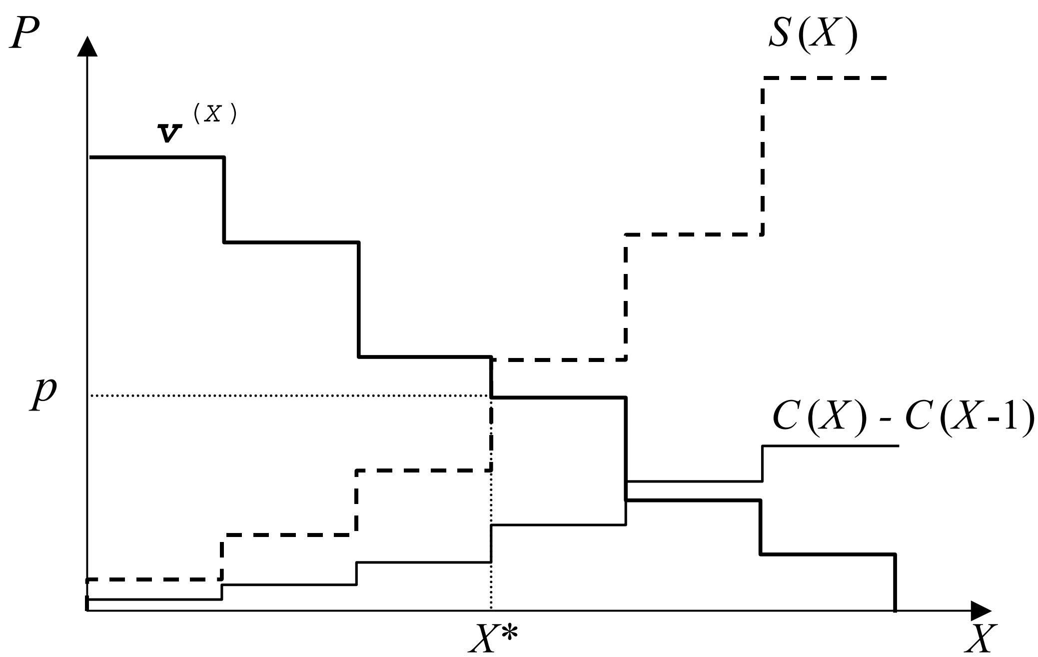

Figure 1: Standard Optimal Auction

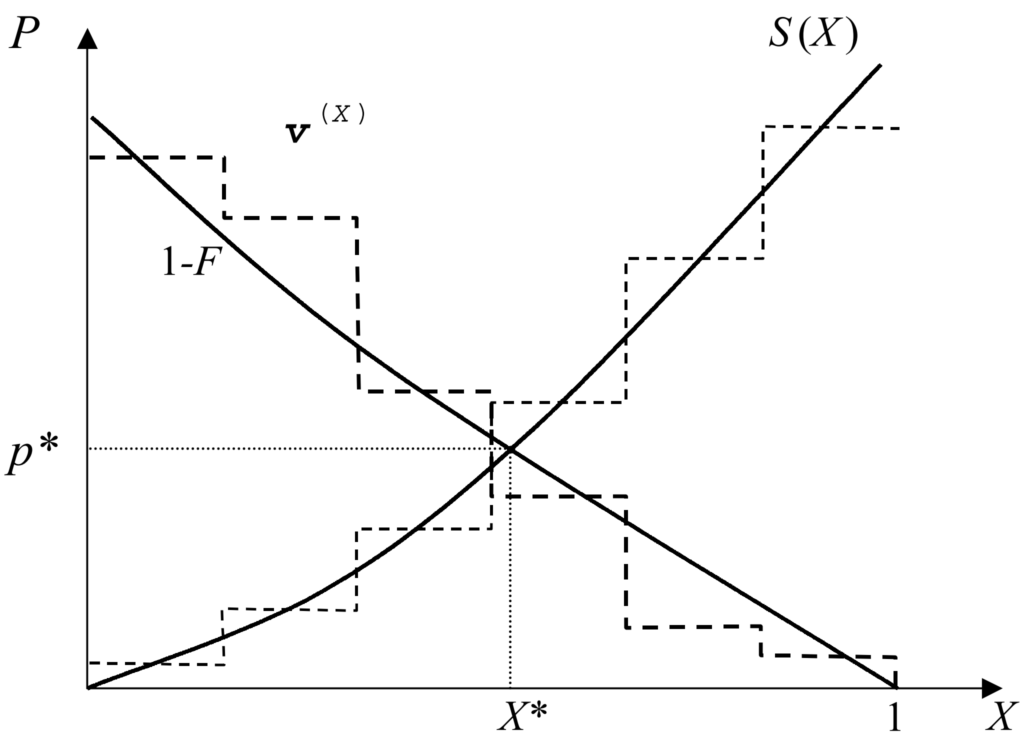

Figure 2: Asymptotics with Known Distribution

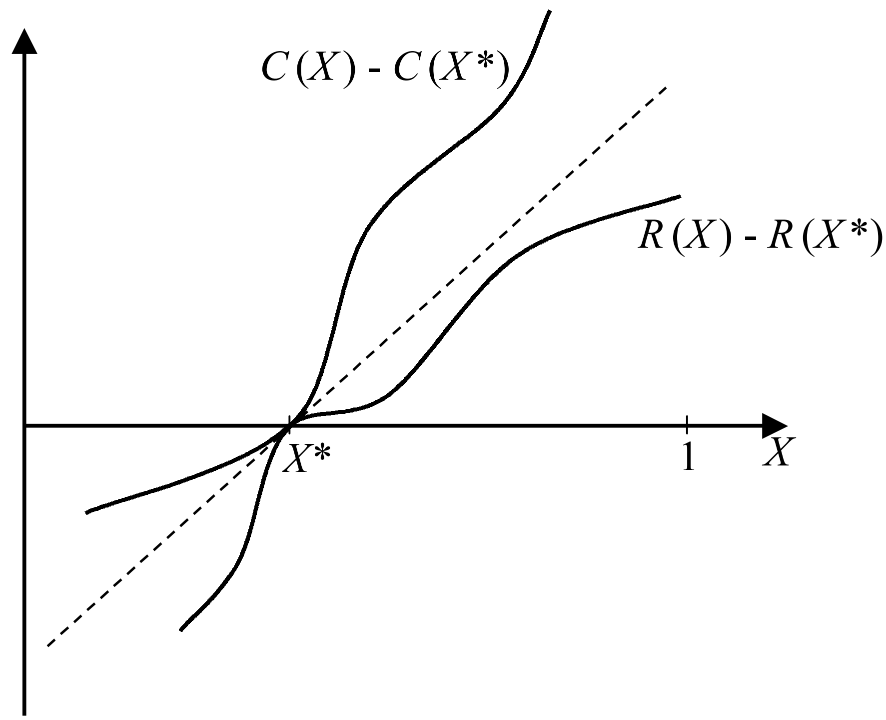

Figure 3: The Separation Condition