Gambling in Contests

Christian Seel*, Philipp Strack

University of Bonn, Lennéstr.43, 53113 Bonn, Germany

This version: January 12, 2011

Abstract This paper presents a strategic model of risk-taking behavior in contests. Formally, we analyze an n-player winner-take-all contest in which each player decides when to stop a privately observed Brownian Motion with drift. A player whose process reaches zero has to stop. The player with the highest stopping point wins. Contrary to the explicit cost for a higher stopping time in a war of attrition, here, higher stopping times are riskier, because players can go bankrupt. We derive a closed-form solution of the unique Nash equilibrium outcome of the game. In equilibrium, the trade-off between risk and reward causes a non-monotonicity: highest expected losses occur if the process decreases only slightly in expectation.

JEL classification: C72; C73; D81

Keywords: Discontinuous games; Contests; Relative performance pay; Risk-taking behavior

---

We would like to thank Paul Heidhues, Benny Moldovanu, and Tymon Tatur for their enduring support of this work. Furthermore, we are grateful to Dirk Bergemann, Norbert Christopeit, Faruk Gul, Sergei Izmalkov, Georg Kirchsteiger, Matthias Lang, Stephan Lauermann, Moritz Meyer-ter-Vehn, Frank Riedel, Xianwen Shi, Nora Szech, Philipp Weinschenk, and many others for helpful discussions. Financial support from German Science Foundation is gratefully acknowledged.

*Corresponding author at: Department of Economics, University of Bonn, Lennéstr. 43, 53113 Bonn, Germany.

E-mail addresses: cseel@uni-bonn.de (C.Seel), philipp.strack@uni-bonn.de (P.Strack).

1. Introduction

To provide more excitement for the players, the (online) gambling industry introduced casino tournaments. The rules are simple: all participants pay a fixed amount of money prior to the tournament—the "buy-in"—that enters into the prize pool. In return, they receive chips, which they can invest in the casino gamble throughout the tournament. At the end of the tournament, the player who has most chips wins a prize, which is the sum of the buy-ins minus some fee charged by the organizers. Benefits are two-sided: players restrict their maximal loss to the buy-in and enjoy a new, strategic component of the game; the casino makes a sure profit through the fee it charges.

The observability of each other's chip stacks throughout the tournament depends on the provider. The no-observability case is a good illustration of our model—in equilibrium, players use the gamble even though it has a negative expected value.[1]

In the model, each player decides when to stop a privately observed Brownian Motion (X_t) with (usually negative) constant drift coefficient μ, constant diffusion coefficient σ, and initial endowment x_0. If a player becomes bankrupt, i.e., $X_t = 0$, she has to stop. The player who stops at the highest value wins a prize.

Instead of an explicit cost for a higher contest success (e.g., Lazear and Rosen, 1981, Hillman and Samet, 1987), here, higher prizes are riskier. In equilibrium, players maximize their winning probability rather than the expected value of the process. Hence, they do not stop immediately even if the underlying process is decreasing in expectation. Intuitively, if all other players stop immediately, it is better for the remaining player to play until she wins a small amount or goes bankrupt, since she can ensure to win an arbitrarily small amount with a probability arbitrarily close to one.

In the unique equilibrium outcome, expected losses are non-monotonic

[1] Several online casinos use a leaderboard for the chip stacks. In most cases, however, it updates with a delay to create more tension. In this variant, players should only play close to the end of the contest to veil their realizations. The resulting equilibrium distributions are equivalent to the no-observability case.

in the expected value of the gamble—a more favorable gamble can lead to higher expected losses. Intuitively, this results from the trade-off between risk and reward: if the gamble has only a slightly negative expected value, the relatively high probability of winning makes people stop later, which increases expected losses. If the principal—who might have imperfect information about drift—obtains wins or losses of the players, contests are not a reliable compensation scheme, because even with a slightly negative drift, the principal incurs a large loss.

The formal analysis proceeds as follows. Proposition 1 derives a necessary formula for an implied stopping chance $F(x)$ in the symmetric equilibrium of an n-player game that pinpoints the unique candidate equilibrium distribution. To do so, we exploit that each player has to be indifferent whether to stop or to continue at any point of her support at any point in time.

For the two-player case, Proposition 2 derives the equilibrium stopping time that induces $F(x)$ explicitly. It involves mixing whether to stop with a chance that depends on the current state X_t. Proposition 4 extends Proposition 1 and 2 to a two-player game with asymmetric starting values.

For more than two players, Proposition 3 ensures the existence of a stopping time that induces $F(x)$. Its proof relies on a result in probability theory on the Skorokhod embedding problem. This literature—initiated by Skorokhod (1961, 1965)—analyzes under which conditions a stopping time of a stochastic process exists that embeds, i.e., induces, a given probability distribution; for an excellent survey article, see Obłój (2004). In the proof of Proposition 3, we verify a sufficient condition from Pedersen and Peskir (2001). This whole approach is new to game theory, and the main technical contribution of this paper.

Proposition 5 provides the main characterization result: the general shape of the expected value of the stopped processes is quasi-convex, falling, then rising in the drift μ and in the variance σ. In particular, highest expected losses occur if the process decreases only slightly in expectation.

Apart from casino tournaments, this paper provides a stylized model for the following applications. First, consider a private equity fund that invests in start-up companies. The value of the fund is mostly private information until maturity, because start-ups do not trade on the stock market and the composition of the fund is often unknown. The model analyzes a competition between fund managers in which, at maturity, the best performing manager gets a prize—a bonus or a job promotion.

In this application, there are several possible reasons for a downward drift. For instance, there may be no good investment opportunities in the market. Moreover, the downward drift may capture the cost of paying an expert to search for possible investments. The model predicts that the return on investment is very sensitive to the profitability of investment opportunities. In particular, a slightly negative drift is most harmful for the investors. In this case, contestants behave as if they were risk-loving, which a payment based on absolute success could avoid.

As a second example, consider a competition in a declining industry. In a duopoly, for instance, firms compete to survive and get the monopoly profit. Fudenberg and Tirole (1986) model the situation as a war of attrition—only the firm who stays alone in the market wins a prize, but both incur costs until one firm drops out.

In an interpretation of our model, managers of both firms decide if they want to make risky investments—into R&D or stocks of other firms. Investments are costly, but could improve the firm's value. When the duopoly becomes unprofitable, the firm with the higher value wins—either by a takeover battle or because the other firm cannot compete in a prize war—and its manager keeps his job.

Our model predicts that managers choose very risky strategies. In particular, investors lose most money in expectation if investment opportunities have a slightly negative expected value, which is consistent with being in a declining industry. This effect increases in the asymmetry of the firms' values. Intuitively, to satisfy the indifference condition for the stronger firm, the weaker firm has to make up for its initial disadvantage by taking higher risks.

1.1. Related Literature

Hvide (2002) investigates whether tournaments lead to excessive risk-taking behavior. He modifies Lazear and Rosen (1981) by assuming that players bear costs to raise their expected value, but can raise their variance without costs. In equilibrium, they choose maximum variance and low effort. Similarly, Anderson and Cabral (2007) scrutinize an infinite competition in which two players, who observe each other, can update their binary choice of variance continuously. In their model, flow payoffs depend on the difference in contest success. In equilibrium, both players choose the risky strategy until the lead of one player is above a threshold; in this case, the leader switches to the save option.

In the literature on races, players balance a higher effort cost against a higher winning probability. Moscarini and Smith (2007)—building on a discrete time model of Harris and Vickers (1987)—analyze a two-person continuous-time race with costly effort choice. In equilibrium, effort is increasing in the lead of a player up to some threshold above which the laggard resigns; for an application to political economy, see also Gul and Pesendorfer (2010). These papers assume full observability of each other's contest success over time. In our model, however, stopping decisions and realizations of the rivals are unobservable.

Regarding the assumptions on information and payoffs, the model most resembles a silent timing game—as first explored in Karlin (1953), and most recently, in Park and Smith (2008). The latter paper also generalizes the all-pay war of attrition, and so assumes that later stopping times cost linearly more. Contrary to a silent timing game, in the present paper, players do not only possess private information about their stopping decision, but also about the realization of their stochastic process.

Finally, the paper relates to the finance literature on gambling for resurrection; e.g., Downs and Rocke (1994). In this literature, managers take unfavorable gambles for a chance to save their firms from bankruptcy. Here, however, players take high risks to veil their contest outcomes.

We proceed as follows. Section 2 introduces the model. Section 3 derives the unique equilibrium distribution. In Section 4, we state the main characterization result, Proposition 5, and discuss its implications. Section 5 concludes.

2. The Model

There are n agents $i \in \{1, 2, \ldots, n\} = N$ who face a stopping problem in continuous time. At each point in time $t \in \mathbb{R}_+$, agent i privately observes the realization of a stochastic process $X^i = (X^i_t)_{t \in \mathbb{R}_+}$ with

$$X^i_t = x_0 + \mu t + \sigma B^i_t \,.$$

The constant $x_0 > 0$ denotes the starting value of all processes; see Section 4.3 for heterogeneous starting values. The drift $\mu \in \mathbb{R}$ is the common expected change of each process X^i_t per time, i.e., $\mathbb{E}(X^i_{t+\Delta} - X^i_t) = \mu\Delta$. The noise term is an n-dimensional Brownian motion (B_t) scaled by $\sigma \in \mathbb{R}_+$.

2.1. Strategies

A strategy of player i is a stopping time τ^i. This stopping time depends only on the realization of his process X^i_t, as the player only observes his own process.[2] Mathematically, the agents' stopping decision until time t has to be $\mathcal{F}^i_t$-measurable, where $\mathcal{F}^i_t = \sigma(\{X^i_s : s < t\})$ is the sigma algebra induced by the possible observations of the process X^i_s before time t. We restrict agents' strategy spaces in two ways. First, we require finite expected stopping times, i.e., $\mathbb{E}(\tau^i) < \infty$. Second, a player has to stop in case of bankruptcy. More formally, we require $\tau^i \leq \inf\{t \in \mathbb{R}_+ : X^i_t = 0\}$ a.s.. To incorporate mixed strategies, we allow for randomized stopping times—progressively measurable functions $\tau^i(\cdot)$ such that for every $r^i \in [0, 1]$, $\tau^i(r^i)$ is a stopping time. Intuitively, agents can draw a random number r^i from the uniform distribution on $[0, 1]$ before the game and play a stopping strategy $\tau^i(r^i)$.

2.2. Payoffs

The player who stops his process at the highest value wins a prize, which we normalize to one without loss of generality. Ties are broken randomly.

[2] The equilibrium of the model would be the same if the stopping decision was reversible and stopped processes were constant.

Formally,

$$\pi^i = \frac{1}{k}\,\mathbf{1}_{\{X^i_{\tau^i} = \max_{j\in N} X^j_{\tau^j}\}}\,,$$

where $k = |\{i \in N : X^i_{\tau^i} = \max_{j\in N} X^j_{\tau^j}\}|$. Hence, the game is a constant sum game. All agents maximize their expected payoff, i.e., the probability to win the contest. This optimization is independent of their risk attitude.

2.3. Condition on the Parameters

To ensure equilibrium existence in finite time stopping strategies, we henceforth impose a technical condition that places a positive upper bound on μ—for a discussion, see Section 3.2.

Assumption 1. $\mu < \log(1 + \frac{1}{n-1})\frac{\sigma^2}{2x_0}$.

3. Equilibrium Analysis

In this section, we first derive the unique candidate equilibrium distribution. Second, we prove equilibrium existence—this is not trivial as the game has discontinuous payoffs and infinite strategy spaces. Our proof shows that there exists a stopping time inducing the candidate equilibrium distribution. We close the section with an extension to asymmetric starting values.

3.1. The Equilibrium Distribution

Every strategy of agent i induces a (potentially non-smooth) cumulative distribution function (cdf) $F^i : \mathbb{R}_+ \to [0,1]$ of his stopped process, where $F^i(x) = \mathbb{P}(X^i_{\tau^i} \le x)$.

The probability of a tie is non-zero only if the distributions of at least two agents have a mass point above zero or the distributions of all agents have a mass point at zero or both.[3] The next lemma proves otherwise.

Lemma 1 (No Mass Points)**.** *In equilibrium, for every $x > 0$, no agent $i \in N$ has a mass point at x, i.e., $\mathbb{P}(X^i_{\tau^i} = x) = 0$. At least one agent has no mass point at zero.*

[3] As common in economic literature, we do not consider the mathematical problem of an accumulation of mass points (Cantor Construction); we thus assume that either there is only a finite number of mass points or they have no accumulation point.

We omit the proof and present a verbal argument instead, because the proof is simply a specialization of the now standard logic in static game theory with a continuous state space; e.g., Burdett and Judd (1983). As usual, mixed strategies in a competitive game can have no interior mass point at the same point in the state space (here, the same x), since this would create a profitable deviation in one direction: With a slightly higher x, one raises one's win chance a boundedly positive probability with an arbitrarily small loss, since one beats everyone with lower x and the one player with mass at x; however, one agent can have a mass point at zero, since any other player who can pass him would have already been bankrupt.

Lemma 1 renders the tie-breaking rule obsolete, because it implies that the probability of a tie is zero. Denote the winning probability of player i if he stops at $X^i_{\tau_i} = x$ by $u^i(x)$, where $u^i(x) : \mathbb{R}_+ \to [0,1]$. As there are no mass points away from zero, we can express $u^i(x)$ in terms of the other agents' cdf's.

$$\begin{aligned} u^i(x) &= \mathbb{P}(x > \max_{j\neq i} X^j_{\tau^j}) + \frac{1}{k}\underbrace{\mathbb{P}(x = \max_{j\neq i} X^j_{\tau^j}\})}_{=0} \\ &= \prod_{j\neq i} \mathbb{P}(X^j_{\tau^j} \leq x) = \prod_{j\neq i} F^j(x) \end{aligned} \tag{1}$$

We call $u^i(\cdot)$ the utility function of agent i given the distributions of other agents. These utility functions are helpful to derive the equilibrium—a point where each player maximizes $\mathbb{E}(u^i(X^i_{\tau^i}))$.

Denote the right endpoint of the support of the distribution of player i by $\overline{x}^i = \sup\{x : F^i(x) < 1\}$ and the left endpoint by $\underline{x}^i = \inf\{x : F^i(x) > 0\}$. The right endpoint has to be finite, because agents can only use strategies that stop almost surely in finite time. The following results establish necessary conditions on u^i and the distribution functions in equilibrium; the proofs are in the appendix.

Lemma 2 (Strict Monotonicity)**.** *The utility u^i of every agent $i \in N$ is strictly increasing on the interval $[\underline{x}^i, \overline{x}^i]$.*

Lemma 3 (Indifference)**.** *For each player i, the utility $u^i(X_t^i)$ is a local martingale on the interior of the support of his distribution, i.e., $X_t^i \in (\underline{x}^i, \overline{x}^i) \Rightarrow \mathbb{E}(\mathrm{d}u^i(X_t^i)|\mathcal{F}_t^i) = 0$.*

Lemma 4 (Symmetry of the Equilibrium Distributions)**.** *The support of the cdf of each player is identical and starts at zero.*

All players share the same utility function. Hence, Lemma 3 and 4 directly imply the following corollary:

Corollary 1. *The unique equilibrium distributions are atomless and symmetric.*

As the utility u^i does not depend on time ($\frac{\partial u^i}{\partial t} = 0$), by Itô's lemma (Revuz and Yor, 2005, p.147) the expected change in utility per marginal unit of time is

$$\begin{aligned} \mathbb{E}(\mathrm{d}u^i(X_t^i)|\mathcal{F}_t^i) &= \mathbb{E}\left((\mu u^{i\prime}(X_t^i) + \frac{\sigma^2}{2} u^{i\prime\prime}(X_t^i))\mathrm{d}t + u^{i\prime}(X_t^i)\sigma \mathrm{d}B_t | \mathcal{F}_t^i\right) \\ &= \mu u^{i\prime}(X_t^i) + \frac{\sigma^2}{2} u^{i\prime\prime}(X_t^i)\mathrm{d}t\,. \end{aligned}$$

By Lemma 3, this equation is equal to zero for all x on the support of F^i, which yields the following ordinary differential equation:

$$0 = \mu u^{i\prime}(x) + \frac{\sigma^2}{2} u^{i\prime\prime}(x)\,.$$

For $\mu \neq 0$, all solutions to this equation are of the form $u^i(x) = \alpha + \beta \exp(\frac{-2\mu x}{\sigma^2})$ for all constants $\alpha, \beta \in \mathbb{R}$. To fix α and β, we use two constraints on u^i. First, all players win with probability $\frac{1}{n}$ in equilibrium (Corollary 1). In particular, they do so when they stop immediately (Lemma 3). Second, the value of the cdf at zero is zero, because the support is atomless (Corollary 1). Thus, we get:

$$\begin{aligned} \frac{1}{n} = u^i(x_0) &= \alpha + \beta \;\; \exp(\frac{-2\mu x_0}{\sigma^2}) \\ 0 = u^i(0) &= \alpha + \beta\,. \end{aligned}$$

This system of equations uniquely determines α and β, and thereby also u^i as

$$u^i(x) = \min\left\{1, \frac{1}{n}\frac{\exp(\frac{-2\mu x}{\sigma^2}) - 1}{\exp(\frac{-2\mu x_0}{\sigma^2}) - 1}\right\}.$$

It remains to construct the corresponding equilibrium distributions. For this purpose, we insert the symmetry property of the equilibrium (Corollary 1) into equation (1) to get

$$u^i(x) = \prod_{j\neq i}^{n} F^j(x) = F(x)^{n-1} \Rightarrow F(x) = \sqrt[n-1]{u^i(x)}\,.$$

Hence, we characterize the unique candidate for an equilibrium distribution as follows (for an illustration, see Figure 1):

Proposition 1. *Assume $\mu \neq 0$. A strategy profile is a Nash equilibrium, if and only if each player's strategy induces the cumulative distribution function*

$$F(x) = \min\left\{1, \sqrt[n-1]{\frac{1}{n}\frac{\exp(\frac{-2\mu x}{\sigma^2}) - 1}{\exp(\frac{-2\mu x_0}{\sigma^2}) - 1}}\right\}.$$

Proof. We have already proven that any equilibrium strategy is symmetric and induces the distribution F. To complete the proof, we need to show that no deviation gives a player a winning probability greater than $\frac{1}{n}$. Recall that, by construction of F, $u^i(X_t^i)$ is a supermartingale. By Doob's optional stopping theorem (Revuz and Yor, 2005, p.70), the stopped process $u^i(X_{\tau^i}^i)$ is also a supermartingale. Hence, $\mathbb{E}(u^i(X_{\tau^i}^i)) \leq \mathbb{E}(u^i(x_0^i)) = \frac{1}{n}$. □

To complete the analysis, we scrutinize the special case in which X_t^i is a martingale, i.e., $\mu = 0$. In this case, the first term in the differential equation vanishes. The same calculation as in the case $\mu \neq 0$ yields the unique equilibrium distribution, where

$$F(x) = \min\left\{1, \sqrt[n-1]{\frac{x}{nx_0}}\right\}.$$

$F(x)$ is continuous in μ at $\mu = 0$, because the same formula follows by taking

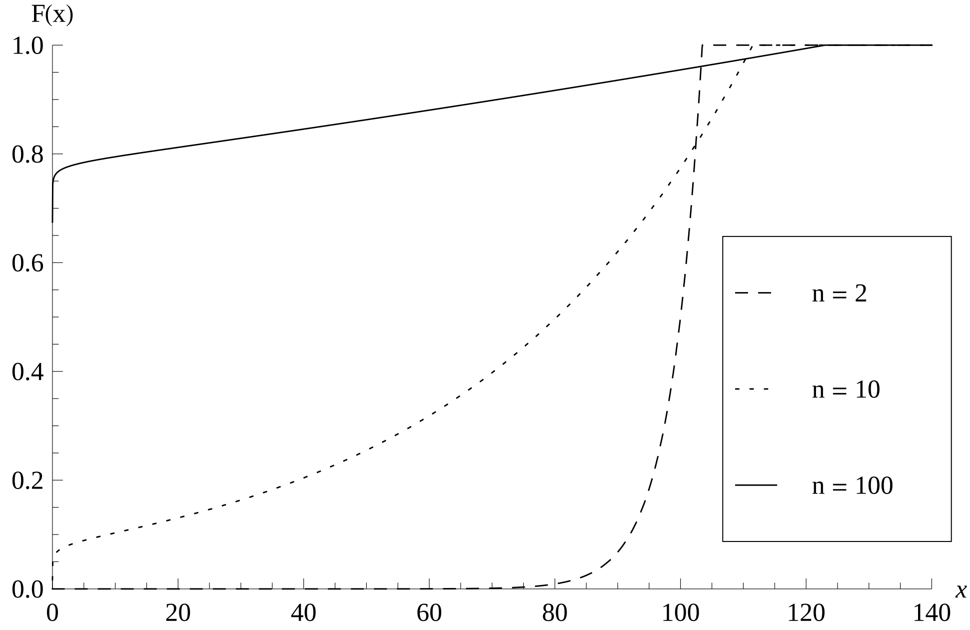

Figure 1: An example ($\mu = -0.1$, $x_0 = 100$, $\sigma = 1$) of the equilibrium cdf's for different sizes of players n.

limits in Proposition 1, using the approximation $e^A = 1 + A + O(A^2)$ for small A.

3.2. Equilibrium Strategies

So far, we have been silent about the existence of a finite time stopping strategy τ inducing the equilibrium distribution F. For a given distribution to be implementable in finite time stopping strategies, its right endpoint has to be finite. Recall that

$$1 = F(\overline{x}) = \sqrt[n-1]{\frac{1}{n}\frac{\exp(\frac{-2\mu\overline{x}}{\sigma^2}) - 1}{\exp(\frac{-2\mu x_0}{\sigma^2}) - 1}}\,.$$

Hence, the right endpoint $\overline{x}$ satisfies

$$\overline{x} = \frac{\sigma^2}{-2\mu}\log(n(\exp(\frac{-2\mu x_0}{\sigma^2}) - 1) + 1)\,.$$

Consequently, the right endpoint is finite if and only if $\mu < -\log(1 - \frac{1}{n})\frac{\sigma^2}{2x_0}$, i.e., Assumption 1 holds; otherwise, no equilibrium in finite time stopping strategies exists. Intuitively, if the drift becomes too large, for every point x, the strategy, which stops only at 0 and x, reaches x with a probability higher than $\frac{1}{n}$.

In the next step, we derive mixed strategies inducing the distribution F in the two-player case to convey the main intuition. The construction uses a mixture of deterministic threshold strategies to induce the final distribution. To formalize this intuition, we introduce the martingale transformation $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, where

$$\phi(x) = \frac{\exp(\frac{-2\mu x}{\sigma^2}) - 1}{\exp(\frac{-2\mu x_0}{\sigma^2}) - 1} \,.$$

By Itô's lemma (since $\phi''/\phi' = -2\mu/\sigma^2$), the process $(\phi(X_t^i))_{t\in\mathbb{R}_+}$ is a martingale. In this case, $F(x) = \phi(x)/2$.

Proposition 2 (Equilibrium Strategy for Two Players)**.** *If agent i randomly selects a number $\alpha \in (0, 1]$ from a uniform distribution and stops if*

$$\tau^i = \inf\{t : |\phi(X_t^i) - 1| \geq \alpha\} \,,$$

then the cumulative distribution function induced by this strategy equals F, i.e., $\mathbb{P}(X_{\tau^i}^i \leq x) = F(x)$.

Proof. By the martingale property of $(\phi(X_t^i))_{t\in\mathbb{R}_+}$, we get

$$\mathbb{P}(\phi(X_{\tau^i}^i) = 1 - \alpha) = \mathbb{P}(\phi(X_{\tau^i}^i) = 1 + \alpha) = \frac{1}{2} \,.$$

As α is uniformly distributed on $(0, 1]$ and agent i stops iff $\phi(X_t^i) = 1 \pm \alpha$, the random variable $\phi(X_{\tau^i}^i)$ is uniformly distributed on $[0, 2]$. It follows that

$$\mathbb{P}(X_{\tau^i}^i \leq x) = \mathbb{P}(\phi(X_{\tau^i}^i) \leq \phi(x)) = \frac{\phi(x)}{2} = F(x) \,.$$

□

For more than two players, the feasibility proof requires an auxiliary result from probability theory on the Skorokhod embedding problem. This literature studies whether a distribution is feasible by stopping a stochastic process; in their terminology, there exists an embedding of a probability distribution in the process. Skorokhod (1961, 1965) analyzes the problem of embedding in Brownian motion without drift. In a recent contribution, Pedersen and Peskir (2001) derive a necessary and sufficient condition for general non-singular diffusions. They define the scale function $S(\cdot)$ by

$$S(x) = \int_0^x \exp(-2\int_0^u \frac{\mu(r)}{\sigma(r)}dr)du = -\frac{\sigma^2}{2\mu}(\exp(\frac{-2\mu x}{\sigma^2}) - 1)\,.$$

Lemma 5 (Pedersen and Peskir, 2001, Theorem 2.1.)**.** *Let (X_t) be a non-singular diffusion on $\mathbb{R}$ starting at zero, let $S(\cdot)$ denote its scale function satisfying $S(0) = 0$, and let ν be a probability measure on $\mathbb{R}$ satisfying $|S(x)|\nu(\mathrm{d}x) < \infty$. Set $m = \int_{\mathbb{R}} S(x)\nu(\mathrm{d}x)$. Then there exists a stopping time τ_* for (X_t) such that $X_{\tau_*} \sim \nu$ if and only if one of the following four cases holds:*

1. $S(-\infty) = -\infty$ *and* $S(\infty) = \infty$ *;*
2. $S(-\infty) = -\infty, S(\infty) < \infty$ *and* $m \geq 0$ *;*
3. $S(-\infty) > -\infty, S(\infty) = \infty$ *and* $m \leq 0$ *;*
4. $S(-\infty) > -\infty, S(\infty) < \infty$ *and* $m = 0$ *.*

Hence, to prove feasibility for our distribution F, it suffices to show $m = 0$.

Proposition 3 (Feasibility of the Equilibrium Distribution)**.** *There exists a stopping strategy inducing the distribution $F(\cdot)$ from Proposition 1.*

Proof. To verify the condition in Pedersen and Peskir (2001), we need a process which starts in zero. Thus, we consider the process $\tilde{X}_t = X_t - X_0$. After some transformations, we get $S(x-x_0) = -\frac{\sigma^2}{2\mu}(1-\exp(\frac{2\mu x_0}{\sigma^2}))(\phi(x)-1)$. This gives us

$$\begin{aligned} m &= \int_{\mathbb{R}} S(x - x_0)f(x)\mathrm{d}x \\ &= -\frac{\sigma^2}{2\mu}\left(1 - \exp(\frac{2\mu x_0}{\sigma^2})\right)\left(\int_{\mathbb{R}} \phi(x)f(x)\mathrm{d}x - 1\right)\,. \end{aligned}$$

Consequently, it remains to show $\int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x = 1$.

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x &= \int_0^{\overline{x}} \underbrace{\frac{(n^{-\frac{1}{n-1}})}{n-1}\phi(x)^{-\frac{n-2}{n-1}}\phi'(x)}_{f(x)}\phi(x)\mathrm{d}x \\
&= \int_{\phi(0)}^{\phi(\overline{x})} \frac{(n^{-\frac{1}{n-1}})}{n-1} y^{\frac{1}{n-1}}\mathrm{d}y \\
&= \left[\frac{(n^{-\frac{1}{n-1}})}{n} y^{\frac{n}{n-1}}\right]_{y=\phi(0)=0}^{y=\phi(\overline{x})=n} \\
&= 1\,.
\end{aligned}$$

As $m = 0$, there exists an embedding for the distribution F by Theorem 2.1. in Pedersen and Peskir (2001).[4] □

Proposition 1 and 3 combined yield F as the unique equilibrium distribution of the game.

3.3. An Extension: Asymmetric Starting Values

In this extension, we allow for heterogeneity in the starting values. To get an analytical solution, we restrict attention to the two-player case—without loss of generality $x_0^1 > x_0^2$. The proof of the following proposition is similar to the proof of Proposition 1.

Proposition 4. *In equilibrium, the cdf of the first player is*

$$F^1(x) = \min\left\{1, \frac{1}{2}\frac{\exp(\frac{-2\mu x}{\sigma^2}) - 1}{\exp(\frac{-2\mu x_0^1}{\sigma^2}) - 1}\right\}.$$

The cdf of the second player is

$$F^2(x) = \min\left\{1, \rho + (1-\rho)\frac{1}{2}\frac{\exp(\frac{-2\mu x}{\sigma^2}) - 1}{\exp(\frac{-2\mu x_0^1}{\sigma^2}) - 1}\right\}.$$

[4] An alternative proof of Proposition 3 would verify a result on embedding in Brownian with drift from Grandits and Falkner (2000) for the process $\tilde{X}_t = \frac{X_t - X_0}{\sigma}$.

Proof. The cdf of player 1 is the same as in the symmetric case. Thus, it is feasible by Proposition 2. For player 2, consider the following strategy: First, play until $X_t^2 \in \{0, x_0^1\}$; then use the same stopping strategy as player 1 if he reaches x_0^1. This induces the above cdf, where the constant ρ—probability of absorption in 0—fulfills

$$\rho = \frac{\exp(\frac{-2\mu(x_0^1 - x_0^2)}{\sigma^2}) - 1}{\exp(\frac{-2\mu(x_0^1 - x_0^2)}{\sigma^2}) - \exp(\frac{2\mu x_0^2}{\sigma^2})}.$$

As in the proof of Proposition 1, the expected winning probability for each player in the above equilibrium candidate is the same as if he stops immediately. Furthermore, as $u^i(X_t^i)$ is a supermartingale by construction, Doob's optional stopping theorem implies that the stopped processes $u^i(X_{\tau_i}^i)$ are supermartingales for any τ_i. Hence, no player can do better than to stop immediately, which yields the equilibrium payoff. We show uniqueness of the equilibrium in the appendix. □

Compared to the symmetric case, the player with the lower starting value takes more risks here. In particular, he loses everything with probability ρ and takes the same gamble as player 1 with probability $1 - \rho$. Asymmetry in the contest leads to higher percentage losses for a negative drift, because the handicapped player takes higher risks to compensate his initial disadvantage.

4. Comparative Statics

This section analyzes how changes in the parameters affect the expected value of the stopped processes. To determine the expected value, we first calculate the density from the cdf in Proposition 1:

$$f(x) = \frac{2\mu}{n(n-1)\sigma^2} \sqrt[\frac{2-n}{n-1}]{\frac{\exp(\frac{-2\mu x}{\sigma^2}) - 1}{n(\exp(\frac{-2\mu x_0}{\sigma^2}) - 1)}} \frac{\exp(\frac{-2\mu x}{\sigma^2})}{1 - \exp(\frac{-2\mu x_0}{\sigma^2})}.$$

In what follows, we restrict attention to the two-player case for tractability; in the appendix, we state the formula for the expected value for n players. We use the density f to derive the expected value of the stopped processes

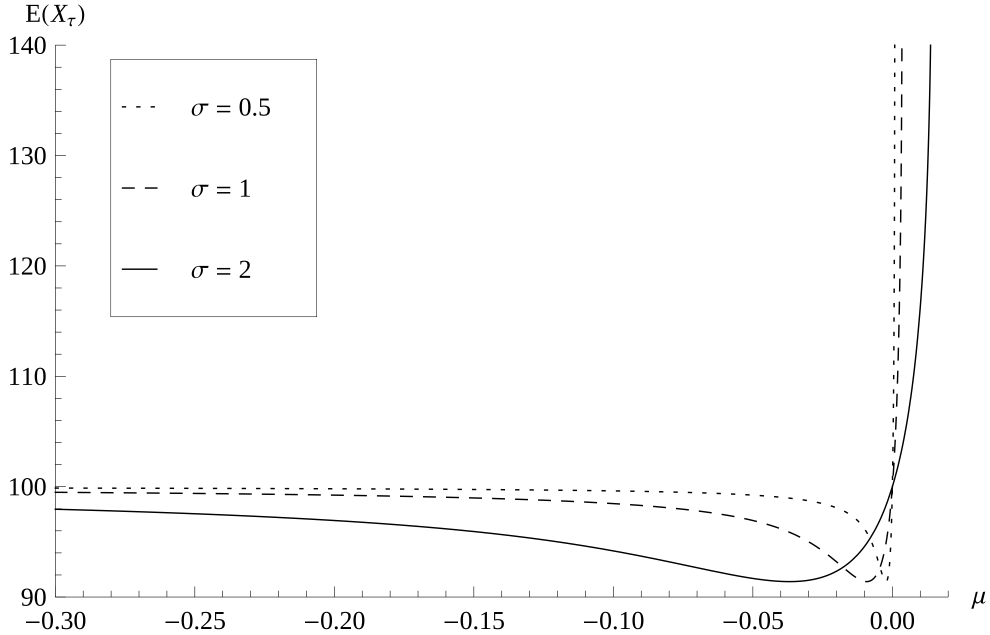

Figure 2: An example ($n = 2$, $x_0 = 100$) of the expected value of the stopped processes $\mathbb{E}(X_\tau)$ depending on the drift μ for different values of variance σ.

for two players:

$$\begin{aligned}\mathbb{E}(X_\tau) &= \mathbb{E}_f(x) = \int_0^{\overline{x}} x f(x)\mathrm{d}x \\ &= \frac{\sigma^2}{2\mu} + (1 + \frac{1}{2(\exp(-\frac{2\mu x_0}{\sigma^2}) - 1)})(x_0 - \frac{\sigma^2 \log(2 - \exp(\frac{2\mu x_0}{\sigma^2}))}{2\mu}) .\end{aligned}$$

The explicit formula of the expected value allows us to characterize its shape in the following proposition—the proof is in the appendix.

Proposition 5. $\mathbb{E}(X_\tau)$ *is quasi-convex, falling, then rising in* μ*. If* $\mu < 0$*,* $\mathbb{E}(X_\tau)$ *is quasi-convex, falling, then rising in* σ*.*

Hence, an increase in the drift does not imply an increase in the expected value of the stopped processes. Intuitively, for $\mu < 0$, there are two opposing effects: an increase in the drift lowers the expected losses per time but increases the expected stopping time. Similarly, as the variance increases,

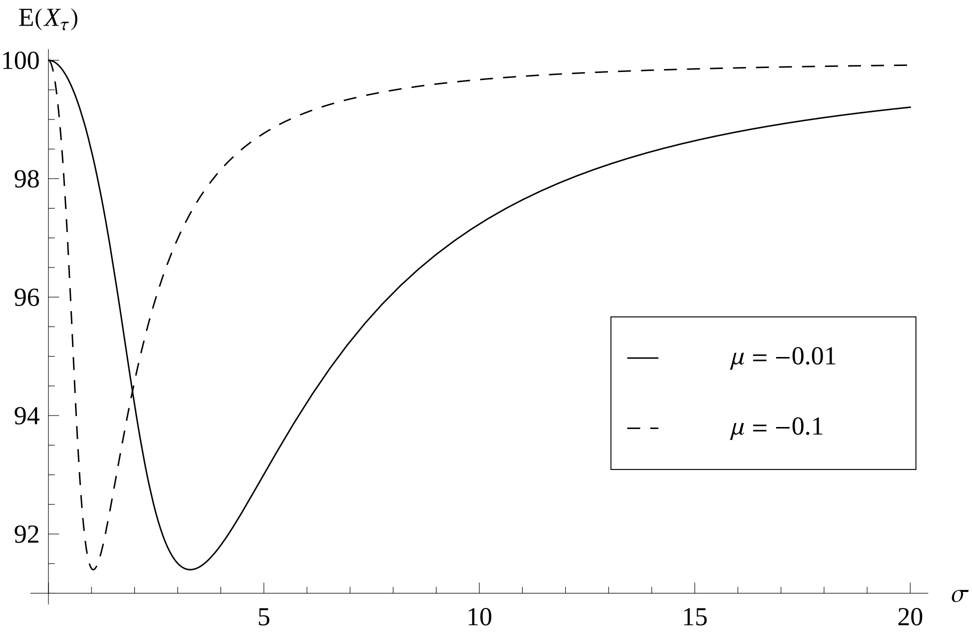

Figure 3: An example ($n = 2$, $x_0 = 100$) of the expected value of the stopped processes $\mathbb{E}(X_\tau)$ depending on the variance σ for different values of drift μ.

the gamble gets more attractive, but it also takes less time to implement the equilibrium distribution.

From an economic point of view, Proposition 5 illustrates a drawback of relative performance payments in risky environments: even if risky investment opportunities have only a slightly negative expected value, the principal loses a lot in expectation. Intuitively, contestants only care about outperfoming each other and thus behave as if they were risk-loving. A simple linear compensation scheme based on absolute performance would avoid this drawback.

4.1. A Comparison to Related Models

In the static two-player contest of Lazear and Rosen (1981), contest success depends on the effort choice and the realization of a random variable. In their framework, contests are suitable to induce the optimal amount of effort. If, in our two-player model, agents had to specify a fixed date at which they stop, they would stop immediately for negative values of the drift. Hence, to obtain our results, we need a dynamic decision problem for each player.

The equilibrium distributions in the present paper are similar to those of all-pay auctions with complete information (e.g., Hillman and Samet, 1987, or Baye et al., 1996).[5] In both settings, the joint equilibrium distribution of the other players makes each player indifferent. The trade-off between a higher risk and a higher chance to win the prize thus serves as an implicit cost. In contrast to the all-pay auction, all players participate actively in the contest in any equilibrium.

5. Conclusion

We have studied a new continuous' time model of contests. Contrary to the previous literature, players face a trade-off between a higher winning probability and a higher risk. If there are no good investment opportunities available, e.g., in a declining industry, contestants behave as if they were risk-loving—they invest in projects with negative expected returns. According to our main characterization result, Proposition 5, this problem is most severe for the natural case in which the drift is close to zero.

From a technical point of view, this paper has developed a new method to verify equilibrium existence. The approach via Skorokhod embeddings seems promising to analyze other models without observability, because there are many sufficient conditions available in the probability theory literature.

6. Appendix

Proof of Lemma 2: Assume, by contradiction, there exists an interval $I = (a, b) \subset [\underline{x}^i, \overline{x}^i]$ such that $u^i(x) = \prod_{j \neq i} F^j(x)$ is constant for all $x \in I$. We distinguish three cases:

(i) For all player's $j \neq i$, $F^j(a) = 1$. Hence, by optimality, player i stops with probability 1 whenever at $\max_{j \neq i} \overline{x}^j$. This implies $\max_{j \neq i} \overline{x}^j \leq a \leq \underline{x}^i$ and player i wins for sure. Player j can deviate profitably and stop only if she hits 0 or $\overline{x}^i$, which contradicts the equilibrium assumption.

[5] Complete information about valuations in the all-pay auction corresponds to complete information about starting values in this paper.

(ii) There exists a player $j \neq i$ with $F^j(b) = 0$. Hence, in equilibrium, no player ever stops in the interval $(0, \underline{x}^j)$, but at least two players stop with positive probability in every ϵ-ball around $\underline{x}^j$. To stop at $\underline{x}^j$ (with $u^i(\underline{x}^j) = 0$ by Lemma 1) is strictly worse than to continue until $X^i_t \in \{0, \max_j \overline{x}^j\}$. By continuity (Lemma 1), the argument extends to an ϵ-neighborhood of $\overline{x}^j$. This contradicts the equilibrium assumption of weak optimality of stopping in $(\underline{x}^j, \underline{x}^j + \epsilon)$.

(iii) No player $j \neq i$ stops in I, but (i) and (ii) do not hold. Hence, player i does not stop in I. Denote by $\tilde{x}$ the infimum of points above b at which a player stops. At $\tilde{x}$ (and, by continuity at an ϵ-neighborhood of $\tilde{x}$), it is strictly better to continue until $X^j_t \in \{\frac{b+a}{2}, \max_j \overline{x}^j\}$ than to stop, which contradicts the equilibrium assumption. □

Proof of Lemma 3: We define $\Phi(x) = \prod_{i=1}^n F^i(x) = F^i(x)u^i(x)$. Denote the set of players who stop at x by $M(x) \subseteq N$, i.e.,

$$M(x) = \{i \in \{1, \dots, n\} : (F^i)'(x) \neq 0\}\,.$$

By Lemma 2, $|M(x)| \geq 2$ for all $\min_{i \in N} \underline{x}^i < x < \max_{i \in N} \overline{x}^i$. For notational convenience, we omit the point x, at which all functions are evaluated, i.e., we write F^i, M instead of $F^i(x), M(x)$. Furthermore, we write $\mathbb{E}(\mathrm{d}u^i(x))$ shorthand for $\mathbb{E}(\mathrm{d}u^i(X^i_s)|\mathcal{F}^i_s)$ given $X^i_s = x$. For every agent $k \notin M$, we have:

$$\begin{aligned}
|M|\Phi' = \sum_{i \in M} (F^i u^i)' = \sum_{i \in M} \left(F^i u^{i\prime} + F^{i\prime} u^i\right) &= \sum_{i \in M} F^i u^{i\prime} + \underbrace{\sum_{i \in N} F^{i\prime} u^i}_{\Phi'} \\
\Leftrightarrow \quad (|M| - 1)\Phi' &= \sum_{i \in M} F^i u^{i\prime} \\
\Rightarrow \quad (|M| - 1) F^k u^{k\prime} &= \sum_{i \in M} F^i u^{i\prime} \\
\Rightarrow \quad (|M| - 1) F^k u^{k\prime\prime} &= \sum_{i \in M} \left(F^i u^{i\prime\prime} + F^{i\prime} u^{i\prime}\right)\,.
\end{aligned}$$

We calculate the expected change in winning probability of player k if he continues to play for an infinitesimally short time $\mathbb{E}(\mathrm{d}u^k)$:

$$\begin{aligned}(|M|-1)F^k\mathbb{E}(\mathrm{d}u^k) &= (|M|-1)F^k(\mu u^{k\prime} + \frac{\sigma^2}{2}u^{k\prime\prime}) \\ &= \mu(|M|-1)F^k u^{k\prime} + \frac{\sigma^2}{2}(|M|-1)F^k u^{k\prime\prime} \\ &= \mu\sum_{i\in M} F^i u^{i\prime} + \frac{\sigma^2}{2}\sum_{i\in M}\Big(F^i u^{i\prime\prime} + F^{i\prime}u^{i\prime}\Big) \\ &= \sum_{i\in M}\underbrace{(\mu u^{i\prime} + \frac{\sigma^2}{2}u^{i\prime\prime})}_{=\mathbb{E}(\mathrm{d}u^i)=0}F^i + \sum_{i\in M}\underbrace{F^{i\prime}u^{i\prime}}_{>0} \\ &> 0\,.\end{aligned}$$

As agent $i \in M$ stops with strictly positive probability in any neighborhood of x, he is indifferent between the strategy that stops at x and any other strategy that stops in a small neighborhood of x. Thus, $\mathbb{E}(\mathrm{d}u^i(x)) = 0$.
So far, we have shown that $\mathbb{E}(\mathrm{d}u^i(x)) = 0$ if $i \in M(x)$ and $\mathbb{E}(\mathrm{d}u^i(x)) > 0$ if $i \notin M(x)$. For every agent i, there exists an interval $I \subset [\underline{x}^i, \overline{x}^i]$ such that $i \in M(x)$ for every $x \in I$. Whenever $X^i_t = x \in I$, agent i is indifferent between the strategy that stops immediately and the strategy $\tau = \inf\{t : t \in \{\underline{x}^i, \overline{x}^i\}\}$. Formally,

$$\begin{aligned}0 &= u^i(x) - \mathbb{E}(u^i(X_\tau)) \\ &= u^i(x) - \mathbb{E}(u^i(x) + \int_t^\tau \mu u^{i\prime}(X^i_s) + \frac{\sigma^2}{2}u^{i\prime\prime}(X^i_s)\mathrm{d}s + \int_t^\tau u^{i\prime}(X^i_s)\sigma \mathrm{d}B_s) \\ &= \mathbb{E}(\int_t^\tau \mu u^{i\prime}(X^i_s) + \frac{\sigma^2}{2}u^{i\prime\prime}(X^i_s)\mathrm{d}s) = u^i(x) + \mathbb{E}(\int_t^\tau \mathbb{E}(\mathrm{d}u^i(X^i_s)))\,.\end{aligned}$$

The process enters every interval and $\mathbb{E}(\mathrm{d}u^i(x))$ is non-negative for all $x \in [\underline{x}^i, \overline{x}^i]$. Hence, the expectation $\mathbb{E}(\int_t^\tau \mathbb{E}(\mathrm{d}u^i(X^i_s)))$ can only be zero if $\mathbb{E}(\mathrm{d}u^i(x)) = 0$ almost surely.

□

Proof of Lemma 4: By contradiction, assume $\max_i \underline{x}^i \neq 0$. Thus, to stop at $X^i_t = \underline{x}^i$ (and, by continuity in a neighborhood of this point) is strictly worse than to continue until $X_t^j \in \{0, \max_i \overline{x}^i\}$; this contradicts optimality.

Assume there exists players i and j such that $\overline{x}^i > \overline{x}^j$. Assume player j reaches his right endpoint at time t, $X_t^j = \overline{x}^j$. By the same argument as in Lemma 3, the continuation strategy $\tau = \inf\{s \geq t : X_s^j \in \{\overline{x}^j - \epsilon, \overline{x}^i\}\}$ is strictly better than to stop at $\overline{x}^j$, which contradicts optimality.

□

Proof of Proposition 4*:* To prove uniqueness, note that Lemma 1-4 do not rely on any symmetry arguments and do still hold. Hence, the equation $u^i(x) = F^j(x)$ fixes the above construction uniquely given the right endpoint. The minmax property (constant sum game) implies that each player must receive the same payoff in any equilibrium. Thus, the local martingale condition uniquely determines $\overline{x}$. By Lemma 1, only one agent might set a mass point at 0. Feasibility implies that the agent with the lower starting value sets the mass point at zero and uniquely determines the size of the mass point. □

Formula for the Expected Value in the n-Player Case:
Let *Hyp* denote the Gauss hypergeometric function.

$$
\begin{aligned}
\mathbb{E}(x) &= \int_0^{\overline{x}} x f(x)\mathrm{d}x = (\overline{x}F(\overline{x}) - 0F(0)) - \int_0^{\overline{x}} F(x)\mathrm{d}x \\
&= \overline{x} - \int_0^{\overline{x}} \sqrt[n-1]{\frac{1}{n}\frac{\exp(-2\mu x) - 1}{\exp(-2\mu x) - 1}}\mathrm{d}x \\
&= \overline{x} + \frac{\sqrt[n-1]{1 - \exp(-2\mu\overline{x})}}{2\mu}(n-1)Hyp(\frac{1}{n-1}, \frac{1}{n-1}, \frac{n-2}{n-1}, \exp(2\mu\overline{x}))\,.
\end{aligned}
$$

Proof of Proposition 5. We apply the monotone transformation $y = \exp(\frac{2\mu x_0}{\sigma^2})$ to $\mathbb{E}(X_\tau)$ to get

$$
\begin{aligned}
\mathbb{E}(X_\tau) &= \frac{x_0}{\log(y)} + (1 + \frac{y}{2(1-y)})(x_0 - \frac{x_0 \log(2-y)}{\log(y)})\,, \\
&= x_0 \left(\frac{1}{\log(y)} + (1 + \frac{y}{2(1-y)})(1 - \frac{\log(2-y)}{\log(y)})\right)\,.
\end{aligned}
$$

for $y \neq 1$. This expression is convex if and only if it is convex for $x_0 = 1$. Assumption 1 implies $y \in (0, 2)$.

$$\begin{aligned}
\frac{\partial^2 \mathbb{E}(X_\tau)/x_0}{\partial y^2} &= \frac{4(-2+y)(-1+y)^3 + 2(-1+y)^2\,(2-5y+2y^2)\log(y)}{2(-2+y)(-1+y)^3 y^2 \log(y)^3} \\
&+ \frac{y^2\,(3-4y+y^2)\log(y)^2 - 2(-2+y)y^2\log(y)^3}{2(-2+y)(-1+y)^3 y^2 \log(y)^3} \\
&- \frac{(-2+y)\log(2-y)(2(-2+y)(-1+y)^2 - 2y^2\log(y)^2)}{2(-2+y)(-1+y)^3 y^2 \log(y)^3} \\
&- \frac{(-2+y)\log(2-y)\log(y)\,(-2+7y-6y^2+y^3)}{2(-2+y)(-1+y)^3 y^2 \log(y)^3}
\end{aligned}$$

with the continuous extension $\frac{\partial^2 \mathbb{E}(X_\tau)/x_0}{\partial y^2} = \frac{1}{6}$ at $y = 1$. Simple algebra shows that nominator and denominator are negative on $y \in (0, 2), y \neq 1$. Hence, the function is convex on $(0, 2)$. As y is monotone increasing in μ, $\mathbb{E}(X^i_{\tau^i})$ is quasi-convex in μ. As y is also monotone increasing (decreasing) in σ for $\mu < 0$ ($\mu > 0$), $\mathbb{E}(X^i_{\tau^i})$ is quasi-convex (quasi-concave) in σ if $\mu < 0$ ($\mu > 0$).

It remains to show that $\mathbb{E}(X_\tau)$ is first decreasing, then increasing. For $\mu \to -\infty$ and $\mu \to 0$, $\mathbb{E}(X_\tau) \to x_0$. For any negative value of μ, the expected value of the stopped processes is smaller than x_0, because the process is a supermartingale. Hence, by quasi-convexity, $\mathbb{E}(X_\tau)$ has to be first decreasing, then increasing.

□

References

[1] A. Anderson and L. M. B. Cabral. Go for broke or play it safe? Dynamic competition with choice of variance. *RAND Journal of Economics*, 38:593–609, 2007.

[2] M. R. Baye, D. Kovenock, and C. de Vries. The all-pay auction with complete information. *Economic Theory*, 8:291–305, 1996.

[3] K. Burdett and K. L. Judd. Equilibrium price dispersion. *Econometrica*, 51:955–969, 1983.

[4] G. W. Downs and D. M. Rocke. Conflict, agency, and gambling for resurrection: The principal-agent problem goes to war. *American Journal of Political Science*, 38:362–380, 1994.

[5] D. Fudenberg and J. Tirole. A theory of exit in duopoly. *Econometrica*, 54:943–960, 1986.

[6] P. Grandits and N. Falkner. Embedding in Brownian motion with drift and the Azéma-Yor construction. *Stochastic Processes and their Applications*, 85:249–254, 2000.

[7] F. Gul and W. Pesendorfer. The war of information. *Working Paper*, 2010.

[8] C. Harris and J. Vickers. Racing with uncertainty. *Review of Economic Studies*, 54:1–21, 1987.

[9] A. L. Hillman and D. Samet. Dissipation of contestable rents by small numbers of contenders. *Public Choice*, 54:63–82, 1987.

[10] H. Hvide. Tournament rewards and risk taking. *Journal of Labor Economics*, 20:877–898, 2002.

[11] S. Karlin. Reduction of certain classes of games to integral equations. in: H. Kuhn & A. Tucker (Eds.), 'Contributions to the Theory of Games, II.', Vol.28 of *Annals of Mathematical Studies*, Princeton University Press, Princeton, pp.125-128, 1953.

[12] E. P. Lazear and S. Rosen. Rank-order tournaments as optimum labor contracts. *Journal of Political Economy*, 89:841–864, 1981.

[13] G. Moscarini and L. Smith. Optimal dynamic contests. *Working Paper*, 2007.

[14] J. Obłój. The Skorokhod embedding problem and its offspring. *Probability Surveys*, 1:321–392, 2004.

[15] A. Park and L. Smith. Caller number five and related timing games. *Theoretical Economics*, 3:231–256, 2008.

[16] J. L. Pedersen and G. Peskir. The Azéma-Yor embedding in non-singular diffusions. *Stochastic Processes and their Applications*, 96:305–312, 2001.

[17] D. Revuz and M. Yor. Continuous Martingales and Brownian Motion. Springer Verlag, 2005.

[18] A. V. Skorokhod. Issledovaniya po teorii sluchainykh protsessov (stokhasticheskie differentsialnye uravneniya i predelnye teoremy dlya protsessov markova). *Izdat Kiev University*, 1961.

[19] A. V. Skorokhod. Studies in the theory of random processes. *Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co.*, 1965.