Optimal Dynamic Auctions for Durable Goods: Posted Prices and Fire-sales*

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Abstract

We consider a seller who wishes to sell K goods by time T. Potential buyers enter IID over time and are patient. At any point in time, profit is maximized by awarding the good to the agent with the highest valuation exceeding a cutoff. These cutoffs are characterized by a one-period-look-ahead rule and are deterministic, depending only on the number of units left and the time remaining. The cutoffs decrease over time and in the inventory size, with the hazard rate of sales increasing as the deadline approaches. In the continuous time limit, the optimal allocation can be implemented by posted-prices with an auction at time T. Unlike the cutoffs, the prices depend on the history of past sales.

1 Introduction

We consider a seller who wishes to sell K goods by time T. Potential buyers enter the market over time with privately known values and, once they arrive, prefer to obtain the good sooner rather than later. At each point in time, the seller thus chooses whether to sell today or incur a costly delay and wait for new buyers - a tradeoff shared by many real-life problems. For example, when an airline sells tickets for a flight, a car dealer tries to clear its inventory, or an owner sells her house, they face the tradeoff between lowering the price today or waiting for new entrants.

There is a substantial literature on revenue management that analyses how such sellers should price over time (see the book by Talluri and van Ryzin (2004)). It is estimated that

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these techniques have led to a substantial increase in profits for airlines (Davis (1994)), retailers (Friend and Walker (2001)) and car manufacturers (Coy (2000)). However, these models typically assume that buyers are impatient, exiting the market if they do not immediately buy. In this paper, we derive the optimal mechanism when buyers are patient and forward-looking. This seems natural when considering markets such as airline tickets and cars, where buyers can easily time their purchases. It is also becoming more important as buyers use price prediction tools to aid such inter-temporal arbitrage (e.g. bing.com, where searches for flights results contain predictions about changes in prices).[1]

The practical problems that we model have two key properties. First, the pricing problem is non-stationary. In our examples, the good may expire at a fixed date (e.g. plane tickets, advertising slots), become much less valuable (e.g. seasonal clothing), or the number of interested buyers may decline over time (e.g. a house). Second, once a buyer arrives, he prefers to purchase the good sooner rather than later. In the case of clothing or a house, he has more days to enjoy the good; in case of a flight or advertising slots, he has more time to plan complementary activities such as hotels or production schedules.

In our model, we capture the non-stationary nature of the problem by assuming that the good expires after time T (or equivalently, that there is no more entry after T). Buyers arrive stochastically over time according to an IID process; motivated by online markets, we assume the number of entrants is not observed by other buyers. Upon entering, each buyer has unit demand, draws a private value from a common distribution, and calculates whether to buy today or wait at the risk of a stock-out. Finally, we model impatience by assuming symmetric proportional discounting (we later allow for inventory costs).

We start our analysis in Section 3 with the one-unit case. We show the profit-maximizing mechanism allocates the good to the agent with the highest valuation exceeding a cutoff. The cutoffs are determined by a *one-period-look-ahead* policy, whereby the seller is indifferent between serving the cutoff type today and waiting one more period, potentially selling to a new entrant.[2] The optimal cutoffs are independent of the valuations of old buyers and take a very stark form: they are constant in periods $t < T$, and drop sharply in period T to the static monopoly price. As a result, a buyer who arrives at time t either buys immediately or waits for the "fire-sale" in period T.

In discrete time, the optimal mechanism can be implemented using a sequence of second-price auctions with deterministic reserve prices. Taking the continuous time limit, assuming new buyers enter according to a Poisson process, the seller can use deterministic posted prices

[1]For evidence of forward looking behaviour see Zeithammer (2006) on eBay auctions, Hartmann (2006) on golf rounds, Chevalier and Goolsbee (2009) on textbooks or Gowrisankaran and Rysman (2009) on camcorders.

[2]In a general dynamic program, the seller would be indifferent between serving the cutoff type today and delaying. Under the one-period-look-ahead policy, the profits from delaying equal the profits from selling tomorrow.

and a fire-sale at date T. These prices fall faster than the rate of interest. Intuitively, the prices are set so that the cutoff type is indifferent between buying today and waiting a little. Since the agent forgoes one period's enjoyment of the good, the price has to drop at least at the speed of interest, but since he is also risking the arrival of new competition, the price has to drop even more. Prices also fall faster as the deadline approaches and fall faster when the buyer faces more competition from potential entrants. Observe that this implementation, via posted prices and a fire-sale, works with very little information: the seller does not have to observe when buyers arrive or have any idea of their values.

In Section 4, we suppose the seller has K-goods, with the one-unit case being the first step of our inductive solution. When there are k units left, one can think of the units being awarded sequentially within a period, with the k^{th} unit being awarded to the remaining agent with the highest valuation, subject to his valuation exceeding a cutoff x_t^k. As in the one-unit case, the optimal allocations are characterised by a one-period-look-ahead policy. In addition, the cutoffs are *deterministic*, depending on the number of units and time remaining, but not on the number of buyers, their values of when previous units were sold. Intuitively, when the principal is indifferent between selling to the highest value agent and delaying, the decision to delay does not affect when lower value agents buy. Hence raising these agents' values raises the profits from selling and delaying equally, and does not affect the cutoff type. Finally, the optimal cutoffs fall over time and decrease in the number of remaining units. Intuitively, if the seller delays awarding the k^{th} unit by one period then she can allocate it to the highest value entrant rather than the current leader. As the game proceeds, the current leader is increasingly likely to be awarded the good eventually, decreasing the option value of delay and causing the cutoff to fall over time. Similarly, when there are more goods remaining, the current leader is more likely to be awarded the good eventually, again reducing the option value of delay and causing the cutoff to fall in k.

In the continuous time limit, since cutoffs are deterministic, the optimal mechanism can be implemented by a sequence of posted prices p_t^k and a fire-sale for the remaining units at time T. Unlike the cutoffs, the prices do depend on when previous units were sold. If they were sold earlier, the current cutoff type has more potential competition, leading to higher prices. Overall, the price-path exhibits a slow decline, with occasional upward jumps when sales occur. When there is no sale, the price will fall because the cutoffs decline and the deadline approaches. The rate of decline is then determined by rate at which the cutoff drops, the interest rate and the probability the cutoff type will lose the good if he delays a little, either to an existing buyer or a new entrant. When a sale does occur, the cutoff to allocate one of the remaining units jumps upwards, as does the price.

We next consider three basic extensions of the model. In Section 5 we suppose time-preference comes from inventory costs rather than proportional discounting. If the costs are (weakly) convex in time, the cutoffs and deterministic, (weakly) decline over time and are therefore determined by a one-period-look-ahead policy. The supporting prices decline over time, despite the lack of discounting, because a buyer who delays risks the good being bought by a competing agent.

In Section 6 we study the effect of inter-temporal changes in the distribution of the number of entrants. For example, a seller of a house may face a stock of pre-existing buyers as well as a flow of entrants. We show that if the number of entrants falls over time, in the sense of first-order stochastic dominance, then the seller's option value of waiting also falls. As a result, the cutoff values decrease over time, so that an agent who enters in period t may end up buying in any of the subsequent periods. Conversely, if the number of entrants increases over time, then the cutoffs rise until period $T-1$, so that an agent who enters in period t either buys immediately or waits until the final period. In this case, the one-period-look-ahead policy fails, with the principal being indifferent between serving the cutoff type in period t and waiting until the end of the game to serve this agent. In either case, we show how to implement the optimal allocations with deterministic posted prices and a fire-sale at time T.

In contrast to the impatient buyer model (e.g. Vulcano, van Ryzin, and Maglaras (2002)), we assume a buyer's valuation at time t is independent of when he enters the market. In Section 7, we bridge this gap and consider two models of partially patient agents. In the first, agents' values decline deterministically after they enter the market. When there is one unit and constant entry, we show that cutoffs decline over time but that, because of the declining valuations, an agent who enters at time t either buys immediately or waits until the final period. These cutoffs can be implemented through posted prices and a biased auction in period T. The second model allows buyers to exit randomly. Unlike our other results, the optimal cutoffs depend on the valuations of all entering agents. As a result, they cannot be implemented through simple posted-prices or auctions.

1.1 Literature

There are a number of papers that examine how to sell to patient buyers entering over time. Our results are related to a classic result on "asset selling with recall" (e.g. Bertsekas (1995, p. 177)). Bertsekas derives the welfare-maximising policy with one good, when one agent enters each period and his value is publicly known. We derive the profit-maximising policy for many goods, when several agents enter each period and their values are privately known. We also show how to implement the optimal mechanism in both continuous and discrete time.

Wang (1993) supposes that the seller has one object and that buyers arrive according to a Poisson distribution and experience a fixed per–period delay cost. Wang shows that with an infinite horizon, a profit-maximising mechanism is a constant posted-price. Gallien (2006) characterises the optimal sequence of prices when agents arrive according to a renewal process over an infinite time horizon. Assuming inter-arrival times have an increasing failure rate, Gallien proves that agents will buy when they enter the market (or not at all). In contrast to both Wang (1993) and Gallien (2006), we find that the optimal mechanism may induce delay of purchases on the equilibrium path.

Pai and Vohra (2008) consider a model without discounting where agents arrive and leave the market over time, and partially characterize the profit-maximising mechanism. Mierendorff (2009) considers a two-period version of a similar model and provides a complete characterisation of the optimal contract.[3] In a separate line of work, Said (2009) characterises the optimal dominant strategy mechanism where agents are patient but goods are nonstorable, and describes a dynamic open-auction implementation.

There is also a classic literature studying the sequential allocation of goods to impatient buyers. Karlin (1962) analyses the problem of allocating multiple goods to buyers who arrive sequentially but only remain in the market for one period. In the optimal policy, a buyer is awarded a unit if their valuation exceeds a cutoff. This cutoff is decreasing in the number of units available and increasing in the time remaining. These results have been extended in a number of ways. Derman, Lieberman, and Ross (1972) allow for heterogeneous goods. Albright (1974) allows for random arrivals with positive discount rates. More recently, a number of studies allow buyers' valuations to be private information. Gershkov and Moldovanu (2009a) solve the profit-maximising policy, while Gershkov and Moldovanu (2009b) allow the seller to learn about the distribution of valuations over time, introducing correlations in buyers' valuations. Vulcano, van Ryzin, and Maglaras (2002) suppose N agents enter each period, and allow the seller to hold an auction.

Finally, the paper is related to the durable goods literature. Stokey (1979) characterises the optimal strategy for a seller with infinite supply who faces a fixed distribution of buyers. Conlisk, Gerstner, and Sobel (1984) suppose a homogenous set of buyers enters each period, while Board (2008) allows the entering generations to differ.

2 Model

Basics. A seller has K goods to sell. Time is discrete and finite, $t \in \{1, \ldots, T\}$. Time-preference comes from a common discount factor $\delta \in [0, 1)$.[4] In Section 5 we show our results

[3] There are a number of papers on similar themes. Shen and Su (2007) summarize the operations research literature. For example, Aviv and Pazgal (2008) suppose a seller has many goods to sell to agents who arrive over time and are patient. Aviv and Pazgal restrict the seller to choosing two prices which are independent of the past sales. In economics, Board (2007) assumes a seller sells a single unit to agents whose values vary over time. Hörner and Samuelson (2008) consider a seller with no commitment power who sells a single unit to N agents by setting a sequence of prices.

[4] Our results go through if $\delta = 1$, replacing δ^τ with $\mathbf{1}_{\tau \le T}$ in the proofs. High value buyers will still buy before low value buyers because of the risk of stock-out.

extend to inventory costs.

Entrants. At the start of period t, N_t agents/buyers arrive. We initially assume N_t is IID random variable; this is generalised in Section 6. N_t is observed by the seller, but not by other agents.[5,6]

Preferences. After he has entered the market, an agent wishes to buy a single unit. The agent is endowed with a privately-known valuation, v_i, drawn IID with density $f(\cdot)$, distribution $F(\cdot)$ and support $[\underline{v}, \overline{v}]$. If the agent buys at time t for price p_t, his utility is $(v - p_t)\delta^t$. Let v_t^k denote the k^{th} highest order statistic of the agents entering at time t. Similarly, let $v_{\leq t}^k$ denote the k^{th} highest order statistic of the agents who have entered by time t.

Mechanisms. Each agent makes report $\tilde{v}_i$ when he enters the market. A mechanism $\langle P_{i,t}, TR_i \rangle$ maps agents' reports into an allocation rule $P_{i,t}$ describing the probability agent i is awarded a good in period t, and a transfer TR_i expressed in time-0 prices. A mechanism is *feasible* if (a) $P_{i,t} = 0$ before the agent enters, (b) $\sum_t P_{i,t} \in [0, 1]$; (c) $\sum_i \sum_t P_{i,t} \leq K$; and (d) $P_{i,t}$ is adapted to the seller's information, so $P_{i,t}$ can vary only with the reports of agents that have entered by t.[7]

Agent's Problem. Suppose agent i enters the market in period t_i. Upon entering the market, the agent chooses his declaration $\tilde{v}_i$ to maximise his expected utility,

$$u_i(\tilde{v}_i, v_i) = E_0 \left[\sum_{s \geq 1} v_i \delta^t P_{i,s}(\tilde{v}_i, v_{-i}) - TR_i(\tilde{v}_i, v_{-i}) \middle| v_i \right] \tag{2.1}$$

where E_t denotes the expectation at the start of period t, before agents have entered the market. A mechanism is incentive compatible if the agent wishes to tell the truth, and is individually rational if the agent obtains positive utility.

Seller's Problem. The seller chooses a feasible mechanism to maximise the net present value

[5] The assumption that the seller can observe N_t is for definiteness: the optimal allocation and implementation are identical if the seller cannot observe N_t. The assumption that an agent cannot observe N_t is motivated by anonymous markets, such as large retailers and online sellers. If N_t is publicly observed, the optimal allocations are unaffected although, when implementing this allocation, the optimal price at time t is a function of $\{N_1, \ldots, N_t\}$. See footnote 11.

[6] For simplicity, we assume the buyer does not know the number of units remaining; when implementing the optimal allocation, we attain the same profits when agents know the number of units available, so the seller does not benefit from hiding his remaining inventory.

[7] This formulation ignores the correlation between allocations, for a fixed set of reports. We can model such correlation by considering allocation function $P_{i,t}(v, \omega) \in \{0, 1\}$ where $\omega \in \Omega$ is a random variable. Since the optimal mechanism is deterministic, the correlation plays no role.

of profits

$$\Pi_0^K = E_0 \left[\sum_i TR_i(v_i, v_{-i}) \right] \tag{2.2}$$

subject to incentive compatibility and individual rationality.

Some remarks regarding interpretation are pertinent. First, time T can be viewed is the date at which the good expires (e.g. a plane ticket) or the last time agents enter the market, since no sales will occur after this point. Second, we adopt a durable-goods utility specification, interpreting the discount rate as the rate of time preference. If instead the discount rate is the degree agents' values fall over time (e.g. values for summer clothes will be lower in July than in June), then utility is given by $v\delta^t - \tilde{p}_t$. Under this new specification, the analysis is unchanged with prices given by $\tilde{p}_t = \delta^t p_t$.

2.1 Preliminaries

When an agent enters the market, he chooses his declaration $\tilde{v}_i$ to maximise his utility (2.1). As shown in Mas-Colell, Whinston, and Green (1995, Proposition 23.D.2), an allocation rule is incentive compatible if and only if the discounted allocation probability

$$E_0 \left[\sum_{s \geq 1} \delta^s P_{i,s}(v_i, v_{-i}) \right] \tag{2.3}$$

is increasing in v_i. Using the envelope theorem and integrating by parts, expected utility is then

$$E_0[u_i(v_i, v_i)] = E_0 \left[\sum_{s \geq 1} \delta^t P_{i,s} \frac{1 - F(v_i)}{f(v_i)} \right]$$

where we use the fact that an agent with value $\underline{v}$ earns zero utility in any profit-maximising mechanism. Profit (2.2) equals welfare minus agents' utilities,

$$\Pi_0^K = E_0 \left[\sum_i \sum_{s \geq 1} P_{i,s} \delta^s m(v_i) \right] \tag{2.4}$$

where the *marginal revenue* of agent i is given by $m(v_i) := v_i - (1 - F(v_i))/f(v_i)$. Throughout we assume $m(v)$ is increasing in v, implying that the seller's optimal mechanism is characterised by cutoff rules, and allowing us to ignore the monotonicity constraint (2.3).

Suppose the seller has k goods at time t. Write *continuation profits* before the period-t

entrants have entered by[8]

$$\Pi_t^k := E_t\left[\sum_i \sum_{s\geq t} \hat{P}_{i,s}\delta^{s-t}m(v_i)\right]. \tag{2.5}$$

where $\hat{P}_{i,s}$ is the allocation function given the principal has k goods in period t. Let the expected continuation profits after period-t entrants have entered be denoted by $\tilde{\Pi}_t^k$. When $k=1$, we omit the superscript.

3 Single Unit

We first derive the optimal solution when the firm has one unit to sell. By the principle of optimality, the seller maximises continuation profits in every state. At time t, profit is

$$\begin{aligned}\Pi_t &= \max_{\hat{P}_{i,t}} E_t\left[\sum_i \hat{P}_{i,t}m(v_i) + \left(1-\sum_i \hat{P}_{i,t}\right)\delta\Pi_{t+1}\right] \\ &= \max_{\hat{P}_{i,t}} E_t\left[\sum_i \hat{P}_{i,t}(m(v_i)-\delta\Pi_{t+1})\right] + E_t[\delta\Pi_{t+1}]\end{aligned} \tag{3.1}$$

Equation (3.1) implies that the good is allocated to maximise the flow profit minus the opportunity cost of allocating the good, $\delta\Pi_{t+1}$. As a result, when the good is awarded, it will be given to the agent with the highest marginal revenue (and the highest valuation).

We can now think of the highest current valuation, v, as a state variable. Let $\Pi_t(v)$ be the profit just before entry in time t, so that

$$\begin{aligned}\Pi_t(v) &= E_t\big[\max\{m(v), m(v_t^1), \delta\Pi_{t+1}(\max\{v, v_t^1\})\}\big] \quad \text{for } t<T \\ \Pi_T(v) &= E_T\big[\max\{m(v), m(v_T^1), 0\}\big]\end{aligned} \tag{3.2}$$

The following result shows that the optimal cutoffs can be characterised by a simple one-period-look-ahead rule.

Proposition 1. *Suppose $K=1$ and N_t are IID. The optimal mechanism awards the good to the agent with the highest valuation exceeding a cutoff. The cutoffs $\{x_t\}$ are uniquely determined*

[8]While we call Π_t^k continuation profits, this includes the impact of time t decisions on the willingness to pay of agents who buy in earlier periods.

by:

$$\begin{aligned} m(x_t) &= \delta E_{t+1}[\max\{m(v^1_{t+1}), m(x_t)\}] \qquad \textit{for } t < T \\ m(x_T) &= 0 \end{aligned} \tag{3.3}$$

Consequently, the cutoffs are constant in periods $t < T$.

Proof. The proof is by induction. In period $t = T$, then $m(x_T) = 0$. In period $t = T - 1$, the seller should be indifferent between selling to agent x_{T-1} today and waiting one more period and getting a new set of buyers. Hence

$$m(x_{T-1}) = \delta E_T[\max\{m(v^1_T), m(x_{T-1})\}].$$

Continuing by induction, fix t and suppose x_s, as defined by (3.3), are optimal for $s > t$. If $v < x_t$ then

$$m(v) < \delta E_{t+1}[\max\{m(v^1_{t+1}), m(v)\}]$$

so the seller strictly prefers to wait one period rather than sell to type v today. Conversely, if $v > x_t$ then

$$m(v) > \delta E_{t+1}[\max\{m(v^1_{t+1}), m(v)\}]. \tag{3.4}$$

Since N_t is IID, (3.4) implies that $v > x_{t+1}$ so type v will buy tomorrow if he does not buy today. Hence

$$\Pi_{t+1}(v) = E_{t+1}[\max\{m(v^1_{t+1}), m(v)\}]$$

and (3.4) implies that the seller strictly prefers to sell to type v today rather than waiting. Putting this together, x_t is indeed the optimal cutoff. □

Proposition 1 uniquely characterises the optimal cutoffs, and shows they are constant in all periods prior to the last. The intuition is as follows. At the cutoff the seller is indifferent between selling to the agent today and delaying one period and receiving another draw. This indifference rule relies on the assumption that if type x_t does not buy today, then he will buy tomorrow. This is satisfied because the seller faces exactly the same tradeoff tomorrow and therefore is once again indifferent between selling and waiting.

The optimal cutoffs are deterministic, depending on the number of periods remaining, but not on the number of agents who have entered in the past and their valuations. While the value of the second highest agent may affect the seller's realised revenue, it does not alter the seller's expected revenue and hence the optimal cutoff. Since cutoffs are deterministic the seller can implement the optimal mechanism without observing the number of arrivals, as we show below.

Proposition 1 is very different from the optimal mechanism when buyers are impatient (e.g. Vulcano, van Ryzin and Marglaras (2002)). In this case, the optimal cutoffs are fully forward-looking, and fall over time as the seller becomes increasingly keen to sell the good. In contrast, when agents are patient, the allocations are determined by a one-period-look-ahead rule.[9]

Finally, let us assess the welfare consequences of Proposition 1. Using an analogous proof, one can show that the welfare-maximising mechanism awards the good to the agent with the highest value exceeding a cutoff given by $x_t^W = E_{t+1}[\delta \max\{v_{t+1}^1, x_t^W\}]$ for $t < T$, and $x_T^W = 0$. If $(1 - F(v))/vf(v)$ is decreasing in v, then the profit-maximising cutoffs exceed the welfare-maximising cutoffs for all t, implying that a profit-maximising seller awards the good later than is efficient (and sometimes never at all).[10]

3.1 Implementation through Sequential Second-Price Auctions

The optimal mechanism allocates the good to the agent with the highest valuation exceeding a cutoff. Corollary 1 shows that this allocation can be implemented through a sequence of second-price auctions with declining reserve prices.

Denote the cutoff in periods $t < T$ by x^* and let $\theta := \delta \Pr(v_t^1 < x^*)$ be an agent's effective discount rate, taking into account the possibility that, if he delays, the good may be sold to a new entrant.

Corollary 1. *Suppose $K = 1$ and N_t is IID. The profit-maximising allocation can be implemented by a sequence of second-price auctions with deterministic reserve prices R_t satisfying $R_T = m^{-1}(0)$ and*

$$R_t = (1 - \theta^{T-t})x^* + \theta^{T-t} E_0 \left[\max\{v_{\leq T}^2, m^{-1}(0)\} \middle| N_1 \geq 1, v_{\leq T}^1 = x^* \right] \qquad \textit{for } t < T. \tag{3.5}$$

Proof. See Appendix A.1. □

The reserve prices are constructed so that the marginal agent is indifferent between buying today and delaying. When making this calculation, we must condition on the buyer existing; since N_t is IID, we assume the buyer enters in period 1.

The sequential second-price auction has several interesting features. First, while cutoffs are constant in periods $t < T$, the reserve prices decline. When the an agent delays he forgoes one

[9]This assumes T is finite. When $T = \infty$, the cutoffs are determined by (3.3) and are therefore constant in all periods. An agent therefore either buys immediately or never, and we can assume that buyers are impatient without loss of generality (Gallien (2006)).

[10]Proof: Since $(1 - F(v))/vf(v)$ is decreasing in v, $m(v)/v$ is increasing in v and $m(v)/m(x) \geq v/x$ for $v \geq x$ if $m(x) > 0$. If $x_t^W > x_t$, then

$$1 = E_{t+1}\left[\delta \max\left\{\frac{m(v_{t+1}^1)}{m(x_t)}, 1\right\}\right] \geq E_{t+1}\left[\delta \max\left\{\frac{v_{t+1}^1}{x_t}, 1\right\}\right] > E_{t+1}\left[\delta \max\left\{\frac{v_{t+1}^1}{x_t^W}, 1\right\}\right] = 1$$

yielding the required contradiction.

period's enjoyment of the good, so the price has to drop at least at quickly as the discount factor, but since he is also risking the arrival of new competition, the price has to fall faster.

Second, agents below x^* abstain, even though their valuations may exceed the reserve price. Such an agent wishes to delay in order to take advantage of the fire-sale in period T.

Third, the reserve prices are deterministic. Intuitively, if an agent has value above x^*, he bids his value, either wins or loses the good, and the game ends. If an agent has value below x^*, he abstains and does not reveal his valuation, so there is no new information arriving to the market to affect the optimal reserve price.[11]

Fourth, the reserve prices depend on the expected number of agents who will enter in the future. This means that as the distribution of N_t grows in the usual stochastic order then (a) the cutoff x^* rises, and (b) the probability of stocking out grows and $x^* - R_t$ shrinks.

Finally, while we assume that the seller uses a second-price auction, we could equally well use a different auction format. One possibility is to use an English auction each period. As in the second-price auction, an agent bids his value, conditional on participation. Given reserve prices (3.5), type x^* is again the lowest type participating. A second possibility is to use a first-price auction in periods $t < T$ and a second-price auction in period T, with the reserve prices given by (3.5). In periods $t < T$, agents below x^* abstain, while those above x^* adopt an increasing bidding strategy with $b(x^*) = R_t$. This implements the same allocation as sequential second-price auctions and, by revenue equivalence, raises the same revenue. In period T, agents have different beliefs about the distribution of types for new and old bidders, so a second-price auction can be used to attain an optimal allocation (a first-price auction would not be efficient).

3.2 Implementation in Continuous Time

The optimal mechanism becomes particularly simple to implement as time periods become very short. Suppose agents enter the market according to a Poisson process with arrival rate λ,[12] and let r be the instantaneous discount rate. Taking the limit of equation (3.3), the optimal allocation at $t < T$ is given by

$$rm(x^*) = \lambda E\big[\max\{m(v) - m(x^*), 0\}\big] \tag{3.6}$$

where v is distributed according to $F(\cdot)$. Equation (3.6) says the seller equates the flow profit from the cutoff type (the left-hand-side) and the option value of waiting for a new entrant (the right-hand-side). At time T, the optimal cutoff is given by $m(x_T) = 0$. See Figure 1 for an

[11] This relies of the fact that one buyer cannot observe the arrival of others (as in online marketplaces). If $\{N_1, \dots, N_t\}$ is publicly observed then the optimal allocations are identical and the reserve is $R_t = (1-\theta^{T-t})x^* + \theta^{T-t} E_0\left[\max\{v^2_{\le T}, m^{-1}(0)\}\big|\{N_1, \dots, N_t\}, v^1_{\le T} = x^*\right]$, which changes with the observed number of agents that have arrived.

[12] In discrete time, this means N_t is distributed according to a Poisson distribution with parameter λ.

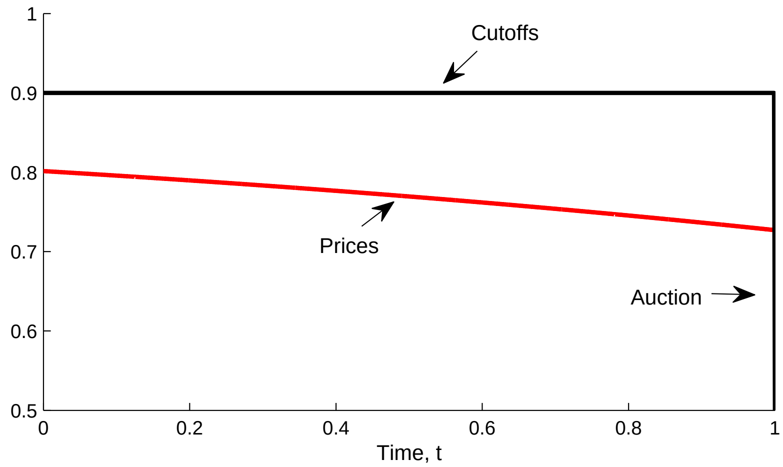

Figure 1: **Optimal Cutoffs and Prices with One Unit in Continuous Time.** When there is one unit, the optimal cutoffs are constant when $t < T$ and drop at time T. The price path is decreasing and concave, with an auction occurring at time T. In this figure, agents enter continuously with Poisson parameter $\lambda = 5$ have values $v \sim U[0,1]$, so the static monopoly price is 0.5. Total time is $T = 1$ and the interest rate is $r = 1/16$.

illustration.

The optimal allocation can be implemented by a deterministic sequence of prices with a fire-sale at time T. In the last period, the seller uses a second-price auction with reserve $R_T = m^{-1}(0)$. At time $t < T$ the seller chooses a price p_t, which makes type x^* indifferent between buying and waiting. The final "buy-it-now" price, denoted by $p_T = \lim_{t \to T} p_t$, is chosen so type x^* is indifferent between buying at price p_T and entering the auction. That is,

$$p_T = E_0 \left[\max\{v^2_{\leq T}, m^{-1}(0)\} \middle| N_0 = 1, v^1_{\leq T} = x^* \right]$$

Note that the buyer conditions on his own existence; since arrivals are independent, we assume that the buyer arrives at time 0 without loss of generality.

When $t < T$, type x^* is indifferent between buying now and waiting dt. This yields

$$(x^* - p_t) = (1 - r dt - \lambda dt)(x^* - p_{t+dt}) + \lambda dt (x^* - p_{t+dt}) F(x^*)$$

Rearranging and letting $dt \to 0$,

$$\frac{dp_t}{dt} = -(x^* - p_t)\big(\lambda(1 - F(x^*)) + r\big).$$

Fixing the cutoffs, prices fall faster if (a) prices are lower, (b) the arrival rate is higher, (c) there is a high probability a new arrival will buy the good, or (d) the interest rate is higher.

The first property implies that p_t is convex in t, so prices fall faster as the deadline approaches (see Figure 1).

4 Multiple Units

In this section we suppose the seller has K goods to sell. Using the principle of optimality, the seller maximises continuation profits at each point in time. Consider period t and suppose the seller has k units.

Lemma 1. *The seller allocates goods to high value agents before low value agents.*

Proof. Suppose in period t the seller sells to agent j but does not sell to agent i, where $v^i \geq v^j$. To be concrete, suppose the seller eventually sells to agent i in period $\tau > t$, where we allow $\tau = \infty$. Now suppose the seller leaves all allocations unchanged but switches i and j. This increases profit by $(1 - \delta^{\tau-t})(m(v^i) - m(v^j))$, contradicting the optimality of the original allocation. □

Using Lemma 1, we need only keep track of the k highest remaining valuations. At the start of time t suppose the seller has agents with valuations $\{y^1, \dots, y^k\}$, where $y^i \geq y^{i+1}$. Profit is described by the Bellman equation[13]

$$\tilde{\Pi}_t^k(y^1, \dots, y^k) = \max_{j \in \{0,\dots,k\}} \left[\sum_{i=1}^{j} m(y^i) + \delta \Pi_{t+1}^{k-j}(y^{j+1}, \dots, y^k) \right]$$

where $\Pi_{t+1}^k := E_{t+1}[\tilde{\Pi}_{t+1}^k]$. The Bellman equation says the seller receives the marginal revenue from the units she sells today plus the continuation profits from the remaining units. The seller's optimal strategy is thus to sell the first object to the highest value agent, subject to his value exceeding cutoff x_t^k. She then sells the second object to the second highest value agent, subject to his value exceeding cutoff x_t^{k-1}, and so forth. We can thus think of the items being awarded sequentially within a period.[14]

The following Lemma shows that when $\{x_t^k\}$ are decreasing in k we can treat each unit separately, comparing the j^{th} cutoff to the corresponding agent's valuation.

Lemma 2. *Fix t and suppose $\{x_t^k\}$ are decreasing in k. Then unit j is allocated to agent i at time t if and only if*
(a) v_i exceeds the cutoff x_t^j.
(b) v_i has the $(k-j+1)^{th}$ highest valuation of the currently present agents.

[13] When $j = 0$ the first term in the summation is zero.

[14] Formally, a *cutoff* x_t^k is defined as the value of y^1 such that the seller is indifferent between selling today and waiting.

Proof. Suppose agent i is allocated good j, then (a) and (b) are satisfied.

Suppose (a) and (b) are satisfied. Then there are $(k-j)$ agents with higher valuations than i. Since the cutoffs are decreasing in k, these valuations exceed their respective cutoffs. Hence object j is allocated to agent i. $\square$

Proposition 2 shows the cutoffs are decreasing in k, and explicitly solves for the optimal cutoffs. In the period $t = T$, the seller wishes to allocate the goods to the k highest value buyers, subject to these values exceeding the static monopoly price. Hence,

$$m(x_T^k) = 0 \qquad \text{for all } k. \tag{4.1}$$

Next, consider period $t = T-1$. If she allocates the k^{th} good she gets $m(x_{T-1}^k)$. The opportunity cost is to wait one period and award the good either to agent x_{T-1}^k or the k^{th} highest new entrant. Hence,[15]

$$m(x_{T-1}^k) = \delta E_{T-1}\left[\max\{m(x_{T-1}^k), m(v_T^k)\}\right] \tag{4.2}$$

In periods $t \leq T-1$, the seller is indifferent between selling to the cutoff type today and waiting one more period. If she sells today, she only sells one unit since $\{x_t^k\}$ are decreasing in k. If she waits, she sells at least one unit tomorrow by the one-period-look-ahead policy. We thus have:

$$\begin{aligned} m(x_t^k) &+ \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1, \dots, v_{t+1}^{k-1})\right] \\ &= \delta E_{t+1}\left[\max\{m(x_t^k), m(v_{t+1}^1)\}\right] + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{x_t^k, v_{t+1}^1, \dots, v_{t+1}^k\}_k^2)\right]. \end{aligned} \tag{4.3}$$

where the notation $\{x_t^k, v_{t+1}^1, \dots, v_{t+1}^k\}_k^2$ represents the ordered vector of the 2^{nd} to k^{th} highest choices from $\{x_t^k, v_{t+1}^1, \dots, v_{t+1}^k\}$. Notably, equation (4.3) is independent of the state $\{y^2, \dots, y^k\}$ for reasons explained below.

Proposition 2. *Suppose the seller has K units to sell and N_t are IID. The optimal allocation awards unit k at time t to the agent with the highest value exceeding a cutoff x_t^k. The cutoffs are characterised by equations (4.1), (4.2) and (4.3). These cutoffs are deterministic, and decreasing in t and k.*

Proof. See Appendix A.2 and A.3. $\square$

Proposition 2 has a number of important consequences. First, the cutoffs are uniquely determined. Intuitively, the sooner an agent buys a good the more his value affects overall

[15] To be more formal, if $y^1 > v_T^k$, the seller loses $y^1(1-\delta)$ by delaying. If $y^1 < v_T^k$, the seller loses $y^1 - \delta v^k$ by delaying. The seller is thus indifferent if y^1 satisfies (4.2).

profit. Hence the left hand sides of (4.2) and (4.3) have a steeper slope than the right hand sides.

Second, the cutoffs are independent of the current state $(y^2, \dots, y^k)$. Intuitively, at the cutoff, the seller is indifferent between selling to y^1 and waiting. In either case the allocation to $(y^2, \dots, y^k)$ is unaffected since, in any future state, this decision does not affect their rank in the distribution of agents available to the seller. This fact is used in equation (4.3), where we set $y^j = 0$ for $j \geq 2$. Moreover, since cutoffs are deterministic, we do not have to elicit the values of agents and, in continuous time, can implement the optimal allocations through posted prices (see below).

Third, the cutoffs increase when there are fewer units available (see Figure 2). Intuitively, if the seller delays awarding the k^{th} unit by one period then she can allocate it to the highest value entrant, rather than agent y^1. When there are more goods remaining, agent y^1 is more likely to be awarded the good eventually, reducing the option value of delay and decreasing the cutoff.

Fourth, the cutoffs for the last unit are identical to the one unit case and are therefore constant in periods $t \leq T-1$. The other cutoffs are decreasing over time (see Figure 2). The intuition, as above, is based on the fact that if the seller delays awarding the k^{th} unit by one period then she can allocate it to the highest value entrant, rather than agent y^1. As the game progresses, agent y^1 is more likely to be awarded the good eventually, reducing the option value of delay and decreasing the cutoff. Figure 2 shows that the cutoffs decrease rapidly as $t \to T$. Figure 3 shows the corresponding hazard rate of sale. The hazard rate with one unit remaining stays low until $t = T$, at which point it jumps to infinity (because of the fire sale). The hazard rate with two units remaining is qualitatively similar: it is low initially and rapidly rises as we approach T, and therefore still resembles a fire-sale.[16]

4.1 Implementation in Continuous Time

Suppose agents enter according to a Poisson process with parameter λ.[17] In period T, the optimal cutoffs are given by $m(x_T^k) = 0$. In period $t < T$, equation (4.3) becomes

$$rm(x_t^k) = \lambda E\Big[\max\{m(v) - m(x_t^k), 0\} + \Pi_t^{k-1}\big(\min\{v, x_t^k\}\big) - \Pi_t^{k-1}(v)\Big] \tag{4.4}$$

where v is drawn from $F(\cdot)$. Equation (4.4) states the seller is indifferent between selling today and delaying a little. The cost of delay is the forgone interest (the left-hand side); the benefit is the option value of a new buyer entering the market (the right-hand side). When compared to the single unit case (3.6), we see that delay leads to a higher marginal revenue tomorrow, if a

[16]We can also bound the k^{th} unit cutoff from above and below in periods $t < T$ by $\overline{x}^k$ and $\underline{x}^k$ as determined by $m(\overline{x}^k) = \delta E_{t+1}[\max\{m(\overline{x}^k), m(v_{t+1}^1)\}]$ and $m(\underline{x}^k) = \delta E_{t+1}[\max\{m(\underline{x}^k), m(v_{t+1}^k)\}]$.

[17]Li (2009) extends our results by providing an implementation in discrete time.

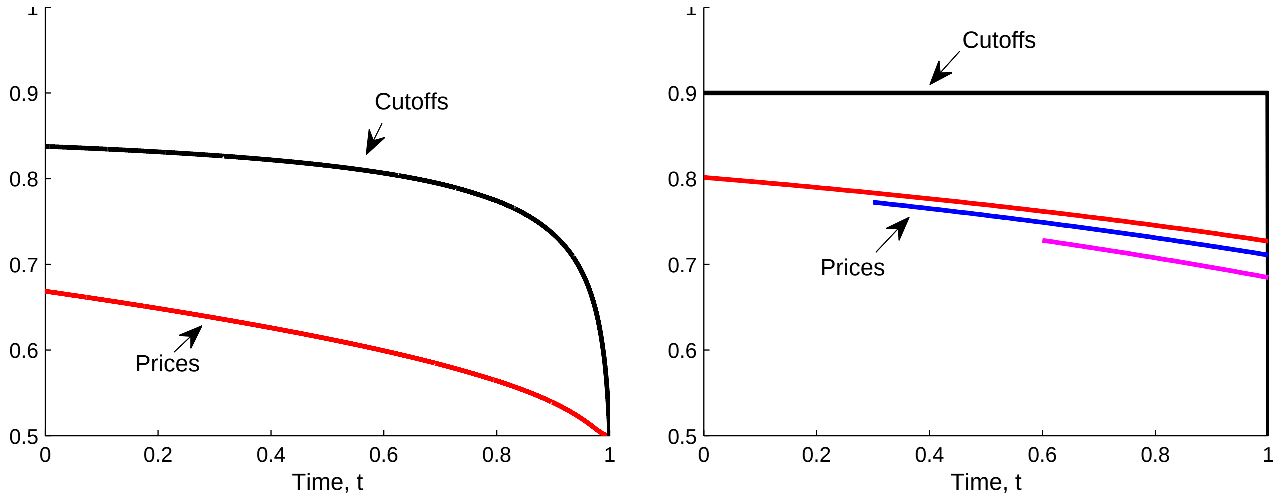

Figure 2: **Optimal Cutoffs and Prices with Two Units.** The **left** panel shows the optimal cutoffs/prices when the seller has two units remaining. The **right** panel shows the optimal cutoffs/prices when the the seller has one unit remaining. The three price lines illustrate the seller's strategy when it sells the first unit at times $t = 0$, $t = 0.3$ and $t = 0.6$. In this figure, agents enter continuously with $\lambda = 5$ and have values $v \sim U[0, 1]$. Total time is $T = 1$ and the interest rate is $r = 1/16$.

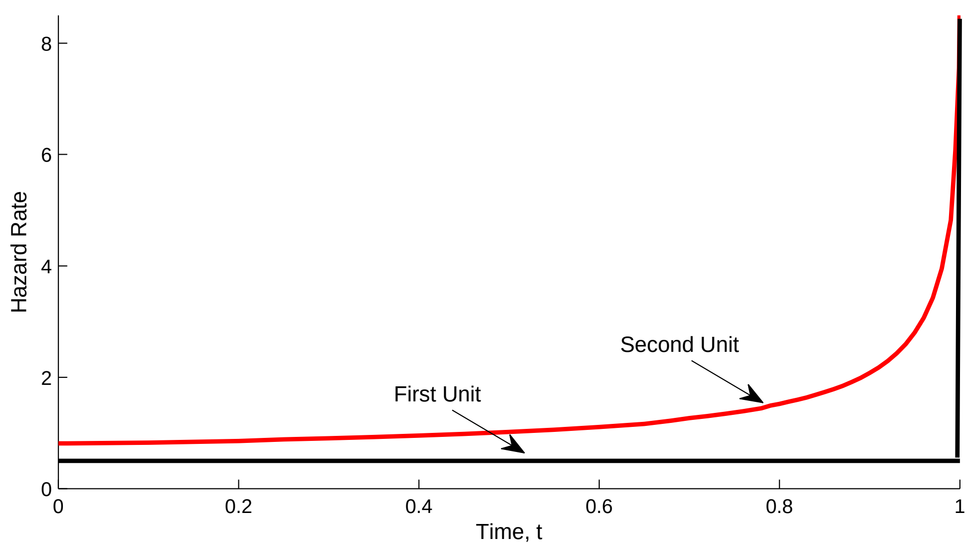

Figure 3: **Hazard Rates with Two Units** This figure shows the probability the first/second unit is sold at time $t + dt$, conditional on there being one/two units remaining at time t. We assume agents enter continuously with Poisson parameter $\lambda = 5$ and have values $v \sim U[0, 1]$. Total time is $T = 1$ and the interest rate is $r = 1/16$.

new agent enters, and a lower state variable in the continuation game. While the continuation value depends on the values of the highest $k-1$ agents, the difference in continuation values only depends on the highest value (Lemma 4). This enables us to write Π_t^{k-1} as a function of one variable and, when computing the cutoffs, assume there is only one buyer present. Using Lemmas 5–7, equation (4.4) implies that x_t^k is uniquely determined, and decreasing in k and t. When $k=1$, x_t^1 is constant for all $t<T$, and jumps down discontinuously at $t=T$. For $k\geq 2$, $\Pi_t^{k-1}(v)\to m(v)$ as $t\to T$, so (4.4) implies that $x_t^k\to m^{-1}(0)$, as shown in Figure 2.

We can implement the optimal allocations with prices $\{p_t^k\}$ and a fire-sale at time T for the last unit. We first wish to understand the limit of prices as $t\to T$, giving us a boundary point. For $k\geq 2$, $m(x_t^k)\to 0$ and hence the prices converge to $m^{-1}(0)$. For $k=1$, the seller can use a second-price auction with reserve $m^{-1}(0)$ at time T. When $t<T$, the price converges to

$$p_T = E_0\left[\max\{v_{\leq T}^2, m^{-1}(0)\}\middle| v_{\leq T}^1 = x^*, s_T(x^*)\right]$$

where x^* is the constant cutoff with one unit remaining, and $s_T(x)$ denotes the last time the cutoff went below x. Note that p_T depends on when other agents purchased units. In particular, the earlier those units were purchased, the more competition agent with type x^* expects at time T, and the higher is p_T (see Figure 2).

In earlier periods, the prices are such that the cutoff type is indifferent between buying now and waiting a little. If he waits, the price is a little lower; however the agent forgoes some utility, and the good may be taken by a new buyer or a buyer with a slightly lower valuation. Equating these terms yields[18]

$$\frac{dp_t^k}{dt} = \left[\frac{dx_t^k}{dt}(t-s_t(x_t^k))\lambda f(x_t^k) - \lambda(1-F(x_t^k))\right]\left[x_t^k - p_t^k - U_t^{k-1}(x_t^k)\right] - r\left(x_t^k - p_t^k\right) \tag{4.5}$$

where $U_t^{k-1}(x_t^k)$ is the buyer's utility at time t when there are $k-1$ goods left, conditional on $v_{\leq t}^1 = x_t^k$ and the history of the price path.[19] Fixing the cutoffs, prices fall faster if (a) the arrival rate is higher, (b) there is a high probability a new arrival will buy the good, or (c) the interest rate is higher, as in Section 3.2. In addition, equation (4.5) shows that prices fall faster if (d) the cutoffs fall quickly, or (e) there is a high probability a second agent has a value just below x_t. This second effect means that prices drop faster if buyers think they have more competition from existing buyers. Overall, the price path falls smoothly over time, but jumps up with every sale.

[18]For a derivation see Appendix A.5.

[19]Note: Utilities can be expressed in terms of future allocations via the envelope theorem.

5 Extension: Inventory Costs

In some applications the cost of delay is likely to be a function only of time, rather than proportional to values. (e.g. floor space in a shop). Suppose these costs are given by a convex function c_t for $t \in \{1, \ldots, T+1\}$ and let $\Delta c_t := c_{t+1} - c_t$ be the cost of a one period delay. A buyer's utility is given by (2.1), where we set $\delta = 1$. Adapting (2.4), the firm's profits are given by

$$\Pi_0^K = E_0 \left[\sum_i \sum_{s \geq 1} P_{i,s}[m(v_i) - c_t] + \left(K - \sum_i \sum_{s \geq 1} P_{i,s} \right) (-c_{T+1}) \right]$$

We can now state the analogue of Proposition 1.

Proposition 3. *Suppose $K = 1$ and c_t is convex. The optimal cutoffs x_t are uniquely determined by*

$$\begin{aligned} m(x_t) &= E_{t+1}[\max\{m(v_{t+1}^1), m(x_t)\}] - \Delta c_t \qquad \text{for } t < T \\ m(x_T) &= -\Delta c_T \end{aligned} \tag{5.1}$$

These cutoffs are decreasing over time.

Proof. Since Δc_t is increasing in t the cutoffs, as defined by (5.1), are decreasing in t. The rest of the proof is the same as Proposition 1. □

In continuous time, these allocations can be implemented by a deterministic price sequence p_t and a fire-sale at date T.[20] Suppose buyers enter with Poisson arrival rate λ and the inventory cost function $c(t)$ is differentiable, increasing and weakly convex.[21] The optimal cutoff at time $t < T$ is given by

$$c'(t) = \lambda E\big[\max\{m(v) - m(x_t), 0\}\big]$$

where v is drawn according to $F(\cdot)$. At time T, the optimal cutoff is given by $m(x_T) = -c'(T)$.

The agent's utility (2.1) is not affected by the inventory costs, so the implementation is the same as before. At time T, the seller can use a second-price auction with reserve $R_T = m^{-1}(-c'(T))$. The final "buy-it-now" price is given by

$$p_T = E_0\left[\max\{v_{\leq T}^2, m^{-1}(-c'(T))\}\big| N_0 = 1, v_{\leq T}^1 = \overline{x}_T\right]$$

[20] In discrete time, the optimal cutoffs can be implemented through a sequence of second-price auctions with deterministic reserve prices, as in Section 3.1.

[21] If $c(t)$ has kinks then x_t will sometimes jump down, requiring the use of an auction.

where $\overline{x}_T := \lim_{t\to T} x_t$. In earlier periods, prices are determined by

$$\frac{dp_t}{dt} = -(x_t - p_t)\left(-\frac{dx_t}{dt}\lambda t f(x_t) + \lambda_t(1 - F(x_t))\right)$$

which is similar to equation (4.5).

For simplicity, we have assumed there is only one good. When $K \geq 1$ and the per-unit inventory cost c_t is convex, the proof of Proposition 2 can be adapted to show the optimal cutoffs x_t^k are characterised by the one-period look ahead rule:[22]

$$\begin{aligned} m(x_t^k) &+ E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1, \ldots, v_{t+1}^{k-1})\right] \\ &= E_{t+1}\left[\max\{m(x_t^k), m(v_{t+1}^1)\}\right] + E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{x_t^k, v_{t+1}^1, \ldots, v_{t+1}^k\}_k^2)\right] - \Delta c_t \end{aligned}$$

for $t < T$, with $m(x_T^k) = -\Delta c_T$. These cutoffs are deterministic and decreasing in t and k. In continuous time, the optimal allocation can be implemented by posted prices plus an auction for the last unit in period T. The continuous time cutoffs are determined by

$$c'(t) = \lambda E\Big[\max\{m(v) - m(x_t^k), 0\} + \Pi_t^{k-1}\big(\min\{v, x_t^k\}\big) - \Pi_t^{k-1}(v)\Big]$$

where v is drawn from $F(\cdot)$. Prices are then determined by (4.5) with $r = 0$ and the auction for the last unit as above.

6 Extension: Varying Entry

This section analyses the optimal mechanism when the expected number of entrants varies over time. In Section 6.1 we suppose fewer agents enter over time, as the stock of potential entrants is used up. In Section 6.2 we suppose more agents enter over time, as word of the market's existence spreads. This analysis forms a bridge between models with no entry (e.g. Harris and Raviv (1981)) and the constant entry model in Section 3.

6.1 Decreasing Entry

We first show that, when entry is decreasing, the IID allocations and prices are easily generalized. In particular, the cutoffs are determined by a one-period-look-ahead policy and are deterministic.

[22]The proof of Proposition 2 has to be slightly adjusted. First, equations (A.5) and (A.6) have to be adjusted to include inventory costs; similarly equation (A.15) in Lemma 7. Second, in Lemmas 5-7, δ^τ should be replaced by $\mathbf{1}_{\tau\leq T}$.

Proposition 4. *Suppose $K = 1$ and N_t is decreasing in the usual stochastic order. Then the optimal cutoffs are characterised by (3.3). These cutoffs are decreasing over time.*

Proof. Since N_t is decreasing in the usual stochastic order, v_t^1 is decreasing in the usual stochastic order and x_t, as defined by (3.3), is decreasing in t. The rest of the proof is the same as Proposition 1. $\square$

In continuous time, these allocations can be implemented by a deterministic price sequence p_t and a fire-sale at date T.[23] Suppose buyers enter with Poisson arrival rate λ_t, which is decreasing in t. The optimal cutoff at time $t < T$ is given by

$$rm(x_t) = \lambda_t E\big[\max\{m(v) - m(x_t), 0\}\big]$$

where v is drawn according to $F(\cdot)$. At time T, the optimal cutoff is given by $m(x_T) = 0$.

At time T, the seller can implement the optimal allocation through a second-price auction with reserve $R_T = m^{-1}(0)$. The final "buy-it-now" price is given by

$$p_T = E_0\left[\max\{v^2_{\leq T}, m^{-1}(0)\}\big|N_0 = 1, v^1_{\leq T} = \overline{x}_T\right]$$

where $\overline{x}_T := \lim_{t\to T} x_t$. In earlier periods, prices are determined by

$$\frac{dp_t}{dt} = -(x_t - p_t)\left(-\frac{dx_t}{dt}\left(\int_0^t \lambda_\tau \, d\tau\right) f(x_t) + \lambda_t(1 - F(x_t)) + r\right) \tag{6.1}$$

which is similar to equation (4.5).

To illustrate, suppose a seller puts her house on the market. There is an initial stock of buyers who have a high probability of seeing the newly listed house, plus a constant inflow of new buyers (where $T = \infty$). In the optimal mechanism, there is an introductory period where cutoffs and price fall quickly, with some buyers strategically waiting. In the limit, where existing buyers see the new house immediately, the seller reduces prices instantly in the form of a Dutch auction. After this introductory period, prices coincide with the cutoffs, and are constant over time, so that no buyer ever delays.

For simplicity, we have assumed there is only one good. When $K \geq 1$, Proposition 2 applies to the decreasing entry case, and optimal cutoffs are characterised by equations (4.1), (4.2) and (4.3).[24] As before, these cutoffs are deterministic, and decreasing in t and k. In continuous time, the optimal allocation can be implemented by posted prices plus an auction for the last

[23] In discrete time, the optimal cutoffs can be implemented through a sequence of second-price auctions with deterministic reserve prices, as in Section 3.1. In order to do this, however, agents' time of entry must not lead to different beliefs about the number of competitors, which is satisfied if N_t comes from a Poisson distribution.

[24] Proof: Replace Lemma 7 with Lemma 7′ in Appendix A.4.

unit in period T. The continuous time cutoffs are determined by (4.4), replacing λ with λ_t. Similarly, the price path is determined by (4.5), again replacing λ with λ_t.

6.2 Increasing Entry

When the number of entrants increases over time, the one-period-look-ahead policy fails. Intuitively, because the number of entrants is rising, the seller wishes to increase the cutoff. If the seller does not serve a cutoff type x_t in period t, she will therefore not return to that agent until period T. As a result the optimal allocations depend on the number of entrants in all future periods, not just the adjacent period.

Recursively define the following functions:

$$\begin{aligned} \pi_t(v) &= E_t\big[\max\{m(v_t^1), \delta\pi_{t+1}(\max\{v, v_t^1\})\}\big] \qquad \text{for } t < T \\ \pi_T(v) &= E_T\big[\max\{m(v), m(v_T^1), 0\}\big] \end{aligned} \tag{6.2}$$

This looks similar to equation (3.2), but is simpler because an agent who does not receive the good at time t need not be considered again until period T.

Proposition 5. *Suppose $K = 1$ and N_t is increasing in the usual stochastic order. Then the optimal cutoffs are given by*

$$\begin{aligned} m(x_t) &= \delta\pi_{t+1}(x_t) \qquad \textit{for } t < T \\ m(x_T) &= 0. \end{aligned} \tag{6.3}$$

These cutoffs are increasing over time, for $t < T$.

Proof. See Appendix A.6. □

When the number of entrants increases over time, the optimal cutoffs (6.3) also increase. As a result, an agent either buys when he enters the market or waits until the final period. This means that, unlike the one-period-look-ahead policies in Propositions 1–4, the optimal cutoffs depend on the future of the game. Consequently, today's cutoff increases if either the game becomes longer, or the future number of entrants rises.

In continuous time, these allocations can be implemented by a deterministic price sequence p_t and a fire-sale at date T.[25] Suppose buyers arrive with Poisson arrival rate λ_t, which is increasing in t. We can define functions corresponding to (6.2) using the end point $\pi_T(v) = v$

[25] In discrete time, the optimal cutoffs can again be implemented through a sequence of second-price auctions with deterministic reserve prices, as in Section 6.1.

and the differential equation

$$r\pi_t(v) = \frac{d\pi_t(v)}{dt} + \lambda_t E\Big[\max\big\{m(v'), \pi_t(\max\{v, v'\})\big\} - \pi_t(v)\Big] \tag{6.4}$$

where v' is the value of the new entrant and is drawn from $F(\cdot)$. Equation (6.4) says that asset value of profits are determined by the increase in their value and the option value from new entrants arriving. We can now define the optimal cutoffs. At time T, the optimal cutoff is given by $m(x_T) = 0$. At time $t < T$, the optimal cutoff is given by $m(x_t) = \pi_t(x_t)$.

At time T, the seller can implement the optimal allocation through a second-price auction with reserve $R_T = m^{-1}(0)$. For $t < T$, the prices are determined so that buyer x_t is indifferent between buying in period t and waiting until the fire-sale. That is,

$$(x_t - p_t) = e^{-r(T-t)} \Pr(v^1_{\geq t} \leq x_t) E\big[x_t - \max\{v^2_{\leq T}, m^{-1}(0)\}\big| N_0 = 1, v^1_{\leq T} = x_t\big] \tag{6.5}$$

where $v^1_{\geq t}$ is the highest order statistic of the buyers who have entered after time t. Let

$$\psi_t := e^{-r(T-t)} \Pr(v^1_{\geq t} \leq x_t) = e^{-r(T-t)} e^{-(\int_t^T \lambda_\tau \, d\tau)(1-F(x_t))}.$$

Note that ψ_t increases in t, and that $\psi_T = 1$. Prices are then given by

$$p_t = (1 - \psi_t)x_t + \psi_t E[\max\{v^2_{\leq T}, m^{-1}(0)\}|N_0 = 1, v^1_{\leq T} = x_t], \tag{6.6}$$

Over time, the optimal posted prices will tend to rise and then fall. Intuitively, as t grows so the cutoff increases, increasing the first term in (6.6). However, as $t \to T$, the fire-sale at T comes closer, decreasing agents willingness to delay and increasing the weight on the second term in (6.6). If we take $T \to \infty$, then the right hand side of (6.5) converges to zero and $p_t \to x_t$ for all t. This follows from the fact that a buyer who delays at time t must wait until period T to have another opportunity to buy.

7 Extension: Partially Patient Agents

One limitation of our analysis is that we do not allow for heterogeneity in the timing of buyers' demands. That is, a type-v agent who enters in period 1 has the same valuation in period t as a type-v agent who enters in period t. This is a problematic assumption for some applications, since buyers may exit the market (for example, a customer may buy another airline ticket), or buyers' valuations may decline relative to the entrants (for example, a customer's value for a seasonal piece of clothing declines after his vacation). In this section we consider these two perturbations of the model: In Section 7.1 we suppose buyers' values decline deterministically relative to those of new entrants; In Section 7.2 we assume that buyers exit stochastically. These results highlight the difficulties these considerations create and bridge our results with the analysis of impatient agents (e.g. Vulcano, van Ryzin, and Maglaras (2002)).

7.1 Declining Values

For the first model, assume that an agent with value v who enters in period t and buys in period s receives utility

$$\delta^s \beta^{s-t} v \tag{7.1}$$

where $\beta \in [0,1]$. When $\beta = 1$ this coincides with the model in Section 3; when $\beta = 0$ this coincides with the model of impatient agents. Following the derivation in Section 2, profits are given by

$$\Pi_0 = E_0 \left[\sum_i \sum_{s \geq 1} P_{i,s} \delta^s \beta^{s-t_i} m(v_i) \right]$$

It will be convenient to think of the state variable as the highest marginal revenue, $\hat{m}$, rather than the highest valuation. Recursively define the following functions:

$$\begin{aligned} \pi_t(\hat{m}) &= E_t\big[\max\{m(v_t^1), \delta\pi_{t+1}(\max\{\beta\hat{m}, m(v_t^1)\})\}\big] \qquad \text{for } t < T \\ \pi_T(\hat{m}) &= E_T\big[\max\{\beta\hat{m}, m(v_T^1), 0\}\big] \end{aligned} \tag{7.2}$$

This looks similar to equation (3.2), but is simpler because, if the seller delays at time t then she does not return to that buyer until period T.

Proposition 6. *Suppose $K = 1$, N_t is IID, and agents have declining values (7.1). At time $t < T$, the optimal mechanism awards the good to the highest value agent who enters in time t, if this value exceeds a cutoff x_t defined by*

$$m(x_t) = \delta\pi_{t+1}^t(m(x_t)). \tag{7.3}$$

These cutoffs have the property that $m(x_t) \geq m(x_{t+1}) \geq \beta m(x_t)$ for $t < T-1$. At time T, the good is awarded to the agent with the highest discounted marginal revenue, $\beta^{T-t_i} m(v_i)$, providing it is positive.

Proof. See Appendix A.7. □

Proposition 6 tells us that, when agents are only partially patient, the one-period-look-ahead policy fails to hold. In particular, an agent either buys when he enters the market or waits until period T. As in models with impatient agents, the cutoffs fall over time as the seller's options shrink. However, since the seller can always return to an old agent, the rate of decline is bounded below by β.

From equation (7.2) one can see that the seller's profits are increasing in β. That is, the seller prefers agents to be forward looking. While it may seem counterintuitive that allowing inter-temporal arbitrage benefits the seller, delay allows the seller to merge buyers from different cohorts and obtain a more efficient allocation. This suggests that if the seller can run an optimal mechanism, she should embrace price prediction sites such as bing.com, rather than viewing them as a threat to inter-temporal price discrimination.

We now turn to implementation. In period $t < T$, the optimal mechanism awards the good to the entrant with the highest value exceeding the cutoff. If the generalized failure rate $vf(v)/(1-F(v))$ is creasing in v then $m(x_{t+1}) \geq \beta m(x_t)$ implies that $x_{t+1} \geq \beta x_t$ so the time-t cutoff type will not wish to buy at time $t+1$. As a result, we can implement the optimal mechanism via simple second-price auctions with appropriate reserve prices, despite the new and old entrants being asymmetric. Similarly, in continuous time the optimal mechanism can be implemented via posted prices.

At time T, the optimal mechanism allocates the good to the agent with the highest $\beta^{T-t_i} m\left(v_i\right)$ while a second-price auction would allocate it to the one with the highest $\beta^{T-t_i} v_i$. If the generalized failure rate is decreasing in v, $\beta^{T-t_1} v_1 = \beta^{T-t_2} v_2$ implies $\beta^{T-t_1} m(v_1) \geq \beta^{T-t_2} m(v_2)$ for $t_2 > t_1$. As a result, allocation is biased towards agents who enter the market earlier. Intuitively, allocating the object to an older buyer gives away fewer information rents because they have a higher valuation relative to their cohort. We can thus implement the optimal allocation by having agents register with the seller when they arrive in the market. The seller can then give a "discount voucher" to an agent who arrives early. For example, if $v \sim U[0,1]$ then the seller should give an agent who registers in period t a discount of $(1-\beta^{T-t})/2$.[26]

7.2 Disappearing Buyers

Another natural way to model the heterogeneity in agents' timing decisions is to allow buyers to exit probabilistically over time. If entry and exit times are private information of the buyers, the optimal mechanisms are very complicated, as discussed by Pai and Vohra (2008) and Mierendorff (2009). Even if we simplify the model to assume that agents have no private information about their exit times and assume each agent exits the game with probability ρ, the optimal allocations become much more complicated. In particular, the following example illustrates that the striking feature of our model — that the optimal cutoffs are deterministic — does not hold in a general model with random exits.

Suppose time is discrete, $T = 2$ and $K = 1$. Suppose there are two entrants in period

[26] Proof: The seller wishes award the good to the agent who maximises $\beta^{T-t}(2v-1)$, or equivalently $v\beta^{T-t} + (1-\beta^{T-t})/2$.

$t = 1$ and one more entering at $T = 2$, and values are distributed uniformly over $[0, 1]$, so that $m(v) = 2v - 1$. Solving for optimal cutoffs, $x_2 = \frac{1}{2}$ and x_1 is given by:

$$m(x_1) = \delta E_{v_3}[\max\{m(x_1), m(v_3)\}] \tag{7.4}$$

Next, suppose agents independently exit with probability ρ.[27] How does the optimal mechanism change? Without loss, suppose that $v_1 \geq v_2$. Then it is optimal to sell the good to agent 1 if and only if

$$m(v_1) \geq \delta E_{v_3}\left[(1-\rho)\max\{m(v_1), m(v_3)\} + \rho(1-\rho)\max\{m(v_2), m(v_3), 0\} + \rho^2 \max\{0, m(v_3)\}\right]$$

This expression is much more complicated than (7.4) because we need to take into account the risk of losing either of the two agents. Importantly, the possibility that agent 1 will exit and agent 2 will stay, makes the decision of whether to award the good to agent 1 today depend on the value of agent 2! For $\delta = 1$ and $\rho = \frac{1}{9}$ the optimal cutoff for agent 1 as a function of agent 2 value is:

$$x_1(v_2)\begin{cases} \approx 0.91 & \text{for } v_2 > 0.91 \\ = \frac{9}{8} - \frac{1}{24}\sqrt{79 - 64v_2^2} & \text{for } v_2 \in [0.5, 0.91] \\ \approx 0.79 & \text{for } v_2 < 0.5 \end{cases}$$

Hence the optimal cutoffs depend on the values of all players that have entered, and are not deterministic. While we can implement such an allocation through a direct revelation mechanism, it seems unlikely that any natural indirect mechanism, such as auctions or posted prices, will work.

8 Conclusion

We have characterized the optimal mechanism for a seller who wishes to sell K goods to buyers who enter the market over time and are patient. We have also shown that the optimal mechanism is deterministic and can be implemented by a sequence of prices with an auction for the final good at time T.

A major motivation for the paper was to introduce forward-looking buyers into a standard revenue management model. When compared to a model with myopic buyers, this change has three major consequences: first, cutoffs are determined by a simple one-period-look-ahead rule rather than a fully forward-looking optimal stopping problem; second, prices depend on the history of sales, since this affects the competition faced by current buyers; third, the seller should hold an auction at the end (or reduce prices rapidly) to harvest delaying buyers. The

[27] If exits are perfectly correlated (e.g. the good expires) then the expiration probability can be incorporated into the discount rate, and the analysis of Sections 3–4 is unchanged.

importance of these differences depends on the environment. The assumption of myopic buyers is without loss if entry is IID and either there are infinite periods (Gallien (2006)), or markets are large (Segal (2003)), since a constant price is optimal under either scenario. This means that properly modelling patient buyers is most important where the seller's options decline over time, or where the market is small. It suggests, for instance, that airline companies should be more concerned with forward-looking customers on their small flights than on their large ones.

A Omitted Proofs

A.1 Proof of Corollary 1

In period $t = T$, the optimal reserve price is $R_T = m^{-1}(0)$ and it is a weakly dominant strategy for an agent to bid his valuation, $b_T(v) = v$.

Consider period $t < T$. As we verify below, if we set R_t according to (3.5) then an agent with type x^* is indifferent between buying today and delaying, conditional on having the highest valuation. Since $\delta < 1$, types $v \geq x^*$ prefer to buy today and bid at least R_t, while types $v < x^*$ prefer to delay and do not bid. For an agent who bids above the reserve price, it is a weakly dominant strategy to bid his valuation.

We now verify the appropriate reserve price is defined by (3.5). Consider period $T-1$ and suppose $v^1_{\leq T-1} = x^*$. The reserve is determined by the indifference condition

$$(x^* - R_{T-1}) = \delta E_0 \left[\left(x^* - \max\{v^2_{\leq T-1}, v^1_T, m^{-1}(0)\} \right) \mathbf{1}_{v^1_T < x^*} \middle| N_1 \geq 1, v^1_{\leq T-1} = x^* \right]. \tag{A.1}$$

Rearranging (A.1), the reserve price is

$$\begin{aligned} R_{T-1} &= (1-\theta)x^* + \theta E_0 \left[\max\{v^2_{\leq T-1}, v^1_T, m^{-1}(0)\} \middle| N_1 \geq 1, v^1_{\leq T-1} = x^*, v^1_T < x^* \right] \\ &= (1-\theta)x^* + \theta E_0 \left[\max\{v^2_{\leq T}, m^{-1}(0)\} \middle| N_1 \geq 1, v^1_{\leq T} = x^* \right]. \end{aligned} \tag{A.2}$$

Next, consider period $t \leq T-2$. Type x^* should indifferent between buying and waiting. If he buys in period t he pays the reserve price, R_t; if he waits, we assume he buys in period $t+1$, since x^* is constant.[28] Hence the period-t reserve is determined by the AR(1) equation

$$(x^* - R_t) = \delta E_0 [(x^* - R_{t+1}) \mathbf{1}_{v^1_{t+1} < x^*} | N_1 \geq 1, v^1_{\leq t} = x^*] \tag{A.3}$$

Rearranging (A.3),

$$R_t = (1-\theta)x^* + \theta R_{t+1} \tag{A.4}$$

Using (A.2) and (A.4) the reserve price is given by (3.5).

A.2 Proof of Proposition 2

At time $t = T$ and $t = T-1$ the cutoffs are given by (4.1) and (4.2), as argued in the text. We now claim that $\{x^k_t\}$ are deterministic and decreasing in t and k. This is true for the last two periods. We now continue by induction.

[28] Since the cutoffs are constant, we can equally well assume that, if type x^* waits at time t, then he waits until the period $t = T$.

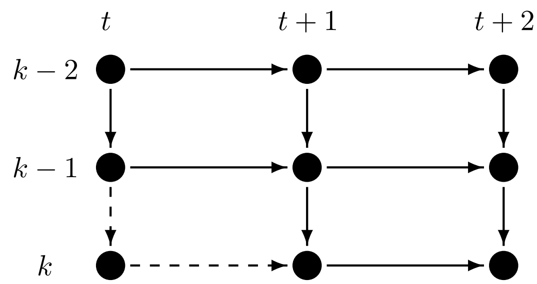

Figure 4: **Proof of Proposition 2.** This figure shows the order of cutoffs in induction step, where an arrow from $(k-1,t)$ to $(k-1,t+1)$ indicates that $x_t^{k-1} \geq x_{t+1}^{k-1}$. We know the relations indicated by dark arrows hold, and wish to prove the relations indicated by dashed arrows. Cases (a) and (b) correspond to the two ways these inequalities may not hold.

Definitions. At time t, suppose the state is $(y^1, y^2, \ldots, y^k)$. If the seller sells one unit today then continuation profit is

$$\begin{aligned}\Pi_t^k(\text{sell 1 today}) &= m(y^1) + \delta \Pi_{t+1}^{k-1}(y^2, \ldots, y^k) \\ &= m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{y^2, \ldots, y^k, v_{t+1}^1, \ldots, v_{t+1}^k\}_{k-1}^1)\right] \end{aligned} \tag{A.5}$$

If the seller sells one or more units tomorrow then she will obtain

$$\Pi_t^k(\text{sell tomorrow}) = \delta E_{t+1}\left[\max\{m(y^1), m(v_{t+1}^1)\}\right] + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{y^1, y^2, \ldots, y^k, v_{t+1}^1, \ldots, v_{t+1}^k\}_k^2)\right] \tag{A.6}$$

Denote the difference function by

$$\Delta\Pi_t^k(y^1, \ldots, y^k) = \Pi_t^k(\text{sell 1 today}) - \Pi_t^k(\text{sell tomorrow}).$$

As shown in Lemma 4 in Appendix A.3, $\Delta\Pi_t^k$ is independent of $\{y^2, \ldots, y^k\}$, so we can write it as a function of y^1 only.

Monotonicity in k and t. Since selling the last unit is identical to selling a single unit, the cutoffs are determined by (3.3) and obey $x_t^1 \geq x_{t+1}^1$. We now proceed by induction (see Figure 4). By contradiction, let $k \geq 2$ be the smallest number that either (a) $x_t^k > x_t^{k-1}$ or (b) $x_t^k < x_{t+1}^k$. If there are multiple t's that satisfy either condition, pick the higher number. [29]

Case (a). Consider good k in period t. We have

$$x_t^k > x_t^{k-1} \geq x_{t+1}^{k-1} \geq x_{t+1}^k, \tag{A.7}$$

so the k^{th} cutoff is decreasing in t. At the cutoff the seller is indifferent between selling today and waiting. If she sells today she earns $\Pi_t^k(\text{sell today}) \geq \Pi_t^k(\text{sell 1 today})$. If she waits then

[29] Since $x_t^{k-1} \geq x_{t+1}^{k-1} \geq x_{t+1}^k$, these two cases are mutually exclusive.

(A.7) implies that she chooses to sell good k tomorrow and $\Pi_t^k(\text{wait}) = \Pi_t^k(\text{sell tomorrow})$. The indifference condition therefore implies that, at the cutoff,

$$\Delta\Pi_t^k(x_t^k) \leq 0. \tag{A.8}$$

Consider the cutoff for good $k-1$ in period t. Since $\{x_t^j\}_{j<k}$ are decreasing in k, $\Pi_t^{k-1}(\text{sell today}) = \Pi_t^{k-1}(\text{sell 1 today})$. Since $x_t^{k-1} \geq x_{t+1}^{k-1}$, $\Pi_t^{k-1}(\text{wait}) = \Pi_t^{k-1}(\text{sell tomorrow})$. As a result,

$$\Delta\Pi_t^{k-1}(x_t^{k-1}) = 0. \tag{A.9}$$

We therefore conclude that

$$0 \geq \Delta\Pi_t^k(x_t^k) > \Delta\Pi_t^k(x_t^{k-1}) \geq \Delta\Pi_t^{k-1}(x_t^{k-1}) = 0. \tag{A.10}$$

yielding the required contradiction. In equation (A.10), the first inequality comes from (A.8). The second comes from $x_t^k > x_t^{k-1}$ and Lemma 5, which says that $\Delta\Pi_t^k(x)$ is strictly increasing in x. The third inequality comes from Lemma 6, which says that $\Delta\Pi_t^k(x)$ is increasing in k. The final equality comes from (A.9).

Case (b). Consider good k in period t. We have

$$x_t^k < x_{t+1}^k \leq x_{t+1}^{k-1} \leq x_t^{k-1}$$

so $\{x_t^j\}_{j\leq k}$ are decreasing in k. At the cutoff the seller is indifferent between selling today and waiting. If the seller sells today she earns $\Pi_t^k(\text{sell today}) = \Pi_t^k(\text{sell 1 today})$, since $\{x_t^j\}_{j\leq k}$ are decreasing in k. If she waits then she obtains $\Pi_t^k(\text{wait}) \geq \Pi_t^k(\text{sell tomorrow})$. The indifference condition implies that, at the cutoff,

$$\Delta\Pi_t^k(x_t^k) \geq 0 \tag{A.11}$$

Consider the cutoff for good k in period $t+1$. Since $\{x_t^j\}_{j\leq k}$ are decreasing in k, $\Pi_{t+1}^k(\text{sell today}) = \Pi_{t+1}^k(\text{sell 1 today})$. Since $x_{t+1}^k \geq x_{t+2}^k$, $\Pi_{t+1}^k(\text{wait}) = \Pi_{t+1}^k(\text{sell tomorrow})$. As a result,

$$\Delta\Pi_{t+1}^k(x_{t+1}^k) = 0. \tag{A.12}$$

We therefore conclude that

$$0 \leq \Delta\Pi_t^k(x_t^k) < \Delta\Pi_t^k(x_{t+1}^k) \leq \Delta\Pi_{t+1}^k(x_{t+1}^k) = 0. \tag{A.13}$$

yielding the required contradiction. In equation (A.13), the first inequality comes from (A.11). The second comes from $x_t^k < x_{t+1}^k$ and Lemma 5, which says that $\Delta\Pi_t^k(x)$ is strictly increasing in x. The third inequality comes from Lemma 7, which says that $\Delta\Pi_t^k(x)$ is increasing in t. The final equality comes from (A.12).

Summary. Given that $\{x_t^k\}$ are decreasing in k and t the optimal cutoffs are given by $\Delta\Pi_t^k(x_t^k) = 0$. Using Lemma 4 we can assume $y^j = 0$ for $j \geq 2$ and write this as (4.3).

A.3 Lemmas for Proof of Proposition 2

Lemma 3. *Fix t and suppose $\{x_s^k\}_{s\geq t+1}$ are decreasing in k. Suppose $y^1 \geq y^2 \geq \ldots \geq y^k$, and let $y^{j-1} \geq \tilde{y}^1 \geq y^j$. Then the difference*

$$\tilde{\Pi}_{t+1}^k(y^1, y^2, \ldots, y^k) - \tilde{\Pi}_{t+1}^k(\tilde{y}^1, y^2, \ldots, y^k)$$

is independent of $\{y^j, \ldots, y^k\}$.

Proof. Suppose the state is $(y^1, y^2, \ldots, y^k)$ and pick $i \geq j$. Since cutoffs are decreasing in k, Lemma 2 says the good is allocated to value y^i if and only if (a) given previous allocations (including those within the period), y^i has the highest value; and (b) y^i exceeds the current cutoff. Since this rule only depends on the rank of y^i, the allocation rule is the same as in state $(\tilde{y}^1, y^2, \ldots, y^k)$. Hence the difference in continuation profits is independent of y^i, as required. □

Lemma 4. *Fix t and suppose $\{x_s^k\}_{s\geq t+1}$ are decreasing in k. Then $\Delta\Pi_t^k(y^1, \ldots, y^k)$ is independent of $\{y^2, \ldots, y^k\}$.*

Proof. Case 1. Suppose $y^1 \geq v_{t+1}^1$. Then

$$\Pi_t^k(\text{sell 1 today}) = m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{y^2, \ldots, y^k, v_{t+1}^1, \ldots, v_{t+1}^k\}_{k-1}^1)\right]$$
$$\Pi_t^k(\text{sell tomorrow}) = \delta m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{y^2, \ldots, y^k, v_{t+1}^1, \ldots, v_{t+1}^k\}_{k-1}^1)\right]$$

Hence $\Delta\Pi_t^k = (1-\delta)m(y^1)$, which is independent of $\{y^2, \ldots, y^k\}$.

Case 2. Suppose $y^1 < v_{t+1}^k$. Then

$$\Pi_t^k(\text{sell 1 today}) = m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1, v_{t+1}^2 \ldots, v_{t+1}^{k-1})\right]$$
$$\Pi_t^k(\text{sell tomorrow}) = \delta E_{t+1}\left[m(v_{t+1}^1)\right] + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^2 \ldots, v_{t+1}^k)\right]$$

Hence $\Delta\Pi_t^k$ is independent of $\{y^2, \ldots, y^k\}$.

Case 3. Suppose $v_{t+1}^{j-1} > y^1 \geq v_{t+1}^{j}$ for $j \in \{2, \ldots, k\}$. Then

$$\Pi_t^k(\text{sell 1 today}) = m(y^1) + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1, v_{t+1}^2 \ldots, v_{t+1}^{j-1}, \{y^2, \ldots, y^k, v_{t+1}^j, \ldots, v_{t+1}^k\}_{k-j}^1)\right]$$
$$\Pi_t^k(\text{sell tomorrow}) = \delta E_{t+1}[m(v_{t+1}^1)] + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(y^1, v_{t+1}^2 \ldots, v_{t+1}^{j}, \{y^2, \ldots, y^k, v_{t+1}^j, \ldots, v_{t+1}^k\}_{k-j}^1)\right]$$

Since $\{x_s^k\}_{s\geq t+1}$ are decreasing in k, we can apply Lemma 3, implying that $\Delta\Pi_t^k$ is independent of $(y^2, \ldots, y^k)$. □

Lemma 5. $\Delta\Pi_t^k(y^1)$ *is strictly increasing in* y^1.

Proof. Using equation (A.5),

$$\frac{d}{dy^1}\Pi_t^k(\text{sell 1 today}) = m'(y^1)$$

Using equation (A.6) and the envelope theorem,

$$\frac{d}{dy^1}\Pi_t^k(\text{sell tomorrow}) = m'(y^1)\delta^{\tau_1^k(y^1)-t}$$

where $\tau_1^k(y^1)$ is the time y^1 buys when he's in first position at time t and there are k goods to sell. The result follows from the fact that $\tau_1^k(y^1) > t$ and $\delta < 1$. □

Lemma 6. *Fix* t *and suppose* $\{x_s^k\}_{s\geq t+1}$ *are decreasing in* k. *Then* $\Delta\Pi_t^k(y^1)$ *is increasing in* k.

Proof. Let $\{y^1, \ldots, y^k\}$ and $\{\tilde{y}^1, \ldots, \tilde{y}^k\}$ be arbitrary vectors, where $y^j \geq \tilde{y}^j$ for each j. Using equation (2.5),

$$\begin{aligned}
\tilde{\Pi}_{t+1}^k(y^1, \ldots, y^k) - \tilde{\Pi}_{t+1}^k(\tilde{y}^1, \ldots, \tilde{y}^k) &\geq \tilde{\Pi}_{t+1}^k(y^1, \ldots, y^{k-1}, \tilde{y}^k) - \tilde{\Pi}_{t+1}^k(\tilde{y}^1, \ldots, \tilde{y}^{k-1}, \tilde{y}^k) \\
&= \delta^{-(t+1)}\int_{\{\tilde{y}^1,\ldots,\tilde{y}^{k-1}\}}^{\{y^1,\ldots,y^{k-1}\}} (m'(z^1)\delta^{\tau_1^k(z^1)}, \ldots, m'(z^{k-1})\delta^{\tau_{k-1}^k(z^{k-1})})d(z^1, \ldots, z^{k-1}) \\
&\geq \delta^{-(t+1)}\int_{\{\tilde{y}^1,\ldots,\tilde{y}^{k-1}\}}^{\{y^1,\ldots,y^{k-1}\}} (m'(z^1)\delta^{\tau_1^{k-1}(z^1)}, \ldots, m'(z^{k-1})\delta^{\tau_{k-1}^{k-1}(z^{k-1})})d(z^1, \ldots, z^{k-1}) \\
&= \tilde{\Pi}_{t+1}^{k-1}(y^1, \ldots, y^{k-1}) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1, \ldots, \tilde{y}^{k-1}) \qquad \text{(A.14)}
\end{aligned}$$

The first line comes from the fact that $y^k \geq \tilde{y}^k$. The second line use the envelope theorem, where τ_j^k is the stopping time of the agent in the j^{th} position when there are k objects for sale. The third line follows from the fact that stopping times increase when the seller has one less object since $\{x_s^k\}_{s\geq t+1}$ are decreasing in k. The final line again uses the envelope theorem.

Looking at equations (A.5) and (A.6), observe that the vector

$$\{y^2, \ldots, y^k, v_{t+1}^1, \ldots, v_{t+1}^{k-1}\}_{k-1}^1$$

is pointwise larger than the vector

$$\{y^1, y^2, \ldots, y^k, v_{t+1}^1, \ldots, v_{t+1}^{k-1}\}_k^2.$$

The result follows from equation (A.14). □

Lemma 7. *Fix t and suppose $\{x_s^k\}_{s\geq t+1}$ are decreasing in s and k. Then $\Delta\Pi_{t+1}^k(y^1) \geq \Delta\Pi_t^k(y^1)$.*

Proof. If $y^1 \geq v_{t+1}^1$ then $\Delta\Pi_t^k(y^1) = (1-\delta)m(y^1)$ is independent of t, as shown in Lemma 4. We thus assume $y^1 < v_{t+1}^1$. Lemma 4 implies that the values below y^1 do not affect $\Delta W_t^k(y^1, \ldots, y^k)$, so we can set $y^j = 0$ for $j \geq 2$. For shorthand, write

$$\tilde{\Pi}_{t+1}^{k-1}(z) := \tilde{\Pi}_{t+1}^{k-1}(z, v_{t+1}^2 \ldots, v_{t+1}^{k-1}).$$

By definition of $\Delta\Pi_t^k$ we thus have,

$$\Delta\Pi_t^k(y^1) = m(y^1) - \delta E_{t+1}[m(v_{t+1}^1)] + \delta E_{t+1}[\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1)] \tag{A.15}$$

where $\tilde{y}^1 := \max\{y^1, v_{t+1}^k\}$. Using the envelope theorem,

$$\tilde{\Pi}_{t+1}^{k-1}(v^1) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1) = \delta^{-1}\int_{\tilde{y}^1}^{v_{t+1}^1} m'(z)\delta^{\overline{\tau}^{k-1}(z)-t}\, dz$$

where $\overline{\tau}^k(z)$ is the time the object is allocated to type z, holding $\{v_{t+1}^2, \ldots, v_{t+1}^{k-1}\}$ constant. As t increases the cutoff x_t^k decreases and $\overline{\tau}^k(z) - t$ falls. Hence $\delta^{\overline{\tau}^k(z)-t}$ and $\tilde{\Pi}_{t+1}^{k-1}(v_{t+1}^1) - \tilde{\Pi}_{t+1}^{k-1}(\tilde{y}^1)$ increases. Since N_t is IID, $\Delta\Pi_t^k$ increases, as required. □

A.4 Extending Proposition 2 to Decreasing Entry

Lemma 7′. *Fix t and suppose $\{x_s^k\}_{s\geq t+1}$ are decreasing in s and k. Then $\Delta\Pi_{t+1}^k(y^1) \geq \Delta\Pi_t^k(y^1)$.*

Proof. Let $\hat{v}_{t+2}^j$ be the order statistics at time $t+2$ if the number of bidders N_{t+2} were drawn from the distribution of entrants at time $t+1$. Define

$$\begin{aligned}\Delta\hat{\Pi}^k_{t+1}(y^1) &= m(y^1) + \delta E_{t+2}\left[\tilde{\Pi}^{k-1}_{t+2}(\{y^2,\ldots,y^k,\hat{v}^1_{t+2},\ldots,\hat{v}^k_{t+2}\}^1_{k-1})\right] \\ &\quad - \delta E_{t+2}\left[\max\{m(y^1), m(\hat{v}^1_{t+2})\}\right] + \delta E_{t+2}\left[\tilde{\Pi}^{k-1}_{t+2}(\{y^1,y^2,\ldots,y^k,\hat{v}^1_{t+2},\ldots,\hat{v}^k_{t+2}\}^2_k)\right]\end{aligned}$$

where we have replaced v^j_{t+2} with $\hat{v}^j_{t+2}$, for $j \in \{1,\ldots,k\}$. Since N_t is decreasing in the usual stochastic order, $\hat{v}^j_{t+2}$ exceeds v^j_{t+2} in the usual stochastic order. Since each entrant buys earlier under "sell tomorrow", this change increases Π^k_t(sell tomorrow) more than Π^k_t(sell 1 today). Hence $\Delta\Pi^k_{t+1}(y^1) \geq \Delta\hat{\Pi}^k_{t+1}(y^1)$.[30]

We now prove that $\Delta\hat{\Pi}^k_{t+1}(y^1) \geq \Delta\Pi^k_t(y^1)$. Since $\hat{v}^j_{t+2}$ and v^j_{t+1} have the same distribution, we can assume that $\hat{v}^j_{t+2} = v^j_{t+1}$ for each j. As in the proof of Lemma 7, consider the case where $y^1 < v^1_{t+1}$ and let $\tilde{y}^1 = \max\{y^1, v^k_{t+1}\}$. We then have,

$$\Delta\Pi^k_t(y^1) = y^1 - \delta E_{t+1}[m(v^1_{t+1})] + \delta E_{t+1}[\tilde{\Pi}^{k-1}_{t+1}(v^1_{t+1}) - \tilde{\Pi}^{k-1}_{t+1}(\tilde{y}^1)].$$

Using the envelope theorem,

$$\tilde{\Pi}^{k-1}_{t+1}(v^1) - \tilde{\Pi}^{k-1}_{t+1}(\tilde{y}^1) = \delta^{-1}\int_{\tilde{y}^1}^{v^1_{t+1}} m'(z)\delta^{\overline{\tau}^{k-1}(z)-t}\,dz$$

where $\overline{\tau}^k(z)$ is the time the object is allocated to type z, holding $\{v^2_{t+1},\ldots,v^{k-1}_{t+1}\}$ constant. Since the cutoff types are decreasing in k, agent z buys the first time (a) he has the highest valuation, and (b) his type exceeds the cutoff. Since (a) future order statistics are decreasing in t, and (b) future cutoffs decrease in t, $\overline{\tau}^k(z) - t$ decreases in t. Hence $\delta^{\overline{\tau}^k(z)-t}$ and $\tilde{\Pi}^{k-1}_{t+1}(v^1_{t+1}) - \tilde{\Pi}^{k-1}_{t+1}(\tilde{y}^1)$ increases in t. Since $\hat{v}^1_{t+2}$ and v^1_{t+1} have the same distribution, $\Delta\hat{\Pi}^k_{t+1}(y^1) \geq \Delta\Pi^k_t(y^1)$, as required. □

A.5 Derivation of Equation (4.5)

Suppose the highest agent at time t, y^1, has value equal to the cutoff, x^k_t. Let y^2 be the value of the second highest agent, and denote its distribution function conditional on the entire history of cutoffs by H_t. Cutoffs are decreasing over time so if y^1 delays, agent y^2 may buy. Given that arrivals are independent, this occurs with probability

$$1 - \Pr(y^1 \leq x^k_{t+dt} | y^1 = x^k_t, N_t = 1) = 1 - H_t(x^k_{t+dt}).$$

[30] This is analogous to Lemma 6.

Note that $H_t(x_t^k) = 1$ and the density is

$$h_t(x_t^k) = \Pr(y^2 = x_t^k | \text{ past cutoffs}) = \lambda(t - s_t(x_t^k))f(x_t^k)$$

where $s_t(x)$ is the last time the cutoff went below x. Prices are determined by the cutoff type's indifference condition,

$$\begin{aligned}(x_t^k - p_t^k) &= (1 - rdt - \lambda dt)\left[x_t^k - p_{t+dt}^k\right] H_t(x_{t+dt}^k) + (1 - rdt - \lambda dt)U_t^{k-1}(x_t^k)\left[1 - H_t(x_{t+dt}^k)\right] \\ &\quad + (\lambda dt)(x_t^k - p_{t+dt}^k)F(x_t^k) + (\lambda dt)U_t^{k-1}(x_t^k)(1 - F(x_t^k))\end{aligned}$$

Rearranging and letting $dt \to 0$ yields (4.5).

A.6 Proof of Proposition 5

In period T, the seller awards the good to the agent with the highest value, subject to his marginal revenue exceeding zero, implying that $m(x_T) = 0$. We next claim that x_t are weakly increasing for $t < T$. Suppose, by contradiction, that there exists $t < T-1$ such that $x_t > x_{t+1}$. Then the cutoff x_t is given by

$$m(x_t) = \delta E_{t+1}[\max\{m(x_t), m(v_{t+1}^1)\}] \tag{A.16}$$

This follows from the fact that type x_t will buy in period $t+1$ if he does not buy in period t. Now consider period $t+1$ and suppose the seller faces a buyer of value x_{t+1}. If the seller delays she obtains at least $\delta E_{t+2}[\max\{m(x_{t+1}), m(v_{t+2}^1)\}]$. Indifference therefore implies that

$$m(x_{t+1}) \geq \delta E_{t+2}[\max\{m(x_{t+1}), m(v_{t+2}^1)\}] \tag{A.17}$$

Since N_t is increasing in the usual stochastic order, v_{t+2}^1 is larger than v_{t+1}^1 in the usual stochastic order, so (A.16) and (A.17) imply $x_{t+1} \geq x_t$, yielding a contradiction.

Fix $t < T$. If the seller sells to type x_t, she obtains $m(x_t)$. If the seller delays, she obtains $\delta\pi_{t+1}(x_t)$, as defined by (6.2), where we use the fact that x_t will not buy in period $t+1$ because the cutoffs are increasing. The seller is indifferent between selling to type x_t and delaying, yielding (6.3), as required.

A.7 Proof of Proposition 6

We first show that for $t < T-1$, $m(x_{t+1}) \geq \beta m(x_t)$. By contradiction, let t be the last time this inequality is not satisfied, so $m(x_{t+1}) < \beta m(x_t)$. It follows that, if the seller chooses not to sell to type x_t at time t then she will sell at time $t+1$. Hence the time-t cutoff is determined by

$$m(x_t) = \delta E_{t+1}[\max\{\beta m(x_t), m(v^1_{t+1})\}] \tag{A.18}$$

where the left-hand side is the payoff today, and the right-hand side is the payoff from delaying using the fact that $m(x_{t+1}) < \beta m(x_t)$. At time $t+1$ the cutoff satisfies

$$m(x_{t+1}) \geq \delta E_{t+2}[\max\{\beta m(x_{t+1}), m(v^1_{t+2})\}] \tag{A.19}$$

where the right-hand side is lower bound on the value from delaying. From (A.18) and (A.19), $m(x_{t+1}) \geq m(x_t)$, which contradicts the assumption that $m(x_{t+1}) < \beta m(x_t)$.

Since $m(x_{t+1}) \geq \beta m(x_t)$, we know that if type x_t does not obtain a good in period t then he will not obtain one until period T. The resulting indifference equation yields equation (7.3).

Finally, since the seller can always choose allocate the good by period $T-1$, the profit function obeys $\pi_t(v) \geq \pi_{t+1}(v)$. Equation (7.2) thus implies that x_t decreases over time.

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