Economies with Asymmetric Information and Individual Risk *

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Abstract

In economies with asymmetric information agents have private information on economically relevant variables: on individual states (economies with private information), on the action taken (moral hazard), on their type (adverse selection). We analyze competitive equilibria of these economies in the tradition of Prescott and Townsend ([9]). It is known that economies with adverse selection behave differently from the others: we clarify how and why.

Equilibria typically fail to exist for adverse selection, while they always exist for the other classes, even when there are aggregate states affecting all types.

The reason of this difference is in the structure of the incentive compatible trades. To see it, we first provide a unified treatment of the different types of economies. Once this is done, it becomes clear that all these economies share the property that efficient outcome require personalized (type dependent) prices, since they are all economies with individual risk (as in Malinvaud). Type dependent prices are not problematic in economies where types are known, since agents may be restricted to trade in the market corresponding to their type. In Adverse Selection Economies, this requirement becomes an additional constraint: agents of one type must not find convenient to trade in any other market. This requirement cannot be typically satisfied in economies where equilibria are efficient.
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1 Introduction

Different notions of equilibrium have been used to analyze economies with asymmetric information. This paper is in the line of work initiated by Prescott and Townsend ([9]). At the core of this project is the idea of extending the general equilibrium analysis of existence and Pareto optimality to the study of economies with asymmetric information. The problem is then to model asymmetric information

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economies as standard competitive economies where equilibria exist under classical assumptions and equilibrium allocations are efficient.

There are three main difficulties to be solved in carrying out this program. The first difficulty is that there are several types of economies where these problems arise: these include economies with moral hazard, with private information and with adverse selection. In the paper we express these different models as special cases of a general class of economies. This will make the comparison, and the characterization of what is specific of each economy particularly easy. A characteristic which is common to all these economies is that the feasibility constraint is aggregate: the precise form of this constraint is in the equation (3) below, but the crucial feature is that an increase in a personal state contingent consumption of individuals of different types has a different effect on feasibility, because the probability of that personal state is different for different types. Economies with individual risk (as introduced by Malinvaud, see ([7]), ([8])) are a special case of this general class. They are a special case because there is no private information. But in all the economies we mentioned the feasibility constraints are defined as averages of different terms that depend on types. A crucial feature of economies with individual risk is that efficiency requires personalized prices. The reason for this is briefly reviewed in section 2.4 below.

Incentive Compatibility Constraints

The second difficulty is represented by the nature of the exchange: private information on some variable makes *ex-ante* plans manipulable. This problem may be overcome by the restriction of the consumption set to the set of incentive compatible allocations. This restriction however makes the consumption set non-convex. Prescott and Townsend ([9]) first suggested that this non-convexity can be overcome by the introduction of lotteries over goods. The utility function of the individuals defined on the consumption set is extended by expected utility to the set of probability measures over it. This achieves two results. First, since the set of incentive compatible lotteries may strictly contain the convex hull of incentive compatible deterministic allocations, the introduction of lotteries may enhance the economic efficiency of equilibrium allocations. The second result is that the set of incentive compatible lotteries is convex (and, if expected utility is used, preferences are linear). However, even when the allocations are restricted to be deterministic, the large numbers can be used as a convexifying devices, (as in the classical analysis of non-convexities of Hildenbrand, ([6]), thereby playing the same technical role of lotteries.

The view that the difficulties for the classical theorems of Competitive General equilibrium (in particular existence and optimality) stem from the lack of convexity of the consumption set and by the particular form of the feasibility constraint is clearly wrong. Once the consumption set is identified with incentive compatible lotteries (or the large numbers exploited as a convexifying device) personalized, linear prices, as in the individual risk economy, clear the markets at efficient level of exchange. Under classical assumptions, in moral hazard and private information economies, competitive equilibria exist and yield efficient allocations. The introduction of aggregate uncertainty is purely notational and does not alter the analysis. The last finding is opposite to what claimed in Bennardo and Chiappori ([1]). In light of this paper, their conclusion is wrong.

Personalized prices and Adverse Selection Economies

A third difficulty however also exists, and it is clear in Adverse Selection Models. The difficulty is produced by two requirements colliding. The special form of the feasibility constraint requires personalized prices for efficiency: there is one price for each type. As long as type is public information, any asymmetric information other than Adverse Selection does not create special problems. Existence and efficiency of equilibria can be established, even in models with aggregate states.

But in model of Adverse Selection, types are private information. If prices are different for different types, any agent can trade in any of the markets he chooses. In other words, the competitive markets cannot solve the private information problem, and cannot force agents to reveal their private information. This second difficulty does not seem to have solution in the framework of a Competitive Equilibrium model. If one insists with the requirement that the Competitive Equilibria is constrained efficient, then Competitive equilibria do not exist. The last assertion needs some further qualification. Once the consumption set is identified with the set of lotteries, linearity in prices does not imply linearity in commodities, but rather in probabilities. Thus prices can be non-linear in commodity bundles. In pure exchange economies, non-linear prices support a plethora of, potentially inefficient, allocations. There are two substantially equivalent ways to get around this problem, i.e., to obtain efficient and determinate equilibrium allocations. Both of them work for moral hazard and private information and both of them fail for adverse selection economies.

First, the price domain is restricted to linear prices satisfying the two classical requisites of the Individual Risk analysis: I) prices for the delivery of commodities contingent on aggregate state, but independent of individual accidents, are linear and type invariant, and II) prices for the delivery of commodity bundles contingent on aggregate and personal states are obtained by multiplying the prices of point I) by the type dependent probability (eventually, conditional on some taken action) of the personal state. Alternatively, following Prescott and Townsend, the competitive model is augmented by the introduction of a firm. The latter is a profit-maximizing clearing-house that, exploiting the law of large numbers, supplies feasible measures over joint allocations. If the firm is allowed to supply signed measures, the zero profit condition, necessary for equilibrium, delivers the same price restrictions exogenously imposed with the first approach. Most importantly and independently of price linearity, if the production set of the firm contains the intersection of the feasible and incentive compatible sets, equilibrium allocations are efficient. This aspect collides with the nature of the private information in adverse selection economy and makes the equilibrium set potentially empty. This is independent of whether the economy is modeled as an economy with externalities, as in Bisin and Gottardi, ([2]), or as a competitive economy, as in Rustichini and Siconolfi ([13]).

Two points are essential: 1) the firm makes at equilibrium zero profits, and 2) within the incentive compatible and feasible set, different types have, typically, different preferred allocations. Types are private information so that they can try to buy their preferred allocation on every market. If they do not succeed, these allocations must be too expensive, but then the profit-maximizing firm should produce them. Thus, an equilibrium cannot exist. Of course, the existence of an equilibrium can be restored, by either imposing additional restrictions on trade or suppressing both the firm and the price linearity, as in Gale ([5]) or Dubey and Geanakoplos ([4]). However the price to be paid is loss of efficiency of the equilibrium allocations.

Outline

In section 2 we set up the basic model, in enough generality to include the different classes of economies with asymmetric information. The sections 3, 4 and 5 deal with economies where types are publicly observed. The section 6 analyzes economies with Adverse Selection.

2 Economies with Asymmetric Information

To keep the technical side of the analysis simple we study finite economies: the states and the trades are a finite set.

2.1 The economy

Individuals in the economy belong to one of a finite *set of types*, $I \equiv \{1, \ldots, i, \ldots, n\}$. For each type there is a large population of size λ^i, with $\lambda^i > 0$ and $\sum_{i \in I} \lambda^i = 1$. Each individual chooses an *action*, a, out of a finite set A. For example, in models of principal agent the action is the effort of the agent.

Individuals face aggregate and personal uncertainty. Ω is a finite set of *states of nature* that affect every agent in the economy. Each state ω occurs with a fixed probability $\rho(\omega)$. For example in insurance models a state may be a flood, or an earthquake. A *personal state* s out of a finite set S is realized, one for each individual. In insurance models an s may be an accident.

The probability of such realization depends on type, action and state of nature, as we now describe. For every finite set Y, $\Delta(Y)$ denotes the set of probability vectors on Y. For each type, action and aggregate state, there exists a probability vector over the set of personal states: that is, a $q^i(\cdot; a, \omega) \in \Delta(S)$ is given for every $i \in I$ and every $(a, \omega) \in A \times \Omega$. Individuals exchange goods according to a finite set of *individual net trades*, X, which is independent of types, states and actions; $0 \in X$, so no transfer is always an option. X is a subset of the Euclidean space R^L, where $L \geq 1$ is the number of physical commodities. The preferences of type i are represented by a utility function

$$v^i : A \times S \times \Omega \times X \to R, \text{ for every } i \in I.$$

2.2 Contracts

The set of *net trade policies* is the set Z of state contingent net trades. It is the finite set of maps $z : S \times \Omega \to X$. The set of *contracts* is the set C of pairs of action and net trade policy. A contract $c = (a, z)$ assigns an action a and stipulates the provision of a state contingent net trade z, which describes for every realization of the pair of states (s, ω) a net trade vector $z(s, \omega) \in X$. The set of contracts is finite, is type invariant and so it is its cardinality. The utility function v^i induces a utility function u^i over C that takes the expected utility form

$$u^i(c) = u^i(a, z) \equiv \sum_{\omega \in \Omega} \rho(\omega) \sum_{s \in S} q^i(s; a, \omega) v^i(a, s, \omega, z(s, \omega)).$$

Lotteries on deterministic contracts are also traded: a lottery τ is an element of $\Delta(C)$. A different description of a lottery is given by a pair (τ_1, τ_2) where $\tau_1 \in \Delta(A)$ and τ_2 is a vector $(\tau_2(\cdot; a)_{a \in A})$

of conditional probabilities on X, one for each action. The two descriptions are equivalent: for every $\tau \in \Delta(C)$ there is a pair (τ_1, τ_2) , and vice versa. A *lottery profile* is a vector

$$\sigma = (\sigma^1, \ldots, \sigma^i, \ldots, \sigma^n),$$

assigning the same lottery σ^i to each individual of type i. Individual utility functions are extended over the set of lotteries $\Delta(C)$, by assuming that they are linear in lotteries. Let U^i denote the row vector of dimension $1 \times C$ with entries $u^i(a, z)$, for every $(a, z) \in C$. Using the convention that individual lotteries are column vectors, the utility of an individual of type i generated by a lottery σ^i is

$$U^i \sigma^i \equiv \sum_{(a,z) \in C} u^i(a, z) \sigma^i(a, z). \tag{1}$$

All the economies we consider are economies with individual risk (as in the classical analysis of Malinvaud (see ([7]), ([8])), we model the individual risk with the variable s). The models differ for the information publicly available. This information may be different for three variables: the action a, the personal state s, and the type i. By making each of these variables private information of the individuals we obtain different types of economies.

2.3 Feasible Lottery Profiles

By the law of large numbers, a fraction $q^i(s; a, \omega)$ of type i individuals that have adopted the action a is at each aggregate state ω in personal state s. Thus a lottery profile $\sigma = (\sigma^i)_{i \in I}$ is *feasible* if for every commodity ℓ and aggregate state ω the sum of net trades is not positive:

$$\sum_{i \in I} \lambda^i \sum_{(a,z) \in Z} \sigma^i(a, z) \sum_{s \in S} q^i(s; a, \omega) z_\ell(s, \omega) \leq 0. \tag{2}$$

To have a more compact notation, for $(a, z) \in C$, $\omega \in \Omega$ and $i \in I$, let

$$T^i((a, z); \omega) \equiv \sum_{s \in S} q^i(s; a, \omega) z(s, \omega)$$

be the column vector of dimension $L \times 1$ of type i aggregate net trade in state ω generated by (a, z). Let $T^i(\omega)$ to be the matrix of dimension $L \times \sharp C$ whose columns are the vectors $T^i((a, z); \omega)$, $(a, z) \in C$, and finally let T^i be the matrix of dimension $L \sharp \Omega \times \sharp C$ obtained by stacking together the matrices $T^i(\omega)$, $\omega \in \Omega$. Thus, the feasibility condition (2) can be rewritten as:

$$\sum_{i \in I} \lambda^i T^i \sigma^i \leq 0. \tag{3}$$

where $0 \in R^{L\Omega}$ and, for $\omega \in \Omega$,

$$\sum_{i \in I} \lambda^i T^i(\omega) \sigma^i \equiv \sum_{i \in I} \lambda^i \sum_{(a,z) \in C} \sigma^i(a, z) \sum_{s \in S} q^i(s; a, \omega) z(s, \omega) \in R^L.$$

2.4 Prices in Economies with Individual Risk

We begin with a simple exposition adapted to our problem, of the main idea of economies with individual risk. We use the simplest setup to keep the main idea in focus. Consider an economy where all information is public, so there are no incentive compatibility constraints. An efficient lottery profile is determined as solution of the problem (with $\alpha \in \Delta(I)$):

$$\max_{\{\sigma=(\sigma^i)_{i\in I}\}} \sum_{i\in I} \alpha^i U^i \sigma^i \tag{4}$$

subject to

$$\sum_{i\in I} \lambda^i T^i \sigma^i \leq 0.$$

Let π^i be the multiplier for the constraint that the vector σ is a probability vector, and $\nu \equiv (((\nu_\ell(\omega))_{\ell=1}^L)_{\omega\in\Omega}$ the multipliers for the feasibility constraint. For $\nu(\omega) \equiv (\nu_\ell(\omega))_{\ell=1}^L$, the first order conditions give

$$u^i(c) \leq \pi^i + \lambda^i \sum_{\omega} v(\omega) T^i(c, \omega), \tag{5}$$

with equality for every c for which the optimal $\hat{\sigma}$ has $\hat{\sigma}_c > 0$. If we compare the conditions (5) with the first order condition of the individual of type i in the competitive economy, we find that prices that support the efficient allocation have to be different for every type, that is have to be i dependent; and of the form

$$p^i = \lambda^i \nu T^i.$$

In conclusion, it is clear that in economies with individual risk prices have to be type dependent. A vector of *type dependent prices* is a vector $p \equiv (p^i)_{I\in I}$: each p^i is a vector in R^C, where the entry $p^i(a, z)$ defines the value of the contract (a, z).

2.5 Information and Time

The complete time sequence of events is the following. The type i of each individual is revealed to him, and the public information available on the type is revealed. Then individuals trade, and get a lottery τ. The action a is chosen according to the lottery τ_1, and this outcome is communicated to him. The individual chooses the action b, possibly different from a. The public information available on the chosen action b is revealed. Then first the state of nature ω is determined, and the personal state is realized, according to $q(\cdot; a, \omega)$. The personal state s is communicated, and the public information on s is revealed. Then individuals report the personal state t and type j, possibly different from s and i respectively. Finally the aggregate state is communicated, and the transfer made.

2.6 Four Types of Asymmetric Information

The public information is, in different models, either completely revealing (the variable is observed) or completely non revealing. When a variable is publicly observed, individuals have to be truthful: the action chosen is the prescribed action if action is observed, the reported personal state is the true state if the state is observed, and the reported type is the true type if the type is observed. We now describe in detail four different types of economies: for each one there is a corresponding incentive compatibility constraint on the set of lottery profiles.

2.6.1 Individual Risk

This is the basic model. In this economy all variables (action chosen by the individual, personal states and types) are observed, so there is no incentive compatibility constraint. For each type i the set of incentive compatible lotteries is

$$IC(IR)^i \equiv \Delta(C), \text{ for } i \in I$$

and the set of incentive compatible strategy profiles is the product over types of the set of lotteries $IC(IR) = \times_{i \in I} IC(IR)^i$. It is well known (see ([7]), ([8]), ([3])) that an equilibrium with *type dependent prices*, where each type i faces a different price p^i, exists.

2.6.2 Moral Hazard

In these economies the type and personal states are observed, while the action is private information. The set of incentive compatible lotteries for type i, $IC(MH)^i$, is the set of lotteries such that the individual of each type indeed prefers the action assigned by the lottery to any other action. Recall that the prescribed action is communicated, and the action chosen, before the net trade policy is revealed: so the individual can make his choice of action depend only on the prescribed action. Let $\Phi(MH)$ be the finite set of all functions from A to A. Each ϕ corresponds to a deviation from the prescribed action. For $\phi \in \Phi(MH)$, define the new vector $U^i(\phi)$ by

$$U^i(\phi)(a, z) = u^i(\phi(a), z).$$

Then, the set of incentive compatible lotteries for type i can be written as:

$$IC(MH)^i \equiv \{\tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi(MH)\}, \tag{6}$$

and it is non-empty, closed and convex subset of $\Delta(C)$. The set of incentive compatible lottery profiles is the product of the set of incentive compatible lotteries for the different types: that is $IC(MH) \equiv \times_{i \in I} IC(MH)^i$.

2.6.3 Private Information

In these economies the type and action are observed, the realization of the personal state is not. Individuals can misreport the personal state realization, given the information they have on the action. So for any function $\phi : A \times S \to S$ they can transform a contract (a, z) into a contract $(a, z(\phi(a, \cdot)))$ where $z(\phi(a, \cdot))(\omega, s) \equiv z(\omega, \phi(a, s))$. Let $\Phi(PI)$ be the finite set of all functions from S to S. For any $\phi \in \Phi(PI)$, let

$$U^i(\phi)(a, z) = u^i(a, z(\phi(a, \cdot))).$$

The set of incentive compatible lotteries for type i can be written as:

$$IC(PI)^i \equiv \{\tau : (U^i(\phi) - U^i)\tau \leq 0, \text{ for every } \phi \in \Phi(PI)\}. \tag{7}$$

This set too is non-empty, closed and convex. The set of incentive compatible lottery profiles is the product of the set of incentive compatible lotteries for the different types: that is $IC(PI) \equiv \times_{i \in I} IC(PI)^i$.

2.6.4 Adverse Selection

In these economies the type of the individual is not observed. The incentive compatibility constraint is a joint restriction on the set of lottery profiles. A lottery profile $\sigma = (\sigma^i)_{i \in I}$ is incentive compatible if out of the set of lotteries $(\sigma^i)_{i \in I}$ each type i chooses the lottery σ^i, that is if for every type i

$$U^i(\sigma^j - \sigma^i) \leq 0, \text{ for every } j \in I \tag{8}$$

The set $IC(AS)$ of incentive compatible lottery profiles is the set of σ's that satisfy the condition (8) for every i. This set is non-empty, closed and convex: but there is no meaningful way to define a set of incentive compatible lotteries for a single type, as we have done for $IC(MH)^i$ and $IC(PI)^i$. To simplify notation, we write the incentive compatibility constraints as $B^i(k)\sigma^i \leq 0$, for $k \in \{IR, MH, PI\}$. The matrices $B^i(k)$ are defined by the coefficients in the incentive compatibility constraints (6) for the Moral Hazard, by (7) for the Private information economy. The constraints are vacuous for Individual Risk economies. For the economies with Adverse Selection, we write the incentive compatibility constraints as $B\sigma \leq 0$, where the coefficient in inequalities (8) define the entries of the matrix B. The concepts we have defined extend naturally to deterministic contracts. A deterministic contract is incentive compatible if the corresponding degenerate lottery assigning that contract for sure is incentive compatible.

3 Competitive Equilibria with observed types

We analyze first the economies in the set of economies with observed types, which includes economies with Individual Risk, Moral Hazard and Private Information, modeling them as pure exchange general equilibrium environments.

3.1 The Pure Exchange Economy

In the pure exchange economy individual trade over lotteries is constrained to take place in the incentive compatible set. Each individual of type i is constrained to trade in the i-th market, that is to choose a lottery in the set $IC(k)^i$, for $k \in \{IR, MH, PI\}$, and pay it at the price p^i. Thus the consumption set of type i individuals in the k economy is

$$\Delta(C) \cap \{\tau : B^i(k)\tau \leq 0\}.$$

The domain of individual prices is the set of linear functionals over lotteries, described by vectors in R^C. Even though prices are linear in lotteries, they do not need to be linear or even affine in commodities. Thus, for instance, if both (a, z) and $(a, 2z)$ are in C, $p^i(a, 2z)$ may be different from $2p^i(a, z)$, and $p^i(a, 0)$ maybe different from 0. We narrow down the price domain by following the personalized pricing rule in the classical analysis of the Individual Risk Economies (([7]), ([8]), ([3])). Prices in this restricted domain are defined by two elements. First let $\theta \in R^{L\Omega}$ be a type invariant, commodity price contingent on the state of nature: $\theta_{\ell\omega}$ is the type-invariant price of one unit of commodity ℓ to be delivered in aggregate state ω, independently of the realization of individual states. Then each pricing of the state contingent commodities extends to a price of pairs of action-contingent commodities (that is contingent on aggregate and personal states) according to the rule

$$p^i_{\ell s\omega}(a) = q^i(s; a, \omega)\theta_{\ell\omega}.$$

Hence the restricted price domain is:

$$P = \{(p^i)_{i\in I} \in R^{CI} : p^i = \theta T^i \text{ for } \theta \in R^{L\Omega}_+ \setminus \{0\}\}. \tag{9}$$

We call "fair" a price in P, because the type i's personalized value of a contract $(a, z) \in C$,

$$p^i(a, z) = \sum_{\ell=1}^{L} \sum_{\omega\in\Omega} \theta_{\ell\omega}\rho(\omega) \sum_{s} q^i(s; a, \omega)z(s, \omega)$$

is proportional to the effective net trades it generates. The latter depends on the action a as well as on the type dependent probability vectors q^i.

Definition 1 *A competitive equilibrium for a pure exchange economy $k \in \{IR, MH, PI\}$ is a pair $(\hat{p}, \hat{\sigma})$ of price and allocation, with $\hat{p} \in P$, such that:*

1. *The lottery profile $\hat{\sigma}$ is feasible, according to the definition (3);*
2. *For every type i, the lottery $\hat{\sigma}^i$ is the solution of the consumer problem:*

$$\max_{\tau\in\Delta(C)\cap\{\tau':B^i(k)\tau'\leq 0\}} U^i\tau, \text{ subject to } \hat{p}^i\tau \leq 0. \tag{10}$$

As usual, the existence of a competitive equilibrium and the First Fundamental Theorem of Welfare Economics require some minimal assumption. Given the linearity of the preferences, we just need to make sure that local non satiation and the minimum wealth conditions, for $p \in P$, are satisfied. These conditions are stated formally in the following two assumptions. For two vectors x and y we write $x \gg y$ to indicate that x is strictly larger than y in every component.

Assumption 2 ***(Local non satiation)*** *There is a net trade $\hat{z}^1 \gg 0$, $\hat{z}^1 \in X$, and an action $\hat{a} \in A$ such that for every $i \in I$, every $(\omega, s) \in \Omega \times S$, and every $(a, z) \in Z$, if $(a, z(s, \omega)) \neq (\hat{a}, \hat{z}^1)$, then:*

$$v^i(\hat{z}^1, \hat{a}, s, \omega) > v^i(z(s, \omega), a, s, \omega).$$

Assumption 3 ***(Minimum wealth condition)*** *There is a net trade $\hat{z}^2 \in X$ with $\hat{z}^2 \ll 0$.*

The vector of net trades defined in the two conditions of the two assumptions (3) and (2) do not depend on either state of nature or personal state. We define two net trade policies, still denoted by $\hat{z}^k$, for $k = 1, 2$, by $\hat{z}^k(\omega, s) \equiv$

z^k. Take any action $a \in A$: the deterministic contracts $(a, \hat{z}^k)$, for $k = 1, 2$, are incentive compatible contract in the Private Information economy, since different reports on the personal state do not affect the net trade. Similarly, for some $a^k \in A$, $k = 1, 2$, the contracts $(a^k, \hat{z}^k)$ are incentive compatible in the Moral Hazard economy. The condition of local non-satiation implies that the economy is not satiated within the set of incentive compatible and feasible allocations. The minimum wealth condition guarantees that the minimum wealth condition is satisfied for $p \in P$, since $0 > p^i \hat{z}^2$. If the latter condition holds, the optimal consumption problem of an individual of type i satisfies all the assumptions of the maximum theorem and, hence, the demand for lottery of any type i individual is a non-empty, compact-valued and upper-hemi continuous correspondence for all $p \in P$. Since the utility is linear in lotteries, it is convex-valued as well.

Theorem 4 *Under assumption (3), a competitive equilibrium exists for every economy with observed types.*

Proof. Let $\Theta = \{\theta \in R_+^{L\Omega} : \sum_{\omega\in\Omega} \sum_{\ell=1}^{L} \theta_{\ell\omega} = 1\}$. For $\theta \in \Theta$, the pricing map p_θ is defined as:

$$p_\theta^i \equiv \theta T^i \tag{11}$$

Also let the correspondence Φ from the product $\Theta \times \Delta(C)^I$ to itself be defined by $\Phi = \Phi_1 \times \Phi_2$ where

$$\Phi_1(\theta, \sigma) \equiv \text{ argmax }_{\eta\in\Theta} \sum_i \lambda^i (p_\eta^i \sigma^i) \tag{12}$$

and

$$\Phi_2^i(\theta, \sigma) \equiv \text{ argmax }_{\xi\in\Delta(C)\cap\{\tau: B^i(k)\tau\leq 0\}} U^i\xi \text{ subject to } p_\theta^i \xi \leq 0 \tag{13}$$

The conditions of Kakutani's fixed point theorem are satisfied. Take a fixed point $(\hat{\theta}, \hat{\sigma})$: we claim that the pair $(p_{\hat{\theta}}, \hat{\sigma})$ is a competitive equilibrium. The optimality of consumer's choice follows from the definition of Φ_2^i for every i, so the condition (10) is satisfied. Also the allocation $\hat{\sigma}$ is feasible. The budget constraint for every type i insures that

$$\hat{\theta} T^i \hat{\sigma}^i \leq 0 \tag{14}$$

Suppose now that for some pair (ℓ, ω)

$$\sum_{i\in I} \lambda^i \sum_{s,a,z} z_\ell(s, \omega) q^i(s; a, \omega) \hat{\sigma}^i(a, z) > 0$$

Then $\max_{\tau\in\Theta} \sum_i \lambda^i (p_\tau^i \hat{\sigma}^i) = \sum_i \lambda^i \hat{\theta} T^i \hat{\sigma}^i > 0$, a contradiction with (14). ■

If the local non-satiation condition in assumption (2) holds then at any feasible allocations each individual is locally non-satiated and, at equilibrium, the budget constraints of the individuals are satisfied with an equality. Thus, each competitive allocation $\hat{\sigma}$ is constrained Pareto optimal, that is there is no other feasible and incentive compatible allocation τ such that,

$$U^i(\tau^i) \geq U^i(\hat{\sigma}^i) \text{ for all } i, \text{ with at least one strict inequality.}$$

This is proved in the next theorem.

Theorem 5 *Under assumption (2), competitive allocations of the exchange economy are constrained efficient.*

Proof. Suppose, by contradiction, that a feasible and incentive compatible allocation $\tau = (\tau_i)_{i \in I}$ Pareto dominates the equilibrium allocation. By the local non-satiation assumption (2), at the competitive price $\hat{p}$, $\hat{p}^i \tau^i \geq 0$, for all i , with at least one strict inequality. Since $\hat{p} \in P$, $\hat{p}^i = \hat{\theta} T^i$. Multiply the budget constraint of every type by λ^i and add to get:

$$\hat{\theta} \sum_i \lambda^i T^i \tau^i > 0$$

and therefore τ is not feasible. ∎

4 The unconstrained economy

In the economic environment studied so far the consumption set of the individuals is identified with the set of incentive compatible lotteries over contracts in C, i.e., with $\Delta(C) \cap \{\sigma^i : B^i \sigma^i \leq 0\}$. This definition makes economies with moral hazard and private information special cases of standard general equilibrium economies with individual risk: this is the substance of theorems (4) and (5). This, however, requires an exogenous restriction on consumption sets. Is the price system alone, in absence of incentive compatibility restrictions on trade, able to coordinate efficiently environments with asymmetric information? This question naturally leads to an alternative formulation of the competitive economy, which does not have these restrictions on the consumption sets. We call this new environment the unconstrained economy. In these economies the consumption set of the individuals is $\Delta(C)$.

Quite naturally we are led to the following definition of equilibrium. For convenience we denote by $\Phi(IR)$ the set of misrepresentation functions for the Individual Risk economies, which is of course the empty set, since there is nothing to misrepresent.

Definition 6 *An equilibrium of the unconstrained pure exchange economy with observed types is a pair $(\hat{p}, \hat{\sigma})$, with $\hat{p} \in R^C_+$, such that:*

1. $(id, \hat{\sigma}^i) \in \arg\max_{\phi \in \Phi(k), \sigma \in \Delta(C)} U^i(\phi)\sigma^i$ *subject to* $\hat{p}^i \sigma^i \leq 0$ *, for* $k \in \{IR, MH, PI\}$*,*
2. *markets clear:* $\Sigma_i \lambda^i T^i \hat{\sigma}^i \leq 0$*.*

Given the freedom of choice of the price maps an equilibrium exists. For instance, whenever $0 \in X$, the price maps $p^i(a,z) = z' z$ support as a competitive equilibrium the allocation $(\delta^i_{(\hat{a}^i,0)})_{i \in I}$, for $\hat{a}^i \in \arg\max_{a \in A} U^i \delta_{(a,0)}$. It is clear that this unrestricted notion of equilibrium is of little interest. Thus, we are led to investigate how much we can support by imposing fairness to the price system. In defining a fair price system for the unconstrained economy we face a difficulty. Within the incentive compatible set the behavior of the unconstrained and exchange economy is identical. Therefore, the problem is to price lotteries outside the incentive compatible set. The difficulty here stems from the fact that a lottery is priced through the values assigned to the deterministic contracts by the price system, i.e., by the linearity in probabilities of the prices. We define a *modified fair price region. Modified* stands for the fact that the market assigns prices to non incentive compatible contracts according to how they are optimally manipulated. With this price restriction we obtain equilibrium allocations that satisfy a weak notion of constrained Pareto optimality: they are efficient within the set of probability distributions over deterministic, incentive compatible contracts. Now the details. For each deterministic contract $c \in C$, let $\phi^{*i}(c)$ be the set of utility maximizing manipulations of individual i, i.e.

$$\phi^{*i}(c) = \arg\max_{\phi \in \Phi(k)} U^i(\phi)\delta_c$$

Let c_ϕ denotes the manipulation of a given contract c by the function $\phi \in \Phi_k$. Evidently, for $c \in C$, c_ϕ, $\phi \in \phi^{*i}(c)$, is an incentive compatible contract for i. Let $C_k^i(IC)$ be the set of incentive compatible deterministic contracts, i.e., $C_k^i(IC) = \{c \in C : id \in \phi^{*i}(c)\}$, a non-empty set. Let f^i be an arbitrary selection that assigns to $c \in C \backslash C_k^i(IC)$ a contract $f^i(c) \in c_{\phi^{*i}(c)}$, and assigns to any $c \in C_k^i(IC)$, c itself. So we define

$$f^i(c) = c \text{ for } c \in C_k^i(IC) \tag{15}$$

and

$$f^i(c) = c' \in c_{\phi^{*i}(c)}, \text{ for } \in C \backslash C_k^i(IC)$$

Evidently, whenever individual preferences are strict, $\phi^{*i}(c)$ is a singleton and, therefore, $f^i(c) = c_{\phi^{*i}(c)}$ is unique. Let $f = (f^i)_{i \in I}$ be a selection profile and $F = \{f : \text{ for every } i, f^i \text{ satisfies (15)}\}$. The modified fair price region for the unconstrained economy is defined as:

$$P^U = \{p : p^i(c) = \theta T^i(f^i(c)), f \in F \text{ and } \theta \in R_+^{\Omega L} \backslash \{0\}\}.$$

Proposition 7 *Under assumption (3), there always exists an equilibrium of the unconstrained economy $(\hat{p}, \hat{\sigma})$, $\hat{p} \in P^U$, such that there is no feasible allocation $\bar{\sigma}$, with the support of $\bar{\sigma}^i$, $S(\bar{\sigma}^i) \subset C_k^i(IC)$, for all i, that Pareto dominates $\hat{\sigma}$.*

The proof is in the appendix, section 8.1.

As we have already said there might not exist price systems $p \in R^C$ able to decentralize constrained efficient allocation. Thus in absence of significant restrictions, it is impossible to improve upon the last proposition. In order to illustrate the problem, let us define

$$V^i(\sigma^i) = \max_{\phi \in \Phi(k)} U^i(\phi)\sigma^i, \sigma^i \in \Delta(C) \text{ and } k \in \{MH, PI\}.$$

V^i is, by construction, the upper envelope of the (finite) family of linear maps $U^i(\phi)\sigma^i$ and, thus, may fail to be concave. Consider an equilibrium allocation of the constrained economy, σ^*. By theorem (5), σ^* is a constrained efficient allocation. However, if σ^{*i} is, for some i in a non concave area of the map V^i, the unconstrained economy may fail to have linear prices supporting σ^* as a competitive equilibrium. For this to happen, two conditions are necessary : i) the incentive compatibility constraints are strictly binding and ii) the support of the lottery $\sigma^{*i}, S(\sigma^{*i})$, contains non-incentive compatible, deterministic contracts, i.e., $S(\sigma^{*i}) \cap C_k^i(IC) \neq S(\sigma^{*i})$, for some type i.

If $S(\sigma^{*i}) \cap C_k^i(IC) = S(\sigma^{*i})$, by the last theorem, there exists an equilibrium and, thus, $U^i\sigma^i$ coincides with the concave regularization of V^i at σ^{*i}. While if the incentive compatibility constraints are not binding, there obviously exists an equilibrium. The example that follows is clearly robust and shows an unconstrained economy with private information without constrained efficient equilibrium allocations. The example may be easily reformulated for a moral hazard scenario.

Example 8 *There are unconstrained economies without constrained efficient equilibrium allocation.*

The proof is in the Appendix, section (8.2).

5 The Production Economy

In the previous sections we have shown the existence of competitive equilibria supported by fair prices in the pure exchange economy Although intuitively appealing, those price restrictions are arbitrary. On the other hand, some restriction is necessary. Without restricting the price domain, any feasible and incentive compatible allocation, $(\hat{\sigma}^i)_{i\in I}$, is price supportable as a competitive equilibrium allocation: it suffices to define $\hat{p}^i(a,z) = u^i(a,z) - U^i\hat{\sigma}^i$. At $\hat{p}$, every lottery preferred to $\hat{\sigma}^i$ is more expensive and, thus, $\hat{p}$ supports $\hat{\sigma}$ as a competitive. Hence, the question is to find economically meaningful ways to restrict endogenously the set of prices compatible with equilibrium. We introduce a firm, which is a price taking profit maximizing institution. Its production set, Y, coincides with the set of collections of individual vectors $\beta = (\beta^i)_{i\in I} \in R^{CI}$ that are feasible, that is,

$$Y \equiv \{\beta = (\beta^i)_{i\in I} \in R^{CI} : \sum_i T^i\beta^i \leq 0\}. \tag{16}$$

Note that the β's are not required to be non negative, or to add to one. This the formalization of a competitive economy with asymmetric information in Prescott and Townsend ([9]) and ([**?**]). The next step is to adjust the definition of equilibrium to take into account the presence of the firm. We call this concept, *strong equilibrium.*

Definition 9 *A strong equilibrium of the k economy (where k is one of $\{IR, MH, PI\}$) is a pair $(\hat{p}, \hat{\sigma})$ such that:*

1. $\hat{p} = (\hat{p}^i)_{i\in I}$, *with* $\hat{p}^i \in R^C$*;*

2. *For every type i, the lottery $\hat{\sigma}^i$ is the solution of the consumer problem:*

$$\max_{\tau\in\Delta(C)\cap\{\tau' : B^i(k)\tau'\leq 0\}} U^i\tau, \text{ subject to } \hat{p}^i\tau \leq 0.$$

3. $(\lambda^i\sigma^{*i})_{i\in I}$ *is profit maximizing, i.e.,*

$$(\lambda^i\sigma^{*i})_{i\in I} \in \arg\max_{\beta}\{\sum_i \hat{p}^i\beta^i \text{ subject to } \beta \in Y.\}$$

Since the production set of the firm is linear, in both economies, at equilibrium profits must be zero. Thus, the Farkas's alternative theorem provides an immediate characterization of the price restriction induced by the presence of the firm:

$$\sum_i \hat{p}^i\beta^i > 0 \text{ and } \sum_i T^i\beta^i \leq 0$$

does not have a solution if and only if

$$\hat{p}^i = \theta T^i$$

for all $i \in I$ and $\theta \geq 0$. Once local non satiation is taken into account, at equilibrium $\theta > 0$ so that the existence of a strong equilibrium restricts $p \in P$. This is stated formally in the next theorem.

Theorem 10 *Under assumption (2), the set of strong equilibria of the economy is identical to the set of equilibria of the pure exchange economy.*

Proof. It is obvious that a strong equilibrium is an equilibrium of the pure exchange economy. Hence, we just have to show the converse statement. Consider an equilibrium of the pure exchange economy, $(\hat{p}, \hat{\sigma})$. By definition, $\hat{p} \in P$ and by local non satiation, $\hat{p}^i \hat{\sigma}^i = 0$, for all i. Thus, $(\lambda^i \hat{\sigma}^i)_{i \in I} \in Y$, and it yields the maximum profit (equal to zero). ■

Thus, the efficiency of competitive equilibria requires either that we restrict *ex-ante*, in the pure exchange economy, the price domain or equivalently that we introduce a profit maximizing firm. If we insist in looking for equilibria with type independent prices, we must bear the cost of banning heterogeneity from the model. Every time, the linear functionals defining the feasible set are $i-dependent$, competitive equilibrium calls for personalized pricing. This has nothing to do with asymmetric information, but rather with the particular form of the feasibility requirements in large economies: It is a lesson that we have learned from the model of individual risk of Malinvaud. This is not a problem for competitive analysis, once the competitive model is enriched to account for personalized pricing. Most importantly, all the economies considered in our analysis have the same form of the feasible set.

5.1 Moral hazard with aggregate states.

As an application of the previous result, we consider the model of Bennardo and Chiappori (see ([1])). Their model is a special case of the Moral Hazard Economy that we have described, with the additional restriction that $\Omega \equiv \{1, 2\}$, that there is only one type (as standard in Moral Hazard problems) and one physical commodity, i.e., $L = 1$. The equilibrium concept is our strong equilibrium.

Proposition 11 *The economy of Bennardo and Chiappori has a competitive equilibrium.*

Proof. Since there is only one type we drop the index i in p^i and T^i. Bennardo and Chiappori consider a Constrained Pareto Efficient allocation $\hat{\sigma}$, such that

$$T(1)\hat{\sigma} < 0 \text{ and } T(2)\hat{\sigma} = 0$$

and claim that no Walrasian equilibrium exists which supports this allocation. We prove that an equilibrium exists, constructing it along the lines of the proof of the theorem. The equilibrium price $\hat{p}$ is conjectured to be

$$\hat{p} = T(2) \tag{17}$$

that is $\hat{\theta} = (0, 1)$. We claim that $(\hat{p}, \hat{\sigma})$ is a competitive equilibrium. If not, then σ^* be a solution of the consumer problem at price $\hat{p}$, such that:

$$\hat{p}\sigma^* = T(2)\sigma^* \leq 0 \tag{18}$$

and $U\hat{\sigma} < U\sigma^*$. But then the allocation

$$\sigma^\epsilon \equiv (1-\epsilon)\hat{\sigma} + \epsilon\sigma^*$$

is incentive compatible (because this set is convex, and the two allocations $\hat{\sigma}$ and σ^* are in the set), feasible in the state 2 (since both allocations are), and feasible at 1 for ϵ small enough, but positive (since $T(1)\hat{\sigma} < 0$. For this positive ϵ the corresponding allocation σ^ϵ is feasible, and $U\sigma^\epsilon > U\hat{\sigma}$, contradicting the hypothesis that $\hat{\sigma}$ is constrained Pareto efficient, a contradiction. ∎

6 Equilibrium with Adverse Selection

In economies with observed types the restriction on lottery profiles induced by the incentive compatibility constraints are defined by restrictions on the consumption sets of each type. The equilibrium is defined by requiring that each individual of type i chooses a lottery in the set of incentive compatible allocations for that type. On the contrary, in economies with adverse selection the incentive compatibility constraints restrict allocation jointly across types: so in the definition of competitive equilibrium one cannot constrain the individuals of a particular type to choose a lottery in the set of incentive compatible lotteries for that type.

6.1 Weak equilibria

In the next definition, we let each consumer choose the type he declares.

Definition 12 ***Weak Competitive Equilibria*** *A weak competitive equilibrium for an economy with Adverse Selection is a pair $(\hat{p}, \hat{\sigma})$ of price and lottery profile such that:*

1. *$\hat{\sigma}$ is feasible (according to equation (3));*

2. *For every type i, the lottery $\hat{\sigma}^i$ is the solution of the consumer problem:*

$$\max_{\tau \in F(\hat{p})} U^i \tau, \tag{19}$$

where U^i is defined in equation (1), and

$$F(p) \equiv \cup_{j \in I} \{\tau : \hat{p}^j \tau \leq 0, \} \tag{20}$$

A large number of allocations are typically weak equilibrium allocations. Given the linearity in lotteries of both the utility function and the pricing system, the characterization of weakly supporting prices is nothing else than the solution to a linear system of inequalities. The theorems of the Alternative provide the technical tool for the characterization. It suffices to say that any feasible, deterministic and incentive compatible allocation $(\delta_{(a^i,z^i)})_{i \in I}$ is a weak equilibrium allocation. The price $\bar{p} = p^i$, for all i, defined as $\bar{p}(a,z) = \max_i (U^i(\delta_{(a^i,z^i)}) - U^i(\delta_{a,z}))$, is a supporting price.

6.2 Strong Equilibria

There are many, alternative modeling strategies for describing general equilibrium economies with adverse selection. In ([15]), we analyze all of them. It suffices to say that when the model has non-empty set of equilibria, equilibrium allocations may be inefficient, while when the model produces efficient equilibria, the equilibrium set may be empty. Both phenomena are robust. Here, we focus on the modeling strategy used so far for economies $k \in \{IR, MH, PI\}$. Thus, we look for price restrictions or strong notions of equilibrium that support constrained efficient allocations of the adverse selection economy. By theorem (10), the two paths are equivalent and we just investigate the search for a strong equilibrium. Two possibilities seem natural. In both cases the equilibria are Pareto efficient. In both cases, equilibria typically do not exist.

In a first definition of strong equilibrium, used in Rustichini and Siconolfi ([14]), the incentive compatibility constraints are removed from the consumption set of the individuals. Prices are type dependent, $p = (p^i)_{i \in I}$, and are linear over individual lotteries. Since the type is private information, consumers are free to select the price at which they optimally trade. Since individual trades are not restricted in the incentive compatible set, the firm in order to correctly compute profits (as well as the excess demand generated by its supply) restricts supply within the incentive compatible set. Thus, we define the production set of the firm, Y^* as the set of feasible and incentive compatible probability vectors, that is,

$$Y^* = \{\beta \in \Delta(C)^I : B\beta \leq 0 \text{ and } \sum_i \lambda^i T^i \beta^i \leq 0\}, \tag{21}$$

Note that we have restricted the component β^i to be lotteries on C rather than vectors in R^C. This is made necessary by the incentive compatibility constraints. With adverse selection, the latter are of the from $U^i\sigma^i \geq U^i\sigma^j$, for all i and j. The inequalities are meaningful only if σ^i is a probability vector, for $i \in I$.

Definition 13 ***Strong Competitive Equilibrium.*** *A strong competitive equilibrium is a pair of prices and a lottery profile $(\hat{p}, \hat{\sigma})$ such that:*

1. *$(\hat{\sigma}^i)_{i \in I}$ is an optimal solution to the firm profit maximization problem:*

$$\max_{\beta \in \Delta(C)^I} \sum_{i \in I} \lambda^i p^i \beta^i, \beta \in Y^* \tag{22}$$

2. *$(i, \hat{\sigma}^i)$ is an optimal solution to*

$$\max_{(j,\tau) \in I \times \Delta(C)} U^i \tau, \text{ subject to } p^j \tau \leq 0 \tag{23}$$

A second definition recognizes that the nature of the incentive compatible constraints induces a (consumption) externality. The economy is represented as a Lindhal economy with a profit maximizing firm. Individuals select joint lotteries $\mu = \mu(j)_{j \in I} \in \Delta(C)^I$, where $\mu(j)$ is the lottery assigned to type j individuals. Only trade in incentive compatible lottery profiles is allowed. As in any Lindhal equilibrium, a price q is an array of personalized prices linear in lotteries, $q = (q^1, \ldots, q^n)$, with $q^i = (q^i(j))_{j \in I} \in R^{CI}$. Since types are private information, the consumers are free to select the price at which they optimally trade. The domain of the consumer utility function is the set of individual contracts and not of their collection. Thus the value of the utility function of an individual of any type acquiring the vector $\xi \in \Delta(C)^I$ in the j-th market is determined by the $j-th$ component $\xi(j)$: an individual consumes a lottery designed for the stated type.

The production set of the firm is the set Y^*. There are two good reasons to do so. First, as we are going to prove later, with this formalization, the set of strong Lindhal equilibrium allocations contains the set of strong competitive allocations. This property allows for a unique non existence argument, i.e., we do not need distinct arguments for distinct specifications of the economy. Second, in the spirit of Prescott and Townsend ([9]) and Bisin and Gottardi ([2]), we could remove the incentive compatible constraints from Y^* (as well as the sign restrictions on β). Independently of the definition adopted, at equilibrium, allocations must be both incentive compatible and probability vectors, i.e., elements of Y^*. Thus, the set of strong Lindhal equilibria of the economy with no incentive compatibility and no sign restrictions in the production set is a subset of the strong equilibria of our Lindhal economy. Hence, by showing that the latter is empty we show that so it is the former.

Definition 14 ***(Strong Lindhal Equilibrium)***[1] *A Strong Lindhal equilibrium is a pair of prices and allocation* $(\hat{q}, \hat{\rho})$, *with* $q = (q^i)_{i\in I}$, $q^i = (q^i(j))_{j\in I}$, $\rho = (\rho^i)_{i\in I}$, *and* $\rho^i = (\rho^i(j))_{j\in I}, \rho^i(j) \in \Delta(C)$, *such that:*

1. *for some* $\hat{\beta} \in \Delta(C)^I$, *and for every* i, $\hat{\rho}^i = \hat{\beta}$;
2. $\hat{\beta}$ *is an optimal solution to*
$$\max_{\beta\in Y^*}(\sum_{i\in I} \lambda^i q^i)\beta; \tag{24}$$
and
3. $(i, \hat{\rho}^i)$ *is an optimal solution to*
$$\max_{(j,\xi)\in I\times\Delta(C)^I} U^i\xi(j), \text{ subject to } \sum_{k\in I} q^j(k)\xi(k) \leq 0, B\xi \leq 0 \tag{25}$$

Both notions of equilibrium require the introduction of personalized prices, but trades takes place over joint lotteries in the Lindhal notion and over individual lotteries in the competitive one. Furthermore, at equilibrium (with both notions), the firm makes zero profits. This is an immediate consequence of the definition of the production set and the market clearing condition.

The set of strong Lindhal equilibrium allocations contains the set of strong competitive equilibrium allocations. Thus if the former is empty so it is the latter. We map a competitive equilibrium price

[1] In adverse selection economies, the externality is generated by the incentive compatibility constraints. We have formulated the latter in terms of net trades, rather than consumption bundles. Thus, we view adverse selection economies as economies with "net trade externalities." Obviously, the initial endowments of a type over both its own as well as others net trades are zero. This explains the adopted formulation of the budget set. In ([2]), adverse selection economies are viewed as consumption externalities and, hence, budget constraints take a different form. However, the non-existence argument is clearly independent of the form in which the budget constraints are written.

p^* supporting the equilibrium allocation σ^* into a Lindhal equilibrium price π_{p^*} supporting the same allocation. It suffices to define q_{p^*} as:

$$q^i_{p^*}(j) = 0, \text{ if } i \neq j \text{ and } q^i_{p^*}(i) = p^{*i} \text{ for all } i \in I.$$

Since $(\sigma^{*i})_{i\in I}$ is incentive compatible, at prices q_{p^*} the choice q_{p^*}, (i, σ^{*i}) is an optimal solution to the individual programming problem of the Lindhal economy. Furthermore, at q_{p^*}, the profit of the firm are:

$$\sum_i \lambda^i q^i_{p^*} \beta^i = \sum_i \lambda^i p^{*i} \beta^i.$$

Thus, since the production sets of the Lindhal and of the competitive economies are identical, $(\sigma^{*i})_{i\in I}$ is a profit maximizing choice at q_{p^*}. As already mentioned, by removing, in the Lindhal economy, the incentive compatibility set from Y^* we narrow down the equilibrium set.

A strong Lindhal equilibrium allocation, and, hence, a strong competitive allocation, is a constrained Pareto optimum. Otherwise, at the equilibrium prices, individuals could not afford the superior allocation because of its greater cost in all markets. Thus, the firm would not be maximizing its profits at the equilibrium prices.

6.3 Non-existence of Equilibria in a standard example

We now show that strong competitive equilibria, and, hence, strong Lindhal equilibria may not exist. The failure of existence is robust. All of the economies of adverse selection used in applications have preferences represented by von-Neumann and Morgestern utility functions with type independent cardinality indexes and type invariant endowments. None of these economies has a strong equilibrium. A general argument for the nonexistence is in our paper on adverse selection ([14]). Here, we just provide an example, which although simple, provides the key elements of the argument.

6.4 A Rotschild-Stiglitz economy

Consider the text book example of an Adverse Selection economy as described in Rotschild and Stiglitz ([13]). There are two types; there are no actions, nor states of aggregate uncertainty, while each type has two individual states, called α and β, with $q^1(\alpha) > q^2(\alpha)$. There is only one commodity, so $L = 1$. Endowments are not dependent on types; they are higher in state α, with $e(\alpha) > e(\beta) = 0$. The utility functions v^i and u^i are independent of the type, so they are denoted by v and u. They are strictly concave. The utility associated to a net trade $z = (z(\alpha), z(\beta))$ can be written as

$$U^i \delta_z = q^i(\alpha)u(z(\alpha) + e(\alpha)) + (1 - q^i(\alpha))u(z(\alpha)).$$

It is well known ([13]) that in this class of economies, constrained efficient allocations are deterministic. We call an allocation which is type-invariant and feasible a *pooling allocation.*

First we prove that there exists a pooling constrained efficient allocation, denoted by z^p. Consider first the problem:

$$\max_{z=(z^1,z^2)\in IC(AS)} \sum_{i=1,2} \lambda^i [q^i(\alpha)u(z^i(\alpha) + e(\alpha)) + (1 - q^i(\alpha))u(z^i(\beta)] \tag{26}$$

subject to

$$\sum_{i=1,2} \lambda^i [q^i(\alpha) z^i(\alpha) + (1 - q^i(\alpha)) z^i(\beta)] = 0.$$

Since u is strictly concave, the problem (26) has a unique solution. Given the strict monotonicity of the objective function, a solution to the problem is a constrained Pareto optimum. Let z' be such optimal solution. The allocation $\bar{z}$, defined by

$$\bar{z}(s) = (\lambda^1 q^1(s) z'^1(s) + \lambda^2 q^2(s) z'^2(s))/(\lambda^1 q^1(s) + \lambda^2 q^2(s)), s = \alpha, \beta,$$

is feasible and, by the strict concavity of u, yields a non inferior value of the objective function. Thus, the original solution z' must be type invariant, that is pooling, and this is the pooling constrained efficient allocation z^p. Uniqueness follows from the convexity of the constraint set and the strict concavity of the objective function.

A particular family of pooling allocation, that we call $i-$pooling allocations, plays an important role in the argument. Let $\bar{q} = \lambda^1 q^1(\alpha) + \lambda^2 q^2(\alpha)$. The $i-$pooling allocation $\bar{z}^i$ is the solution to the following programming problem:

$$\max q^i(\alpha) u(z(\alpha) + e(\alpha)) + (1 - q^i(\alpha)) u(z(\beta)) \tag{27}$$

subject to

$$\bar{q} z(\alpha) + (1 - \bar{q}) z(\beta) = 0. \tag{28}$$

The allocation $\bar{z}^i$ is feasible because it satisfies the constraint (28), and is incentive compatible because it is pooling. Furthermore, since $q^1(\alpha) \neq q^2(\alpha)$,

$$\text{for all } i, U^i \delta_{\bar{z}^i} > U^i \delta_{z^p}. \tag{29}$$

Then, since z^p is a constrained optimum, the latter inequalities implies that for each constrained efficient allocation $z^* = (z^{*1}, z^{*2})$ there exists a type i such that $U^i \delta_{\bar{z}^i} > U^i \delta_{z^{*i}}$. This is the reason for the lack of strong Lindhal equilibria (and, thus, of strong competitive equilibria).

By contradiction, suppose that one, $(\hat{p}, \hat{\sigma})$, exists. Then, by the argument we have just seen, there exists a type i such that $U^i \delta_{\bar{z}^i} > U^i \hat{\sigma}^i$. Define now a new allocation $\sigma(\bar{z}^i)$ as follows: $\sigma(j)(\bar{z}^i) = \delta_{\bar{z}^i}$, for all $j \in I$. The allocation $\sigma($

z^i) is incentive compatible and feasible because the deterministic allocation $\bar{z}^i$ is feasible. Hence, $\sigma(\bar{z}^i) \in Y^*$. Thus, by revealed preferences, $\hat{p}^j \sigma^j(\bar{z}^i) > 0$, for all j, because the type i can trade in every market j. Multiplying these inequalities and adding, $(\sum_j \lambda^j \hat{p}^j) \sigma(\bar{z}^i) > (\sum_i \lambda^i \hat{p}^i) \hat{\sigma} = 0$. Therefore, since $\sigma(\bar{z}^i) \in Y^*$, at $\hat{p}$, $\hat{\sigma}$ is not a profit maximizing choice. Hence a strong Lindhal equilibrium (and, therefore, a strong competitive equilibrium) does not exist. ∎

6.5 An incomplete markets model

The economies we have described so far have the classical form of budget set, defined by a single budget constraint. We prove here that any constrained efficient allocation is an equilibrium of a competitive economy, where agents are not constrained to trade in subsets of the commodity space $\Delta(C)$, but are facing multiple budget constraints with monetary transfers. For every vector of price and monetary transfers, the zero transfer allocation is feasible for each consumer, so voluntary participation is guaranteed. So according to this definition a competitive equilibrium exists, and the second welfare theorem holds.

Let the social planner problem be defined, for every $\alpha \in \Delta(I)$ by:

$$\max_{\sigma=(\sigma^i)_{i\in I}\in\Delta(C)^I} \sum_{i\in I} \alpha^i U^i \sigma^i \tag{30}$$

subject to the feasibility constraint:

$$\sum_{i\in I} \lambda^i T^i \sigma^i \leq 0 \tag{31}$$

the incentive compatibility constraint

$$U^i\sigma^i \geq U^i\sigma^j \text{ for every } i \text{ and } j; \tag{32}$$

and the individual rationality constraint

$$U^i\sigma^i \geq U^i\delta_{(\hat{a}^i(0),0)}, \text{ for every } i. \tag{33}$$

where $\hat{a}^i(0)$ is the optimal choice of action for a zero transfer, namely:

$$\hat{a}^i(0) \equiv \text{ argmax }_{a\in A} U^i\delta_{(a,0)}$$

Note that for every i the lottery $\delta_{(\hat{a}^i,0)}$ satisfies the constraints (32) and (33).

The program (30) defines a set of solutions $(\hat{\sigma}^i(\alpha))_{i\in I}$ for every α. Choose any vector in this set, and for this vector let

$$\Gamma(\alpha) \equiv \text{ co } \{\hat{\sigma}^1(\alpha), \ldots, \hat{\sigma}^n(\alpha), \delta_{(\hat{a}^1,0)}, \ldots, \delta_{(\hat{a}^n,0)}\}$$

where co defines the convex hull. This is going to be the budget set in the competitive economy that we are going to define.

This set is non empty, convex subset of $\Delta(C)$, and such that

$$\text{for every } i, \hat{\sigma}(\alpha)^i \in \text{ argmax }_{\tau\in\Gamma(\alpha)} U^i\tau. \tag{34}$$

This property follows because the vector $(\hat{\sigma}^i(\alpha))_{i\in I}$ satisfies the two constraints (32) and (33) since it is a solution of the problem (30). This is equivalent to

$$\text{for every } i, \text{ and } \tau \in \{\hat{\sigma}^i(\alpha), i\in I, \delta_{(\hat{a}^i,0)}, i \in I\}, U^i\hat{\sigma}(\alpha)^i \geq U^i\tau.$$

Since $\tau \to U^i\tau$ is linear, this implies the claim.

Now the set $\Gamma(\alpha)$, being a convex non empty subset of $\Delta(C)$, can be written as an intersection

$$\Gamma(\alpha) = \{\tau \in \Delta(C) : \text{ for every } k \in K, p^k\tau \leq y^k\} \tag{35}$$

where K is some index set, and the vectors p^η can be interpreted as prices. By theorem 19. 1, of ([12]), the set $\Gamma(\alpha)$, which is finitely generated, is polyhedral, that is the index set K is finite.

The two equalities (34) and (35) show that the vector of constrained efficient allocations $(\hat{\sigma}^i(\alpha))_{i\in I}$ is the solution of the maximization of an individual consumer choice, subject to a vector of budget constraints, namely for every $k, p^k\tau \leq y^k$.

A comparison of the method of the previous analysis and the classical Negishi proof of existence of competitive equilibrium may clarify the nature of prices in our economies. The idea of the Negishi's proof begins with an efficient allocation, for a given vector of weights on the utility of individuals. The efficient allocations generates (for example, by the Lagrange multipliers of the problem) a vector of prices that support the allocation. At these prices however the allocation itself might not be affordable to agents if the income is the one obtained by the sale of endowments. To make these allocations affordable for every individual in the economy a transfer is necessary. In the modified budget constraint (defined by the prices and the transfers) the no-trade option might not be affordable for some individual.

The second step in Negishi's proof is a fixed point argument, in the space of weights, to determine those efficient allocations that are also supportable with zero transfers: that is, those allocations and prices for which the no-trade option is feasible to each individual.

Our argument above does not have this second step, and still the budget constraint we describe (in the equation (35)) makes the zero-transfer allocations affordable. The reason for the difference is the non-linearity of prices in the variable contract (as we already noted in the first paragraph of section (3.1). The model we described is called an "incomplete market model" because individuals cannot reduce the multiple budget constraint into a single one.

7 Conclusion

In their classical paper ([13]) Rotschild and Stiglitz pose the issue of existence and optimality of equilibrium in economies with Asymmetric Information. They do so adopting a mixed notion of equilibrium (Cournot-Nash, as they say in ([13]), page 633). Prescott and Townsend ([9]) define the problem trying to bring it back into the classical analysis of competitive equilibria. How much of the difficulties raised in ([13]) is solved in this new setup has so far been not completely clear.

In this paper we have first defined a model of economy general enough to include all known cases of economies with Asymmetric Information. This setup allows a direct exam of the specific difficulties in the different classes of economies. Once we do this, the key insight is the following. All these economies are also economies with individual risk (as defined by Malinvaud, ([7]), ([8])). In these economies, efficiency requires prices that are not identical for all consumers, but may depend on the consumer's type. This characteristic of the price system is made necessary by the form of the aggregate feasibility constraint, not by informational asymmetries, that are absent in the economies in ([7]), ([8]). Prices dependent on types do not create a problem to the standard Competitive Analysis in economies where types are publicly observable (as are economies with Moral Hazard and economies with Private Information). But they do create a problem in economies with Adverse Selection. In a market economy a consumer is free to choose the market where he trades, and so in particular he can choose the type he states. This freedom introduces into the economy the non-convexity that the extension to lotteries had eliminated. As a consequence, the two requirements of existence and constrained optimality are, in economies with Adverse Selection, incompatible.

8 Appendix

8.1 Proof of proposition (7)

In order to prove the proposition we associate to each given economy a particular economy of individual risk that we call the pseudo-economy. It is obtained by deleting the non-incentive compatible contracts and by restricting the individual utility functions U^i over the set of incentive compatible deterministic contracts. Thus, in the pseudo-economy, the consumption set of type i individuals is $\Delta(C_k^i(IC))$ and the utility functions is $\hat{U}^i\bar{\sigma}^i$, $\bar{\sigma}^i \in \Delta C_k^i(IC)$, for $\hat{U}^i = \{U^i(c), c \in C_k^i(IC)\}$. The price domain of the pseudo-economy is $\hat{P} = \{p \in R^{C_k^i(IC)} : p^i(c) = \theta \hat{T}^i(c)$, for $\theta \in R_+^{L\Omega} \setminus \{0\}\}$. The pseudo-economy is a standard individual risk economy. Thus, by theorem (4), an equilibrium $(\bar{p}, \hat{\sigma})$, with $\bar{p} \in \hat{P}$, exists and, by theorem (5), $\hat{\sigma}$ is efficient.

The argument now proceeds by showing in two separate claims that the equilibrium sets of the unrestricted economy and of the associated pseudo-economy are, basically, identical. Since the pseudo-economy has efficient equilibrium allocations and has consumption sets $\Delta(C_k^i(IC))$, the equivalence of the equilibrium allocation sets proves the theorem.

Let the pair of prices and joint allocations (q, τ) be an equilibrium of the pseudo economy. An extension of (q, τ) is a pair of prices and joint allocations (σ, p) for the original unrestricted economy such that $(q^i(c), \tau^i(c)) = (p^i(c), \sigma^{*i}(c))$, for $c \in C_k^i(IC)$, and $\tau^i(c) = 0$, $c \in C \setminus C_k^i(IC)$, for all i.

Claim 15 *Let (q, τ) be an equilibrium of the pseudo economy. Then, any extension of (q, τ), (p, σ), with $p \in P^U$, is an equilibrium of the unrestricted original economy.*

Proof: Pick any price extension of q, i.e., pick any selection f^i, satisfying (15) and set

$$p^i(\hat{c}) = q^i(f^i(\hat{c})),$$

We need to show that at prices p^i, the pair (id, σ^i) is an optimal solution of the individual programming problem of the unrestricted economy. Suppose otherwise and let $(\hat{\phi}, \tau^i)$ denote the optimal solution. Define a lottery μ^i with support $S^i \subset C_k^i(IC)$, as follows:

$$S^i = [S(\tau^i) \cap C_k^i(IC)] \cup f^i[S(\tau^i) \setminus C_k^i(IC)]$$

and

$$\mu^i(c) = \sum_{\hat{c} \in f^{i-1}(c)} \tau^i(\hat{c}).$$

By definition $U(\hat{\phi})\tau^i = \sum_{c \in C} U(c_{\hat{\phi}})\tau^i(c)$. Since $f^i(c) \in c_{\phi^{*i}(c)}$, both $U^i(f^i(c))\delta_{f^i(c)} \geq U^i(c_{\hat{\phi}})\delta_c$, for all $c \in C$, as well as $S^i \subset C_k^i(IC)$. Thus, by the definition of μ^i (and since $f^i(c) \in c_{\phi^{*i}(c)}$) :

$$U^i\mu^i - U^i(\hat{\phi})\tau^i = \sum_{c \in C}(U^i(f^i(c)) - U^i(c_{\hat{\phi}}))\tau^i(c) \geq 0$$

By the definition of $\hat{p}$,

$$0 \geq p^i\tau^i = \sum_{c \in C_k^i(IC)} q^i(c)\tau^i(c) + \sum_{c \notin C_k^i(IC).} q^i(f^i(c))\tau^i(c) = q\mu^i.$$

Thus, if σ^i is not optimal at p^i, there exists a lottery μ^i with support $S^i \subset C_k^i(IC)$, budget feasible at prices q and such that $U^i\mu^i > U^i\tau^i$. The last inequality contradicts the definition of σ^i. ■

Let (σ, p) be an equilibrium of the unrestricted economy. Bear in mind that by the definition of equilibrium, individuals do not manipulate, *i.e.*, $\phi^i = id$, for all i. Thus, by definition of non-incentive compatible contract, $\sigma^i(c) = 0$, for $c \in C \backslash C_k^i(IC)$, for all i. A reduction of (σ, p) is a pair of joint allocations and prices of the pseudo economy (τ, q) such that $(\tau^i(c), q^i(c)) = (\sigma^i(c), p^i(c))$. The following claim is obvious and, as explained, concludes the argument.

Claim 16 *Let (σ, p) be an equilibrium of the unrestricted economy. The reduction of (p, σ), (τ, q) is an equilibrium of the pseudo economy.*

■

8.2 Example

Consider a private information economy with $\Omega = L = A = I = 1$ and $S = 2$. For this economy, $C = X \times X$ and thus a contract is a pair of (personal) state contingent consumption goods (z_1, z_2). Assume that $X = (-10, 0, 20, 50)$ and that the utility function of the individuals is

$$U(z_1, z_2) = 5U_1(z_1) + U_2(z_2).$$

U_s are identified by 4 dimensional vectors and they are assumed to be:

$$U_1 = (-10, -5, 20, 30) \text{ and } U_2 = (-10, 0, 20, 80).$$

An allocation can be identified with a pairs of elements in $\Delta(X)$. σ_s denotes the lottery contingent on the declaration of state s. Personal states are equiprobable. Thus, a lottery (σ_1, σ_2) is feasible if

$$E_{\sigma^1} z + E_{\sigma^2} z \leq 0,$$

where $E_{\sigma_s} z = (-10, 0, 20, 50)\sigma_s,\ s = 1, 2$. A constrained efficient allocation (σ_1^*, σ_2^*) is the optimal solution to:

$$\max_{(\sigma_1, \sigma_2)} 5U_1\sigma_1 + U_2\sigma_2$$

subject to

$$E_{\sigma_1} z + E_{\sigma_2} z \leq 0, U_s(\sigma_s - \sigma_{s'}) \geq 0, (s, s') \in \{1, 2\}^2.$$

By performing straightforward computations we get the following characterization of the unique constrained efficient allocation:

$$\sigma_1^* = (9/15, 0, 6/15, 0) \text{ and } \sigma_2^* = (13/15, 0, 0, 2/15);$$

$$U_1(\sigma_1^* - \sigma_2^*) > 0 \text{ and } U_2(\sigma_1^* - \sigma_2^*) = 0.$$

The constrained efficient allocation (σ_1^*, σ_2^*) has two key characteristics :

1. one of the two incentive compatibility constraints is strictly binding and

2. the support of σ^*, $S(\sigma^*) = \{(-10,-10),(-10,50),(20,-10),(20,50)\}$, contains deterministic allocations that are not incentive compatible.

We have defined a price system as a map $\hat{p} : X \times X \to R$. In the context of the economy used in this example, it is convenient and natural to define a price system as a pair of state contingent maps $p_s : X \to R$, $s = 1, 2$. Obviously, any pair (p_1, p_2) defines a map $p : X \times X \to R$. Vice versa, if $\hat{p}^* : X \times X \to R$ is a competitive equilibrium price supporting the allocation (σ_1^*, σ_2^*), the pair (p_1, p_2) defined as $p_s(\hat{z}) = \sum_z \hat{p}^*(\hat{z}, z)\sigma_{s'}(z)$, $z \in X$, $s' \neq s$, supports the same allocation.

In the next claim we use the two conditions 1 and 2 above to show the nonexistence of a supporting price.

Claim 17 (σ_1^*, σ_2^*) *cannot be decentralized as a competitive equilibrium of the unconstrained economy.*

Proof. In the unconstrained economy, the individual solves:

$$\max_{(\sigma_1,\sigma_2)} 5U_1\sigma_1 + U_2\sigma_2 \text{ subject to } p_1\sigma_1 + p_2\sigma_2 \leq 0,$$

If (σ_1^*, σ_2^*) is price supportable, there must exists a price p^* such that:

$$9p_1^*(-10) + 6p_1^*(20) + 13p_2^*(-10) + 2p_2^*(50) = 0$$

Since $U_2(\sigma_1^* - \sigma_2^*) = 0$, the individuals are indifferent at σ_2^*, when $s = 2$ realizes, whether to report $s = 1$ or $s = 2$. Furthermore, at each allocation (σ_1^*, σ_2), with $U_2(\sigma_2 - \sigma_2^*) \leq 0$, by reporting $\phi^*(1) = \phi^*(2) = \{1\}$, individuals guarantee the same level of utility of σ^*, i.e., $U\sigma^* = U(\phi^*)(\sigma_1^*, \sigma_2) = U(\sigma_1^*, \sigma_1^*)$. Consider the lottery $\hat{\sigma}_2 = \delta_{-10}$ together with the report $\phi^*(1) = \phi^*(2) = \{1\}$. Given, the local non satiation of the preferences at (σ_1^*, σ_1^*) it must be

$$p_2^*(\hat{\sigma}_2 - \sigma_2^*) \geq 0, \text{ or, equivalently, } p_2^*(-10) \geq p_2^*(50).$$

However, the last inequality implies that the lottery $\bar{\sigma}_2 = (0,0,0,1)$ is at most as expensive as the lottery σ_2^* , while $U_2(\bar{\sigma}_2 - \sigma_2^*) > 0$. Thus, (σ_1^*, σ_2^*) is not decentralizable. ■

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