Caller Number Five:
*Timing Games that Morph from One Form to Another**

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Abstract

There are two varieties of timing games in economics: In a war of attrition, more predecessors helps; in a pre-emption game, more predecessors hurts. In this paper, we introduce and explore a spanning class with *rank-order payoffs* that subsumes both as special cases. In this environment with unobserved actions and complete information, there are endogenously-timed phase transition moments. We identify equilibria with a rich enough structure to capture a wide array of economic and social timing phenomena — shifting between phases of smooth and explosive entry.

We introduce a tractable general theory of this class of timing games based on potential functions. This not only yields existence by construction, but also affords rapid characterization results. We then flesh out the simple economics of phase transitions: Anticipation of later timing games influences current play — swelling pre-emptive atoms and truncating wars of attrition. We also bound the number of phase transitions as well as the number of symmetric Nash equilibria. Finally, we compute the payoff and duration of each equilibrium, which we uniformly bound. We contrast all results with those of the standard war of attrition.

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1 Introduction

Suppose that a radio call-in show awards Stones tickets to the seventy-seventh caller. If the number of other potential callers is known, and if waiting to call inflicts opportunity costs on listeners, when should they call? Intuitively, players initially strategically benefit from the delay, but eventually succumb to a fear of missing out. How long will the game last? What economic lessons can be gleaned from players' equilibrium timing behaviour?

Timing models in economics fall into one of two opposing camps. In a *war of attrition*, delay is exogenously costly, and each player prefers that others act before him. The situation is reversed in a *pre-emption game*, where the passage of time is exogenously beneficial, and players wish to pre-empt others. There are, however, many important strategic situations where players prefer to be neither first nor last. This new class offers many intriguing implications — like periods of slow entry interspersed with sudden rushes. The motivational radio show example aside, they also are widely applicable. For instance, entry into a growing potential new market is often most profitable for early firms after the leader — who struggle with neither market creation nor brand identification. In stock market run-ups, like the 1990s hi-tech sector, early and late sellers fare most poorly. The social phenomenon of fashionable lateness bespeaks a preference for a middling arrival rank. On the other hand, one seeks to be early or late in rush hour, but not in the middle.

We develop a comprehensive theory for complete information timing games where rewards depend on the players' *ordinal stopping ranks.* For instance, in a many-player war of attrition, the first stopper earns less than the second, who gets less than the third, etc. The reverse holds in a pre-emption game. In either case, rewards are monotonic in the ordinal stopping ranks. Our timing games subsume non-monotonic rank-rewards.

Returning to the motivational radio show example, one might well imagine that players wait to call, and suddenly enter en masse, jamming the phone lines. What in fact happens is more subtle. Since delaying is explicitly costly, agents are initially locked in a war of attrition. Everyone adopts a mixed strategy, and the chance of winning is ever increasing. Ideally each wants to enter when the *probability* that seventy-six have called is maximal. At that moment, everyone else would do likewise, triggering explosive entry. But the story does not end there. Only one stopper can win, which diminishes the value of the expected prize. The pre-emption moment advances backward in time until everyone is indifferent between pre-empting the entry atom and playing with the mass. Thus, the pre-emption atom 'prematurely' truncates the war of attrition phase. Relative to the direct sum of equilibria from two timing games, agents pre-empt earlier and do so with an excessively large mass. In our paper, both time and size of explosive entry moments are endogenous.

Towards a tractable theory, we assume unobservable actions, no time-exogenous payoff growth, and exogenous delay costs — specifically, discounting and direct opportunity costs. Thus, pure strategies are the stopping times, and the solution concept is Nash equilibrium. Players cannot foresee their stopping rank, and thus can only know their expected flow payoff if they stop. As is common in the timing game literature, we focus on symmetric equilibria in mixed strategies, which captures an anonymity of play natural in many contexts. We also exclude strategies explicitly depending on focal calendar times or random coordination devices like sunspots. With this proviso, we solve for the symmetric stationary Nash equilibria.

We show that each such equilibrium tractably admits a unique *potential function* that summarizes play. This function yields the equilibrium expected payoffs by differentiation. One example of a potential function is the greatest convex function below the integrated expected flow payoff, the convex hull. It always exists, and thus we have existence by construction (Theorem 1). In mechanism design, the convex hull is often used to "iron" non-monotonic payoffs; our potential function can be understood as a generalized ironing technique. Our approach to equilibria thus reduces to characterizing functions.

Turning to the equilibrium analysis, a war of attrition phase is intuitively possible only for rising expected payoffs — when strategic and exogenous delay costs conflict. Likewise, pre-emptive behaviour is mandated when expected flow payoffs fall since no conflict is possible. Hence, the slope sign changes of expected payoffs are critical. Theorem 2 bounds this number above by the underlying deterministic rank reward using the classical Descartes Rule of Signs; this provides a simple upper bound on the number of phase transitions that binds for some equilibria.

A phase transition from a war of attrition to a pre-emption game (or back) can only occur if expected flow payoffs before (or after) atomic entry coincide with the atomic payoff. As seen in our radio call-in show example, the slope of expected flow payoffs does not easily determine equilibrium play. Rather, the relation between expected flow and atomic rewards matters — and these jointly relate exactly as do marginals and averages (Lemma 1). Our first major qualitative finding about timing games builds on this insight to deduce that wars of attrition are "prematurely" truncated in equilibrium and pre-emption game atoms are likewise inflated: The war ends before expected flow payoffs peak and starts after they trough (Theorem 3).

With multiple phases possible, there are potentially multiple equilibria. Since wars of attrition and pre-emption games must alternate, the question is whether any consecutive pair of them is played. With J such matched pairs, this choice must be made for all of them. Theorem 4 therefore shows that the number of potential Nash equilibria equals 2^J.

In the benchmark war of attrition, all rents — namely, the greatest minus the least expected flow payoff — are dissipated. Here, this is not true because rank order payoffs are non-monotonic. For fewer rents are lost because the pre-emption games start before the peak flow expected payoff, when expected flow and average atomic payoffs coincide. Theorem 5 instead shows that the maximal payoff burn in the game is captured not by a difference of expected flow payoffs, but by a difference of the greatest backward average payoff and the least forward average payoff. Further, the game's expected payoff is at least the minimum of the forward average payoffs from the game's outset. This contrasts with the war of attrition, where the value is always the least expected flow payoff.

Our model discovers a dichotomy between rank payoffs and the time costs: Since costs play an important role in determining equilibrium strategies, one might think that not much can be said about the equilibrium without specifying them. However, the potential function is derived from the rank payoffs alone, which determines an equilibrium for *any* time costs. Further, the patterns of wars of attrition and explosive phases are identical functions of the stopping probability. For example, if the game starts with an explosive phase, the size of the first atom is independent of costs.

Many details can influence play in a timing game. While stylized, our formalization is the first that generally captures the important class of non-monotonic rank-payoffs, and it does so in a tractable manner. As usual, clarity of purpose most readily emerges in a simple environment, and our greatest simplification is assuming unobservable actions. This affords the general and yet tractable formulation. We conclude by briefly considering observable actions. This results in multiple information sets and greatly enriches the set of supportable equilibria (now subgame perfect). Still, we briefly argue that our main qualitative insight from Theorem 3 remains applicable with a simple refinement.

Our paper provides a useful set of tools for analyzing a richer class of timing games. We strongly believe that this class is economically important. To see this, consider the example of high-tech market entry with a middle mover advantage. A first mover advantage is often presumed in markets for new technologies. If this were always true, one ought not observe entry rushes for ranks three or four. Late rushes, however, are not uncommon. As an example take hybrid-powered vehicles: Honda was first (with its Insight in 1999); only much later Lexus, GM and Ford entered the market almost simultaneously (in 2005). Our model provides one explanation for this entry pattern — even though several companies may already own the technology, they let Honda create the market!

As another example, consider a car sales analyst who must predict customer demand and suppose that customers have middle-mover preferences.[1] It is important the he/she

[1]Mercedes' market research found that European customers decide almost a year ahead of time on

understands equilibrium behaviour. If the analyst ignores the middle-mover advantage and makes predictions based on past sales alone, then the atomic entry would take him/her by complete surprise. Knowing about the middle mover-advantage but ignoring that wars of attrition are truncated, he/she would expect the atom to be later, and, again, would be surprised by the early atom and by its large size.

Timing games are a largely settled corner of game theory. Maynard Smith (1974) first formalized the war of attrition for theoretical biology: Two animals are fighting for a fallen prey, the first to give up loses, and fighting is costly for both. With multiple players, payoffs are increasing in the stopping rank. Recently, Hendricks, Weiss, and Wilson (1988) have characterized continuous time complete information war of attrition-equilibria, while Bulow and Klemperer (1999) analyzed a generalized N-player war of attrition.

The pre-emption game has also been studied widely. An often ignored literature is the early work on tactical duels[2] — most simply, a two player zero-sum timing game, played on a compact (time)-interval. Two duelists shoot at each other with accuracy increasing in proximity, and they may or may not observe the other's shot. The flavour of the modern economics examples is best captured by 'Grab-the-Dollar': A player can either grab the money on the table or wait for one more period; meanwhile, the pot increases by one unit. Players want to be the first to take the money, but would rather grab a larger pot. Recent examples Abreu and Brunnermeier (2003), who model financial bubbles, and Levin and Peck (2003), who look at market entry.

In independent work, Sahuguet (2004) has explored the equilibria in a three player timing game with both pre-emption and attrition structures. His payoffs are not rank-dependent. Amidst this large literature on timing games, we believe that our respective works are the first that are neither just a pre-emption game nor just a war of attrition. We hope that it suggests a wider and richer application of timing games in economics.[3] It offers insight into periodic unexpected rushes of uncertain size, followed by relative quiet.

Overview. In Sections 2 and 3, we outline the unobservable actions model, and derive the key ideas for the equilibrium analysis. In Section 4, we bound the numbers of equilibria and phase transitions, and show how wars of attrition are truncated and pre-emptive atoms inflated in equilibrium. Section 5 bounds the payoffs and game durations of our equilibria. Section 6 lays out the equilibrium analysis for observable actions.

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new car purchases. So customers take their decision with unobservable actions.

[2]In 1949, the RAND Corporation kick-started the study of duels by organizing a conference with leading economists, statisticians, and economists — for an extensive survey see Karlin (1959).

[3]Shinkai (2000) developed a three-player Stackelberg-type game that fits our rank-payoff formulation: In his framework, quantity pre-emption and learning from predecessors' choices interact to effectively form U-shaped rank rewards. Shinkai, however, does not model the timing decision explicitly.

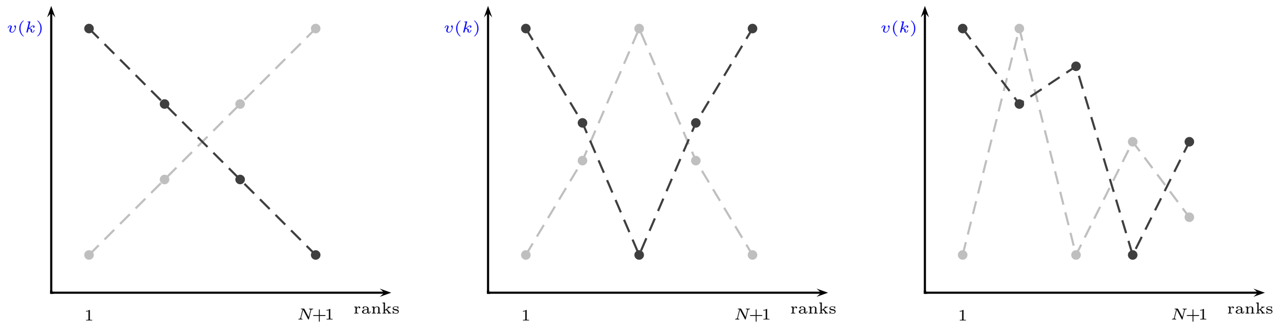

Figure 1: **Plots of rewards structures.** *Left Panel:* A stylized War of Attrition reward structure (gray, higher ranks yield higher rewards), and a stylized pre-emption game reward structure (black, low ranks are better). *Middle Panel:* Hill-shaped reward structure (gray, the some middle rank is best), and an 'avoid-the-crowd' U-shaped reward structure (black, either a very low or very high rank is best). *Right Panel:* Two general reward structures with multiple hills: there are multiple 'locally' optimal ranks.

2 A New Class of Timing Games

Players and Actions. Since we analyze settings where players desire to be neither first nor last, we assume $N+1 \geq 3$ players. Play transpires in continuous time, starting at time $t = 0$. Players are identical and have only two decisions: 'to stop' or 'not to stop'; they may stop only once; a stopping decision is irrevocable. Actions are unobservable.

Strategies. With unobservable actions, there is only one information set. (Such were called *silent* games of timing in Karlin (1959).) A player's strategy specifies the point in time when he will stop. A mixed strategy is, then, a non-decreasing and right-continuous function (cdf) $G : [0, \infty) \to [0, 1]$, where a player stops with chance $G(t)$ by time t.

Rank rewards. Upon stopping, a player receives a one-time lump-sum payment that depends on his ordinal stopping rank. This payment is captured in the *reward-function* $v : \{1, \ldots, N+1\} \to \mathbb{R}_+$. For instance, in a two-player war of attrition, $v(1) = 0$ and the prize is $v(2) > 0$. In the Caller Number Five game, $v(k) = 0$ for all $k \neq 5$, and the prize is $v(5) > 0$. In general, more predecessors helps in a war of attrition— or $v(k) < v(k+1)$ for all k. In a pre-emption game, the situation is reversed, as more predecessors hurts, or $v(k) \geq v(k+1)$ for all k. See Figure 1 for various rank-reward structures.

Payoffs for Simultaneous Stopping. Agents who stop at the same time equally share the available rank rewards. This is tractable and retains a single information set. It also realistically reflects the anonymity of random stopping. Players don't control their rank order among simultaneous stoppers, and therefore all rank order are equally likely.

Assume then that $k \in \{0, \ldots, N\}$ players have stopped, and $j+1 \in \{1, \ldots, N-k+1\}$ players stop together. Then the *atomic rewards* are the average rank reward $\mathsf{A}(k, j) := (v(k+1) + \cdots + v(k+j+1))/(j+1)$. For instance, in a war of attrition, if both agents stop immediately, then their order is randomly determined, and they share the prize equally.

Time costs. We consider two types of explicit costs: Continuous time discounting at the interest rate $r \geq 0$, and exogenous participation costs $c(t)$, with $c(0) = 0$, $\dot{c} > 0$, and $\lim_{t\to\infty} c(t) = \infty$.

Equilibrium. As is well-known, timing games may have asymmetric equilibria. While possibly yielding reasonable predictions in some situations, the literature has not focused on them, for instance, because they cannot capture games with anonymous roles. We follow in this tradition and explore symmetric strategy Nash equilibria. To avoid a continuum of arbitrary outcomes, we also confine attention to equilibria whose cdf G has convex *support* starting at 0. (The support of G is the set of all t with $G(t+\varepsilon) - G(t-\varepsilon) > 0$ for all $\varepsilon > 0$.) This restriction embodies a strong *stationarity* assumption: Strategies with no gaps cannot explicitly depend on calendar time — i.e. apart from how time influences the current cdf $G(t)$ or delay costs $c(t)$. It also rules out any dependence on random events, like sunspots (also incompatible with a unique information set).

3 Equilibrium Analysis

In this section, we outline several tools used in equilibrium analysis: necessary conditions for mixed strategies, atomic entry, potential functions, and general existence.

3.1 First Order Conditions for Smooth Entry

Consider a symmetric strategy $G(t)$. If $G(t) = g$, then k of N players independently have chosen to stop with chance $\binom{N}{k} g^k (1-g)^{N-k}$. Hence, if no one else enters at time t, then one necessarily secures rank payoff $v(k+1)$. Altogether,

$$\text{expected “flow” rewards } \phi(g) := \sum_{k=0}^{N} \binom{N}{k} g^k (1-g)^{N-k} v(k+1).$$

In any mixed strategy equilibrium, an agent must be indifferent about stopping at any point in time, so that expected flow payoffs are constant on the support.

Payoffs are discounted rewards less costs, or $e^{-rt}[\phi(G(t)) - c(t)]$. Assume $\dot{G}$ exists. Then in equilibrium, this is constant, and therefore

$$0 = -\dot{c} - r\ (\phi(G) - c) + \dot{G}\phi'(G) \tag{1}$$

Here $\dot{c} + r(\phi(G) - c)$ and $\phi'(G)$ are *marginal exogenous costs* and *marginal strategic gains* from delay. We can only solve for $\dot{G} \geq 0$ in any equilibrium if $\dot{c} + r(\phi(G) - c)$ and $\phi'(G)$ share the same sign — namely, if $\phi'(G) > 0$. For there must be a strategic incentive to delay, since one advances in the ranks to greater payoffs. This is true in a war of attrition.

3.2 Analogy for Atomic Rewards: Average vs. Marginal Revenue

Suppose that the N other players, acting independently, have stopped with chance $G(t) = g$ by time t, at which time each plays with chance $h-g>0$. Then the chance that players of ranks $k+1,\ldots,k+j$ stop at time t equals a trinomial coefficient $N!/k!j!(N-k-j)!$ times $g^k(h-g)^j(1-h)^{N-k-j}$. The expected payoff in this atom, should one also join, is then

$$\Lambda(g,h) := \sum_{k=0}^{N}\sum_{j=0}^{N-k} \frac{N!}{k!j!(N-k-j)!} g^k(h-g)^j(1-h)^{N-k-j}\mathsf{A}(k,j) \tag{2}$$

Thus, $\Lambda(0,h)$ is the payoff of an *initial atom* of size h, and $\Lambda(g,1)$ the payoff of a *terminal atom* of size $1-g$. When $0<g<h<1$, $\Lambda(g,h)$ is the average payoff in the *on-path atom* from g to h, and $\Phi(g) = \int_0^g \phi(x)dx$ the anti-derivative of $\phi(g)$. This motivates:

Lemma 1 $\Phi(h) - \Phi(g) = (h-g)\Lambda(g,h)$.

While it is possible to prove this algebraically, it is messy (and omitted). Yet it is intuitive: Independently place each of the N other players into the stopped, atom, and remaining categories, with respective weights $(g, h-g, 1-h)$. Thus, $\Lambda(g,h)$ is the expected average rank payoff in the atom category, when one is also included. Next independently assign each of the N other players the status 'stopped' (chance w) and 'not stopped'. Then $\Phi(x)/x$ is the expected average rank payoff in the stopped status with any weight $w \le x$, when one has also stopped. That $\Lambda(g,h) = (\Phi(h)-\Phi(g))/(h-g)$ follows analogously.

This has a nice illustrative analogue in standard producer theory: When AR and MR denote average and marginal revenue, and q is quantity, then $MR - AR = qAR'(q)$. Differentiating Lemma 1 w.r.t. h directly yields $\phi(h) - \Lambda(g,h) = (h-g)\frac{\partial}{\partial h}\Lambda(g,h)$. This admits an analogous interpretation: $h-g$ is the mass of the atom, and corresponds to the quantity. The expectation $\Lambda(g,h)$ aggregates and averages rewards, and $\phi(h)$ is the derivative of aggregated (non-averaged) rewards. Lemma 1 thus implies that $\phi(\cdot)$ crosses $\Lambda(g,\cdot)$ from above at the local interior maxima of Λ, and from below at the minima.

3.3 Potential Functions, Equilibrium, and Existence

A cdf $G:[0,\infty)\to[0,1]$ is a (symmetric stationary Nash) *equilibrium* if
E1: The support of G is a connected interval $[0,T]$ or $[0,\infty)$;
E2: $e^{-rt}[\phi(G(t))-c(t)]$ is the same constant for all times in the support of G with $G(t)<1$;
E3: If $G(t^*)>G(t^*-)$, then $\phi(G(t^*-))=\Lambda(G(t^*-),G(t^*))\ge\phi(G(t^*))$ (equality if $G(t^*)<1$)

Conditions $(E2)$ and $(E3)$ assert that net payoffs are constant along the support of play, and that there is no strict incentive to out-wait all other players.

In a symmetric mixed strategy equilibrium, $[0,1]$ partitions into subintervals having endpoints $0=\xi_0<\xi_1<\cdots<\xi_k=1$, with atomic entry ($G$ jumps) or smooth entry (G' exists) on alternating intervals $[\xi_i,\xi_{i+1}]$. To find an equilibrium cdf $G(t)$, we thus solve (1) subject to the right boundary conditions, determine atomic jumps so that ($E3$) is not violated, and ensure that the boundary conditions reflect the atomic jumps.

Since G is non-decreasing, costs are ever increasing. In equilibrium expected rank payoffs must compensate for these costs and so must also increase. We are now introducing a formulation that is able capture this idea in a very simple manner.

A C^2 function $\Gamma:[0,1]\to\mathbb{R}_+$ *induces* a strategy G for Φ if $\dot{G}=(\dot{c}+r[\Gamma'(G)-c])/\Gamma''(G)$ whenever $\Gamma(G(t))=\Phi(G(t))$, while if $\Gamma\neq\Phi$ on a maximal interval (g,h), then $G(\cdot)$ jumps from g to h. The function $\Gamma:[0,1]\to\mathbb{R}_+$ is a *potential function*[4] w.r.t. Φ if

P1: $\Gamma(0)=0$, $\Gamma(1)=\Phi(1)$, and $\Gamma'(1)\geq\Phi'(1)$;

P2: Γ is monotonically increasing, convex, and continuously differentiable;

P3: at each $\xi\in(0,1)$, either $\Gamma(\xi)=\Phi(\xi)$, or Γ is linear in an interval around ξ.

Note that ($P1$) asserts that $\Gamma(1)$ is the average rank payoff,[5] while $\Gamma'(1)\geq\phi(1)=v(N+1)$.

Lemma 2 (Equivalence Result) *Fix Φ. Any potential function Γ induces a unique equilibrium cdf G, and any equilibrium cdf G is induced by a unique potential function Γ.*

Proof: Fix a potential function Γ. By ($P3$), $[0,1]$ partitions into subintervals with endpoints $0=\xi_0<\xi_1<\cdots<\xi_k=1$, where[6] $\Gamma=\Phi$ or Γ is linear on alternating $[\xi_i,\xi_{i+1}]$.

First, $\Gamma(\xi_i)=\Phi(\xi_i)$ for all i, by ($P1$) or ($P3$). Assume Γ is linear on $[\xi_i,\xi_{i+1}]$. Then

$$\Lambda(\xi_i,\xi_{i+1})\equiv\frac{\Phi(\xi_{i+1})-\Phi(\xi_i)}{\xi_{i+1}-\xi_i}=\frac{\Gamma(\xi_{i+1})-\Gamma(\xi_i)}{\xi_{i+1}-\xi_i}=\begin{cases}\Gamma'(\xi_{i+1}) & = & \Phi'(\xi_{i+1}) & \text{if } \xi_{i+1}<1\\ \Gamma'(\xi_i) & = & \Phi'(\xi_i) & \text{if } \xi_i>0\end{cases}$$

by smoothness ($P2$) and Lemma 1. So ($E2$) obtains: Stoppers earn identical payoffs just *before* atomic entry if $\xi_i>0$, for then $\Lambda(\xi_i,\xi_{i+1})$ equals $\Phi'(\xi_i)=\phi(\xi_i)$, and *after* atomic entry if $\xi_{i+1}<1$, since $\Phi'(\xi_{i+1})=\phi(\xi_{i+1})$. Also, ($E1$) holds, as flow payoffs are positive, by $\Gamma(\xi_{i+1})>\Gamma(\xi_i)$. If $\xi_{i+1}=1$, then $\phi(1)=\Phi'(1)\leq\Gamma'(1)=\Lambda(\xi_i,1)$ by ($P1$); so ($E3$) holds.

Assume $\Gamma=\Phi$ on $[\xi_i,\xi_{i+1}]$, so that $\phi=\Phi'=\Gamma'$ (which exists by ($P2$)). We then needn't worry about ($E3$). Since Γ is convex by ($P2$) and Φ smooth, $\phi'=\Gamma''\geq 0$.

[4]Our phrase "potential function" is in the spirit of the harmonic function for a conservative vector field, which remains constant, and whose derivatives describe the gradient on the vector field. Hart and Mas-Colell (1989) may be the first to use the phrase "potential functions" in game theory; theirs was a function in a transferable utility game, whose differences yielded marginal payoff contributions. In Myerson (1981), what we call a potential function is related to the convex hull of integrated 'virtual valuations' in an auction design problem; the derivatives fix the priority level for allocating the good.

[5]$\Phi(1)=\int_0^1\sum_{k=0}^N\binom{N}{k}x^k(1-x)^{N-k}v(k+1)dx=\sum_{k=0}^N v(k+1)/(N+1)$, using the Beta distribution.

[6]By ($P3$) alone, the number k of such intervals may be infinite; Theorem 2 will rule out $k=\infty$.

Also, ϕ is strictly increasing inside the interval, being a nonconstant polynomial; thus, $(E1)$ holds, as ϕ can only initially vanish. Assume that $G(\underline{t}) = \xi_i$, for some $\underline{t} \geq 0$. Thus, the ODE $\dot{G} = (\dot{c} + r[\phi(G) - c])/\phi'(G)$ in (1) admits the "constant payoff" solution $e^{-rt}[\phi(G(t)) - c(t)] = \phi(0) = \Gamma'(0)$, the initial payoff. (Recall that the support of G includes 0.) Hence, $(E2)$ obtains. Let $C(t) := c(t) + e^{rt}\Gamma'(0)$. Since ϕ is strictly increasing on (ξ_i, ξ_{i+1}), $G(t) = \phi^{-1}(C(t))$ obtains on the domain $(\underline{t}, \bar{t})$, where $\bar{t} = C^{-1}(\xi_{i+1})$.

Next, fix an equilibrium $G(t)$. For $(P3)$ to hold, the potential function inducing this equilibrium is found via: $\Gamma(g) = \Phi(g)$ whenever $G(t)$ is continuous at $G^{-1}(g)$; at any jump from g to h, Γ is the linear function through $(g, \Phi(g))$ and $(h, \Phi(h))$, with slope

$$\frac{\Gamma(h) - \Gamma(g)}{h - g} = \frac{\Phi(h) - \Phi(g)}{h - g} \equiv \Lambda(g, h) \begin{cases} \geq \phi(h) & \text{with equality if } h < 1 \\ = \phi(g) = \Gamma'(g) & \text{if } g > 0 \end{cases} \tag{3}$$

by constant payoffs $(E2), (E3)$. This gives $(P2)$: Γ is differentiable, increasing (by (3) or by $\Gamma' = \phi > 0$), and convex: either Γ is linear, or has slope ϕ, increasing by $(E2)$.

Finally, we show $(P1)$. If $\Gamma = \Phi$ near 1, then $\Gamma(1) = \Phi(1)$ and $\Gamma'(1) = \Phi'(1)$. If $\Gamma = \Phi$ near 0, then $\Gamma(0) = \Phi(0)$. If $G(t)$ starts with a jump from 0 to h, then Γ has a linear segment with slope $\Phi(h)/h$ through $(h, \Phi(h))$. This forces $\Gamma(0) = 0$. If G ends with a jump to 1, then $\Gamma'(1)$ is the final linear slope, i.e. $\Gamma'(1) \geq \phi(1) = \Phi'(1)$ by (3). □

Example 1: Caller Number Two of Three. Assume $N+1 = 3$ and $v = (0, 1, 0)$. Then $\phi(g) = 2g(1-g)$ and $\Phi(g) = g^2(1-2g/3)$. There are exactly two potential functions: First, Γ may initially equal Φ, so that $\Gamma_1(g) = \Phi(g)$ for $g \leq 1/4$ and $\Gamma_1(g) = 3g/8 - 1/24$ for $g > 1/4$. Second, Γ may be initially linear, whereupon it remains linear on $[0, 1]$, by convexity, differentiability and $(P3)$: $\Gamma_2(g) = g/3$. These obey the key properties of smoothness, convexity and boundary values: e.g. $\Gamma_2'(1) = 1/3 > \Phi'(1) = 0$.

Assume delay costs $c(t) = t$ and no discounting. The first equilibrium involves smooth play described by the ODE $0 = -1 + \dot{G}(t)(1 - 2G(t))$ from (1), with solution $G(t) = 1/2 - 1/2\sqrt{1 - 2t}$ until $G(t) = 1/4$. At that point $t = 1/4$, a jump to $G = 1$ occurs. The second equilibrium entails simply a time-0 jump to $G = 1$.

Example 2: U-Shape. Assume $N+1 = 3$ and $v = (1, 0, 1)$. Then $\phi(g) = (1-g)^2 + g^2$ and $\Phi(g) = 1/3g^3 - 1/3(1-g)^3$. Here there is a unique potential function $\Gamma_3(g) = 5g/8$ for $g \leq 3/4$ and $\Gamma_3(g) = \Phi(g)$ for $g > 3/4$. Solving (1) yields $0 = -1 + 2\dot{G}(t)(2G(t) - 1)$, with solution $2G(t) = 1 + \sqrt{1/4 + 2t}$. Continuous play begins at $t = 0$, with $G(0) = 3/4$. Figure 2 illustrates both examples.

In mechanism design problems, non-monotonic payoff functions are often "ironed" to produce a monotonic function (e.g. Baron and Myerson (1982)). Namely, let $\mathsf{vex}(\Phi)$ be

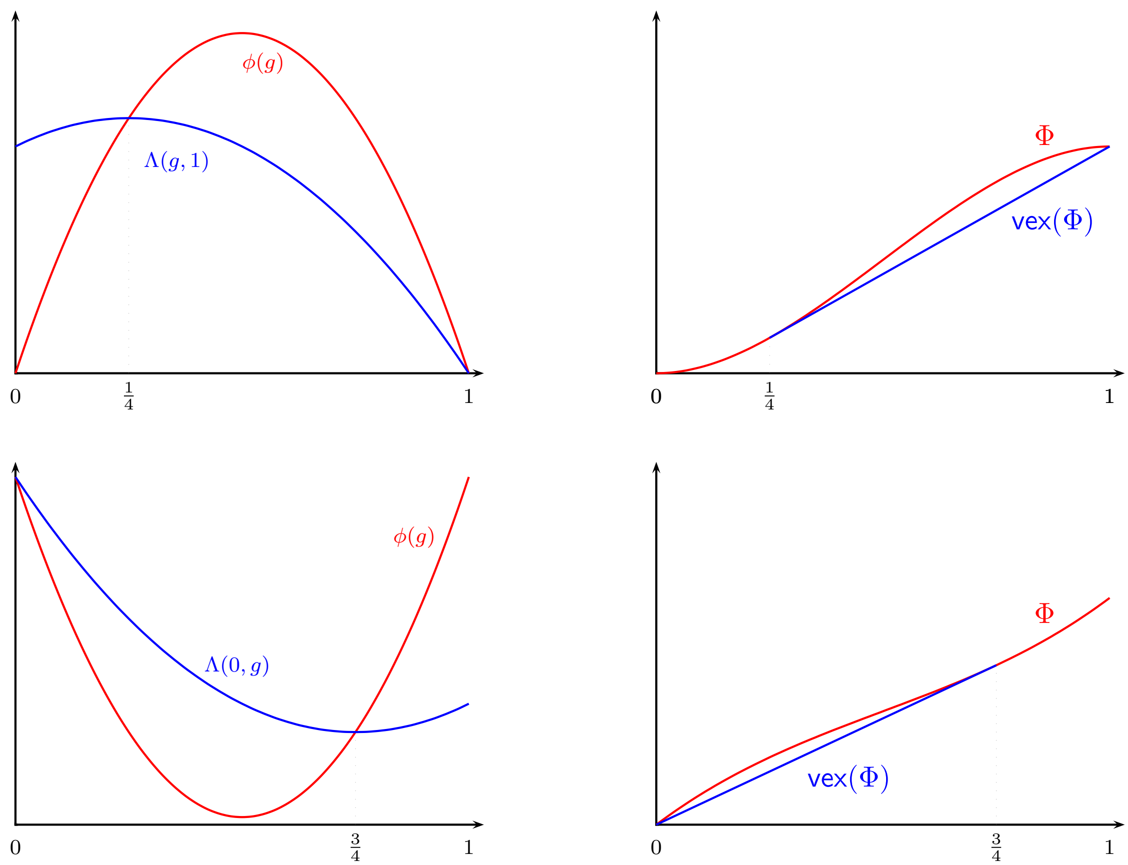

Figure 2: **Examples 1 and 2 from Section 3: Caller 2 of 3 and U-Shape.** The top left panel illustrates the flow payoff $\phi(g)$ and the average payoff $\Lambda(g,1)$ for the Caller 2 of 3 game; the equilibrium is as described in the introduction. The top right panel plots the running integral of payoffs Φ and the potential function $\mathsf{vex}(\Phi)$ identified in the existence Theorem 1. The bottom left figure plots ϕ and $\Lambda(0,g)$ for the U-shaped example, the bottom right figure plots Φ and the unique potential function $\mathsf{vex}(\Phi)$ for this example. The plots also illustrate the theorems that will be introduced in the next sections: Both examples attain the upper bound number of phases (Theorem 2), with two phases. Consistent with Theorem 3, the war of attrition is truncated in each case. Just as in Theorem 4, there are two equilibria in the top game (the potential function for the unit jump is not drawn), and one in the bottom game.

the convex hull of Φ, i.e. the largest convex function with $\mathsf{vex}(\Phi)(g) \leq \Phi(g)$ for every g. The "ironed" function then is the derivative $\mathsf{vex}(\Phi)'(g)$ (see Figure 3). Our potential functions follow a similar idea. Since costs are ever-increasing, so must be the expected rank-payoffs. Rank-payoffs however, may decline, and these non-monotonicities must be ironed away. Our potential function describes exactly how this works: its derivative is the rank payoff, its convexity ensures that equilibrium payoffs increase. If the potential function contains a linear segment, then rank payoffs are constant, and since delay is costly, atomic entry must occur.

Theorem 1 *A symmetric mixed strategy equilibrium exists and ends in finite time.*

Proof: Observe that $\mathsf{vex}(\Phi)$ is a potential function and thus induces an equilibrium.

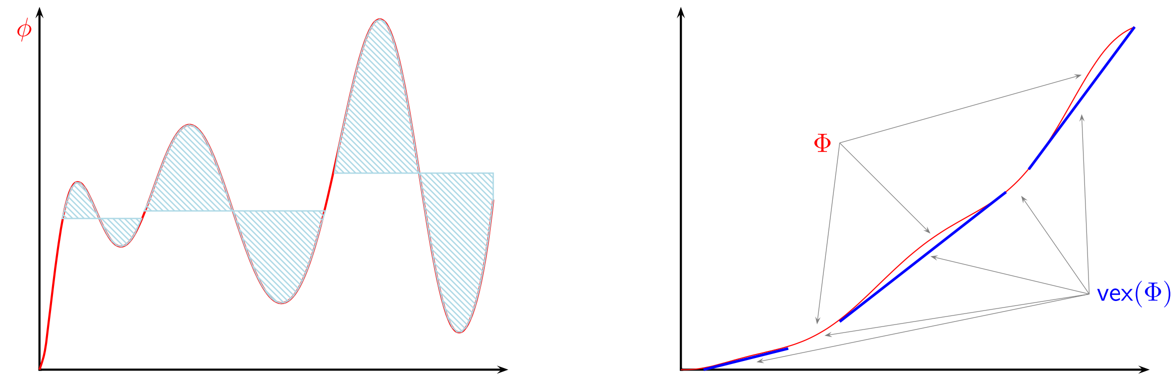

Figure 3: **Ironing ϕ.** The left panel illustrates the ironing procedure on ϕ, the right panel depicts both Φ and the convex hull of Φ, called $\mathsf{vex}(\Phi)$.

In any equilibrium, payoffs are constant on the support at $\phi(0)$. So there exists $\tilde{t} < \infty$ with $\max_g e^{-rt}[\phi(g) - c(t)] < \phi(0)$ after $\tilde{t}$. Delaying beyond $\tilde{t}$ is a dominated strategy, as rewards are discounted or consumed by exogenous delay costs ($\lim_{t\to\infty} c(t) = \infty$). □

In the Caller Number Two of Three example, $\mathsf{vex}(\Phi)(g) = \Gamma_1(g)$. In the U-shape example, $\Gamma_3(g)$ is the unique potential function, and therefore coincides with $\mathsf{vex}(\Phi)(g)$.

4 Behavioural Properties of Equilibria

4.1 Phases and Phases Transitions

We first bound the number of slope-sign changes of the expected flow rewards.

Lemma 3 (Variation Diminishing Property of Expected Rank Rewards) *Let the slope of rank rewards $v(k)$ change sign m times. Then the slope of expected rewards $\phi(g)$ changes sign at most m times, the number of sign-variations in ϕ is smaller by a multiple of two (including 0), and the signs of the first and last slopes of v and ϕ coincide.*

Proof: The derivative of $\phi(g)$ in g can be rearranged as follows:

$$\phi'(g) = \sum_{k=1}^{N} \binom{N}{k} k g^{k-1}(1-g)^{N-k}(v(k+1) - v(k)).$$

The first differences $v(k+1) - v(k)$ swap their sign m times. Scale ϕ' by $g/(1-g)^N$, and let $a_k := k\binom{N}{k}(v(k+1) - v(k))$ and $z := g/(1-g)$. Then

$$\frac{g}{(1-g)^N}\ \phi'(g) = \sum_{k=1}^{N} k\binom{N}{k}(v(k+1) - v(k))\left(\frac{g}{1-g}\right)^k = \sum_{k=1}^{N} a_k z^k =: P(z).$$

Obviously, $P(z(g))$ and $\phi'(g)$ enjoy the same number of sign variations, i.e. positive real roots of P. By *Descartes' Rule of Sign*, this number is at most the number of sign changes of its coefficients $a_0, a_1, \ldots, a_N$. Also, if smaller, it is smaller by a multiple of 2.

Finally, $\phi'(0) = v(2) - v(1)$ and $\phi'(1) = v(N+1) - v(N)$, proving the last clause. $\square$

As noted earlier, this paper subsumes the two classes of timing games. In a *war of attrition*, an exogenous delay cost opposes a strategic incentive to outwait others. The reverse holds in a *pre-emption game*, where delay is exogenously beneficial, and players wish to pre-empt others. We now categorize them by their strategic incentives:

Definition 1 (Phases)

- *If $\dot{G}(t+) > 0$ exists and $\phi'(G(t+)) > 0$ on some $(\underline{t}, \bar{t})$, then there is a* war of attrition*.*
- *If $\dot{G}(t+) > 0$ exists and $\phi'(G(t+)) < 0$ on $(\underline{t}, \bar{t})$, then there is a* slow pre-emption game*.*
- *If G jumps at t, as $G(t) > G(t-)$, then a* pre-emptive atom phase *obtains.*

A *phase transition* occurs at some time t if two distinct timing games obtain in every neighborhood of t. If three games obtain, then there are two phase transitions at t.

Theorem 2 (Phase Transitions)
(a) *Equilibrium play consists solely of an alternating sequence of wars of attrition and pre-emptive atom phases. There are no slow pre-emption games, and pre-emptive atoms subsume entire intervals when ϕ is decreasing.*
(b) *There are at most as many phase transitions as sign changes of $v(k) - v(k-1)$.*
(c) *If ϕ has m alternating slope signs, then the maximal number of phase transitions is $m-1$. This bound is attained in equilibrium iff $\mathsf{vex}(\Phi)$ touches every convex portion of Φ.*

Proof of (a): Expected payoffs are constant along the support of play. Delay is always exogenously costly, and thus in equilibrium, a player's expected payoff from rank rewards rises over time. So whenever $\phi' < 0$, any player must stop at once because delay is costly both strategically and exogenously. So play involves smooth wars of attrition and pre-emptive atoms. Consecutive wars of attrition or pre-emptive atoms can be merged.

Proof of (b): This follows because ϕ cannot have more interior extrema than v, by Lemma 3 — which also showed that the first and last slope signs of v and ϕ match.

Proof of (c): A phase transition occurs iff Γ switches between locally linear and strictly convex ($\Gamma'' > 0$). The smooth Γ only changes slope when $\Gamma = \Phi$. As a non-linear polynomial, Φ is strictly convex at most as many times as Γ, with equality iff ($\star$): Γ touches each convex portion of Φ. As $\mathsf{vex}(\Phi)$ is a potential function, this proves sufficiency. Next, assume ($\star$). The smooth Γ includes the unique supporting tangent line between all consecutive convex portions. The unique such potential function is $\mathsf{vex}(\Phi)$. $\square$

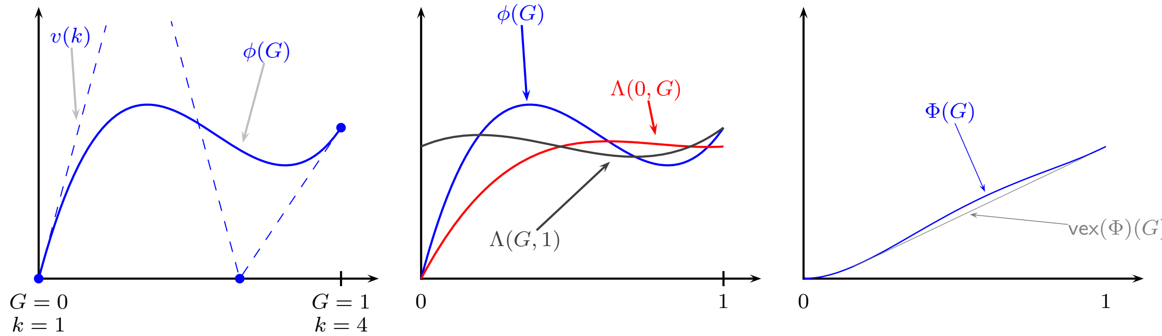

Figure 4: **The Zick-Zack Game**: In this merger of examples 2 and 3, rank payoffs now twice change slope, as $v = (0, \psi, 0, 1)$. If $v(2) = \psi$ (off the graph) is large enough, then the expected flow reward ϕ likewise has both a hill and a valley. Otherwise, ϕ is monotonically increasing. The middle panel plots the expected reward function ϕ and reward functions for initial atoms, $\Lambda(0, g)$ and terminal atoms $\Lambda(g, 1)$. The right panel plots $\Phi(g)$ and $\mathsf{vex}(\Phi)(g)$.

One can show that the maximum number of phase transitions is attained only if both the sequence of minima of $\Lambda(0, g)$ and the sequence of maxima of $\Lambda(g, 1)$ are increasing.

Example 3: Zick-Zack. The left panel of Figure 4 depicts the four player game *Zick-Zack*, with rank rewards $v = (0, \psi, 0, 1)$, with $\psi > 0$. Then

$$\begin{aligned}
\phi(g) &= 3g(1-g)^2 \cdot \psi + g^3 \cdot 1 \quad \text{and} \quad \phi'(g) = 3\psi(2g-1)^2 + 3(1-\psi)g^2, \\
\Phi(g) &= \big(1/4\ (1-(1-g)^4) - g(1-g)^3\big) \cdot \psi + g^4/4.
\end{aligned}$$

Analyzing $\phi(g)$, one can see that $\phi(g)$ is monotonic for $\psi \le 1 =: \underline{\psi}$, even though the underlying rank reward structure v has two slope-sign changes. This illustrates the strict inequality in Lemma 3, by a multiple of two. Then Φ is convex, and thus the unique potential function is $\Phi = \mathsf{vex}(\Phi)$ itself; there are no phase transitions (Theorem 2 (b)).

If $\psi > \underline{\psi}$, then ϕ has two slope-sign-changes, like v. The quartic polynomial Φ thus has two points of inflection, and $\mathsf{vex}(\Phi)$ must contain at least one linear portion. Hence, there can be at most two phase transitions (Theorem 2 (c)).

Next, $\mathsf{vex}(\Phi)$ touches both the first and second convex portions of Φ for $\underline{\psi} \le \psi \le \bar{\psi} := (5 + \sqrt{33})/4$. By Theorem 2 ($c$), the associated equilibrium has the maximum number of phase transitions (two): war of attrition, pre-emptive atom, and then war of attrition.

We have shown that having unobserved actions smoothes out rank payoffs in ϕ, and thereby reduces the number of phase transitions. But our equilibria are not merely the "direct sum" of the constituent timing games, as computed from the derivative ϕ' of the smoothed payoff function: It is not true that a war of attrition obtains iff $\phi' > 0$ and a pre-emption game obtains iff $\phi' < 0$. First of all, ϕ may be non-monotonic and yet there may be a unique pre-emptive atom — for instance, with $v = (2, 0, 1)$. More subtly, the slope ϕ' does not by itself determine the current timing game, because the relation of marginal and average rewards, ϕ and Λ, is critical. Pre-emptive atoms subsume intervals when ϕ is decreasing, by Theorem 2-(a). The reverse is not true, as we now flesh out.

Theorem 3 (Truncation and Atom-Inflation) *Pre-emptive atoms are inflated and wars of attrition truncated: Any pre-emptive atom subsumes at least some portion of the adjacent intervals where ϕ is increasing, and where a war of attrition is played.*

Proof: A linear portion of a potential function Γ must be a common tangent to distinct convex portions of Φ, and corresponds to a pre-emptive atom phase. If this tangent joins non-adjacent convex portions, then the atom is perforce inflated, as it subsumes at least one entire war of attrition phase. It therefore suffices to consider a common tangent τ of adjacent convex portions. Without inflation, such τ must touch at consecutive points of inflection of Φ, i.e. where $\phi'(g) = 0$. This is impossible, as it would slice through Φ. □

For instance, in the Caller Number Two of Three example, at most one phase transition occurs, since ϕ' changes its sign just once, from positive to negative when $g = 1/2$. Observe that the ODE defining the war of attrition is defined until time $t = 1/2$. While this may be its natural termination point, terminal atomic rewards are too small at that moment. Indeed, the atom would have size $G(1/2) = 1/2$, and $\Lambda(1/2, 1) = 1/3 < \phi(1/2) = 1/2$. Hence, the atom advances until $\phi(g)$ and $\Lambda(g, 1)$ cross. This occurs when $\Lambda(g, 1)$ has a maximum at $g = 1/4$, i.e. $G(3/8) = 1/4$. This is before time $t = 1/2$, hence truncation.

4.2 The Number of Equilibria

Let $\mathcal{E}_m$ denote the set of symmetric stationary Nash equilibria. Given the expected flow rewards ϕ, we can tie down the maximal cardinality of $\mathcal{E}_m$ as follows.

Theorem 4 (How many equilibria?) *Assume ϕ has exactly m alternating slope-signs. If ϕ slopes up at $g=0$, then $|\mathcal{E}_{2k}|, |\mathcal{E}_{2k+1}| \leq 2^k$; if it slopes down, $|\mathcal{E}_{2k-1}|, |\mathcal{E}_{2k}| \leq 2^{k-1}$.*

Proof: The number of equilibria is $2^{|\mathcal{J}_m|}$, where $\mathcal{J}_m$ is the set of up-slopes of ϕ followed by down-slopes. Why? An equilibrium implies a unique set of up-slopes played (the common tangent on pairs of strictly convex portions of Φ is unique). Indeed, an initial down-slope prior to $\mathcal{J}_m$ does not affect the number of equilibria, as the down-slope is skipped in a jump. A terminal up-slope likewise does not affect the number of equilibria, as it will either be skipped by a pre-emptive atom or played with a war of attrition, but not both. So there is a 1-1 map from equilibria $\mathcal{E}_m$ to sets $\mathcal{J}_m$ — hence, the power set enumeration for the upper bound of $|\mathcal{E}_m|$. □

For instance, the standard war of attrition has one slope-sign, and thus has $|\mathcal{E}_{2\cdot 0+1}| = 2^0 = 1$ equilibrium. The U-shaped game has two slopes, and slopes down first, so that it has maximally $|\mathcal{E}_{2\cdot 1}| = 2^{1-1} = 1$ equilibrium. Caller Number Two of Three has $m = 2$ slopes, but slopes up first, so that there are maximally $|\mathcal{E}_{2\cdot 1}| = 2^1$ equilibria.

For *Zick-Zack*, the theorem asserts that the terminal up-slope should not affect the maximum number of equilibria, i.e. still $|\mathcal{E}_{2\cdot 1+1}| \le 2^1$. Why? Clearly, if $\psi \le 1$, then the unique equilibrium is a war of attrition. If $\psi > 1$, then Φ has two points of inflection, and there are three possible potential functions: The first begins with a linear segment τ_0 that touches the second convex portion of Φ and is then strictly convex. The second is strictly convex, ending with a linear portion through $(1, \Phi(1))$. This linear segment τ_1 is tangent to the first convex portion of Φ and must have slope $\Gamma'(1) \ge \Phi'(1)$. The last potential function has a linear segment τ in the interior of $[0, 1]$ which is the unique *common tangent* to the first and second convex portions of Φ.

By construction, each of these potential functions is unique — if it exists. Observe that the tangent τ necessarily first touches Φ at some $g \in (0, 1)$, because $\Phi'(0) = \phi(0) = 0 < \phi(g) = \Phi'(g)$ for $g > 0$. However, its second touch point occurs at some interior $h < 1$ only in some conditions, namely iff $\psi \in [\underline{\psi}, \bar{\psi})$. Moreover, as is geometrically clear, the tangents τ and τ_1 coincide at the very moment that $\psi = \bar{\psi}$. The tangent τ_1 in fact exists for $\psi \ge \underline{\psi}_1 := (11 + 3\sqrt{17})/16$. But its slope only weakly exceeds $\Phi'(1)$ for $\psi \ge \bar{\psi}$, where $\bar{\psi} > \underline{\psi}_1$. Altogether, τ_1 is part of a potential function iff $\psi \ge \bar{\psi}$.

This illustrates why the terminal up-slope in Zick-Zack does not increase the number of equilibria relative to the Caller Number Two of Three game: tangent τ_1 represents a terminal atom skipping the last up-slope, while τ corresponds to an interior atom after which the terminal up-slope is played. Precisely one of the two obtains.

One can finally show that the initial tangent τ_0 exists for $9/5 := \underline{\psi}_0 \le \psi \le 3 := \bar{\psi}_0$. For $\psi > \bar{\psi}_0$, τ_0 is no longer tangent to the second convex portion of Φ. For when $\psi = \bar{\psi}_0$, τ_0 becomes a straight line from the origin to $(1, \Phi(1))$ corresponding to a time zero unit atom. In summary, the maximum number of equilibria (two) is attained iff $\psi \ge \underline{\psi}_0$.[7]

When is the maximum number of equilibria attained?[8] One may be tempted to think it sufficient that $\mathsf{vex}(\Phi)$ touches all convex portions of Φ, as in Theorem 2-(c). But the above analysis of Zick-Zack shows that this is not enough: For $\psi \in [\underline{\psi}, \underline{\psi}_0)$, $\mathsf{vex}(\Phi)$ touches

[7]The $\underline{\psi}_0, \underline{\psi}_1$ thresholds are most easily obtained via $\Lambda(0, g)$ and $\Lambda(g, 1)$: First, $\Lambda(0, g)$ has an interior maximum and minimum for all $\psi \ge \underline{\psi}_0$, and is monotonic for smaller ψ. If a potential function starts with a linear portion, then τ_0 is tangent to $\Phi(g)$ exactly when $\phi(g)$ and $\Lambda(0, g)$ intersect at an interior minimum of $\Lambda(0, g)$. The middle panel of Figure 4 illustrates this point. The threshold $\bar{\psi}_0$ for ψ allows $\phi(g)$ and $\Lambda(0, g)$ to cross at $g = 1$. The computations for $\underline{\psi}_1$ follow similar lines of reasoning using $\Lambda(g, 1)$. Finally, the payoff from the interior maximum of $\Lambda(g, 1)$ coincides with $\phi(1)$ at $\bar{\psi}$.

[8]Detailed sufficient conditions for this are available from the authors upon request.

both convex portions of Φ, and yet the induced equilibrium is unique.

Even when the maximal number of equilibria is attained, no equilibrium need attain the maximal number of phase transitions. In Zick-Zack, both equilibria have only one phase transition for $\psi > \bar{\psi}$, while the most phase transitions is two, by Theorem 2-(c).

5 Equilibrium Payoffs

In a timing game, two questions arise: Which is each player's expected payoff, and how much "rent" is lost by hold-up? In the pure unobserved actions war of attrition, the expected payoff is $\phi(0) = v(1)$, and all rents are dissipated, namely the difference $\phi(1) - \phi(0) = v(N+1) - v(1)$ between highest and lowest rank payoffs — or, the total variation in rank payoffs. Not so here. To simplify matters, we assume no discounting and constant marginal participation costs, $c(t) = t$; rent dissipation thus equals the length of play.

Theorem 5 (Payoffs)
(a) *Fix an equilibrium corresponding to a given potential function* Γ. *Then the expected payoff is* $\Gamma'(0)$, *and the game must end after an elapse time of* $\Gamma'(1) - \Gamma'(0)$.
(b) *The equilibrium with the least expected payoff and maximal length corresponds to* $\mathsf{vex}(\Phi)$. *Thus, the least value is* $\mathsf{vex}(\Phi)'(0)$, *and the greatest length is* $\mathsf{vex}(\Phi)'(1) - \mathsf{vex}(\Phi)'(0)$.

For a given potential function, the equilibrium expected payoff of the game is thus a local minimum of the forward looking average rewards; the game lasts until a local maximum of backward average rewards obtains. Moreover, the least expected payoff of the game is the global minimum of the forward average payoffs, and the maximal time elapse likewise occurs when the global maximum backward average rewards are reached.

Proof of (a): Fix a potential function Γ. Since the mixed strategy ensures a constant payoff along the support of play, the expected payoff of the game is the time zero payoff $\Gamma'(0)$. With unobservable actions, the game ends in finite time by Theorem 1. The length of play depends on the payoffs dissipated — the higher the payoff they can obtain, the longer people are willing to delay. Since expected rank-payoffs must increase along the support of play, the largest rank-payoff $\Gamma'(1)$ obtains when the game ends.

Proof of (b): Suppose, counterfactually, that $\bar{\Gamma}'(0) < \mathsf{vex}(\Phi)'(0)$ for some potential function $\bar{\Gamma}$. Since $\mathsf{vex}(\Phi) \leq \Phi$ everywhere, we have $\mathsf{vex}(\Phi)'(0) \leq \Phi'(0)$, and thus $\bar{\Gamma}'(0) < \Phi'(0)$. Then $\bar{\Gamma}$ is initially linear by ($P3$). But differentiability and ($P3$) jointly imply that $\bar{\Gamma}$ can only change slopes while tangent to $\Phi \geq \mathsf{vex}(\Phi)$. If this happens at $g \in (0,1)$, then $\mathsf{vex}(\Phi)(g) \leq \Phi(g) = \bar{\Gamma}(g) = \bar{\Gamma}'(0)g < \mathsf{vex}(\Phi)'(0)g$. This violates convexity of $\mathsf{vex}(\Phi)$.

Similarly, at $g = 1$ we have $\bar{\Gamma}'(1) \leq \mathsf{vex}(\Phi)'(1)$ for any potential function $\bar{\Gamma}$. □

This result subsumes the standard war of attrition with monotonic rank rewards: When ϕ is monotonic, Φ is globally convex, and the only potential function is Φ itself. The expected payoff is $\Phi'(0) = \phi(0) = v(1)$, and the maximal length is $\Phi'(1) - \Phi'(0) = \phi(1) - \phi(0) = v(N + 1) - v(1)$. In fact, by $(P3)$ and Theorem 5-(b), this is the length of any unobserved actions game where $\mathsf{vex}(\Phi)$ begins and ends on a strictly convex portion.

Since rank payoffs are smoothed in ϕ with unobserved actions, the total variation in $\phi = \Phi'$ is a tighter upper bound on payoff dissipation (eg. Figure 4, left). But war of attrition-phases are truncated, and even this measure is not tight enough. The length and expected payoff depend on the slopes of the initial and terminal tangents τ_0 and τ_1.

In Caller Number Two of Three, $\mathsf{vex}(\Phi)$ is strictly convex for $g \leq 1/4$ and linear with slope $3/8$ for $g > 1/4$. The expected payoff is $\phi(0) = 0$ and the maximum length of the game is $3/8$. In the U-shaped example, $\mathsf{vex}(\Phi)$ is linear with slope $5/8$ for $g < 3/4$ and strictly convex for $g \geq 3/4$. The expected payoff in the game is the first flow payoff in the war of attrition, $\phi(3/4) = 5/8$, and the maximum elapse time equals $\phi(1)-\phi(3/4) = 3/8$.

In Zick-Zack, with rank rewards $(0, \psi, 0, 1)$, $\mathsf{vex}(\Phi)$ is the potential function that starts with a strictly convex portion. Thus, the minimum expected flow payoff is $\phi(0) = 0$. For $\psi \leq \underline{\psi}$, Φ is strictly convex, and the unobserved actions game is then equivalent to a war of attrition. If $\psi \in (\underline{\psi}, \bar{\psi})$, $\mathsf{vex}(\Phi)$ touches both convex portions of Φ and hence $\mathsf{vex}(\Phi)'(0) = \phi(0)$ and $\mathsf{vex}(\Phi)'(1) = \phi(1)$. Thus, the maximum duration is $\phi(1)-\phi(0) = 1$, which is below the total variation ψ in rank payoffs. Finally, for $\psi > \bar{\psi}$, $\mathsf{vex}(\Phi)$ ends with a linear portion, and the terminal payoff is governed by the slope of the tangent τ_1. The maximum duration exceeds $\phi(1) - \phi(0)$, but is still less than the total variation of ϕ.

What about the *most efficient* equilibrium? If $\Phi(1) \geq \Phi'(1)$, then $\Gamma^*(p) = p\Phi(1)$ is a potential function, and clearly corresponds to a time-0 complete atom. But if $\Phi(1) < \Phi'(1)$, then a time-0 jump is no longer an equilibrium. In some of these cases, we can identify the most efficient equilibrium, but we have found no clear theorem. For there are examples where the equilibrium with the greatest expected payoff is not the quickest.

Assuming that $\Phi''(1-) = \phi'(1-) > 0$, for instance, if we can construct a tangent τ^* from the origin to the last convex portion of Φ, tangent at some $\bar{p} \in (0, 1]$, then it is the most efficient equilibrium by both measures: shortest and greatest expected payoff. The shortest equilibrium in Zick-Zack is induced by the potential function Γ with a linear segment at the origin; such a potential function exists when $\psi \geq \underline{\psi}_0$. Since $\Gamma \neq \mathsf{vex}(\Phi)$, its expected payoff is higher. Also, for $\psi < \bar{\psi}_0$, its terminal slope is $\Gamma'(1) = \Phi'(1) = \phi(1) = 1$, which is weakly smaller than $\mathsf{vex}(\Phi)'(1)$. Thus, it is the shortest equilibrium. For $\psi \geq \bar{\psi}_0$, the atom is complete; this equilibrium is the shortest with the maximal expected payoff.

6 Conclusion, and Lessons for Observable Actions

Summary. It is surprising that the timing game literature has so long cleanly partitioned into wars of attrition and pre-emption games. The incentive structure for both varieties of timing games finds a common home in this paper. The resulting equilibria are remarkably rich, with on-path atomic explosions that may be preceded or followed by slow wars of attrition. Further, the two flavours of timing games interact with each other, with anticipation of later phases influencing current play. Thus, the moments for the explosions are chronologically advanced relative to a naïve "direct sum". To facilitate understanding, we have also managed to characterize equilibria via potential functions. This has afforded a strikingly quick existence proof, and easier analyses of these equilibria.

Our theory assumes unobserved actions because the resulting analysis is tractable — but we do believe that it captures many economic situations. Below, we argue that it also provides a benchmark for understanding behaviour with observable actions. Exogenous payoff growth over time, a feature often associated with pure pre-emption games, is an obvious extension that we pursue in future work.

Observed Actions. Once actions are observed, the model grows substantially more complex. Subgame-perfect equilibrium (SPE) is the mandated solution concept. Since players can see the game unfolding, there are now multiple information sets, one for each number of remaining players. There are therefore far more equilibria, since the number of remaining players itself can serve as a coordination device. We shall thus confine attention to symmetric SPE for which players engage in a war of attrition whenever possible, and a pre-emption game only when necessary. This substitutes for the stationarity condition for Nash equilibrium. For intuitively, a pre-emptive atom requires a high degree of coordination, and a war of attrition needs no coordination at all. Despite this refinement which seeks to minimize the role of pre-emption games, we now argue that our main qualitative finding still obtains: wars of attrition are truncated, and pre-emption atoms inflated.

Let $w(k+1)$ be the expected SPE payoff from the subgame after k have stopped.

Lemma 4 *A war of attrition obtains if* $v(k+1) < w(k+2)$ *while a pre-emptive atom of some size* $p \in (0,1]$ *occurs if* $v(k+1) \geq w(k+2)$.

Proof: Any $p < 1$ must equate the expected flow payoffs from the continuation game and the (shared) atomic payoffs:

$$\sum_{i=0}^{N-k} \binom{N-k}{i} p^i (1-p)^{N-k-i} w(k+1+i) = \sum_{i=0}^{N-k} \binom{N-k}{i} p^i (1-p)^{N-k-i} \mathsf{A}(k,i). \qquad (4)$$

Now, the LHS of (4) is flatter than the RHS of (4) at $p = 0$. For comparing slopes yields:

$$(N-k)(v(k+2)-v(k+1))/2+(N-k)(w(k+2)-v(k+1)) < (N-k)(v(k+2)-v(k+1))/2$$

since $w(k+2)-v(k+1) < 0$. Both sides are continuous in p and coincide for $p = 0$. Thus, they either intersect again for some $p \leq 1$, or, if not, the RHS atomic payoff dominates the LHS continuation payoff for all p, and a complete atom must obtain. □

Assuming again a constant cost of delay $c(t) = t$, the expected length of the war is $w(k + 2) - v(k + 1)$, while its expected payoff is $v(k + 1) =: w(k + 1)$. Assume that rank payoffs rise from j to k. We say that a war of attrition is *truncated in time* if its expected duration is less than $v(k) - v(j)$. Call a war of attrition *weakly truncated* (i.e. in ranks) if it nowhere obtains in $\{j, \dots, k\}$, or if it obtains from j' to k' for some $j \leq j' < k' \leq k$. Likewise, if rank payoffs fall from j to k, the pre-emption game is *weakly inflated* (in ranks) if it obtains from j' to k' for some $j' \leq j$ and $k' \geq k$. Once an atom occurs, there is further atomic entry until a war of attrition-subgame is reached.

(♢) All rank payoffs on down-slopes are more valuable than the overall average remaining payoff, or $v(k + 1) > \mathsf{A}(k, N - k)$ whenever $v(k + 1) < v(k)$, for any k.

Theorem 6 *Assume (♢). Wars of attrition are truncated in time, weakly truncated in ranks and pre-emptive atoms are weakly inflated.*

PROOF: As players are symmetric, they cannot expect to gain more than the average remaining rank payoff, $w(k + 1) \leq \mathsf{A}(k, N - k)$. So (♢) implies $v(k + 1) > w(k + 1)$.

A war of attrition along an up-slope from a minimum rank $\underline{k}$ to $\bar{k}$ lasts at most time $w(\bar{k}) - v(\underline{k})$; it is thus truncated in the *time* dimension from the naïve length $v(\bar{k}) - v(\underline{k})$.

Atomic entry obtains whenever $v(k) > w(k + 1)$. Assume that there are subsequent up-slopes of rank-rewards. If the atom is complete, then it is clearly inflated. If the atom is incomplete, then with positive probability play continues on the same down-slope. But then $v(k) > v(k + 1) > w(k + 1)$, and another atom follows immediately. So once atomic entry starts, it stops only when play begins weakly on an up-slope. □

Corollary *Assume (♢). There are at most as many phases as slope signs of $v(k)$.*

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