

Uncertainty and Risk in Financial Markets*

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Abstract

This paper considers a general equilibrium model in which the distinction between uncertainty and risk is formalized by assuming agents have incomplete preferences over state-contingent consumption bundles, as in Bewley (1986). Without completeness, individual decision making depends on a *set* of probability distributions over the state space. A bundle is preferred to another if and only if it has larger expected utility for all probabilities in this set. When preferences are complete this set is a singleton, and the model reduces to standard expected utility. In this setting, we characterize Pareto optima and equilibria, and show that the presence of uncertainty generates robust indeterminacies in equilibrium prices and allocations for any specification of initial endowments. We derive comparative statics results linking the degree of uncertainty with changes in equilibria. Despite the presence of robust indeterminacies, we show that equilibrium prices and allocations vary continuously with underlying fundamentals. Equilibria in a standard risk economy are thus robust to adding small degrees of uncertainty. Finally, we give conditions under which some assets are not traded due to uncertainty aversion.

JEL Codes: D0, D5, D8, G1

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1 Introduction

In his classic work, Knight (1921) argues that there is an important difference between uncertainty and risk, where risk is characterized by randomness that can be measured precisely. Knight also argues that this difference is important in markets. If risk were the only relevant feature of randomness, well-organized financial institutions should be able to price and market insurance contracts that only depend on risky phenomena. Uncertainty, however, creates frictions that these institutions may not be able to accommodate.

Ellsberg (1961) suggests a more precise definition of uncertainty, in which an event is uncertain if it has unknown probability. In particular, Ellsberg's paradox illustrates important consequences of the distinction between uncertainty and risk by showing that individuals prefer gambles with precise probabilities to gambles with unknown odds. Not only are uncertainty and risk distinct characteristics of random environments, but they can also affect individuals' behavior very differently. Such behavior is inconsistent with the expected utility model, and this observation has inspired a significant amount of recent research in economics.

Since uncertainty, as distinct from risk, can exert a significant influence on individual behavior, it should also be a significant determinant of equilibrium outcomes. For example, Knight (1921) claims that risk is insurable through exchange while uncertainty is not. Uncertainty should arguably lead to two notable departures from standard risk-sharing behavior in expected utility models. When uncertainty is prevalent, some insurance markets might break down, resulting in equilibria with no trade. Moreover, indeterminacy may also arise in this setting. Without uncertainty, the probabilities of risky events are known and frictionless markets can precisely price contracts contingent on risky events, at least generically. Even well-functioning markets, however, may not be able to precisely price contracts conditional on uncertain events, since the probabilities of these events are unknown. Such indeterminacy in equilibrium outcomes can generate excess price volatility and predictions that are extremely sensitive to small measurement errors. In this paper, we present a general equilibrium model in which the presence of uncertainty leads directly to these twin predictions.

We consider a general equilibrium model in which the distinction between uncertainty and risk is formalized by assuming agents have incomplete preferences over state-contingent consumption bundles, as in Bewley (1986). Without completeness, individual decision making depends on a *set* of probability distributions over the state space. A bundle is preferred to another if and only if it has larger expected utility for all probabilities in this set. When preferences are complete this set is a singleton, and the model reduces to standard expected utility. Since incompleteness is reflected by multiple probabilities, this approach provides one way to formalize the distinction between risk and uncertainty.

We consider an otherwise standard Arrow-Debreu exchange economy with a complete set of

state-contingent security markets in which preferences might be incomplete.¹ In this setting, we characterize Pareto optima and equilibria, and show that the presence of uncertainty generates robust indeterminacies in equilibrium prices and allocations for any specification of initial endowments. We derive comparative statics results linking the degree of uncertainty with changes in equilibria. Despite the presence of robust indeterminacies, we show that equilibrium prices and allocations vary continuously with underlying fundamentals. Equilibria in a standard risk economy are thus robust to adding small degrees of uncertainty. Finally, we give conditions under which some assets are not traded due to uncertainty aversion.

The basic intuition of our analysis can be traced back to a result in Bewley (1989). He shows that uncertainty makes opportunities for mutually satisfactory trades difficult to find in an exchange economy. A peculiar consequence of uncertainty is that individuals are unwilling to insure each other. This aversion to trade is counterbalanced by the presence of risk aversion, which makes mutual insurance attractive. We show how equilibria can be characterized by the interplay between uncertainty and risk. For example, uncertainty is sometimes so large that no trade results; other times, the desire to insure prevails and there is trade. This trade-off is not captured by the standard expected utility model, where only risk aversion has a role.

The basic tool of our analysis is a characterization of Pareto optimal allocations that is the natural generalization of the corresponding characterization for complete preferences. In the expected utility setting, an (interior) allocation is Pareto optimal when marginal rates of substitution between any two states are equal across individuals. Without completeness, comparisons are carried out “one probability distribution at a time”, with one bundle preferred to another if and only if it has higher expected utility for every probability distribution. Corresponding to each bundle, there is then a *set* of marginal utilities for each individual, and we show that an allocation is Pareto optimal when there exists at least one common element in these sets. In addition, every element in the intersection of these sets yields a price supporting this Pareto optimal allocation. This result leads naturally to a characterization of equilibria: they are simply Pareto optimal allocations at which each individual budget constraint is satisfied at some supporting price.

This characterization illustrates the fundamental role uncertainty plays in hampering trading opportunities and fostering indeterminacies. Roughly, more uncertainty induces a larger set of beliefs, and thus may induce a larger overlap in marginal rates of substitution at a given allocation. For example, the larger is agents’ perception of uncertainty, the easier it is for the initial endowment to be Pareto optimal, and for a range of prices to support it as an equilibrium. Furthermore, even when perceived uncertainty is small and equilibrium involves trade, there may be infinitely many equilibrium allocations and prices. Thus, in our model, there is

¹Our model is an example of the general equilibrium models with unordered preferences developed by Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer and Sonnenschein (1975), and Fon and Otani (1979). In particular, we use their definitions of Pareto optimality and equilibrium, and appeal to their existence results when applicable.

always a trade-off between uncertainty and risk.

An analogous trade-off is absent in other recent work that introduces uncertainty in a general equilibrium framework using decision theories other than Bewley's. Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), Dana (2000) and Dana (2001), Dow and Werlang (1992), Epstein (2001), Epstein and Wang (1994), Liu (1999), Mukerji and Tallon (forthcoming), and Tallon (1998) are some notable examples. They model uncertainty with either Choquet expected utility (CEU), due to Schmeidler (1989), or maxmin expected utility (MEU), due to Gilboa and Schmeidler (1989). Decision makers with CEU preferences evaluate a consumption bundle using its expected utility computed according to a capacity (a non-additive probability), while decision makers with MEU preferences evaluate a consumption bundle using the minimum expected utility over some set of probabilities. Although different in general, these models coincide in an important special case used in most of this work, CEU with a convex capacity. In this case, the Choquet expected utility is equal to the minimum expected utility over a particular set of probabilities. These models are thus similar to incomplete preferences in that uncertainty enters individual decision making through a set of probabilities. In contrast, however, individual behavior in these models can be very different: because individuals only consider the minimum expected utility over this set of priors, behavior displays an extreme form of pessimism. In equilibrium, this pessimism creates strong incentives for mutual insurance.²

Dow and Werlang (1992) consider a portfolio choice problem, and show that there is an interval of (exogenously given) prices at which a decision maker with CEU preferences holding a riskless position neither buys nor sells short a risky asset. Epstein and Wang (1994) extend this partial equilibrium indeterminacy result to a CEU version of Lucas' representative agent asset pricing model. In a series of papers, Anderson, Hansen, and Sargent (2001), Hansen, Sargent, and Tallarini (1999), and Hansen and Sargent (2001) derive similar results using a related representative agent model in which agents' ambiguity is modeled using the framework of robust control theory. These results suggest that despite the strong incentives to insure in the CEU and MEU models, price indeterminacy and no trade may be features of equilibria. Unfortunately, in each of these cases opportunities for mutual insurance are absent.

When these insurance opportunities are present, however, indeterminacies and no trade in equilibrium are typically absent, as Chateauneuf, Dana, and Tallon (2000) and Dana (2001) show. When agents with CEU preferences have a common convex capacity, Chateauneuf, Dana, and Tallon (2000) show that equilibrium allocations are comonotonic, that is, individual consumption vectors are such that the ranking of states from "best" to "worst" in terms of ex post consumption is the same for all agents. Therefore, when there is no aggregate uncertainty, equilibrium allocations must involve full insurance. Moreover, Dana (2001) shows

²A more detailed discussion of the decision theoretic alternatives to incompleteness is contained in Section 2, while Section 6 contains a fuller description of the results derived in these different models.

that no aggregate uncertainty is the only case in which indeterminacy can arise. In this model, generically equilibria are determinate, and coincide with equilibria in a standard expected utility model with fixed priors.³ In Rigotti and Shannon (2001), we show that the generic determinacy of equilibria holds for general MEU preferences, and in particular for CEU preferences with convex but differing capacities.

In our framework, on the other hand, indeterminacies arise for any initial endowments, thus indeterminacy is due solely to uncertainty. The reason for this difference is easy to explain. For example, consider a model with two states. With CEU and MEU preferences, the agent's indifference curves have a kink along the certainty line and nowhere else, because on different sides of this line the minimum expected utility occurs at different probabilities. This kink makes indeterminacies possible, but only in very particular circumstances. With incompleteness, better than sets have a kink at the consumption bundle that is being evaluated, *regardless* of where this bundle is. This kink is a fundamental consequence of uncertainty, and only disappears when preferences are complete and there is no uncertainty.

Our characterization of Pareto optimality is also related to recent work on the absence of betting. Billot, Chateauneuf, Gilboa, and Tallon (2000) show that in an MEU model without aggregate uncertainty, a full insurance allocation is Pareto optimal if and only if individuals have in their sets of priors at least one common probability distribution. Therefore the existence of a common prior is a necessary and sufficient condition for the absence of betting without aggregate uncertainty. We extend and clarify these results. In our framework, an allocation is Pareto optimal if and only if individuals share at least one *marginal utility weighted* probability distribution. Because full insurance means an individual's consumption levels and marginal utilities are identical across states, the set of marginal utility weighted probability distributions is identical to the set of probability distributions in this case. Thus the result of Billot, Chateauneuf, Gilboa and Tallon can be derived from ours. Moreover, the same logic allows us to show that if all individuals are risk neutral, the existence of a common prior is a necessary and sufficient condition for the absence of betting for *any* strictly positive initial endowment.

The paper is organized as follows. The next section describes the basic decision theoretic framework we use, and contrasts this with the CEU and MEU models. Section 3 introduces the general equilibrium model and characterizes Pareto optima and equilibria. Section 4 presents the results on indeterminacy and comparative statics. Section 5 considers the case in which some events are uncertain while others are risky. Section 6 compares our results to the related literature. Section 7 concludes.

³In related work, Liu (1999) shows that indeterminacy obtains in the absence of aggregate uncertainty in a dynamic economy with heterogeneous agents possessing a particular form of MEU preferences.

2 Preliminaries: Incomplete Preferences and Uncertainty

In this section we briefly describe individual behavior under uncertainty when preferences are not necessarily complete. Incompleteness in decision making under uncertainty was first studied by Aumann (1962).⁴ In a series of papers, Bewley (1986), (1987), and (1989), further developed this model, which he called Knightian decision theory. The basic result of Bewley's approach is to modify the standard expected utility framework by replacing the unique subjective probability distribution used in expected utility with a *set* of probability distributions. When an individual's preferences satisfy the completeness axiom, she can compare any two state-contingent consumption bundles; she decides which one is preferred based on their respective expected utilities. Instead if an individual's preferences do not satisfy completeness, she is not necessarily able to compare every pair of consumption bundles. Because incompleteness is reflected by multiplicity of beliefs, she computes many expected utilities for each consumption bundle, and these might not be ranked uniformly.

To formalize this discussion, suppose the state space Ω is finite, and index the states by $s = 1, \dots, S$. Let $x = \{x_1, \dots, x_S\}$ and $y = \{y_1, \dots, y_S\}$ be two contingent consumption vectors. When preferences are not complete but otherwise satisfy standard axioms, Bewley (1986) shows that there exists a closed, convex set of probability distributions Π and a continuous function $u : \mathbf{R}_+ \rightarrow \mathbf{R}$ (unique up to positive affine transformations) such that

$$x \succ y \quad \text{if and only if} \quad E_\pi [u(x)] > E_\pi [u(y)] \quad \text{for all } \pi \in \Pi.$$

Here $E_\pi[\cdot]$ denotes the expected value with respect to the probability distribution π , and $u(\cdot)$ is the usual von Neumann-Morgenstern utility index.⁵ The set of probabilities Π reduces

⁴Recently, Aumann's work has been extended and clarified by Dubra, Maccheroni, and Ok (2001) and Shapley and Baucells (1998).

⁵Bewley's original theorem (1986, Theorem 1.2) is in the Anscombe-Aumann framework (for a Savage version, see Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001)). Bewley assumes that \succ is a transitive and irreflexive strict preference ordering that satisfies non degeneracy, monotonicity, continuity, independence, and completeness over constant acts. More formally, let the set of simple lotteries be Y and the set of acts be $L = Y^S$, while L_c denotes constant acts. Preferences are defined over L and obey the following axioms.

Axiom 1 (*Independence*) For all $f, g, h \in L$, and all $\alpha \in (0, 1)$,

$$f \succ g \quad \Leftrightarrow \quad \alpha f + (1 - \alpha) h \succ \alpha g + (1 - \alpha) h$$

Axiom 2 (*Non-Degeneracy*) For some $f, g \in L$, $f \succ g$.

Axiom 3 (*Completeness over constant acts*) \succ is transitive and irreflexive. Moreover, for any $f \in L_c$ the closure of

$$\{g \in L_c \mid g \succ f\} \cup \{h \in L_c \mid f \succ h\}$$

is L_c .

to a singleton whenever the preference ordering is complete, and the usual expected utility representation obtains in that case. Since the state space is finite, this expression reduces to:

$$x \succ y \quad \text{if and only if} \quad \sum_{s=1}^S \pi_s u(x_s) > \sum_{s=1}^S \pi_s u(y_s) \text{ for all } \pi \in \Pi.$$

This preference order is not represented by a utility function. Instead, comparisons between alternatives are carried out “one probability distribution at a time”, with one bundle preferred to another if and only if it is preferred under every probability distribution considered by the agent.⁶

Bewley (1986) notes that the above representation captures the distinction Knight (1921) draws between risk and uncertainty. An event is risky when the probability is known, and uncertain otherwise. Hence the decision maker perceives only risk when Π is a singleton, and uncertainty otherwise. In this setting, incompleteness and uncertainty are equivalent measures of the same phenomenon. That is, the amount of uncertainty the decision maker perceives is equivalently reflected by the size of the set of priors Π and the degree of incompleteness of the preference order \succ .

This idea can be formalized as follows. Given any preference order \succ , let

$$C_\succ = \{(x, y) \in \mathbf{R}_+^S \times \mathbf{R}_+^S : x \text{ and } y \text{ are comparable under } \succ\}.$$

We say the preference order \succ' is *more complete* than the preference order \succ if $C_\succ \subset C_{\succ'}$ and for every $(x, y) \in C_\succ$, $x \succ y \iff x \succ' y$. Less completeness in terms of the decision maker’s preference order then corresponds to the decision maker’s perception of more uncertainty.⁷

Proposition 1 *Let \succ and \succ' be two preference orders over \mathbf{R}_+^S satisfying Bewley’s axioms such that \succ' is more complete than \succ . Then there are representations of \succ and \succ' involving the same von-Neumann-Morgenstern utility index u and sets of probabilities Π and Π' such that $\Pi' \subset \Pi$.*

Proof: Consider two bundles \bar{x} and \bar{y} (with $\bar{x} \neq \bar{y}$) such that $\bar{x}_s = \bar{x}_{s'}$ and $\bar{y}_s = \bar{y}_{s'}$ for any two states s, s' . Since \succ and \succ' are complete over constant acts, $(\bar{x}, \bar{y}) \in C_{\succ'}$ and

Axiom 4 (*Continuity*) For any $f \in L$, $\{g \in L \mid g \succ f\}$ and $\{h \in L \mid f \succ h\}$ are both open.

Axiom 5 (*Monotonicity*) For all $f, g \in L$, if $f(s) \succ g(s)$ or neither $f(s) \succ g(s)$ nor $g(s) \succ f(s)$ for each $s \in S$, then $f \succ h$ whenever $g \succ h$, and $h \succ g$ whenever $h \succ f$.

⁶There is also a natural notion of indifference in this setting, under which two bundles are indifferent whenever they have the same expected utility for each probability distribution in Π .

⁷A similar result can be found in Ghirardato, Maccheroni, and Marinacci (2001).

$(\bar{x}, \bar{y}) \in C_{\succ}$. Without loss of generality assume $\bar{x} \succ \bar{y}$. Because \succ' is more complete than \succ we must have $\bar{x} \succ' \bar{y}$ as well. Thus \succ and \succ' agree on all constant bundles. Since the von Neumann-Morgenstern index is simply a member of the affine family representing preferences over constant bundles, we can choose this index u to be identical for \succ and \succ' .

Let Π and Π' be sets of probabilities such that (u, Π) represents \succ and (u, Π') represents \succ' . Now consider any two bundles x and y such that $(x, y) \in C_{\succ}$. Since \succ' is more complete than \succ , $(x, y) \in C_{\succ'}$ and $x \succ y \iff x \succ' y$. Thus

$$E_{\pi}[u(x)] > E_{\pi}[u(y)] \text{ for all } \pi \in \Pi \iff E_{\pi}[u(x)] > E_{\pi}[u(y)] \text{ for all } \pi \in \Pi'$$

and

$$E_{\pi}[u(x)] < E_{\pi}[u(y)] \text{ for all } \pi \in \Pi \iff E_{\pi}[u(x)] < E_{\pi}[u(y)] \text{ for all } \pi \in \Pi'.$$

Now we claim $\Pi' \subset \Pi$. If not, then there exists $\pi' \in \Pi' \setminus \Pi$. By the separating hyperplane theorem, there exists $v \neq 0$ such that $\pi' \cdot v > 0 > \pi \cdot v$ for all $\pi \in \Pi$. Choose bundles x and y such that $u(x_s) - u(y_s) = v_s$ for each s . Since $E_{\pi}[u(x)] - E_{\pi}[u(y)] = \pi \cdot v < 0$ for all $\pi \in \Pi$, $(x, y) \in C_{\succ}$ and $y \succ x$. But $E_{\pi'}[u(x)] - E_{\pi'}[u(y)] = \pi' \cdot v > 0$, which is a contradiction. Thus $\Pi' \subset \Pi$. ■

A graph may help clarify how Bewley's representation works. In Figure 1 the axes measure consumption in each of the two possible states. Given a probability distribution over the two states, a standard indifference curve through the bundle y represents all the bundles that have the same expected utility as y according to this distribution. As the probability distribution changes, we obtain a family of these indifference curves representing different expected utilities according to different probabilities. The thick curves represent the most extreme elements of this family, while thin curves represent other possible elements.

A bundle like x is preferred to y since it lies above all of the indifference curves corresponding to some expected utility of y . Also, y is preferred to w since w lies below all of the indifference curves through y . Finally, z is not comparable to y since it lies above some expected values of y and below others. Incompleteness thus induces three regions: bundles preferred to y , dominated by y , and incomparable to y . This last area is empty only if there is a unique probability distribution over the two states and the preferences are complete. Therefore, for any bundle y , the better-than- y set has a kink at y whenever there is uncertainty. This kink is a direct consequence of the multiplicity of probability distributions in Π , and vanishes only when Π is a singleton.

When preferences are not complete, the usual revealed preference arguments do not apply. If x is chosen when y is available, we cannot say x is revealed preferred to y , we can only say y is not revealed preferred to x . In other words, one cannot explain choice among incomparable alternatives. To address this problem, Bewley (1986) introduces a behavioral assumption based

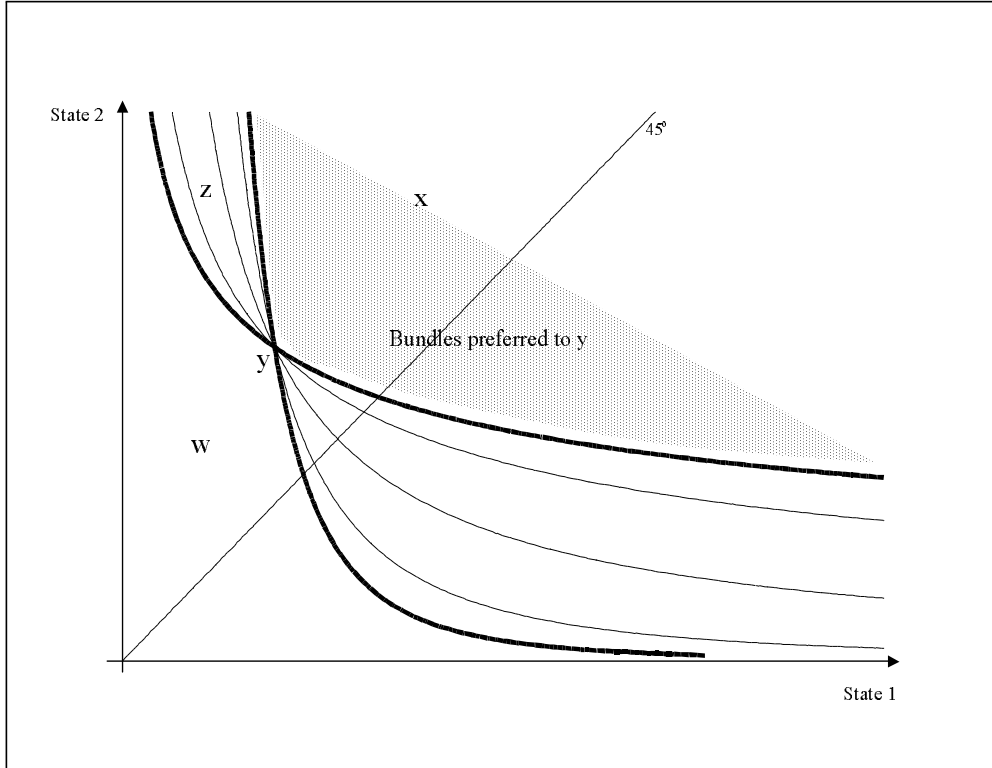


Figure 1: Incomplete Preferences

on the idea of *inertia*. The inertia assumption posits the existence of a *status quo*, imagined to be some planned behavior, that is abandoned only for alternatives that are preferred to it. When the inertia assumption holds, if x is chosen when y is available *and* y is the status quo, then x is revealed preferred to y . If the only alternatives available are not comparable to y , the individual sticks with y . In Figure 1, the inertia assumption implies that if y is the status quo, alternatives like z will not be chosen since they are incomparable to y .

Formally, the inertia assumption as stated in Bewley (1986) requires that if y is the status quo, then x is chosen when y is available only if

$$E_{\pi} [u(x)] > E_{\pi} [u(y)] \text{ for all } \pi \in \Pi.$$

If Π is a singleton, and therefore there is no uncertainty, this notion of inertia implies bundles indifferent to y are not chosen. This type of behavior seems unduly restrictive, since it introduces reluctance to abandon the status quo even when uncertainty is absent. On the other hand, one can easily modify the inertia assumption to correct this problem as follows. If y is

the status quo the *weak inertia assumption* states that x is chosen when y is available only if

$$E_{\pi} [u(x)] \geq E_{\pi} [u(y)] \text{ for all } \pi \in \Pi$$

The inertia assumption is a behavioral statement, rather than a property derived from preferences. Work in both economics and psychology provides significant evidence of behavior under uncertainty that is consistent with such inertia or status quo biases. A classic reference is Samuelson and Zeckhauser (1988), who find evidence of status quo biases in both field and experimental data. In an experimental setting, they find significant status quo bias in investment decisions regarding portfolio composition following a hypothetical inheritance. Moreover, suggestive of the trade-off between uncertainty and risk embedded in our model, they find that this bias varies both with the strength of preference and the number of alternatives: the status quo bias is weaker the stronger is the subjects' preference for the alternatives, and stronger in the face of more alternatives. They find similar evidence in field data on health plan choices and portfolio division in TIAA-CREF plans among Harvard employees. Ameriks and Zeldes (2000) find similar evidence in data from TIAA-CREF and Surveys of Consumer Finance documenting a significant relationship between age and portfolio choices. In fact, almost half of their sample made no change in portfolio composition over the course of the 9 years they observe, while the same period saw drastic changes in the returns to bond and equity holdings. Einhorn and Hogarth (1985) find evidence supporting a status quo bias in initial probability assessments in a number of experiments. See also Fox and Tversky (1995) and Heath and Tversky (1991).

Although there is significant evidence of endowment effects consistent with inertia in a variety of settings, defining a plausible status quo may be difficult in some settings. In a general equilibrium model, however, there is a natural candidate for the status quo: the individual's initial endowment. In our analysis, as the next section makes clear, the (weak) inertia assumption plays the role of a natural equilibrium refinement device. With incompleteness, there may be many equilibria and inertia may select among them.

There are two other prominent models of decision making under uncertainty: Choquet expected utility (CEU), due to Schmeidler (1989), and maxmin expected utility (MEU), due to Gilboa and Schmeidler (1989). Their common premise is Ellsberg's suggestion that the independence axiom may not be a reasonable description of behavior under uncertainty.⁸ Both models retain completeness but relax independence. Since Bewley's model retains independence and relaxes completeness, neither CEU nor MEU is either a generalization or a special case of incomplete preferences. MEU and CEU differ in the particular way in which independence is modified. Understanding their departures from independence is important to understanding both the results obtained in these models and how these results differ from ours, as we discuss in more detail in Section 6. Thus we describe both the CEU and MEU models in some detail here.

⁸Recall that the independence axiom requires that the ranking of any two alternatives is not changed when mixing with a third.

CEU modifies the independence axiom by requiring independence to hold only when comparing comonotonic alternatives, that is, alternatives that rank states in the same way.⁹ Under this assumption, Schmeidler (1989) shows that preferences are represented by their Choquet expected utility, where the expectation is taken with respect to a capacity rather than a probability distribution.¹⁰ In many applications this capacity is also required to be convex, since convexity of the capacity guarantees that preferences are convex.¹¹ This assumption also reflects pessimism on the part of the decision maker facing uncertainty.

On the other hand, MEU requires independence to hold only when comparing constant acts, and adds an axiom called uncertainty aversion.¹² Under these assumptions, Gilboa and Schmeidler (1989) show that preferences are represented by the *minimum* expected utility the decision maker computes using a closed, convex set of probability distributions. As with incomplete preferences, behavior here depends on a set of probabilities. Given this set, an MEU decision maker is always pessimistic, however, and makes decisions based on the worst case scenario.

When the capacity is convex, CEU is equivalent to an MEU representation in which the agent's belief set is given by the core of the capacity. Roughly, the core gives the set of probability distributions consistent with the capacity.¹³ When the capacity is convex, the core is non-empty and the Choquet expected value of a random variable with respect to the capacity is the minimum expected value over distributions in the core. Therefore when the capacity is convex, CEU preferences are simply MEU preferences with belief set given by the core of this capacity. Although different in general, for the version most commonly used in applications, CEU is a special case of MEU.

How do decision makers with CEU and MEU preferences compare to those with incomplete preferences? Figure 2 illustrates for the case in which there are only two states. Consider alternative y . The better-than- y set corresponding to CEU and MEU is the area above thick grey indifference curves, while the better-than- y set under incompleteness is the area above the thin indifference curves. In particular, note that CEU and MEU preferences have a kink at every bundle on the certainty line, and nowhere else. Moreover, with respect to these certain bundles, better-than sets under CEU, MEU, and incompleteness coincide.¹⁴ In general, the

⁹More precisely, two consumption bundles $x = \{x_1, \dots, x_s\}$ and $y = \{y_1, \dots, y_s\}$ are *comonotonic* if for any two states s and t , $(x_s - x_t)(y_s - y_t) \geq 0$.

¹⁰A capacity v on Ω is a mapping $v : 2^\Omega \rightarrow [0, 1]$ such that $v(\emptyset) = 0$, $v(\Omega) = 1$, and for all $A, B \in 2^\Omega$, $A \subset B \Rightarrow v(A) \leq v(B)$.

¹¹A capacity v is convex if for all $A, B \in 2^\Omega$, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$.

¹²Uncertainty aversion requires that for all $f, g \in L$, and all $\alpha \in (0, 1)$, $f \sim g \Rightarrow \alpha f + (1 - \alpha)g \succeq f$.

¹³Formally, the core of a capacity v is defined as

$$\text{core}(v) = \left\{ p \in \Delta : p(A) \geq v(A) \text{ for all } A \in 2^\Omega \right\}.$$

¹⁴To see this, note that if z is a certain bundle, so that $z_s = \bar{z}$ for every s , then the better-than- z set under

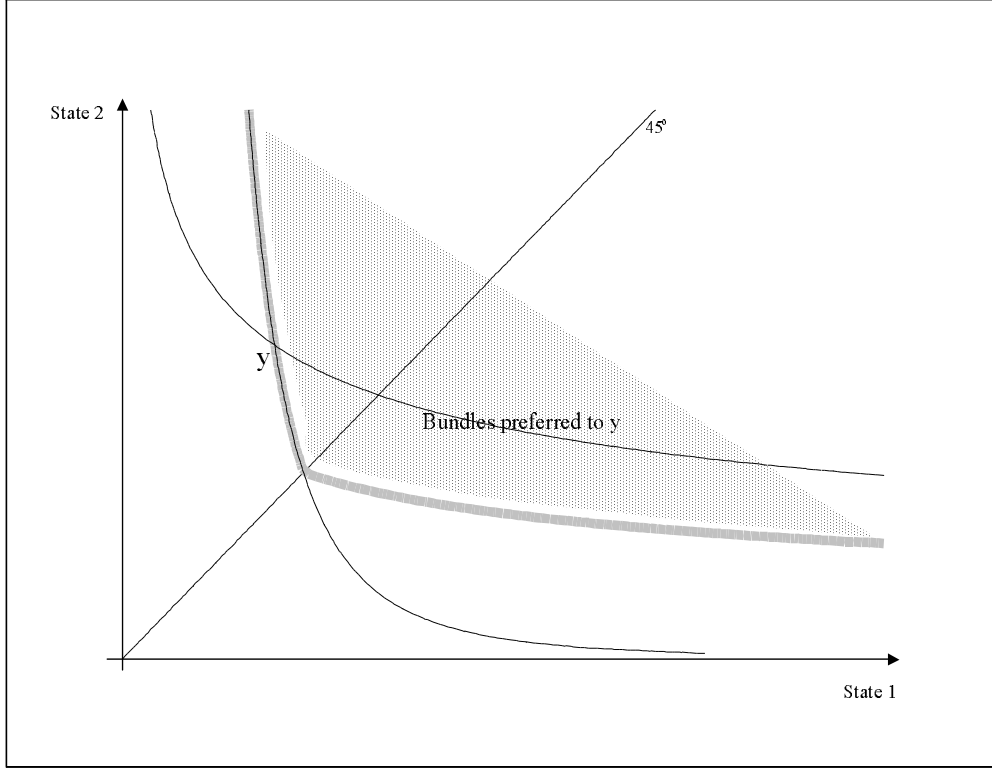


Figure 2: CEU and MEU Preferences

better-than set under incomplete preferences is a subset of the better-than set under MEU.

Finally, we remark that the extreme pessimism implicit in CEU and MEU makes interaction among agents very different from those with incomplete preferences. Intuitively, decision makers who have CEU or MEU preferences care about equating utility across states *in levels*. As we discuss in more detail in Section 6, this makes mutual insurance much more attractive in these models than it is in our framework (or standard expected utility theory), regardless of whether the events being insured against are risky or uncertain. In the next section, we see that this is not the case with incompleteness, which instead make mutually beneficial insurance opportunities harder to find.

each model corresponds to $\{x \in \mathbf{R}_+^S : \sum_s \pi_s u(x_s) \geq u(\bar{z}) \forall \pi \in \Pi\}$.

3 Uncertainty and Complete Markets

In this section, we first describe a simple exchange economy in which agents' preferences over contingent consumption are incomplete. Except for this innovation, the framework we consider is the standard Arrow-Debreu model of complete contingent security markets. The main result of this section is a characterization of Pareto optimal allocations and equilibria. This characterization is based on the fact that incomplete preferences are described by a family of utility functions. The usual condition for Pareto optimality in terms of equating marginal rates of substitution extends naturally using a set-valued notion of marginal rates of substitution. Having obtained a characterization of Pareto optimal allocations, we use a standard argument to characterize equilibria.

There are two dates, 0 and 1. At date 1, there are S possible states of nature, indexed by $s = 1, \dots, S$. There is a single consumption good available at date 1; for simplicity we assume there is no consumption at date 0. At date 0 agents can trade in a complete set of Arrow securities for contingent consumption at date 1. We let Δ denote the standard simplex in \mathbf{R}_+^S .

There are finitely many agents, indexed by $i = 1, \dots, I$. Each agent has an endowment $\omega^i \in \mathbf{R}_+^S$ of contingent consumption at date 1, and a preference order \succ^i over \mathbf{R}_+^S . We use superscripts to denote agents and subscripts to denote states. To model the distinction between risk and uncertainty we follow Bewley (1986) and allow each agent's preference order to be incomplete. We maintain the following assumptions regarding agents' characteristics:¹⁵

For each $i = 1, \dots, I$

(A1) $\omega^i \in \mathbf{R}_{++}^S$

(A2) there exists a closed, convex set $\Pi^i \subset \Delta$ and a C^∞ , concave, strictly increasing function $u^i : \mathbf{R}_+ \rightarrow \mathbf{R}$ such that for any $x, y \in \mathbf{R}_+^S$, $x \succ^i y$ if and only if

$$\sum_{s=1}^S \pi_s u^i(x_s) > \sum_{s=1}^S \pi_s u^i(y_s) \text{ for all } \pi \in \Pi^i.$$

Although preferences are potentially incomplete, there are natural notions of Pareto optimality and equilibrium in this model. Indeed, our model is a special case of equilibrium models with unordered preferences developed in Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer

¹⁵While we work in Bewley's framework, in which each agent has a family of expected utility functions characterizing his incomplete preference order, our results should carry over to a more general setting in which agents' incomplete preferences are generated by a family of utility functions $v(\pi, x)$ over probabilities and contingent consumption bundles that are not necessarily linear in the probability π . For example, it should be possible to incorporate features like probabilistic sophistication, as in Machina and Schmeidler (1992).

and Sonnenschein (1975), and Fon and Otani (1979). Our definitions of Pareto optimality and equilibrium follow theirs.

First, as in standard models, an allocation is Pareto optimal if there is no other allocation that each agent strictly prefers.

Definition A allocation (x^1, \dots, x^I) is *Pareto optimal* if there is no other allocation (y^1, \dots, y^I) such that $y^i \succ^i x^i$ for each $i = 1, \dots, I$.

Thus (x^1, \dots, x^I) is Pareto optimal if there is no allocation (y^1, \dots, y^I) such that for each agent i

$$\sum_{s=1}^S \pi_s u^i(y_s^i) > \sum_{s=1}^S \pi_s u^i(x_s^i) \text{ for all } \pi \in \Pi^i.$$

Notice that if preferences are complete, i.e. if Π^i is a singleton for each i , this corresponds to the standard notion of Pareto optimality.

Similarly, the definition of equilibrium remains unchanged: each agent chooses an element that is maximal in his budget set and all markets clear.

Definition An allocation (x^1, \dots, x^I) and a non-zero price vector $p \in \mathbf{R}_+^S$ are an *equilibrium* if

1. $x \succ^i x^i \Rightarrow p \cdot x > p \cdot \omega^i$ for all i .
2. $p \cdot x^i = p \cdot \omega^i$ for all i .

Although the definition of equilibrium is unchanged, many allocations may be equilibrium allocations when preferences are incomplete. This happens precisely because no assumptions are made about agents' choices between incomparable alternatives. To address this issue, we use an inertia assumption as a possible equilibrium refinement device. Inertia says alternatives incomparable to the status quo are not chosen. We take an individual's initial endowment as his status quo, and apply this reasoning. Given his budget set, an agent will not choose a bundle different from the initial endowment if it is incomparable. This inertia assumption thus leads naturally to the following definition of equilibrium.

Definition An allocation (x^1, \dots, x^I) and a non-zero price vector $p \in \mathbf{R}_+^S$ are an *equilibrium with inertia* if

1. $x \succ^i x^i \Rightarrow p \cdot x > p \cdot \omega^i$ for all i .

2. $p \cdot x^i = p \cdot \omega^i$ for all i .
3. for each i , either $x^i = \omega^i$, or $E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)]$ for each $\pi^i \in \Pi^i$

Equilibrium with inertia considers focal equilibrium allocations in which (some) agents do not trade. With inertia, an equilibrium involves trade only if it is preferred to the status quo by all individuals who engage in trading. Inertia has particularly sharp consequences when the allocation of initial endowments is an equilibrium. In that case, the initial endowment allocation is also the unique equilibrium allocation with inertia provided agents are risk averse and their beliefs have full support, as the following simple lemma demonstrates.

Lemma 1 *Suppose assumptions A1-A2 hold, and in addition that for each i , u^i is strictly concave and $\Pi^i \subset \text{int } \Delta$. If $(\omega^1, \dots, \omega^I)$ is an equilibrium allocation, then it is the unique equilibrium allocation under inertia.*

Proof: Let p be a price vector supporting $(\omega^1, \dots, \omega^I)$ as an equilibrium. Suppose by way of contradiction that there is another equilibrium allocation with inertia, $(x^1, \dots, x^I) \neq (\omega^1, \dots, \omega^I)$. Let T be the set of agents who trade in this equilibrium, so

$$T = \{i : x^i \neq \omega^i\}$$

Then $T \neq \emptyset$, and by inertia $x^i \succeq^i \omega^i$ for each $i \in T$. Since $(\omega^1, \dots, \omega^I)$ is an equilibrium supported by p , we claim that

$$p \cdot x^i > p \cdot \omega^i \text{ for all } i \in T$$

To see this, recall that for each $i \in T$, $E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)]$ for all $\pi^i \in \Pi^i$. Thus for each $n \geq 1$,

$$x^i + \frac{1}{n}\omega^i \succ^i \omega^i$$

by the strict monotonicity of u^i . Then for each $n \geq 1$,

$$p \cdot \left(x^i + \frac{1}{n}\omega^i\right) > p \cdot \omega^i$$

which implies that

$$p \cdot x^i \geq p \cdot \omega^i$$

Now suppose there exists $i \in T$ such that $p \cdot x^i = p \cdot \omega^i$. Fix $\alpha \in (0, 1)$. By the strict concavity of u^i and the fact that $\Pi^i \subset \text{int } \Delta$,

$$E_{\pi^i}[u^i(\alpha x^i + (1 - \alpha)\omega^i)] > E_{\pi^i}[u^i(\omega^i)] \quad \text{for all } \pi^i \in \Pi^i$$

But

$$p \cdot (\alpha x^i + (1 - \alpha)\omega^i) = \alpha p \cdot x^i + (1 - \alpha)p \cdot \omega^i = p \cdot \omega^i$$

which is a contradiction. Thus

$$p \cdot x^i > p \cdot \omega^i \quad \text{for all } i \in T$$

Adding over i yields

$$p \cdot \left(\sum_{i \in T} x^i \right) > p \cdot \left(\sum_{i \in T} \omega^i \right).$$

But this is a contradiction, as

$$\sum_{i \in T} x^i = \sum_{i \in T} \omega^i$$

by feasibility and the definition of T . ■

Our first basic result shows that, as in the standard model with risk alone, equilibria exist, and every equilibrium allocation is Pareto optimal. These results are straightforward applications of work on equilibrium analysis with unordered preferences; see Mas-Colell (1974), Gale and Mas-Colell (1975), and Fon and Otani (1979).

Theorem 1 *Under assumptions A1 and A2, every economy has an equilibrium and every equilibrium allocation is Pareto optimal.*

Proof: To show that an equilibrium exists here, we will appeal to the equilibrium theorem of Gale and Mas-Colell (1975). To this end, for each i define the preference mapping $P^i : \mathbf{R}_+^S \rightarrow 2^{\mathbf{R}_+^S}$ by

$$P^i(x) = \{y \in \mathbf{R}_+^S : y \succ^i x\}.$$

Then P^i is irreflexive, i.e. $x \notin P^i(x)$ for every x , and has non-empty convex values by the concavity and strict monotonicity of u^i . Finally, P^i has an open graph in $\mathbf{R}_+^S \times \mathbf{R}_+^S$. To show this, let $(x, y) \in \text{graph } P^i$. Then

$$\sum \pi_s u^i(y_s) > \sum \pi_s u^i(x_s) \quad \text{for all } \pi \in \Pi^i.$$

Since u^i is continuous, for each $\pi \in \Pi^i$ there exist neighborhoods U_π about π , V_π about y and W_π about x such that

$$\sum \pi'_s u^i(y'_s) > \sum \pi'_s u^i(x'_s) \text{ for all } \pi' \in U_\pi, y' \in V_\pi, x' \in W_\pi.$$

As $\{U_\pi : \pi \in \Pi^i\}$ is an open cover of Π^i and Π^i is compact, we can find a finite subcover $\{U_{\pi^k} : k = 1, \dots, m\}$. Let $V = \cap_k V_{\pi^k}$ and $W = \cap_k W_{\pi^k}$. Then for any $y' \in V$ and $x' \in W$,

$$\sum \pi'_s u^i(y'_s) > \sum \pi'_s u^i(x'_s) \text{ for all } \pi' \in \Pi^i \subset \cup_k U_{\pi^k}.$$

That is, $W \times V \subset \text{graph } P^i$, which establishes that P^i has an open graph. Now by Gale and Mas-Colell's equilibrium theorem, an equilibrium exists.

That every equilibrium allocation is Pareto optimal follows from standard arguments (see Fon and Otani (1979)). ■

Next we establish a parallel result for equilibria with inertia.

Theorem 2 *Under assumptions A1 and A2, every economy has an equilibrium with inertia.*

Proof: Let $X = \{x \in \mathbf{R}_+^S : x \leq 10\omega\}$. Define the following correspondences V^i on $\Pi^1 \times \dots \times \Pi^I \times \Delta \times X^I$

for $i = 1, \dots, I$,

$$V^i(\pi, p, x) = \begin{cases} \{y \in X : p \cdot y < p \cdot \omega^i\} & \text{if } p \cdot x^i > p \cdot \omega^i \\ \{y \in X : p \cdot y < p \cdot \omega^i\} \cap \{y : E_{\pi^i}[u^i(y)] > E_{\pi^i}[u^i(x^i)]\} & \text{if } p \cdot x^i \leq p \cdot \omega^i \end{cases}$$

While for $i = 0$

$$V^0(\pi, p, x) = \{q \in \Delta : q \cdot \sum x^i > q \cdot \sum \omega^i\}.$$

First note that if $V^i(\pi, p, x) = \emptyset$ for each $i = 0, \dots, I$, then (x, p) is an equilibrium in the risk economy with priors given by π .¹⁶ To see this, note that if $V^0(\pi, p, x) = \emptyset$ then $q \cdot \sum x^i \leq q \cdot \sum \omega^i$ for each $q \in \Delta$. Hence $\sum x^i \leq \sum \omega^i$, i.e. x is feasible. Next, if $V^i(\pi, p, x) = \emptyset$, then $p \cdot x^i \leq p \cdot \omega^i$, and thus x^i is maximal in agent i 's budget set since

$$\{y \in X : p \cdot y < p \cdot \omega^i\} \cap \{y : E_{\pi^i}[u^i(y)] > E_{\pi^i}[u^i(x^i)]\} = \emptyset.$$

¹⁶This is just a special case of Gale and Mas-Colell's construction with complete preferences and pure exchange.

Now for each i define the correspondence $W^i : \Pi^1 \times \dots \times \Pi^I \times \Delta \times X^I \rightarrow \Pi^i$ by

$$W^i(\pi^1, \dots, \pi^I, p, x) = \{\hat{\pi}^i \in \Pi^i : E_{\hat{\pi}^i}[u^i(x^i)] < E_{\hat{\pi}^i}[u^i(\omega^i)]\}.$$

Note that each of these might be empty, and that (p, x) is an inertia equilibrium iff there exists some π such that each of these is empty.

So define the correspondence $V : \Pi^1 \times \dots \times \Pi^I \times \Delta \times X^I \rightarrow \Pi^1 \times \dots \times \Pi^I \times \Delta \times X^I$ by $V = (W^1, \dots, W^I, V^0, \dots, V^I)$.

Each component of this correspondence clearly has convex values, and we claim each has an open graph. To show this, fix a point $(\pi, p, x, \pi', q, y) \in \text{graph } V$. Then consider each component of V in turn.

First, for each i , $E_{\pi'}[u^i(x^i)] < E_{\pi'}[u^i(\omega^i)]$, so by continuity we can find neighborhoods N^i of (π, p, x) and M^i of π' such that $E_{\hat{\pi}}[u^i(\hat{x}^i)] < E_{\hat{\pi}}[u^i(\omega^i)]$ for each $(\hat{\pi}, \hat{p}, \hat{x}) \in N^i$ and $\hat{\pi} \in M^i$, i.e. such that $N^i \times M^i \subset \text{graph } W^i$.

Next, consider V^0 . Since $q \cdot \sum x^i > q \cdot \sum \omega^i$, we can find neighborhoods N^0 of (π, p, x) and M^0 of q such that $\hat{q} \cdot \sum \hat{x}^i > \hat{q} \cdot \sum \omega^i$ for every $(\hat{\pi}, \hat{p}, \hat{x}) \in N^0$ and $\hat{q} \in M^0$. Thus $N^0 \times M^0 \subset \text{graph } V^0$.

Finally, consider V^i . If $p \cdot x^i > p \cdot \omega^i$, then $p \cdot y < p \cdot \omega^i$. In this case, clearly there exist neighborhoods L_1^i about (π, p, x) and K_1^i about y^i such that $L_1^i \times K_1^i \subset \text{graph } V^i$. Now suppose $p \cdot x^i \leq p \cdot \omega^i$. Then $p \cdot y < p \cdot \omega^i$ and $E_{\pi^i}[u^i(y^i)] > E_{\pi^i}[u^i(\omega^i)]$. Then again there exist neighborhoods L_2^i of (π, p, x) and K_2^i of y^i such that for every $(\hat{\pi}, \hat{p}, \hat{x}) \in L_2^i$ and $\hat{y}^i \in K_2^i$, $\hat{p} \cdot \hat{y}^i < \hat{p} \cdot \omega^i$ and $E_{\hat{\pi}^i}[u^i(\hat{y}^i)] > E_{\hat{\pi}^i}[u^i(\omega^i)]$. Thus $L_2^i \times K_2^i \subset \text{graph } V^i$. So set $L^i = L_1^i \cap L_2^i$ and $K^i = K_1^i \cap K_2^i$; we have shown that $L^i \times K^i \subset \text{graph } V^i$.

Then by Gale and Mas-Colell's fixed point theorem (Gale and Mas-Colell (1975), p. 10), there exists $(\bar{\pi}, \bar{p}, \bar{x}) \in \Pi^1 \times \dots \times \Pi^I \times \Delta \times X^I$ such that

1. either $\bar{p} \in V^0(\bar{\pi}, \bar{p}, \bar{x})$ or $V^0(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$
2. for each i , either $\bar{x}^i \in V^i(\bar{\pi}, \bar{p}, \bar{x})$ or $V^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$
3. for each i , either $\bar{\pi}^i \in W^i(\bar{\pi}, \bar{p}, \bar{x})$ or $W^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$

Now note that if $\bar{p} \cdot \bar{x}^i > \bar{p} \cdot \omega^i$, then $V^i(\bar{\pi}, \bar{p}, \bar{x}) \neq \emptyset$ and $\bar{x}^i \notin V^i(\bar{\pi}, \bar{p}, \bar{x})$ by construction. So we must have $\bar{p} \cdot \bar{x}^i \leq \bar{p} \cdot \omega^i$. Then clearly $\bar{x}^i \notin V^i(\bar{\pi}, \bar{p}, \bar{x})$, so $V^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$. Next, since $\bar{p} \cdot \bar{x}^i \leq \bar{p} \cdot \omega^i$ for each i , $\bar{p} \cdot \sum \bar{x}^i \leq \bar{p} \cdot \sum \omega^i$. Thus $\bar{p} \notin V^0(\bar{\pi}, \bar{p}, \bar{x})$, which implies that $V^0(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$. Thus for each $i = 0, \dots, I$ we must have $V^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$, from which we conclude that (\bar{p}, \bar{x}) is an equilibrium in the risk economy with priors given by $\bar{\pi}$.

Using the fact that (\bar{p}, \bar{x}) is an equilibrium in the risk economy with priors $\bar{\pi}$, individual rationality implies that $\bar{\pi}^i \notin W^i(\bar{\pi}, \bar{p}, \bar{x})$. Thus $W^i(\bar{\pi}, \bar{p}, \bar{x}) = \emptyset$ for each i . From this we conclude that (\bar{p}, \bar{x}) is an equilibrium with inertia in our original economy. ■

Next we give several useful characterization results. The first is a characterization of Pareto optimal allocations that extends a no-trade result in Bewley (1989). The main idea is to use the set of supports of the preferred set to characterize Pareto optimal allocations and equilibria.

For any fixed bundle $x \in \mathbf{R}_+^S$, incompleteness implies that the set of bundles preferred to x has a kink at x . Nonetheless, we can characterize the possible supports of this set. For this we require the additional assumption that each agent's von Neumann-Morgenstern utility index has no critical points.

(A3) For each i , $u^{ij}(c) > 0$ for every $c > 0$

Now for any consumer i and each $x \in \mathbf{R}_{++}^S$, let

$$\Pi^i(x) = \left\{ \left(\frac{\pi_1 u^{i1}(x_1)}{\sum_t \pi_t u^{it}(x_t)}, \dots, \frac{\pi_S u^{iS}(x_S)}{\sum_t \pi_t u^{it}(x_t)} \right) : \pi \in \Pi^i \right\}.$$

An element $q \in \Pi^i(x)$ is an element of the simplex Δ , i.e. is itself a probability measure, thus this set represents the vectors of “marginal beliefs” induced by the bundle x . As we show below, the set $\Pi^i(x)$ is the set of supports of the better-than- x set. Notice that if x is constant across states, $\Pi^i(x) = \Pi^i$. Also, if Π^i is a singleton, so is $\Pi^i(x)$. Thus when preferences are complete this construction just yields the (normalized) marginal utilities.

Since the set $\Pi^i(x)$ also represents the possible supports of the better-than- x set, an (interior) allocation (x^1, \dots, x^I) is Pareto optimal if and only if there is a common element of the sets $\Pi^i(x^i)$. This result is established formally in the following theorem, adapted from Bewley (1989).¹⁷

Theorem 3 *Under assumptions A1-A3, an interior allocation (x^1, \dots, x^I) is Pareto optimal if and only if*

$$\bigcap_{i=1}^I \Pi^i(x^i) \neq \emptyset.$$

¹⁷Bewley (1989) focuses on the absence of trade when individuals' preferences are incomplete. In particular, Bewley assumes the inertia assumption holds, and therefore that an individual is willing to trade only if she can obtain a strictly preferred consumption bundle. His Propositions 3.1 and 5.1 then provide conditions under which a no-trade result obtains under inertia. In our setting, the same conditions are used to characterize Pareto optimality and equilibrium.

Proof: Define the sets K^i and K as follows:

$$K^i = \left\{ z \in \mathbf{R}^S : \sum_{s=1}^S \pi_s u_s^i(z_s^i + x_s^i) > \sum_{s=1}^S \pi_s u_s^i(x_s^i) \text{ for all } \pi \in \Pi^i \right\}$$

and

$$K = \left\{ z \in \prod_{i=1}^I (K^i \cup \{0\}) : \sum_{i=1}^I z^i = 0 \right\}.$$

Then (x^1, \dots, x^I) is a Pareto optimal allocation if and only if $K = \{0\}$. So we need only show that

$$\bigcap_{i=1}^I \Pi^i(x^i) \neq \emptyset \iff K = \{0\}.$$

Suppose $\bigcap_{i=1}^I \Pi^i(x^i) \neq \emptyset$ and let $p \in \bigcap_{i=1}^I \Pi^i(x^i)$. For each i , let $z^i \in K^i$; then, using the definition of K^i , we have

$$\begin{aligned} 0 &< \sum_{s=1}^S \pi_s [u_s^i(z_s^i + x_s^i) - u_s^i(x_s^i)] \text{ for all } \pi \in \Pi^i \\ &\leq \sum_{s=1}^S \pi_s u^{i'}(x_s^i) z_s^i \text{ for all } \pi \in \Pi^i, \end{aligned}$$

where the second inequality follows from the concavity of u^i . This last inequality implies that

$$\sum_{s=1}^S \frac{\pi_s u^{i'}(x_s^i)}{\sum_t \pi_t u^{i'}(x_t^i)} z_s^i > 0 \text{ for all } \pi \in \Pi^i. \quad (1)$$

Since $p \in \bigcap \Pi^i(x^i)$, for each i there exists $\pi^i \in \Pi^i$ such that

$$p = \left(\frac{\pi_1^i u^{i'}(x_1^i)}{\sum_t \pi_t^i u^{i'}(x_t^i)}, \dots, \frac{\pi_S^i u^{i'}(x_S^i)}{\sum_t \pi_t^i u^{i'}(x_t^i)} \right).$$

Since (1) holds for every $\pi \in \Pi^i$ and every i , we can substitute p in (1) and conclude that

$$\sum_{s=1}^S p_s z_s^i = p \cdot z^i > 0 \text{ for all } i.$$

Summing over individuals, we get

$$\sum_{i=1}^I p \cdot z^i > 0,$$

or

$$p \cdot \left(\sum_{i=1}^I z^i \right) > 0.$$

Since $p \neq 0$ this implies

$$\sum_{i=1}^I z^i \neq 0,$$

and thus $K = \{0\}$.

Now suppose $K = \{0\}$. We must show this implies $\bigcap_{i=1}^I \Pi^i(x^i) \neq \emptyset$. Let $X = \sum_{i=1}^I (K^i \cup \{0\})$. Since $K = \{0\}$, $X \cap \mathbf{R}_-^S = \{0\}$. Let $p \in \mathbf{R}^S$, with $p \neq 0$, separate X and \mathbf{R}_-^S . In particular,

$$p \cdot x \geq 0 \text{ for all } x \in X.$$

Without loss of generality, take p to be normalized so that $\sum_{s=1}^S p_s = 1$. Note that $K_i \subset X$ for all i . Therefore

$$p \cdot x \geq 0 \text{ for all } x \in K_i, \text{ for all } i.$$

We claim that $p \in \bigcap_{i=1}^I \Pi^i(x^i)$. To see this, first note that for all i , $\Pi^i(x^i)$ is compact and convex (see Lemma 3 in the Appendix). Let i be given. Now if $p \notin \Pi^i(x^i)$, there exists $y \neq 0$ such that

$$p \cdot y < b < p^i \cdot y \quad \forall p^i \in \Pi^i(x^i).$$

Moreover, we can take $b = 0$ without loss of generality, as if $b \neq 0$ define \bar{y} such that $\bar{y}_s = y_s - b$ for each s . Then $\bar{y} \neq 0$ and

$$p \cdot \bar{y} = p \cdot y - b < 0 < p^i \cdot y - b = p^i \cdot \bar{y} \quad \forall p^i \in \Pi^i(x^i).$$

Finally, there exists $\alpha > 0$ such that $\alpha \bar{y} \in K_i$ (since $\sum \pi_s^i u_s^i(x_s^i) \bar{y}_s > 0$ for all $\pi^i \in \Pi^i$). So choose $\alpha > 0$ such that $\alpha \bar{y} \in K_i$, and note that

$$p \cdot (\alpha \bar{y}) < 0 < p^i \cdot (\alpha \bar{y}) \quad \forall p^i \in \Pi^i(x^i),$$

contradicting the definition of p . Thus $p \in \Pi^i(x^i)$. Since i was arbitrary, we conclude that $p \in \bigcap_i \Pi^i(x^i)$. ■

In the course of this proof, we have also established the useful and intuitive result concerning the set of supports of an agent's better-than set to which we alluded above: the set of “marginal beliefs” corresponding to a bundle completely characterizes the set of supports of the better-than set at that bundle.

Corollary 1 *Suppose assumptions A1-A3 hold. Let $y \in \mathbf{R}_{++}^S$ and*

$$B^i(y) = \left\{ z \in \mathbf{R}_+^S : \sum \pi_s u^i(z_s) > \sum \pi_s u^i(y_s) \text{ for all } \pi \in \Pi^i \right\}.$$

Then $p \in \Delta$ supports $B^i(y)$ at y if and only if $p \in \Pi^i(y)$. Moreover, if $p \in \Pi^i(y)$ then p strictly supports $B^i(y)$ at y , i.e. for every $z \in B^i(y)$, $p \cdot z > p \cdot y$.

Proof: Note that, adapting the notation of the preceding proof, $B^i(y) = \{y\} + K^i(y)$, where $K^i(y) = \{z : \sum \pi_s u^i(y_s + z_s) > \sum \pi_s u^i(y_s) \text{ for all } \pi \in \Pi^i\}$. Now the results follow by noting that the preceding proof establishes that $p \in \Delta$ supports $K^i(y)$ at 0 if and only if $p \in \Pi^i(y)$, and that each $p \in \Pi^i(y)$ strictly supports $K^i(y)$ at 0. ■

Theorem 3 above can be seen as a generalization of the main result in Billot, Chateauneuf, Gilboa, and Tallon (2000) (for a finite state space). Assuming the economy has no aggregate uncertainty and individuals have MEU preferences, they show that a full insurance allocation is Pareto optimal if and only if all agents share at least one prior. As noted in Section 2, with respect to bundles that involve no uncertainty, better-than sets under MEU and incomplete preferences are the same. Therefore we can apply Theorem 3 to the special case of full insurance allocations to obtain their result as follows.

Corollary 2 *Under assumptions A1-A3, there is a full insurance Pareto optimal allocation if and only if individuals have at least one common probability distribution. In this case every full insurance allocation is Pareto optimal.*

Proof: If x is a full insurance allocation, then x^i is constant in every state. Thus for each i there exists k^i such that $x_s^i = k^i$ for each state s . In this case, the set $\Pi^i(x^i)$ becomes

$$\begin{aligned} \Pi^i(x^i) &= \left\{ \left(\frac{\pi_1 u^i(k^i)}{\sum_t \pi_t u^i(k^i)}, \dots, \frac{\pi_S u^i(k^i)}{\sum_t \pi_t u^i(k^i)} \right) : \pi \in \Pi^i \right\} \\ &= \{(\pi_1, \dots, \pi_S) : \pi \in \Pi^i\} \\ &= \Pi^i. \end{aligned}$$

Therefore by Theorem 3, x is Pareto optimal if and only if

$$\bigcap_i \Pi^i(x^i) = \bigcap_i \Pi^i \neq \emptyset;$$

that is, x is Pareto optimal if and only if agents share a common prior.

If $\bigcap_i \Pi^i \neq \emptyset$, then for any full insurance allocation y ,

$$\bigcap_i \Pi^i(y^i) = \bigcap_i \Pi^i \neq \emptyset.$$

Then by Theorem 3, y is Pareto optimal. ■

As the next corollary shows, if all individuals are risk-neutral then any interior allocation is Pareto optimal provided the agents share at least one prior.¹⁸ This result can be thought of as the extension to an environment where there is uncertainty of the equivalent result under risk. In that case, *any* interior allocation is Pareto optimal when agents have a common (unique) prior. In our case, there must be at least one such prior.

Corollary 3 *Under assumptions A1-A2, if all individuals are risk neutral then an interior allocation is Pareto optimal if and only if individuals have at least one common probability distribution.*

Proof: If agent i is risk neutral, the set $\Pi^i(x^i)$ reduces to

$$\Pi^i(x^i) = \left\{ \left(\frac{\pi_1 k^i}{\sum_t \pi_t k^i}, \dots, \frac{\pi_S k^i}{\sum_t \pi_t k^i} \right) : \pi \in \Pi^i \right\}$$

for some constant k^i . Clearly, this implies

$$\Pi^i(x^i) = \{(\pi_1, \dots, \pi_S) : \pi \in \Pi^i\} = \Pi^i.$$

Therefore, for any allocation x ,

$$\bigcap_i \Pi^i(x^i) \neq \emptyset \quad \text{if and only if} \quad \bigcap_i \Pi^i \neq \emptyset.$$

Again by Theorem 3, an interior allocation x is Pareto optimal if and only if $\bigcap_i \Pi^i \neq \emptyset$. ■

We now turn to characterizations of equilibrium allocations. Theorem 3 shows that an allocation is Pareto optimal if and only if there is a common element in the marginal belief sets of all agents at their bundles in the allocation. As with standard Negishi-type arguments, a Pareto optimal allocation is then an equilibrium if there is a price in this common support set at which each consumer's budget constraint is satisfied.

¹⁸The statement is a direct implication of Proposition 3.1 in Bewley (1989).

Theorem 4 Under assumptions A1-A3, an interior allocation (x^1, \dots, x^I) is an equilibrium allocation if and only if

$$\bigcap_i \Pi^i(x^i) \neq \emptyset$$

and there exists $p \in \bigcap_i \Pi^i(x^i)$ such that $p \cdot x^i = p \cdot \omega^i$ for each i .

Proof: Let (x^1, \dots, x^I) be an interior equilibrium allocation. Then there exists a non-zero $p \in \mathbf{R}_+^S$ such that $p \cdot y > p \cdot x^i$ for every $y \in \mathbf{R}_+^S$ such that $y \succ^i x^i$. Without loss of generality, suppose p is normalized so that $\sum p_s = 1$. Then by Corollary 1, $p \in \Pi^i(x^i)$. Since this holds for each i , $p \in \bigcap_i \Pi^i(x^i)$.

Now suppose there exists $p \in \bigcap_i \Pi^i(x^i)$ such that $p \cdot x^i = p \cdot \omega^i$ for each i . By Corollary 1, for each i , if $y \succ^i x^i$ then $p \cdot y > p \cdot x^i = p \cdot \omega^i$. Thus (x^1, \dots, x^I) is an equilibrium allocation supported by p . ■

As an immediate consequence, we obtain the following characterization of equilibria with inertia.

Corollary 4 Under assumptions A1-A3, an interior allocation $(x^1, \dots, x^I) \neq (\omega^1, \dots, \omega^I)$ is an equilibrium allocation with inertia if and only if for every i such that $x^i \neq \omega^i$, $E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)]$ for each $\pi^i \in \Pi^i$,

$$\bigcap_i \Pi^i(x^i) \neq \emptyset$$

and there exists $p \in \bigcap_i \Pi^i(x^i)$ such that $p \cdot x^i = p \cdot \omega^i$ for each i .

Proof: This follows immediately from Theorem 4 and the definition of equilibrium with inertia. ■

Finally, we can also use these results to characterize when no trade will be the unique equilibrium with inertia.

Corollary 5 Suppose assumptions A1-A3 hold, and in addition that for each i , u^i is strictly concave and $\Pi^i \subset \text{int } \Delta$. Then the no-trade allocation $(\omega^1, \dots, \omega^I)$ is the unique equilibrium allocation with inertia if and only if

$$\bigcap_{i=1}^I \Pi^i(\omega^i) \neq \emptyset.$$

Any price $p \in \bigcap_{i=1}^I \Pi^i(\omega^i)$ supports $(\omega^1, \dots, \omega^I)$.

Proof: This follows immediately from Corollary 4 and Lemma 1. ■

These results can easily be illustrated graphically. Figure 3 depicts a no-trade equilibrium. The initial endowment is ω . Agent 1's preferences are described by the grey indifference curves; bundles preferred to ω^1 are to the northeast of these indifference curves. Agent 2's preferences are described by the black indifference curves; bundles preferred to ω^2 are to the southwest of these indifference curves. In this example, there are no allocations preferred to ω by both individuals, hence the initial endowment is Pareto optimal and an equilibrium, and it is the unique equilibrium with inertia. A range of prices supports this as an equilibrium.

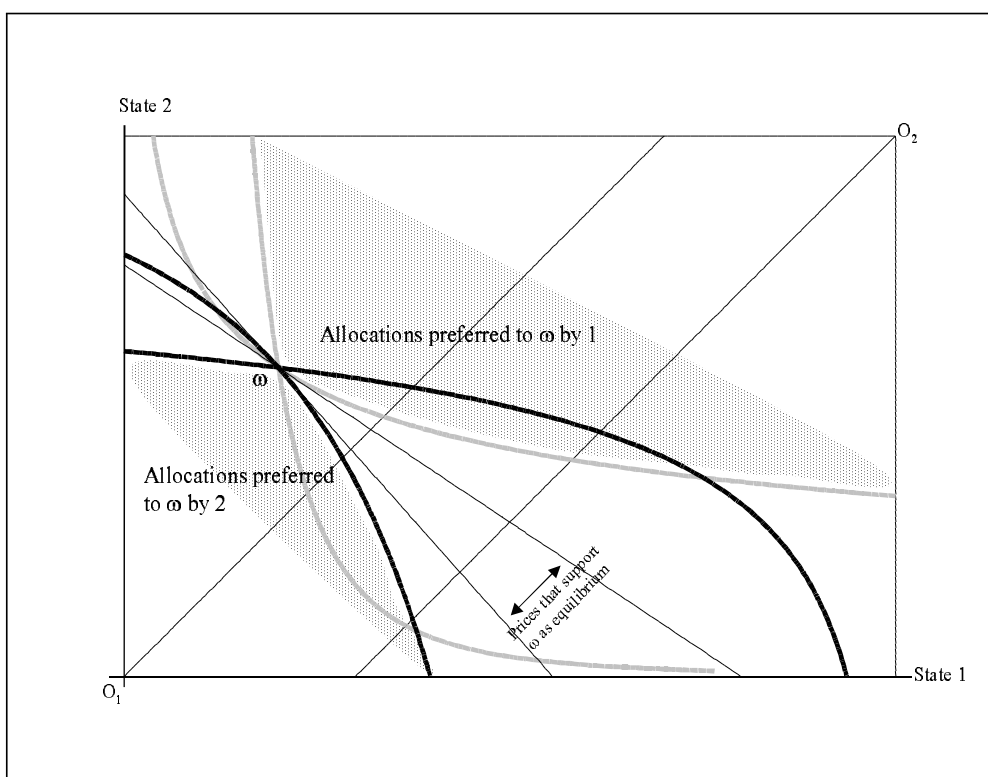


Figure 3: A no-trade equilibrium

Figure 3 helps to represent a basic feature of individuals' interaction in this economy. Uncertainty reduces the set of mutually beneficial trades. In the diagram, the presence of uncertainty in each individual's beliefs is reflected by the kink at the initial endowment. More uncertainty implies a "sharper" kink since more probability distributions are used to determine the set of consumption bundles preferred to the initial endowment. Therefore more uncertainty implies that there are fewer allocations preferred to the initial endowment by all agents. Risk-

aversion also plays a role since the curvature of the utility function determines the shape of the better-than sets. Risk-aversion is reflected in an incentive for individuals to seek mutual insurance, and hence trade away from the initial endowment. The trade-off between these forces determines equilibrium.

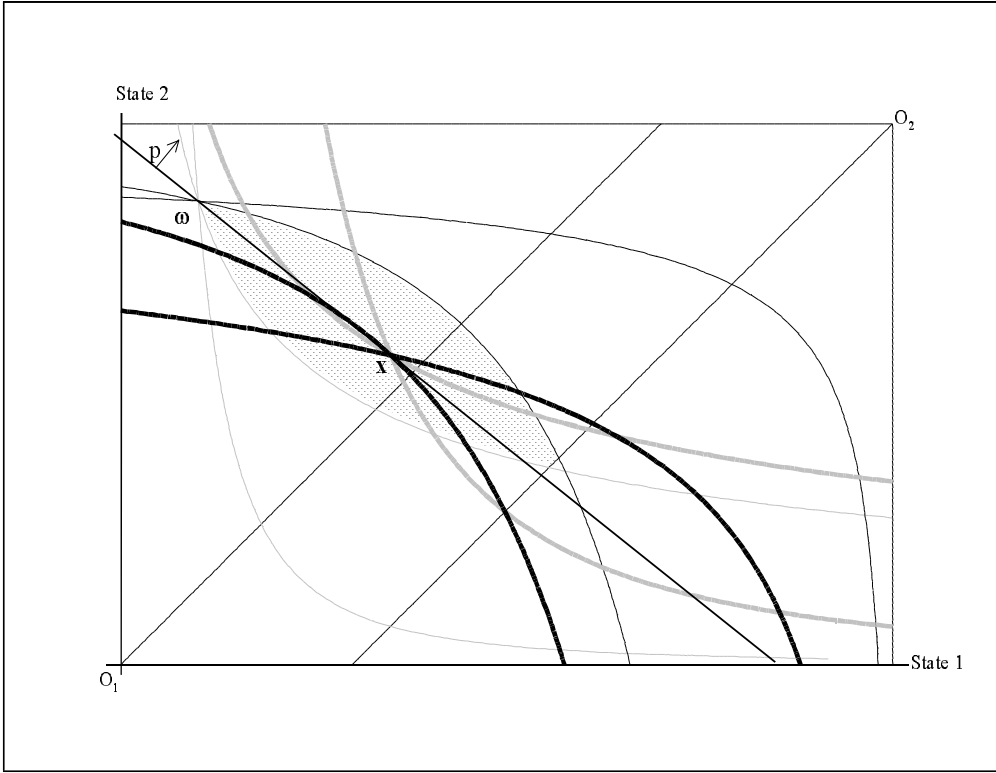


Figure 4: An equilibrium involving trade

Figure 4 depicts an equilibrium with inertia in which there is trade. As before, ω is the initial endowment, agent 1's preferences are represented in grey, while agent 2's preferences are in black. Indifference curves through the initial endowment are illustrated using thin lines. In this case, the set of Pareto improving trades is the lens-shaped set. The allocation x is an equilibrium, supported by p . To see this, note that x is Pareto optimal, since the better-than- x sets do not intersect, and clearly p supports x . Finally, since x^i is preferred to ω^i by both individuals, this is an equilibrium with inertia.

Figures 3 and 4 also help to highlight a relatively simple way to characterize Pareto optimal allocations. As noted in Section 2, we can think of each individual evaluating consumption bundles using a family of utilities. In figures 3 and 4, we display only the most extreme

indifference curves for each individual. When an allocation is Pareto optimal, there must exist members of each individual's family of indifference curves that are tangent at that allocation. A closely related point is made more formally by the following theorem.

Theorem 5 *Under assumptions A1-A3, an interior allocation $x = (x^1, \dots, x^I)$ is Pareto optimal if and only if there exists $\pi^i \in \Pi^i$ for each i such that x solves the social planner's problem*

$$\begin{aligned} \max \quad & \sum_{i=1}^I \lambda^i \sum_{s=1}^S \pi_s^i u^i(x_s^i) \\ \text{s.t.} \quad & \sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i \end{aligned}$$

for some weights $\lambda^i \geq 0$ such that $\sum_i \lambda^i = 1$.

Proof: First note that there exists $\pi^i \in \Pi^i$ for each i such that x solves the social planner's problem

$$\begin{aligned} \max \quad & \sum_{i=1}^I \lambda^i \sum_{s=1}^S \pi_s^i u^i(x_s^i) \\ \text{s.t.} \quad & \sum_{i=1}^I x^i = \sum_{i=1}^I \omega^i \end{aligned}$$

for some weights $\lambda^i \geq 0$ such that $\sum_i \lambda^i = 1$ if and only if x is Pareto optimal in the risk economy with priors (π^1, \dots, π^I) . Then in this case there exists no allocation y such that

$$\sum_s \pi_s^i u^i(y_s^i) > \sum_s \pi_s^i u^i(x_s^i) \quad \text{for all } i = 1, \dots, I$$

which implies that there exists no allocation y such that $y^i \succ^i x^i$ for each i . Thus x is Pareto optimal in the economy with uncertainty.

To establish the converse, note that since x is Pareto optimal, $\bigcap_i \Pi^i(x^i) \neq \emptyset$ by Theorem 3. Thus there exist $\pi^i \in \Pi^i$ for each $i = 1, \dots, I$ such that for each i, j ,

$$\frac{\pi_s^i u^i(x_s^i)}{\sum_t \pi_t^i u^i(x_t^i)} = \frac{\pi_s^j u^j(x_s^j)}{\sum_t \pi_t^j u^j(x_t^j)} \quad \text{for all } s.$$

Then in the risk economy with priors (π^1, \dots, π^I) , x is Pareto optimal. ■

The previous result can also be interpreted as a description of the relationship between Pareto optimal allocations in an economy where uncertainty is present and those in economies where there is only risk. In particular, Theorem 5 shows that an allocation is Pareto optimal in our economy if and only if it is Pareto optimal in a standard risk economy in which agents assign some unique subjective priors π^i chosen from the sets Π^i . In the next section, we expand on this theme.

4 Determinacy and Comparative Statics

In the previous section, we provided a characterization of equilibria with and without inertia. In this section, we analyze the role of uncertainty in determining equilibrium outcomes. We proceed in two directions. First, we investigate how uncertainty affects the determinacy of equilibrium allocations and prices. We find robust indeterminacies for every initial endowment vector. Our results are in fact quite stark: provided there is sufficient overlap in agents' beliefs, there are a continuum of equilibrium allocations and prices, regardless of other features of agents' beliefs, initial endowments, or aggregate endowments. Our second class of results establishes that, despite such robust indeterminacies, the set of equilibria varies continuously with the amount of uncertainty agents perceive. In particular, as uncertainty goes to zero (that is, agents perceive only risk), the equilibrium correspondence converges to an equilibrium of the economy in which there is only risk.

The following is our main indeterminacy result. For this result we will use a standard Inada condition coupled with the requirement that all agents' beliefs have full support to ensure that all equilibria are interior, and hence that we can use the characterization of equilibria in Theorem 4. The addition of these assumptions also further highlights the role of uncertainty as the source of indeterminacies in our model, since these are essentially the assumptions under which generic determinacy would obtain in the absence of uncertainty. Thus here we add

(A4) for each i , $u^i(c) \rightarrow \infty$ as $c \rightarrow 0$

(A5) for each i , $\Pi^i \subset \text{int } \Delta$

Theorem 6 *Suppose assumptions A1-A5 hold, and $\bigcap_i \Pi^i$ has full dimension. For every initial endowment vector $(\omega^1, \dots, \omega^I)$, there is a continuum of equilibrium allocations and prices.*

The idea behind this result is straightforward. First consider an economy which has sufficient uncertainty in which the belief sets of all agents admit a common probability distribution. Consider the risk-only economy defined by this common prior. An equilibrium in this economy is an equilibrium in the economy with uncertainty, but there are infinitely many other equilibria

around it due to uncertainty. This fact, established in the next lemma, immediately implies our indeterminacy result.

Lemma 2 *Suppose assumptions A1-A5 hold, and $\bigcap_i \Pi^i$ has full dimension. Let $\pi \in \text{int} \bigcap_i \Pi^i$ and let x be an equilibrium allocation in the risk economy with common prior π . Let $p \in \Delta$ be the equilibrium price supporting x , so*

$$p_s = \frac{\pi_s u^i(x_s^i)}{\sum_t \pi_t u^i(x_t^i)} \text{ for each } s.$$

Then

- (i) x is an equilibrium allocation and $p \in \bigcap_i \Pi^i(x^i)$
- (ii) there exists a neighborhood V of x such that every neighborhood $V' \subset V$ of x contains infinitely many equilibrium allocations
- (iii) there exists a neighborhood O of p such that every $p' \in O$ is an equilibrium price supporting some $x' \in V$.

Proof: For (i), note that there exist $\lambda^1, \dots, \lambda^I > 0$ such that

$$u^i(x_s^i) = \lambda^i u^I(x_s^I) \text{ for each } s.$$

Then for each $\tilde{\pi} \in \bigcap_i \Pi^i$ and for each i, j ,

$$\frac{\tilde{\pi}_s u^i(x_s^i)}{\sum_t \tilde{\pi}_t u^i(x_t^i)} = \frac{\tilde{\pi}_s u^j(x_s^j)}{\sum_t \tilde{\pi}_t u^j(x_t^j)} \text{ for each } s.$$

Hence $\bigcap_i \Pi^i(x^i) \neq \emptyset$, and $p \in \bigcap_i \Pi^i(x^i)$. Clearly $p \cdot x^i = p \cdot \omega^i$ for each i , which implies that $(x; p)$ is an equilibrium under uncertainty.

For (ii) and (iii), first note that since $\bigcap_i \Pi^i$ has full dimension, so does $\bigcap_i \Pi^i(x^i)$. Moreover, since $\pi \in \text{int} \bigcap_i \Pi^i$, $p \in \text{int} \bigcap_i \Pi^i(x^i)$. Now by Lemma 4 in the Appendix, there exist neighborhoods V of x and O of p such that

$$O \subset \bigcap_i \Pi^i(y^i) \quad \forall y \in V.$$

Without loss of generality, taking a smaller neighborhood if needed, we can ensure that for each $y \in V$, $y^i \succ^i \omega^i$ for each i . Then choose $y \in V$ such that $p \cdot y^i = p \cdot \omega^i$ for each i ; there is a continuum of such allocations. Since $p \in \bigcap_i \Pi^i(x^i)$, y is an equilibrium allocation supported by p .

For (iii), define the correspondence $B : \Delta \rightarrow \mathbf{R}_+^{IS}$ by

$$B(q) = \{y \in \mathbf{R}_+^{IS} : q \cdot y^i = q \cdot \omega^i \forall i\}.$$

To establish (iii), it suffices to show that there exists a neighborhood O of p such that $B(q) \cap V \neq \emptyset$ for all $q \in O$. To that end, we claim that B is lower hemi-continuous at x . To show this, let $q \gg 0$ be arbitrary and take $q^n \rightarrow q$ and $y \in B(q)$. For each n , set

$$y^{i_n} = \frac{1}{q^{1_n}} (q^n \cdot \omega^i - q^n \cdot y^i) e^1 + y^i$$

where $e^1 = (1, 0, \dots, 0)$. For n sufficiently large, $y^{i_n} \geq 0$. Clearly $y^n \rightarrow y$ and

$$\begin{aligned} q^n \cdot y^{i_n} &= q^n \cdot \omega^i - q^n \cdot y^i + q^n \cdot y^i \\ &= q^n \cdot \omega^i. \end{aligned}$$

Thus $y^n \in B(q^n)$ for all n sufficiently large. We conclude that B is lower hemi-continuous. Now note that $B(p) \cap V \neq \emptyset$, as $x \in B(p) \cap V$. Since V is open, the lower hemi-continuity of B implies that there exists a neighborhood O of p such that $B(q) \cap V \neq \emptyset$ for all $q \in O$. ■

This result is illustrated in Figure 5. As usual, ω is the initial endowment, agent 1's preferences are grey, while agent 2's preferences are black. The allocations x and y are both equilibria supported by the same price vector since the corresponding better-than sets do not intersect. Roughly, the kink of the better-than sets of each individual moves along the price line from x to y . Therefore, any allocation between these is also an equilibrium.

Notice that the result in Theorem 6 is independent of the precise nature of agents' beliefs, as long as they have full-dimensional intersection, and of the presence of aggregate uncertainty or certainty. In this sense, uncertainty is a robust source of indeterminacies. In Section 6, we show this is not the case in economies where individuals have MEU or CEU preferences. In those cases, indeterminacies are possible only under very specific assumptions about belief sets and/or initial endowment vectors.

Next we turn to comparative statics analysis, and analyze how the set of equilibria changes with uncertainty. Recall that by Proposition 1 we can think equivalently of varying the degree of completeness of agents' preferences or varying the size of their belief sets. With this in mind, we show first that the set of equilibrium allocations is monotone in beliefs, that is, any equilibrium of an economy with smaller belief sets, hence with more complete preferences, is an equilibrium of the economy with larger belief sets.

Theorem 7 *Suppose assumptions A1-A2 hold. Let (x, p) be an equilibrium in the economy with belief sets (Π^1, \dots, Π^I) . If $\Pi^i \subset \tilde{\Pi}^i$ for each i , then (x, p) is also an equilibrium in the economy with belief sets $(\tilde{\Pi}^1, \dots, \tilde{\Pi}^I)$.*

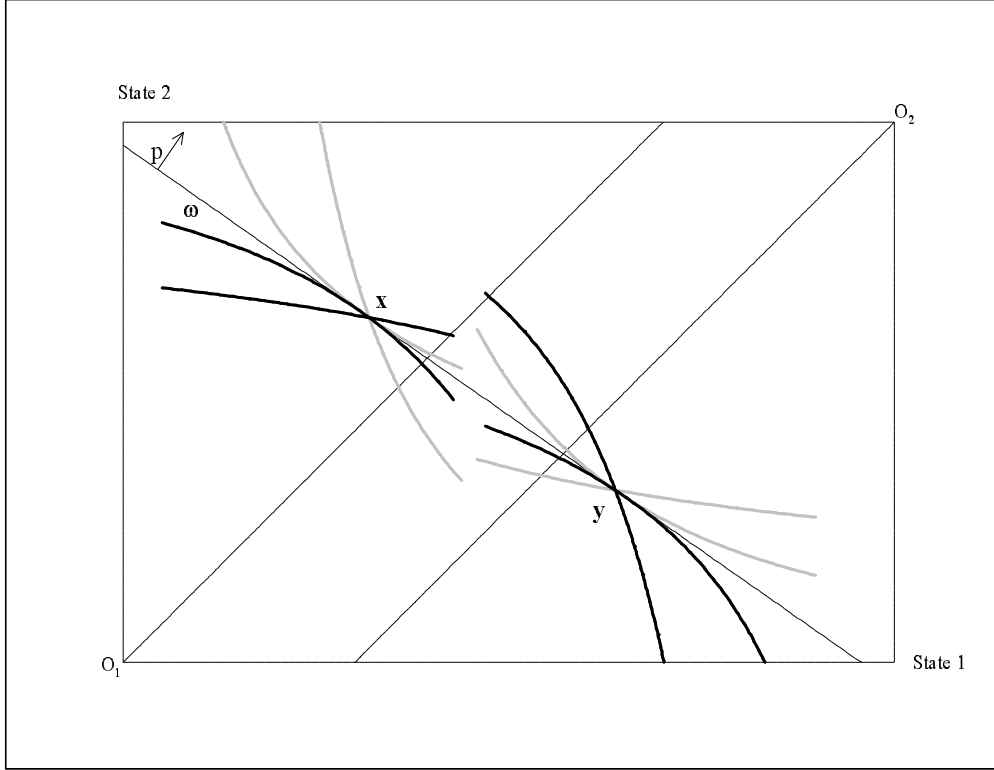


Figure 5: Multiple Equilibria

Proof: Let (x, p) be an equilibrium in the economy with belief sets (Π^1, \dots, Π^J) , and let $\Pi^i \subset \tilde{\Pi}^i$ for each i . Fix j . To see that (x, p) is an equilibrium in the economy with belief sets $(\tilde{\Pi}^1, \dots, \tilde{\Pi}^J)$, it suffices to note that if $\sum \pi_s u^j(y_s) > \sum \pi_s u^j(x_s^j)$ for all $\pi \in \tilde{\Pi}^j$ then $\sum \pi_s u^j(y_s) > \sum \pi_s u^j(x_s^j)$ for all $\pi \in \Pi^j$. Hence $p \cdot y > p \cdot \omega^j$ for any such bundle y , which implies that (x, p) is an equilibrium in the economy with belief sets $(\tilde{\Pi}^1, \dots, \tilde{\Pi}^J)$. ■

The next result establishes that in our model equilibria vary continuously with uncertainty. To make this statement precise, we will need some notion of distance between sets of priors. For this purpose we will use the Hausdorff topology on the set of all subsets of priors 2^Δ . Recall that for arbitrary sets A, B the Hausdorff distance is

$$d(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\}.$$

For compact sets, in particular for the sets of priors to which we restrict attention, this is in fact a metric.

Theorem 8 Under assumptions A1-A2, the equilibrium correspondence $\mathcal{E} : \prod_{i=1}^I 2^\Delta \rightarrow \mathbf{R}_+^{IS} \times \Delta$ is upper hemi-continuous.

Proof: Fix (Π^1, \dots, Π^I) . Let $\Pi^{i_n} \rightarrow \Pi^i$ for each i , and $(x^n, q^n) \in \mathcal{E}(\Pi^{1_n}, \dots, \Pi^{I_n})$ such that $x^n \rightarrow x$ and $q^n \rightarrow q$. For each n ,

$$\begin{aligned} q^n \cdot x^{i_n} &= q^n \cdot \omega^i \quad \forall i \\ \text{and } \sum x^{i_n} &= \sum \omega^i \end{aligned}$$

So

$$\begin{aligned} q \cdot x^i &= q \cdot \omega^i \quad \forall i \\ \text{and } \sum x^i &= \sum \omega^i \end{aligned}$$

Now we must show that $(x, q) \in \mathcal{E}(\Pi^1, \dots, \Pi^I)$. To that end, fix i and let $x \succ^i x^i$. Then for each $\pi^i \in \Pi^i$, $E_{\pi^i}[u^i(x)] > E_{\pi^i}[u^i(x^i)]$. For each $\pi^i \in \Pi^i$ there exists a neighborhood L_{π^i} of π^i and M_{π^i} of x^i such that

$$E_{\hat{\pi}^i}[u^i(x)] > E_{\hat{\pi}^i}[u^i(\hat{x}^i)]$$

for all $\hat{\pi}^i \in L_{\pi^i}$ and $\hat{x}^i \in M_{\pi^i}$. The collection $\{L_{\pi^i} : \pi^i \in \Pi^i\}$ is an open cover of Π^i , and since Π^i is compact we can find a finite subcover $\{L_1, \dots, L_m\}$. Set $L = \cup_{k=1}^m L_k$ and $M = \cap_{k=1}^m M_k$. Then for every $\pi \in L$ and $y \in M$, we have

$$E_\pi[u^i(x)] > E_\pi[u^i(y)]$$

Now since $\Pi^{i_n} \rightarrow \Pi^i$ and $x^{i_n} \rightarrow x^i$, there exists N sufficiently large such that for $n \geq N$, $\Pi^{i_n} \subset L$ and $x^{i_n} \in M$. Thus for $n \geq N$,

$$E_{\pi^{i_n}}[u^i(x)] > E_{\pi^{i_n}}[u^i(x^{i_n})]$$

for all $\pi^{i_n} \in \Pi^{i_n}$. As $(x^n, q^n) \in \mathcal{E}(\Pi^{1_n}, \dots, \Pi^{I_n})$, we conclude that

$$q^n \cdot x > q^n \cdot \omega^i \quad \text{for all } n \geq N$$

Thus $q \cdot x \geq q \cdot \omega^i$. This shows that for every y such that $y \succ^i x^i$, $q \cdot y \geq q \cdot \omega^i$. Now suppose, by way of contradiction, that there exists x such that $x \succ^i x^i$ and $q \cdot x = q \cdot \omega^i$. Choose $\epsilon > 0$ sufficiently small such that $y \equiv x - \epsilon \omega^i \succ^i x^i$. Then $q \cdot (x - \epsilon \omega^i) = (1 - \epsilon)q \cdot \omega^i < q \cdot \omega^i$, a contradiction. Thus if $x \succ^i x^i$, then $q \cdot x > q \cdot \omega^i$. This establishes our claim that $(x, q) \in \mathcal{E}(\Pi^1, \dots, \Pi^I)$. ■

A similar result holds if we restrict attention to equilibria with inertia. We leave the details to the interested reader.

Theorem 8 enables us to derive several important implications. Intuitively, these are simple consequences of the trade-off between risk and uncertainty that characterizes equilibria in our model. At one extreme, for any vector of initial endowments there is a large enough amount of uncertainty such that no-trade is the unique equilibrium allocation with inertia. At the other extreme, when uncertainty becomes smaller and incentives to insure are very strong, all equilibria converge to the equilibrium outcome with only risk. These results are formally established below.

First, the next result shows that in the presence of risk aversion uncertainty can always hamper trade since as it becomes sufficiently large, the no-trade allocation is the unique equilibrium with inertia. Notice that the result would trivially hold if the belief set of each agent is the entire simplex. Instead we show that the initial endowment is an equilibrium allocation even if agents rule out some distributions.

Corollary 6 *Suppose assumptions A1-A3 and A5 hold, and u^i is strictly concave for each i . There exist belief sets $\bar{\Pi}^1, \dots, \bar{\Pi}^I$ with $\bar{\Pi}^i \neq \Delta$ for each i such that the unique equilibrium allocation with inertia in $\mathcal{E}(\bar{\Pi}^1, \dots, \bar{\Pi}^I)$ is $(\omega^1, \dots, \omega^I)$. If $(\Pi^{1n}, \dots, \Pi^{In}) \rightarrow (\bar{\Pi}^1, \dots, \bar{\Pi}^I)$ and (x^n, p^n) is an equilibrium with inertia in $\mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$ for each n , then $x^n \rightarrow (\omega^1, \dots, \omega^I)$.*

Proof: First it suffices to show that there exist belief sets $\bar{\Pi}^1, \dots, \bar{\Pi}^I$ with $\bar{\Pi}^i \subset \text{int } \Delta$ for each i such that $\bigcap_i \Pi^i(\omega^i) \neq \emptyset$. To that end, fix $\pi^1 \in \text{int } \Delta$. For each $i \neq 1$, set

$$\pi_s^i = \lambda^i \frac{\pi_s^1 u^{1'}(\omega_s^1)}{u^{i'}(\omega_s^i)}$$

for each $s = 1, \dots, S$, where

$$\lambda^i = \frac{1}{\sum \frac{\pi_s^1 u^{1'}(\omega_s^1)}{u^{i'}(\omega_s^i)}}.$$

Then $\pi^i \in \Delta$ and by construction,

$$\frac{\pi_s^i u^{i'}(\omega_s^i)}{\sum \pi_t^i u^{i'}(\omega_t^i)} = \frac{\pi_s^1 u^{1'}(\omega_s^1)}{\sum \pi_t^1 u^{1'}(\omega_t^1)}$$

for each i . For each i , choose $\bar{\Pi}^i$ such that $\pi^i \in \bar{\Pi}^i$ and $\bar{\Pi}^i \subset \text{int } \Delta$. Then by construction, for these belief sets $\bigcap_i \Pi^i(\omega^i) \neq \emptyset$. Thus $(\omega^1, \dots, \omega^I)$ is an equilibrium allocation, and by Lemma 1, it is the unique equilibrium allocation with inertia.

Now let $(\Pi^{1n}, \dots, \Pi^{In}) \rightarrow (\bar{\Pi}^1, \dots, \bar{\Pi}^I)$ and (x^n, p^n) be an equilibrium with inertia in $\mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$ for each n . Take any convergent subsequence $(x^{n_k}, p^{n_k}) \rightarrow (x, p)$. By Theorem

8, $(x, p) \in \mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$. Fix i such that $x^i \neq \omega^i$. Then there exists $K > 0$ such that for $k \geq K$, $x^{i_{n_k}} \neq \omega^i$. Since (x^{n_k}, p^{n_k}) is an equilibrium with inertia for each k , we conclude that

$$E_{\pi^i}[u^i(x^{i_{n_k}})] \geq E_{\pi^i}[u^i(\omega^i)] \quad \text{for all } \pi^i \in \Pi^{i_{n_k}}$$

for each $k \geq K$. Letting $k \rightarrow \infty$ yields

$$E_{\pi^i}[u^i(x^i)] \geq E_{\pi^i}[u^i(\omega^i)] \quad \text{for all } \pi^i \in \bar{\Pi}^i$$

Thus (x, p) is an equilibrium with inertia in $\mathcal{E}(\bar{\Pi}^1, \dots, \bar{\Pi}^I)$. By Lemma 1, $x = (\omega^1, \dots, \omega^I)$. Since every convergent subsequence of $\{x^n\}$ must converge to $(\omega^1, \dots, \omega^I)$, we conclude that $x^n \rightarrow (\omega^1, \dots, \omega^I)$. ■

We now show that if individuals' beliefs are close enough to a unique prior, then the equilibria of our economy converge to the equilibria of the risk economy. This result also establishes that the equilibria in the standard risk setting are robust to adding a small degree of uncertainty. This is a consequence of the monotonicity and upper hemi-continuity of the equilibrium correspondence established in Theorems 7 and 8.

Theorem 9 *Suppose assumptions A1-A2 hold. Let $\pi^i \in \Delta$ for $i = 1, \dots, I$, and let $(x, p) \in \mathcal{E}(\pi^1, \dots, \pi^I)$. For any sequence of belief sets $(\Pi^{1n}, \dots, \Pi^{In}) \rightarrow (\{\pi^1\}, \dots, \{\pi^I\})$ such that $\pi^i \in \Pi^{in}$ for each i and n :*

- (i) *there exists a sequence $\{(x^n, p^n)\}$ such that $(x^n, p^n) \in \mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$ for each n and $(x^n, p^n) \rightarrow (x, p)$.*
- (ii) *if $\{(x^n, p^n)\}$ is a sequence such that $(x^n, p^n) \in \mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$ for each n and $(x^n, p^n) \rightarrow (\hat{x}, \hat{p})$, then $(\hat{x}, \hat{p}) \in \mathcal{E}(\{\pi^1\}, \dots, \{\pi^I\})$.*

Proof: Part (i) follows immediately from the monotonicity of the equilibrium correspondence in belief sets established in Theorem 7, by setting $(x^n, p^n) = (x, p)$ for each n . Part (ii) follows immediately from the upper hemi-continuity of the equilibrium correspondence at $(\{\pi^1\}, \dots, \{\pi^I\})$. ■

Finally, as the last corollary demonstrates, if the equilibrium of the risk economy happens to be unique, then all equilibria under uncertainty converge to that equilibrium as uncertainty shrinks.

Corollary 7 *Suppose assumptions A1-A2 hold, and let $\pi^i \in \Delta$ for each i . Assume that (x, p) is the unique equilibrium in the risk economy with priors (π^1, \dots, π^I) . If $(\Pi^{1n}, \dots, \Pi^{In}) \rightarrow (\{\pi^1\}, \dots, \{\pi^I\})$ and $(x^n, p^n) \in \mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$ for each n , then $(x^n, p^n) \rightarrow (x, p)$.*

Proof: By Theorem 8, the equilibrium correspondence is upper hemi-continuous at $(\{\pi^1\}, \dots, \{\pi^I\})$. Let $(\Pi^{1n}, \dots, \Pi^{In}) \rightarrow (\{\pi^1\}, \dots, \{\pi^I\})$ and $(x^n, p^n) \in \mathcal{E}(\Pi^{1n}, \dots, \Pi^{In})$ for each n . Take any convergent subsequence (x^{n_k}, p^{n_k}) . Since the equilibrium correspondence is upper hemi-continuous at $(\{\pi^1\}, \dots, \{\pi^I\})$ and (x, p) is the unique equilibrium given beliefs $(\{\pi^1\}, \dots, \{\pi^I\})$, $(x^{n_k}, p^{n_k}) \rightarrow (x, p)$. Then as every convergent subsequence must converge to (x, p) , we conclude that $(x^n, p^n) \rightarrow (x, p)$. ■

In particular, note that this corollary applies in the case in which there is no aggregate uncertainty, agents are risk averse and have a (unique) common prior, as in this case there is a unique equilibrium in the risk economy with that prior.

5 Uncertainty and Risk

In this section, we derive conditions under which endogenous market incompleteness can arise in our model. In the previous sections, we have shown that no-trade obtains when uncertainty is ubiquitous. In that case, securities markets are extremely incomplete since no exchange takes place in equilibrium. Here, we show that when there is uncertainty only about some events, there may be equilibria in which securities contingent on these events are not traded, while securities contingent on the remaining (risky) events are traded. In this case, a more limited degree of market incompleteness is possible in equilibrium, in that risky securities are traded while uncertain securities are not.

The state space contains two types of events: events that all agents perceive have a unique probability, and events that some perceive do not. To find conditions under which only risky securities are traded in equilibrium we proceed as follows. First, we define an equilibrium concept *as if* trades in securities contingent on uncertain events were not allowed. Then we give conditions under which this constrained equilibrium is an equilibrium when unrestricted trading strategies are allowed. To make this analysis precise, however, we need to introduce some new notation that allows the focus to shift to trade contingent on events rather than states.

To this end, consider the state space S divided into two sets, S_R and S_U . The set S_R contains all of the risky states, those to which each agent assigns a precise probability, and S_U contains all of the uncertain states, those to which some agents assign multiple probabilities. In parallel, the set of all possible events $E \subset S$ is divided between risky events \mathcal{R} and uncertain events \mathcal{U} , where

$$\mathcal{R} \equiv \left\{ E \subset S : \text{for each } i, \sum_{s \in E} \pi_s^i \text{ is unique for all } \pi^i \in \Pi^i \right\}.$$

and

$$\mathcal{U} \equiv \{E \subset S : E \notin \mathcal{R}\}$$

Notice that the union of all uncertain events is an element of \mathcal{R} , and so is the event given by the union of all uncertain states.

In this framework we study the incentives for trade over risky and uncertain events. Given a bundle $x \in \mathbf{R}_+^S$ and an event $E \subset S$, let $x(E) = \{x_s\}_{s \in E}$ denote the vector of consumption in the states contained in the event E . Similarly, given $\pi \in \Delta$, let $\pi(E) = \{\pi_s\}_{s \in E}$ denote the probabilities of the states contained in the event E . With slight abuse of notation, we let $u^i(x(E))$ denote the vector of marginal utilities corresponding to $x(E)$, so $u^i(x(E)) = \{u^i(x_s)\}_{s \in E}$.

Suppose agents can only trade in securities contingent on risky events. Then, we can define an equilibrium subject to this restriction as follows.

Definition An allocation (x^1, \dots, x^I) and a non-zero price vector $p \in \mathbf{R}_+^S$ are an *equilibrium over risky events* if

1. for each i , $x^i(E) = \omega^i(E)$ for each $E \in \mathcal{U}$
2. for each i , if $y(E) = \omega^i(E)$ for each $E \in \mathcal{U}$ and $y \succ^i x^i$, then $p \cdot y > p \cdot \omega^i$
3. for each i , $p \cdot x^i = p \cdot \omega^i$

In an equilibrium over risky events, each agent must consume her endowment when an uncertain event occurs. Therefore, the individuals consider only consumption bundles that respect this restriction. One can see easily that an equilibrium over risky events is, by construction, an equilibrium with inertia.

The next theorem establishes conditions under which an equilibrium over risky events is actually an equilibrium. Intuitively, this can happen whenever individuals' initial endowments in the uncertain events are Pareto optimal contingent on those events. Thus this theorem gives conditions under which markets for trade over uncertain events do not function.

Theorem 10 *Suppose assumptions A1-A3 hold. Let $(x; p)$ be an interior equilibrium over risky events. Then $(x; p)$ is an equilibrium if and only if*

$$\bigcap_i \left\{ \left(\frac{\pi(U) \cdot u^i(x^i(U))}{\sum_{E \in \mathcal{U}} \pi(E) \cdot u^i(x^i(E))} \right)_{U \in \mathcal{U}} : \pi \in \Pi^i \right\} \neq \emptyset.$$

Proof: If x is an equilibrium allocation over risky events, then

$$\bigcap_i \left\{ \left(\frac{\pi(R) \cdot u^i(x^i(R))}{\sum_{E \in \mathcal{R}} \pi(E) \cdot u^i(x^i(E))} \right)_{R \in \mathcal{R}} : \pi \in \Pi^i \right\} \neq \emptyset.$$

By assumption,

$$\bigcap_i \left\{ \left(\frac{\pi(U) \cdot u^i(x^i(U))}{\sum_{E \in \mathcal{U}} \pi(E) \cdot u^i(x^i(E))} \right)_{U \in \mathcal{U}} : \pi \in \Pi^i \right\} \neq \emptyset.$$

Thus for each i there exists $\pi^i \in \Pi^i$ such that

$$\frac{\pi^i(U) \cdot u^i(x^i(U))}{\sum_{E \in \mathcal{U}} \pi^i(E) \cdot u^i(x^i(E))} = \frac{\pi^1(U) \cdot u^1(x^1(U))}{\sum_{E \in \mathcal{U}} \pi^1(E) \cdot u^1(x^1(E))}$$

for each $U \in \mathcal{U}$. Thus for each i there exists $\lambda_{\mathcal{U}}^i > 0$ such that

$$\pi^i(U) \cdot u^i(x^i(U)) = \lambda_{\mathcal{U}}^i \pi^1(U) \cdot u^1(x^1(U))$$

for each $U \in \mathcal{U}$; similarly, for each i there exists $\lambda_{\mathcal{R}}^i > 0$ such that

$$\pi^i(R) \cdot u^i(x^i(R)) = \lambda_{\mathcal{R}}^i \pi^1(R) \cdot u^1(x^1(R))$$

for each $R \in \mathcal{R}$. Now to establish our claim it suffices to show that $\lambda_{\mathcal{R}}^i = \lambda_{\mathcal{U}}^i$ for all i . To that end, note that

$$\bigcup_{s \in S_U} \{s\} = S_U = S \setminus S_R \in \mathcal{R}$$

while for every $s \in S_U$, $\{s\} \in \mathcal{U}$. Thus for every $s \in S_U$,

$$\pi_s^i u^i(x_s^i) = \lambda_{\mathcal{U}}^i \pi_s^1 u^1(x_s^1)$$

which implies that

$$\sum_{s \in S_U} \pi_s^i u^i(x_s^i) = \lambda_{\mathcal{U}}^i \sum_{s \in S_U} \pi_s^1 u^1(x_s^1)$$

or

$$\pi^i(S_U) \cdot u^i(x^i(S_U)) = \lambda_{\mathcal{U}}^i \pi^1(S_U) \cdot u^1(x^1(S_U)).$$

However, since $S_U \in \mathcal{R}$,

$$\pi^i(S_U) \cdot u^i(x^i(S_U)) = \lambda_{\mathcal{R}}^i \pi^1(S_U) \cdot u^1(x^1(S_U))$$

which implies that $\lambda_{\mathcal{R}}^i = \lambda_{\mathcal{U}}^i$ for each i . ■

This result can be viewed as a natural analogue of our equilibrium characterization results in section 3. To make this analogy more precise, define the set $\Pi^i(x^i(U))$ as follows

$$\Pi^i(x^i(U)) = \left\{ \left(\frac{\pi(U) \cdot u^i(x^i(U))}{\sum_{E \in \mathcal{U}} \pi(E) \cdot u^i(x^i(E))} \right)_{U \in \mathcal{U}} : \pi \in \Pi^i \right\};$$

Using this notation, Theorem 10 states that an equilibrium over risky events is an equilibrium if and only if $\bigcap_i \Pi^i(x^i(U)) \neq \emptyset$. In sections 3 and 4, we used the corresponding set $\bigcap_i \Pi^i(x^i)$ to characterize equilibria and derive comparative statics results. Those results could easily be modified so that they apply to equilibria over risky events by using the intersection of these “conditional” belief sets $\bigcap_i \Pi^i(x^i(U))$. For example, an appropriately modified version of Corollary 6 would show that if belief sets over uncertain events are large enough, then all equilibria over risky events are equilibria since the initial endowment is always conditional Pareto optimal. We leave the details of these results to the interested reader.

Theorem 10 proves particularly useful with additional information on the characteristics of the economy. For example, suppose there is no aggregate uncertainty over the uncertain states and therefore there is no aggregate uncertainty over any uncertain event. Then, similar to the conclusion of Corollary 2, an equilibrium over risky events is an equilibrium if and only if agents have at least one prior over uncertain events in common. This result can be illustrated more precisely as follows.

Corollary 8 *Let $\bar{R} \in \mathcal{R}$ be a risky event such that $\bar{R} = U_1 \cup \dots \cup U_m$ for some uncertain events $U_k \in \mathcal{U}$, and such that \bar{R} contains no other proper risky subevent. Suppose there is no idiosyncratic or aggregate uncertainty over the states in \bar{R} , so for each $s, t \in \bar{R}$, $\omega_s^i = \omega_t^i$ for each i . Then an equilibrium over risky events is an equilibrium if and only if agents have one common conditional prior over the uncertain events U_1, \dots, U_m .*

Proof: To establish this claim, consider an equilibrium over risky events (x, p) . Since each agent’s initial endowment is constant across the states in \bar{R} , $x^i(\bar{R})$ will be constant across states for each i as well. Now consider insurance markets for the uncertain events U_1, \dots, U_m . By Theorem 10, (x, p) is an equilibrium if and only if

$$\bigcap_i \left\{ \left(\frac{\pi^i(U_k) \cdot u^i(x^i(U_k))}{\sum_{\ell} \pi^i(U_\ell) \cdot u^i(x^i(U_\ell))} \right)_{k=1, \dots, m} : \pi^i \in \Pi^i \right\} \neq \emptyset.$$

But note that $x^i(U_k)$ is constant for each $s \in U_k \subset \bar{R}$ and for each k , thus for any k ,

$$\frac{\pi^i(U_k) \cdot u^i(x^i(U_k))}{\sum_{\ell} \pi^i(U_\ell) \cdot u^i(x^i(U_\ell))} = \frac{\sum_{s \in U_k} \pi_s^i}{\sum_{\ell} \sum_{t \in U_\ell} \pi_t^i}.$$

Thus there is no trade on these new markets if and only if

$$\bigcap_i \left\{ \left(\frac{\sum_{s \in U_k} \pi_s^i}{\sum_{\ell} \sum_{t \in U_\ell} \pi_t^i} \right)_{k=1, \dots, m} : \pi^i \in \Pi^i \right\} \neq \emptyset.$$

that is, if and only if the agents share a common conditional prior over the uncertain events U_1, \dots, U_m . ■

A simple way to understand this result is to imagine that markets open sequentially. Initially, only markets for trades contingent on risky events are available. After all trades in these markets are completed, and the economy is in an equilibrium over risky events, the possibility to open new markets for insurance over uncertain subevents of some risky event is considered. Is there any trade in these new financial instruments? Using the previous result, we can say the new markets will succeed if and only if there is sufficient disagreement over the likelihoods of uncertain events. Otherwise, the uncertainty motive to avoid trade prevails and financial innovations fail.

6 Comparisons with Existing Literature

In this section, we compare the results of the previous sections regarding the impact of uncertainty on equilibrium predictions with the ones obtained using Choquet expected utility (CEU) or maxmin expected utility (MEU).¹⁹ The heart of this distinction is the following observation. When preferences are incomplete, there is an important trade-off between risk (aversion) and uncertainty (aversion) in exchange situations. This trade-off is absent in CEU and MEU models, because in these cases agents have a strong desire for insurance stemming from their drive to equate utility *in levels* across states at the margin. In the CEU and MEU settings, uncertainty makes agents “more” risk averse.

One of the most important features of previous models of uncertainty aversion has been the possibility of indeterminacy of equilibria. This point was first raised by Dow and Werlang (1992) in a partial equilibrium setting. They study a portfolio choice problem under uncertainty in which the agent is a Choquet expected utility maximizer with convex capacity. They show that if such a decision-maker starts from an initial riskless position, then there is an interval of (exogenously given) prices at which the agent will neither buy nor sell short a given risky asset. This result might suggest that the Choquet expected utility model of uncertainty aversion could generate absence of trade in the face of uncertainty and robust indeterminacies in prices.

¹⁹Most CEU applications restrict attention to the case in which agents use a convex capacity. As discussed in Section 2, this assumption renders CEU equivalent to MEU where the set of probability distributions is the core of the capacity.

Although their conclusions are suggestive, Dow and Werlang (1992) work in a partial equilibrium setting, leaving open the question of whether the endogenously determined *equilibrium* prices display indeterminacy or support no trade. Motivated in part by this work, Epstein and Wang (1994) show that equilibrium prices may also exhibit significant indeterminacy when agents are Choquet expected utility maximizers. They study the extension of Lucas’s asset pricing model to the CEU setting, and show that this representative agent economy may permit a continuum of equilibrium prices. From this, Epstein and Wang conclude that the determination of a particular equilibrium price process is left to “animal spirits” and that significant price volatility might result. Since they work in a representative agent model, however, equilibrium prices are simply prices supporting the representative agent’s initial endowment and no trade in the assets; there are no nontrivial opportunities for risk sharing. Even in their representative agent setting, however, equilibrium indeterminacy is not always obtained. Indeterminacy in equilibrium prices requires asset payoffs to vary across states over which the agent’s endowment is constant. The intuition for this fact can be derived simply from thinking of a two-state picture: with CEU preferences, the representative agent’s indifference curves have a kink along the certainty line and nowhere else. A consumption bundle varying across the states has a unique supporting price, so indeterminacy in supporting prices requires an initial position involving certainty across the states.

The implications of these results for heterogeneous agent economies are even more unclear, as Epstein and Wang (1994) note. The fact that there may be a continuum of prices supporting a given Pareto optimal allocation does not mean there are a continuum of *equilibrium* prices supporting this allocation. Again, a simple Edgeworth box illustrates the problem: unless the endowment is Pareto optimal and hence there is no trade in equilibrium, there is a unique price that both supports the allocation *and* goes through the initial endowment allocation.²⁰

The fact that these indeterminacy results break down in heterogeneous agent CEU economies is further clarified by the work of Dana (2000), Dana (2001), and Chateauneuf, Dana, and Tallon (2000). The latter shows that in the CEU model with common convex capacity, Pareto optimal allocations (and hence equilibrium allocations) are necessarily comonotonic. Since CEU preferences satisfy the independence axiom over comonotonic bundles, these preferences are (locally) like expected utility over the consumption bundles relevant for equilibrium analysis. Moreover, Dana (2000) shows that in the same setting, the degree of equilibrium price indeterminacy is linked to the variation in aggregate endowments across states. If there is no aggregate uncertainty, any prior in the core of the agents’ common capacity is an equilibrium price, and the set of equilibrium allocations is a convex polyhedron. If instead the aggregate

²⁰See also Epstein (2001) for a heterogeneous agent model with this feature. Epstein (2001) argues that uncertainty aversion can help explain the home-bias puzzle in international trade. The paper gives a parametric two-agent, two-period MEU model in which the (robustly) unique equilibrium is consistent with underinvestment in “foreign” securities relative to “home” securities due to greater uncertainty about the returns to “foreign” securities.

endowment is state-revealing (i.e. is a one-to-one mapping on the state space), however, then equilibria correspond to those in a standard risk economy with a fixed common prior, and hence are generically determinate. Since the set of state-revealing endowment vectors is itself an open and dense set in the positive cone, generically there is no indeterminacy in either equilibrium allocations or prices in this case, and generically there are no observable distinctions between these models and standard expected utility. Finally, in Rigotti and Shannon (2001), we show that the generic determinacy of equilibria holds for general MEU preferences, and in particular for CEU preferences with convex but differing capacities.

Mukerji and Tallon (forthcoming) further investigate the possibility that uncertainty aversion in a heterogeneous agent CEU model might lead to an endogenous breakdown in markets for some risky assets. In a model with heterogeneous agents and incomplete markets, they show that the condition identified by Epstein and Wang (1994), that asset payoffs are “idiosyncratic” in that they vary across states over which endowments are constant, is sufficient to guarantee that with sufficiently large uncertainty every equilibrium involves no trade over these assets.

In contrast, we show that when uncertainty is modeled by incomplete preferences, agents’ behavior differs in response to risk or uncertainty, allowing us to capture the intuition that there is a trade-off between risk and uncertainty. Moreover, every initial endowment allocation gives rise to equilibrium price indeterminacy, regardless of whether there is aggregate uncertainty or not, and regardless of agents’ beliefs (provided there is sufficient overlap).

These results can be illustrated using a series of Edgeworth boxes. Figure 6 presents the same situation illustrated in Figure 3 (which depicted a no-trade equilibrium in our setting), but assumes instead that individuals have CEU preferences with common convex capacity. Here the initial endowment vector is ω , and the shaded area represents consumption bundles preferred to this initial endowment vector by both individuals. In this example, there are many such bundles, while instead with incomplete preferences (represented by the thin indifference curves) no-trade is an equilibrium, and is the unique equilibrium with inertia. As is evident here, individuals with CEU preferences have a strong incentive to insure, regardless of whether they are facing risk or uncertainty.

Figure 6 also illustrates equilibria in the CEU model with common convex capacity. As Chateauneuf, Dana, and Tallon (2000) show, Pareto optimal and equilibrium allocations are comonotonic; in the diagram, the comonotonic allocations are those between the two certainty lines. As the picture illustrates, allocations that are not comonotonic cannot be Pareto optimal. With only two states, aggregate uncertainty implies immediately that the aggregate endowment is state-revealing, since it differs across the two states. Thus in this case we see a simple example of Dana’s (2000) results: the equilibria correspond to equilibria in a standard risk economy with a fixed common prior.

In contrast, Figure 7 illustrates the case of no aggregate uncertainty. Here with CEU



Figure 6: Incentives to Insure with MEU Preferences

preferences Pareto optimal allocations involve full insurance, so are on the (common) certainty line. As Dana (2000) shows, every prior in the intersection of the cores of the agents' capacities is an equilibrium price supporting some equilibrium allocation. Moreover, Billot, Chateauneuf, Gilboa, and Tallon (2000) show that full insurance is Pareto optimal in this setting with differing capacities whenever individuals' beliefs have at least one prior in common. This result can also be derived as a simple consequence of necessary conditions for Pareto optimality. Pareto optimality requires the existence of a common support to each agent's preferred set. With CEU preferences, the set of supports at a bundle on the certainty line is simply the set of priors in core of the agent's capacity. Thus an allocation involving full insurance can be Pareto optimal only if there is some common prior in each agent's set of priors.

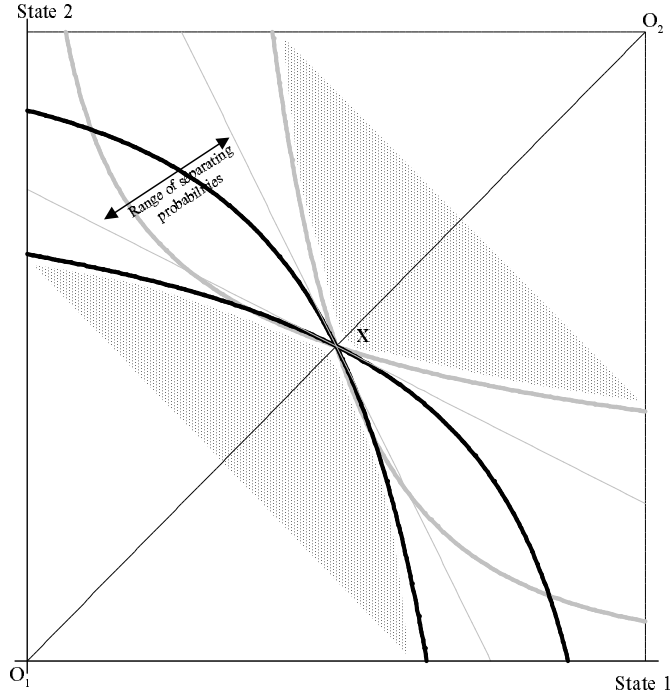


Figure 7: No Aggregate Uncertainty

7 Conclusions

In this paper, we have developed a simple Arrow-Debreu economy in which Knightian uncertainty is modeled using incomplete preferences. This model exhibits an important trade-off between risk and uncertainty that in turn generates two intuitive features of equilibria: robust indeterminacies in both equilibrium allocations and prices and no trade in equilibrium for sufficient degrees of uncertainty. These results are in sharp contrast with what obtains using other approaches to Knightian uncertainty, such as Choquet expected utility and maxmin expected utility. In those cases, generically equilibria may be determinate and identical to equilibria in a standard expected utility model with fixed priors. This contrast highlights the danger in drawing inferences about equilibrium behavior from features of individual choices in a decision-theoretic framework. In this respect, the cautiousness implicit in preferences in our model allows incentives for mutual insurance to depend on this trade-off between risk and

uncertainty, which is entirely novel in an equilibrium setting.

An added benefit of this model is its relative tractability. Despite the apparent complications arising from incomplete preferences, we are able to give a simple characterization of Pareto optimal allocations and equilibria using agents' sets of normalized gradients generated by their sets of priors. In a straightforward generalization of standard results, optimal allocations here are simply those at which *some* selection of priors yields equality of marginal rates of substitution across agents.

Our work suggests several natural and promising areas for further exploration. One that we have briefly discussed here is the connection between uncertainty and incomplete markets, and the extent to which Knightian uncertainty aversion can explain the endogenous failure of some insurance and asset markets. A second is to cast the model in a dynamic framework and examine the interplay between spot markets and asset markets in the face of uncertainty about fundamentals as well as future prices.

A Appendix

Lemma 3 *Under assumptions A1-A3, $\Pi^i : \mathbf{R}_{++}^S \rightarrow 2^{\mathbf{R}_+^S}$ is a closed-, convex-valued, continuous correspondence.*

Proof: Fix $x \in \mathbf{R}_{++}^S$. First, we show that $\Pi^i(x)$ is convex for every x . To that end, pick a $\lambda \in [0, 1]$ and $\pi, \hat{\pi} \in \Pi^i$. Then define

$$\gamma = \frac{1}{\sum_t \pi_t u^{it}(x_t)} \text{ and } \hat{\gamma} = \frac{1}{\sum_t \hat{\pi}_t u^{it}(x_t)}.$$

One can easily verify:

$$\begin{aligned} \frac{\lambda \pi_s u^{is}(x_s)}{\sum_t \pi_t u^{it}(x_t)} + \frac{(1-\lambda) \hat{\pi}_s u^{is}(x_s)}{\sum_t \hat{\pi}_t u^{it}(x_t)} &= \lambda \gamma \pi_s u^{is}(x_s) + (1-\lambda) \hat{\gamma} \hat{\pi}_s u^{is}(x_s) \\ &= [\lambda \gamma + (1-\lambda) \hat{\gamma}] \left(\pi_s \frac{\lambda \gamma}{\lambda \gamma + (1-\lambda) \hat{\gamma}} + \hat{\pi}_s \frac{(1-\lambda) \hat{\gamma}}{\lambda \gamma + (1-\lambda) \hat{\gamma}} \right) u^{is}(x_s) \\ &= \frac{\bar{\pi}_s u^{is}(x_s)}{\sum_t \bar{\pi}_t u^{it}(x_t)} \end{aligned}$$

where

$$\bar{\pi}_s = \pi_s \frac{\lambda \gamma}{\lambda \gamma + (1-\lambda) \hat{\gamma}} + \hat{\pi}_s \frac{(1-\lambda) \hat{\gamma}}{\lambda \gamma + (1-\lambda) \hat{\gamma}}$$

and

$$\begin{aligned}
\frac{1}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} &= \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} \frac{1}{\gamma} + \frac{(1-\lambda)\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} \frac{1}{\widehat{\gamma}} \\
&= \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} \sum_t \pi_t u^{i'}(x_t) + \frac{(1-\lambda)\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} \sum_t \widehat{\pi}_t u^{i'}(x_t) \\
&= \sum_t \left(\pi_t \frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} + \widehat{\pi}_t \frac{(1-\lambda)\widehat{\gamma}}{\lambda\gamma + (1-\lambda)\widehat{\gamma}} \right) u^{i'}(x_t) \\
&= \sum_t \bar{\pi}_t u^{i'}(x_t)
\end{aligned}$$

Now since Π^i is convex, $\bar{\pi} \in \Pi^i$. Thus $\sum_t \bar{\pi}_t u^{i'}(x_t) \in \Pi^i(x)$, which establishes that $\Pi^i(x)$ is convex.

To see that $\Pi^i(x)$ is closed, let $q^n \in \Pi^i(x)$ such that $q^n \rightarrow q$. Since $q^n \in \Pi^i(x)$, there exists $\pi^n \in \Pi^i$ such that

$$q_s^n = \frac{\pi_s^n u^{i'}(x_s)}{\sum_t \pi_t^n u^{i'}(x_t)}$$

Since Π^i is compact, choose a convergent subsequence $\{\pi^{n_k}\}$, converging to $\pi \in \Pi^i$. Then $q^{n_k} \rightarrow q$, and

$$q_s = \frac{\pi_s u^{i'}(x_s)}{\sum_t \pi_t u^{i'}(x_t)}$$

Thus $q \in \Pi^i(x)$.

Next we establish that $\Pi^i(\cdot)$ is upper hemi-continuous at x . Let $x^n \rightarrow x$ and $q^n \in \Pi^i(x^n)$ such that $q^n \rightarrow q$. Then for each n there exists $\pi^n \in \Pi^i$ such that

$$q_s^n = \frac{\pi_s^n u^{i'}(x_s^n)}{\sum_t \pi_t^n u^{i'}(x_t^n)}$$

As above, take $\{\pi^{n_k}\}$ to be a convergent subsequence, converging to $\pi \in \Pi^i$. Since $q^{n_k} \rightarrow q$ and $x^{n_k} \rightarrow x$, the continuity of $u^{i'}$ implies that

$$q_s = \frac{\pi_s u^{i'}(x_s)}{\sum_t \pi_t u^{i'}(x_t)}$$

Thus $q \in \Pi^i(x)$.

Finally, to see that $\Pi^i(\cdot)$ is also lower hemi-continuous at x , let $x^n \rightarrow x$ and $q \in \Pi^i(x)$. Then there exists $\pi \in \Pi^i$ such that

$$q_s = \frac{\pi_s u^{i'}(x_s)}{\sum \pi_t u^{i'}(x_t)}$$

For each n define q^n by

$$q_s^n = \frac{\pi_s u^{i'}(x_s^n)}{\sum \pi_t u^{i'}(x_t^n)}$$

Then $q^n \in \Pi^i(x^n)$ for each n and clearly $q^n \rightarrow q$. Thus $\Pi^i(\cdot)$ is lower hemi-continuous at x . As x was arbitrary, the proof is completed. \blacksquare

Lemma 4 *Suppose assumptions A1-A5 hold. If x is an equilibrium allocation for which $\bigcap_i \Pi^i(x^i)$ has full dimension, and p is an equilibrium price supporting x such that $p \in \text{int } \bigcap_i \Pi^i(x^i)$, then there exists a neighborhood $O \subset \bigcap_i \Pi^i(x^i)$ of p and a neighborhood W of x such that for every $\tilde{x} \in W$, $O \subset \bigcap_i \Pi^i(\tilde{x}^i)$. In particular, $\bigcap_i \Pi^i(\tilde{x}^i) \neq \emptyset$ for all $\tilde{x} \in W$.*

Proof: Let x and p satisfy the hypotheses. Choose a neighborhood O of p such that $C(O) \subset \text{int } \bigcap_i \Pi^i(x^i)$, where

$$C(O) \equiv \left\{ p \in \Delta : p_s \geq \min_{\pi \in \bigcap_i \Pi^i(x^i)} \pi_s, s = 1, \dots, S \right\}$$

For each $s = 1, \dots, S$, let

$$m_s = \min_{\pi \in \bigcap_i \Pi^i(x^i)} \pi_s$$

and let $\{e^s\}$ be the extreme points of $C(O)$, thus

$$e_t^s = \begin{cases} m_t & \text{if } t \neq s; \\ 1 - \sum_{t \neq s} m_t & \text{if } t = s. \end{cases}$$

For each s , define

$$W_s \equiv \left\{ \pi \in \text{int } \bigcap_i \Pi^i(x^i) : \pi_t < e_t^s \text{ for } t \neq s \right\}.$$

Note that W_s is non-empty and open for each s .

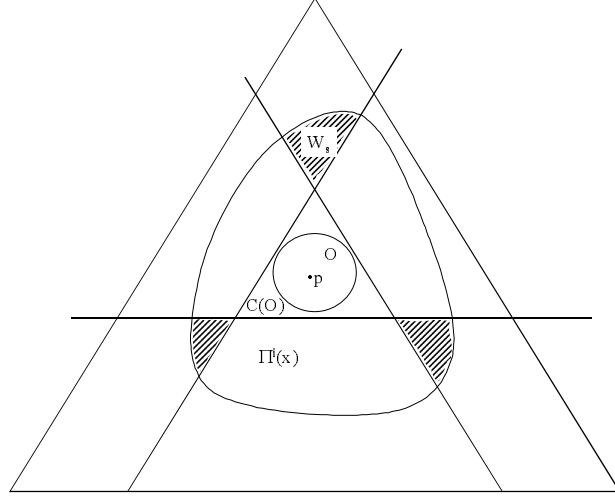


Figure 8: Construction of W_s

Now fix i . Since $\Pi^i(\cdot)$ is lower hemi-continuous, for each s there exists a neighborhood V_s of x^i such that

$$\Pi^i(x) \cap W_s \neq \emptyset$$

for all $x \in V_s$. Let $V = \cap V_s$. Then for every $x \in V$,

$$\Pi^i(x^i) \cap W_s \neq \emptyset \quad \forall s$$

Fix $x \in V$. For each s , choose $q^s \in \Pi^i(x) \cap W_s$. Since $\Pi^i(x)$ is convex,

$$\text{con} \{q^1, \dots, q^S\} \subset \Pi^i(x)$$

where $\text{con}\{z^1, \dots, z^S\}$ denotes the convex hull of the set $\{z^1, \dots, z^S\}$. But by construction, since $q^s \in W_s$ for each s ,

$$O \subset C(O) \subset \text{con} \{q^1, \dots, q^S\} \subset \Pi^i(x)$$

Thus $O \subset \Pi^i(x)$ for all $x \in V$.

As i was arbitrary, there exists a neighborhood V of the allocation x such that $O \subset \bigcap_i \Pi^i(y^i)$ for all $y \in V$. ■

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