Asset Pricing and Wealth Distribution with Heterogeneous Investment Returns

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Abstract

I obtain a closed-form solution for a general equilibrium model with incomplete markets and heterogeneous agents that allow for an arbitrary number of assets, an arbitrary number of aggregate states, and arbitrary shock distributions for asset returns. Agent heterogeneity has non-trivial implications for asset prices and risk premia. In a stationary equilibrium the conditional distribution of consumption and wealth given initial wealth is double Pareto, which has two power law tails. If the initial wealth is lognormal, the stationary unconditional distribution of consumption and wealth is double Pareto-lognormal, which is empirically supported. The baseline model extends to the case with an arbitrary number of neoclassical firms, labor-leisure decisions, and bequest. In the presence of factor obsolescence or factor-augmenting technological change, there is some risk sharing in equilibrium but the equilibrium is generically constrained inefficient.

1 Introduction

Solving for a general equilibrium in closed-form is generally perceived as challenging even for a representative-agent model, hence the usual approach for dealing with a general equilibrium model with heterogeneous agents is numerical.[1] However, if possible solving for the equilibrium analytically is desirable since it makes us better grasp the structure of the model, gives us more freedom to parameterize, and allows us to estimate it. The purpose of this paper is to show that, under some (but general enough) circumstances, we can solve for the equilibrium.

The model is an otherwise standard stochastic growth model, but agents with constant relative risk aversion (CRRA) preferences can invest their wealth in an arbitrary number of investment projects whose returns are either common across all agents (aggregate shocks) or conditionally independent across

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[1] See, for example, Huggett (1993, 1997), Aiyagari (1994), Den Haan (1996), and Krusell and Smith (1997, 1998), among many others.

agents (idiosyncratic shocks).[2] For instance, think about human capital investment or entrepreneurs investing capital in their own investment projects. Because agents' portfolio choice or consumption rate is independent of wealth, the wealth distribution is not a relevant state variable for describing the equilibrium, which makes the model highly tractable. In this regard the model builds on the framework of Krebs (2003a,b, 2006) but extends the scope in a few directions. First, I allow for an arbitrary number of assets, an arbitrary number of aggregate states, and arbitrary shock distributions for asset returns. Second, I obtain a closed-form formula for pricing any asset, including the risk-free asset. Third, I show that an economy with an arbitrary number of neoclassical firms is mathematically equivalent to an economy without firms (but possibly with more assets). However, when there is factor obsolescence or factor-augmenting technological change, the equilibrium is generically constrained inefficient (as opposed to the model without firms which is constrained efficient). Fourth, the computational algorithm for obtaining the equilibrium is constructive and guaranteed to converge. Fifth, the model can be extended to include labor-leisure choice and bequest. Finally, I characterize the stationary distributions of consumption and wealth in equilibrium. The resulting consumption and wealth distributions obey the "double power law" (Toda, 2011a) and fit well to actual data (Toda and Walsh, 2011).

Obtaining closed-form solution for the general equilibrium with heterogeneous agents is of course not my invention. Constantinides and Duffie (1996) construct the households' income process (which is a logarithmic random walk) that can explain any asset prices; Krebs (2003a,b, 2006) characterizes the optimal consumption rule when households rent physical and human capital to firms; Angeletos and Calvet (2005, 2006) and Angeletos (2007) use CARA preferences that make the wealth distribution irrelevant; and Krusell et al. (2011) introduce maximally tight borrowing constraints that leave the economy in autarky and study the asset pricing implications.

The rest of the paper is organized as follows. Section 2 studies the single agent problem without firms, which is the mathematical basis of all subsequent analysis. In Section 3 I define the equilibrium concept and give a constructive existence proof. Section 4 studies the asset pricing implications of the heterogeneous agent model. Sections 5 and 6 introduce firms, labor-leisure choice, and bequest. Appendix A contains results on power law distributions.

2 Single agent problem

We first solve a single agent optimal consumption/portfolio problem similar to the classic paper of Samuelson (1969). The additional aspects are that (i) we allow for an arbitrary number of assets whose returns distributions depend on the state of the economy, which follows an arbitrary finite state Markov chain, (ii) we allow for arbitrary constraints on relative positions, and (iii) we obtain an explicit formula for the optimal consumption/portfolio rule.

[2] Thus the shocks are "rate-of-return" shocks as in Krebs (2003a,b, 2006), Angeletos and Calvet (2005, 2006), Angeletos (2007) as opposed to endowment shocks as in Bewley (1983, 1986), Huggett (1993, 1997), Aiyagari (1994), Krusell and Smith (1997, 1998), Krueger and Lustig (2010), and Krusell et al. (2011).

The consumer has a utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\beta > 0$ is the subjective discount rate[3] and the period utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ is constant relative risk aversion (CRRA) with coefficient $\gamma > 0$. If $\gamma = 1$, it is the usual log utility. We assume that the state of the economy is described by a finite state Markov chain on $\{1, 2, \dots, S\}$ with transition probabilities $\Pr(s \to s') = \pi_{ss'}$. The state at period t is denoted by s_t, and $s^t = (s_0, s_1, \dots, s_t)$ denotes the history up to period t. There are J assets indexed by $j = 1, 2, \dots, J$. Let $\mathbf{R}_{t+1} = (R^1_{t+1}, \dots, R^J_{t+1})$ be the vector of asset returns between periods t and $t+1$. We assume that the distribution of the asset returns vector $\mathbf{R}_{t+1}$ conditional on all past values of $\mathbf{R}$ and history $s^t = (s_0, s_1, \dots, s_t)$ depends only on the current state s_t and is time homogeneous. In particular, for any Borel measurable function f we have

$$\mathrm{E}\left[f(\mathbf{R}_{t+1}) \,\middle|\, \mathbf{R}_t, \dots, \mathbf{R}_0, s^t\right] = \mathrm{E}\left[f(\mathbf{R}_{t+1}) \mid s_t\right].$$

If there is only one state ($S = 1$), this assumption implies that $\{\mathbf{R}_t\}_{t=1}^{\infty}$ is i.i.d.

Let $\theta = (\theta^1, \dots, \theta^J)$ be the portfolio shares of the consumer and $R_{t+1}(\theta) = \sum_j R^j_{t+1}\theta^j$ be the return on the portfolio. By definition, $\sum_j \theta^j = 1$. Of course, the consumer is short in asset j if $\theta^j < 0$ and long if $\theta^j > 0$. The consumer may or may not be constrained in the position he takes, but we assume that he can only be constrained in the relative size of the position (*i.e.*, the portfolio shares θ) and not in the absolute size. Let $\Theta \subset \mathbb{R}^J$ be the constraint set. The following proposition solves the consumer's optimum consumption/portfolio problem.

Theorem 2.1. *Suppose that the portfolio constraint $\Theta \subset \mathbb{R}^J$ is compact, the return on portfolio $R(\theta) = \sum_j R^j\theta^j$ is positive almost surely for any $\theta \in \Theta$ in any state s, and*

$$\frac{1}{\beta} > \begin{cases} \max_{\theta\in\Theta} \mathrm{E}\left[R(\theta)^{1-\gamma} \,\middle|\, s\right] & (\gamma \le 1) \\ \min_{\theta\in\Theta} \mathrm{E}\left[R(\theta)^{1-\gamma} \,\middle|\, s\right] & (\gamma > 1) \end{cases} \tag{2.1}$$

for all $s \in \{1, 2, \dots, S\}$. Then there exists an optimal consumption/portfolio rule, which is given below:

1. *when $\gamma = 1$ the optimal portfolio rule is $\theta_s = \arg\max_{\theta\in\Theta} \mathrm{E}\left[\log R(\theta) \mid s\right]$ and the optimal consumption rule is $c(w, s) = (1-\beta)w$,*

2. *when $\gamma \neq 1$ the optimal consumption rule is $c(w, s) = a_s^{-\frac{1}{\gamma}} w$, where $a_s > 1$ and the optimal portfolio rule θ_s satisfy*

$$a_s^{\frac{1}{\gamma}} = 1 + \left(\beta\,\mathrm{E}\left[a_{s'}R(\theta_s)^{1-\gamma} \,\middle|\, s\right]\right)^{\frac{1}{\gamma}}, \tag{2.2a}$$

$$\theta_s = \arg\max_{\theta\in\Theta} \frac{1}{1-\gamma}\,\mathrm{E}\left[a_{s'}R(\theta)^{1-\gamma} \,\middle|\, s\right]. \tag{2.2b}$$

[3]I follow Kocherlakota (1990) for not restricting the analysis to discount factors smaller than 1.

In particular, if $\gamma \neq 1$ but there is only one state ($S = 1$) the optimal portfolio rule is $\theta^* = \arg\max_{\theta \in \Theta} \frac{1}{1-\gamma} \mathrm{E}[R(\theta)^{1-\gamma}]$ and the optimal consumption rule is $c(w) = (1 - (\beta k)^{\frac{1}{\gamma}})w$, where $k = \mathrm{E}[R(\theta^*)^{1-\gamma}]$.

Proof.

Case 1: $\gamma = 1$. Fix a portfolio $\{\theta_t\}_{t=0}^{\infty}$ and consider the optimization problem

$$\max_{\{c_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t \quad \text{subject to} \quad w_{t+1} = R_{t+1}(\theta_t)(w_t - c_t), \forall t.$$

By (2.1) we have $\beta < 1$. The first-order condition (Euler equation) with respect to c_t is

$$\frac{1}{c_t} = \mathrm{E}\left[\beta \frac{1}{c_{t+1}} R_{t+1}(\theta_t) \,\middle|\, s_t\right],$$

which is satisfied by $c_t = (1-\beta)w_t$ by inspection of the budget constraint. Then

$$\mathrm{E}[\beta^t u'(c(w_t))w_t] = \frac{\beta^t}{1-\beta} \to 0$$

as $t \to \infty$, so the transversality condition is satisfied. Since the Euler equation and the transversality condition are sufficient for optimality, the consumption rule $c_t = (1-\beta)w_t$ is optimal for any portfolio choice. Under this consumption rule, the budget constraint becomes $w_{t+1} = \beta R_{t+1}(\theta_t)w_t$, so the expected utility is

$$\begin{aligned}
&\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log[(1-\beta)w_t] \\
&= \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log\left[(1-\beta)w_0 \prod_{\tau=0}^{t-1} \beta R_{\tau+1}(\theta_\tau)\right] \\
&= \frac{\log[(1-\beta)w_0]}{1-\beta} + \sum_{t=0}^{\infty} \sum_{s^t} \Pr(s^t) \sum_{\tau=t}^{\infty} \beta^\tau \,\mathrm{E}\left[\log(\beta R_{t+1}(\theta_t) \mid s_t]\right).
\end{aligned}$$

By inspection of the right-hand side, the optimal portfolio is clearly

$$\theta_s = \arg\max_{\theta \in \Theta} \mathrm{E}\left[\log R(\theta) \mid s\right],$$

which exists because $R(\theta) > 0$ almost surely, $R(\theta)$ is continuous in θ, and the portfolio constraint Θ is compact.

Case 2: $\gamma \neq 1$. Since for any $c \geq 0$ we have $u(c) \geq 0$ if $\gamma < 1$ and $u(c) \leq 0$ if $\gamma > 1$, for any (not necessarily feasible) consumption plan $\{c_t\}$ the expected utility $\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ exists in $[0, \infty]$ if $\gamma < 1$ and in $[-\infty, 0]$ if $\gamma > 1$. Hence the value function (the supremum of expected utility over budget-feasible consumption plans)

$$V(w,s) = \sup_{\{c_t, \theta_t\}} \left\{ \beta^t \,\mathrm{E}_0 \sum_{t=0}^{\infty} \frac{c_t^{1-\gamma}}{1-\gamma} \,\middle|\, s_0 = s,\ (\forall t)\ w_{t+1} = R_{t+1}(\theta_t)(w_t - c_t) \right\}$$

is well-defined.

Step 1. $V(w,s)$ is finite and homogeneous of degree $1-\gamma$ in w.

Letting

$$C = \begin{cases} \beta \max_s \max_{\theta\in\Theta} \mathrm{E}\left[R(\theta)^{1-\gamma} \,\middle|\, s\right], & (\gamma < 1) \\ \beta \max_s \min_{\theta\in\Theta} \mathrm{E}\left[R(\theta)^{1-\gamma} \,\middle|\, s\right], & (\gamma > 1) \end{cases} \tag{2.3}$$

by (2.1) we have $C < 1$. If $\gamma < 1$, since $c_t \le w_t$ and $w_{t+1} \le R_{t+1}(\theta_t) w_t$, we obtain

$$\begin{aligned} 0 \le V(w,s) &\le \sup_{\{\theta_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{w_t^{1-\gamma}}{1-\gamma} \\ &\le \sup_{\{\theta_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{w_0^{1-\gamma}}{1-\gamma} \prod_{\tau=0}^{t-1} R_{\tau+1}(\theta_\tau)^{1-\gamma} \le \frac{w_0^{1-\gamma}}{1-\gamma} \sum_{t=0}^{\infty} C^t < \infty. \end{aligned}$$

If $\gamma > 1$, let $0 < \epsilon < 1$ and $c_t = \epsilon w_t$. Since $w_{t+1} = (1-\epsilon) R_{t+1}(\theta_t) w_t$, we obtain

$$\begin{aligned} 0 \ge V(w,s) &\ge \sup_{\{\theta_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(\epsilon w_t)^{1-\gamma}}{1-\gamma} \\ &\ge \sup_{\{\theta_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(\epsilon w_0)^{1-\gamma}}{1-\gamma} \prod_{\tau=0}^{t-1} [(1-\epsilon) R_{\tau+1}(\theta_\tau)]^{1-\gamma} \\ &\ge \frac{(\epsilon w_0)^{1-\gamma}}{1-\gamma} \sum_{t=0}^{\infty} [(1-\epsilon)^{1-\gamma} C]^t > -\infty \end{aligned}$$

for small enough $\epsilon > 0$. In either case $V(w,s)$ is finite.

If $\{c_t\}$ is budget feasible with initial wealth w, so is $\{\lambda c_t\}$ with initial wealth λw, where $\lambda > 0$. By the homotheticity of the utility function we obtain $V(\lambda w, s) \ge \lambda^{1-\gamma} V(w,s)$. Letting $w' = \lambda w$ and $\lambda' = 1/\lambda$ we obtain $(\lambda')^{1-\gamma} V(w', s) \ge V(\lambda' w', s)$, so $V(\lambda w, s) = \lambda^{1-\gamma} V(w,s)$ for all w and $\lambda > 0$. Therefore $V(w,s)$ is homogeneous of degree $1-\gamma$ in w.

Step 2. The value function $V(w,s)$ satisfies the Bellman equation

$$V(w,s) = \sup_{\substack{c\ge 0 \\ \theta\in\Theta}} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \, \mathrm{E}\left[V(R(\theta)(w-c), s') \mid s\right] \right\}. \tag{2.4}$$

Suppose the initial state is $s_0 = s$ and fix $c_0 = c$ and $\theta_0 = \theta$. Taking the supremum of expected utility with respect to $\{c_t, \theta_t\}_{t=1}^{\infty}$, we obtain

$$V(w,s) \ge u(c) + \beta \, \mathrm{E}\left[V(R(\theta)(w-c), s') \mid s\right].$$

Taking the supremum with respect to c, θ, we obtain one direction of (2.4). For the other direction, since $V(w,s)$ is finite, for any $\epsilon > 0$ we can take a plan $\{c_t, \theta_t\}_{t=0}^{\infty}$ such that

$$V(w,s) \le \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) + \epsilon.$$

Fixing (c_0, θ_0) but taking the supremum with respect to $\{c_t, \theta_t\}_{t=1}^{\infty}$, we obtain

$$V(w,s) \le u(c_0) + \beta \, \mathrm{E}\left[V(R(\theta_0)(w-c), s') \mid s\right] + \epsilon.$$

Taking the supremum with respect to c_0, θ_0 and letting $\epsilon \to 0$, we obtain (2.4).

Step 3. The value function takes the form $V(w,s) = a_s \frac{w^{1-\gamma}}{1-\gamma}$ *for some* $a_s > 0$. *The supremum in the Bellman equation* (2.4) *is attained by* $c = a_s^{-\frac{1}{\gamma}} w$ *and* θ_s *defined by* (2.2b).

The first claim is trivial by Steps 1 and 2. Substituting $V(w,s) = a_s \frac{w^{1-\gamma}}{1-\gamma}$ into the Bellman equation (2.4) and assuming that the supremum is attained, we get

$$a_s \frac{w^{1-\gamma}}{1-\gamma} = \max_{\substack{c \geq 0 \\ \theta \in \Theta}} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \operatorname{E}\left[a_{s'} R(\theta)^{1-\gamma} \,\middle|\, s \right] \frac{(w-c)^{1-\gamma}}{1-\gamma} \right\}. \tag{2.5}$$

Let $A_s(\theta) = \beta \operatorname{E}\left[a_{s'} R(\theta)^{1-\gamma} \,\middle|\, s \right]$. The first-order condition of the maximization in (2.5) with respect to c is

$$c^{-\gamma} = A_s(\theta)(w-c)^{-\gamma} \iff c = \frac{1}{1 + A_s(\theta)^{\frac{1}{\gamma}}} w.$$

Substituting this into (2.5) and comparing the coefficients of $\frac{w^{1-\gamma}}{1-\gamma}$, we get (after some algebra) $a_s = (1 + A_s(\theta_s)^{\frac{1}{\gamma}})^\gamma$ and $c(w,s) = a_s^{-\frac{1}{\gamma}} w$, where θ_s is the optimal portfolio, or

$$x_s = \begin{cases} 1 + \left(\beta \max_{\theta \in \Theta} \operatorname{E}\left[x_{s'}^\gamma R(\theta)^{1-\gamma} \,\middle|\, s \right] \right)^{\frac{1}{\gamma}}, & (\gamma < 1) \\ 1 + \left(\beta \min_{\theta \in \Theta} \operatorname{E}\left[x_{s'}^\gamma R(\theta)^{1-\gamma} \,\middle|\, s \right] \right)^{\frac{1}{\gamma}}, & (\gamma > 1) \end{cases} \tag{2.6}$$

where $x_s = a_s^{\frac{1}{\gamma}}$. (2.6) is equivalent to (2.2). To show that (2.6) has a solution, let $\mathbf{x} = (x_1, \dots, x_S) \in \mathbb{R}_+^S$ and define the mapping $T : \mathbb{R}_+^S \to \mathbb{R}_+^S$ by the right-hand side of (2.6). Clearly T is monotone, *i.e.*, $\mathbf{x} \leq \mathbf{y}$ implies $T\mathbf{x} \leq T\mathbf{y}$. Let C be as in (2.3) and $M = 1/(1 - C^{\frac{1}{\gamma}})$. By condition (2.1), we have $C < 1$ and hence $M > 0$. If $x_s \leq M$ for all s by (2.6) and the definition of T we obtain

$$(T\mathbf{x})_s \leq 1 + (CM^\gamma)^{\frac{1}{\gamma}} = M,$$

so $T : [0, M]^S \to [0, M]^S$. Define $\mathbf{x}^{(1)} = \begin{cases} M\mathbf{1}, & (\gamma < 1) \\ \mathbf{0}, & (\gamma > 1) \end{cases}$ and $\mathbf{x}^{(n+1)} = T\mathbf{x}^{(n)}$ for all n. Since T is monotone and T is a self map, $\left\{ \mathbf{x}^{(n)} \right\}$ monotonically converges to a fixed point $\mathbf{x}$ of T. Hence (2.6) has a solution $\mathbf{x}$. Since $T\mathbf{y} \geq \mathbf{1}$ for any $\mathbf{y} \in \mathbb{R}_+^S$ and $T\mathbf{y} \gg \mathbf{1}$ for any $\mathbf{y} \in \mathbb{R}_{++}^S$, it follows that $\mathbf{x} \gg \mathbf{1}$, hence $a_s > 1$ for all s. If (2.6) has another solution $\mathbf{x}'$, if $\gamma < 1$ applying T repeatedly to $\mathbf{x}' \leq M\mathbf{1}$ we obtain $\mathbf{x}' \leq \mathbf{x}$.[4] The reverse inequality holds if $\gamma > 1$. Therefore among all functions of the form $a_s \frac{w^{1-\gamma}}{1-\gamma}$ satisfying the Bellman equation (2.4), the one obtained here attains the supremum.

Step 4. The consumption plan $\{c_t\}$ *generated by the policy found in Step 3 attains* $V(w,s)$, i.e., *the optimal consumption rule is* $c(w,s) = a_s^{-\frac{1}{\gamma}} w$ *and the optimal portfolio rule is given by* (2.2b).

[4] Actually applying Minkowski's inequality to (2.6), we can show that T is a contraction mapping if $\gamma < 1$ (but not necessarily if $\gamma > 1$). Therefore for $\gamma < 1$ the fixed point is unique, and the sequence $\left\{ \mathbf{x}^{(n)} \right\}$ converges to the fixed point regardless of the initial value.

Let $\{c_t, \theta_t\}$ be generated by the policy found in Step 3 and $\{w_t\}$ be the associated wealth. Iterating the Bellman equation (2.4), for any $T > 0$ and s, letting $s_0 = s$ we obtain

$$V(w,s) = \mathrm{E}_0 \sum_{t=0}^{T-1} \beta^t u(c_t) + \mathrm{E}_0[\beta^T V(w_T, s_T)].$$

Since $\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ exists by the remark at the beginning of the proof, letting $T \to \infty$ we obtain

$$V(w,s) \le \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) + \limsup_{T\to\infty} \mathrm{E}_0[\beta^T V(w_T, s_T)].$$

Since by the definition of $V(w,s)$ we have $V(w,s) \ge \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$, the consumption plan $\{c_t\}$ attains the supremum if and only if the transversality condition $\limsup_{t\to\infty} \mathrm{E}_0[\beta^t V(w_t, s_t)] \le 0$ holds. If $\gamma > 1$, this is trivial because $V(w_t, s_t) \le 0$. If $\gamma < 1$, since $V(w,s) = a_s \frac{w^{1-\gamma}}{1-\gamma}$ and $w_{t+1} \le R_{t+1}(\theta_t) w_t$, letting $a = \max_s a_s$, by (2.3) we obtain

$$\begin{aligned} \mathrm{E}_0[\beta^t V(w_t, s_t)] &\le \sup_{\{\theta_t\}} \mathrm{E}_0 \left[\beta^t a \frac{w_t^{1-\gamma}}{1-\gamma} \right] \\ &\le \sup_{\{\theta_t\}} \mathrm{E}_0 \left[a\beta^t \frac{w_0^{1-\gamma}}{1-\gamma} \prod_{\tau=0}^{t-1} R_{\tau+1}(\theta_\tau)^{1-\gamma} \right] \le a \frac{w_0^{1-\gamma}}{1-\gamma} C^t \to 0. \end{aligned}$$

In particular, when there is only one state ($S = 1$), by (2.2b) the optimal portfolio rule is $\theta^* = \arg\max_{\theta\in\Theta} \frac{1}{1-\gamma} \mathrm{E}[R(\theta)^{1-\gamma}]$. Letting $a = a_1$, $x = x_1$, and $k = \mathrm{E}[R(\theta^*)^{1-\gamma}]$, it follows from (2.6) that $x = \frac{1}{1-(\beta k)^{\frac{1}{\gamma}}}$. Hence the optimal consumption rule is $c(w) = \frac{w}{x} = (1 - (\beta k)^{\frac{1}{\gamma}})w$. $\square$

The result $c(w) = (1 - (\beta k)^{\frac{1}{\gamma}})w$ when asset returns are i.i.d. is intuitive. It implies that the saving rate $(\beta k)^{\frac{1}{\gamma}}$ is higher when patient (higher β) or there is more risk when $\gamma > 1$. To see that there is a precautionary saving motive, let $\mathbf{R}'$ be a mean-preserving spread of asset returns $\mathbf{R}$ (more precisely, $\mathbf{R}' - \mathbf{R}$ is mean zero and independent of $\mathbf{R}$). Since $R^{1-\gamma}$ is convex in R when $\gamma > 1$, conditioning on $\mathbf{R}$ by Jensen's inequality we obtain

$$\mathrm{E}[R(\theta)^{1-\gamma}] \le \mathrm{E}[R'(\theta)^{1-\gamma}] \Longrightarrow k \le k',$$

where $k = \min_{\theta\in\Theta} \mathrm{E}[R(\theta)^{1-\gamma}]$ and k' is defined similarly. Therefore the saving rate $(\beta k)^{\frac{1}{\gamma}}$ is higher with higher risk.

Theorem 2.1 is silent about the uniqueness of the optimal portfolio rule. Since we have not assumed the convexity of the portfolio constraint set Θ, there may well be multiple optima (however, given a portfolio rule the optimal consumption plan is unique). Of course, if we assume that Θ is convex, the solution is unique since the preference is strictly concave and all constraints are convex.

Theorem 2.1 is similar to the result of Krebs (2006) but more general. First, Krebs assumes that the state of the economy (in my notation $\{1, 2, \dots, S\}$ *and* the investment returns $\{R^1, R^2, \dots, R^J\}$) jointly follow a finite state Markov chain, but I allow for arbitrary (not necessarily finitely supported) distributions for investment returns. Second, in Krebs's model the consumer chooses the allocation of resources between two investments (physical and human capital), but in my model the number of investments is arbitrary. Third, Krebs deals with an arbitrary neoclassical production function with two inputs (physical and human capital), but for simplicity I consider only linear stochastic savings technologies. This specialization is not restrictive, however, as I prove in Section 5 that introducing an arbitrary number of neoclassical firms is mathematically equivalent to the baseline model discussed here. Fourth, my condition (2.1) is similar to condition (9) on (Krebs, 2006, p. 515),[5] but condition (2.1) is weaker since for $\gamma > 1$ I take the *minimum* with respect to the portfolio share θ (which corresponds to the capital-to-labor ratio $\tilde{k}$ in Krebs's notation), not the *maximum*. Fifth, Krebs uses the Brouwer fixed point theorem to prove the existence of equilibrium but I use the simple fact that the sequence obtained by the repeated application of a monotone self map on a compact set converges to a fixed point. Therefore my proof is constructive, which is advantageous for numerical implementation. Finally, because I take the value function approach it is clear how the optimal portfolio is determined by (2.2b), but Krebs takes the Euler equation approach.[6]

3 General equilibrium

Having solved the single agent problem in the previous section, in this section we solve for the general equilibrium when the economy is populated by many ex ante identical agents. It turns out that because agents have homothetic preferences and the portfolio constraint involves only ratio variables, every agent behaves in the same way regardless of the wealth level. Therefore the wealth distribution is not a relevant state variable for the description of the recursive equilibrium and the computation of the equilibrium reduces to solving the single agent problem.

3.1 Description of the economy

There is an arbitrary number of ex ante identical consumers with utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} [\beta(1-\delta)]^t u(c_t), \tag{3.1}$$

where $0 < \beta < 1$ is the subjective discount rate, $0 \le \delta < 1$ is the death probability between any two consecutive periods, and the period utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ is constant relative risk aversion (CRRA) with $\gamma > 0$. If $\gamma = 1$, it is the usual log utility. The initial wealth of the consumer is an i.i.d. draw from some initial distribution Ψ_0. To keep the analysis simple, we assume that personal

[5]Condition (2.1) is also similar to Equation (11) on (Kocherlakota, 1990, p. 45), but the latter contains neither uncertainty nor a portfolio choice since it considers a deterministic endowment economy.

[6]Krebs's proof of the transversality condition seems to be wrong, however: in deriving equation (A.9) on page 521, Krebs takes the maximum of the left-hand side with respect to S, but fails to take the maximum of the right-hand side at the same time which also depends on S.

wealth is destroyed upon death. We can interpret this assumption as a 100% inheritance or estate tax which is spent by the government on useless projects. When a consumer dies, another consumer is born with initial wealth drawn from the same initial distribution Ψ_0 independently of any other variables. Hence the population is constant overtime. The state of the economy follows a Markov chain on $\{1, 2, \dots, S\}$ with transition probability $\Pr(s \to s') = \pi_{ss'}$.

There are J assets indexed by $j = 1, 2, \dots, J$. The vector of asset returns between periods t and $t+1$ are denoted by $\mathbf{R}_{t+1} = (R^1_{t+1}, \dots, R^J_{t+1})$. We interpret a risky asset as a linear stochastic savings technology (hence the supply of assets is not fixed but automatically meets the demand): if a consumer invests a unit of good in asset j at the end of time t, he will receive R^j_{t+1} at the beginning of time $t+1$. We denote the portfolio share (relative position) in asset holdings by a vector $\theta = (\theta^1, \dots, \theta^J) \in \mathbb{R}^J$. A consumer's portfolio share is constrained to be in the set $\Theta \subset \mathbb{R}^J$, which can be interpreted as a constraint on leverage or other institutional constraints (limits on shortsale, restrictions on access to certain capital markets, etc.). The distributional assumptions on $\mathbf{R}_{t+1}$ is the same as in Section 2: the distribution of $\mathbf{R}_{t+1}$ conditional on past returns and states depends only on the current state s_t.

Each asset falls into the category of either *public* (the return is common across all individuals) or *private* (the return is independent across all individuals conditional on the current state, but not necessarily independent from the returns of public assets). For instance, the stock market, the risk-free asset, or Arrow securities can be interpreted as public, whereas the human capital, a house, or a car can be interpreted as private. Markets are incomplete in the sense that there is no insurance for private assets, which can be interpreted as due to a high monitoring cost.[7] Let $\mathcal{J}^{\text{pub}}, \mathcal{J}^{\text{priv}}$ be the set of indices of the public and private assets, and $\mathcal{J} = \{1, 2, \dots, J\} = \mathcal{J}^{\text{pub}} \cup \mathcal{J}^{\text{priv}}$. Some public assets (*e.g.*, the risk-free asset, Arrow securities, or any asset that is not backed by a physical asset) might be in zero net supply; let $\mathcal{J}^0 \subset \mathcal{J}^{\text{pub}}$ be the set of indices of those public assets.

3.2 Existence of recursive equilibrium

The most natural equilibrium concept for the economy described above is the recursive competitive equilibrium. Let Z, z denote the aggregate (public) and individual (private) state variables. A recursive competitive equilibrium consists of a law of motion for the public and private states Z, z, a consumption rule $c(z, Z)$, a portfolio rule $\theta(z, Z)$, and pricing kernels $\left\{Q^j(Z)\right\}_{j \in \mathcal{J}^{\text{pub}}}$ such that

1. $c(z, Z)$ and $\theta(z, Z)$ are optimal subject to individual budget constraints,
2. for $j \in \mathcal{J}^0$ the net supply of asset j is zero, and
3. the law of motion of Z and z are determined by individual decisions.

Usually the public state Z consists of the entire wealth distribution and the current state $s \in \{1, 2, \dots, S\}$, a high-dimensional object (*e.g.*, Krusell and Smith (1998)). However, the following theorem shows that the description and the existence of the recursive competitive equilibrium is straightforward.

[7] Meh and Quadrini (2006) endogenize market incompleteness but for simplicity we assume that the market incompleteness is exogenous.

Theorem 3.1. *Let $\Theta^0 = \left\{\theta \in \Theta \,\middle|\, \forall j \in \mathcal{J}^0, \theta^j = 0\right\}$ be the portfolio constraint with holdings in assets in zero net supply restricted to be zero. Suppose that condition* (2.1) *holds with β replaced by $\beta(1-\delta)$ and Θ replaced by Θ_0. Then a recursive equilibrium exists and can be constructed as follows.*

1. *The public state is $s \in \{1, 2, \dots, S\}$ and the private state is individual wealth w,*

2. *the optimal consumption/portfolio rule is given by Theorem 2.1, where Θ is replaced by Θ^0, and*

3. *the pricing kernels are derived by the Euler equation.*

Furthermore, the equilibrium is constrained efficient.

Proof. Let $c(w, s)$ and θ_s be the consumption/portfolio rule given by Theorem 2.1, where Θ is replaced by Θ^0. For $j \in \mathcal{J}^0$, define the pricing kernel of asset j by the Euler equation as in 4.1. Then by construction the Euler equation holds for every asset $j \in \mathcal{J}^0$. By the optimality of $c(w, s)$ and θ_s under the constraint Θ^0, the Euler equation holds for every asset $j \notin \mathcal{J}^0$. Hence the Euler equation holds for every asset $j \in \mathcal{J}$. Furthermore, the transversality condition holds by the proof of Theorem 2.1. Since the Euler equation and the transversality condition are sufficient for optimality, the consumption/portfolio rule $c(w, s)$ and θ_s is optimal even without the restriction $\theta \in \Theta^0$. Since the individual holdings in asset j is zero for $j \in \mathcal{J}^0$ by construction, the markets of assets with zero net supply clear. Therefore we obtain a recursive competitive equilibrium.

The definition of constrained efficiency is that we cannot make everybody at least as well off and somebody better off by changing only the asset holdings. Since θ_s is optimal for individual optimization problem independent of other agents' portfolio choice, we cannot make anybody better off by changing only asset holdings. Hence the equilibrium is constrained efficient. □

3.3 Computation of equilibrium

Since the existence theorem of the recursive competitive equilibrium (Theorem 3.1) is constructive, it is straightforward to compute the equilibrium: the computation reduces to obtaining the optimal consumption/portfolio rule described in Theorem 2.1. If $\gamma = 1$ (log utility) or $S = 1$ (i.i.d. asset returns), we have explicit formula as in Theorem 3.1. If $\gamma \neq 1$ and $S > 1$, do the following:

1. Pick an initial value of $\mathbf{a} = (a_1, \dots, a_S)'$, say $\mathbf{1} = (1, \dots, 1)'$.

2. Find $\theta \in \Theta^0$ that solves (2.2b) with Θ replaced by Θ^0.

3. Update $\mathbf{a}$ by (2.2a), using $\theta \in \Theta^0$ obtained in Step 2.

4. Repeat Steps 2–3 until convergence is obtained, which is guaranteed by the proof of Theorem 2.1.

This way we obtain an optimal consumption/portfolio rule for all cases.

The only non-trivial part in the above algorithm is the optimization with respect to $\theta \in \Theta^0$. Since the objective function contains an expectation of a function of θ (that is, $\log R(\theta)$ or $R(\theta)^{1-\gamma}$), the optimization cannot be performed by routine programs unless we make special distributional assumptions under which the expectation can be evaluated analytically. One practical solution is to always use the multinomial distribution for asset returns. Since any distribution can be approximated by a multinomial distribution, and for a multinomial distribution the computing the expectation is straightforward, for most applications this solution should suffice.

An alternative is to randomly generate a large number of asset returns and replace the expectation by the sample mean. For instance, if $\gamma = 1$ instead of maximizing $\mathrm{E}[\log R(\theta)]$ we maximize $\frac{1}{N}\sum_{n=1}^{N} \log R_n(\theta)$, where $\{R_n\}_{n=1}^{N}$ are randomly generated asset returns. Since the objective function is strictly concave in θ, the law of large numbers guarantees the convergence.

In the simulation of Toda and Walsh (2011) we took the second approach since we assumed lognormal asset returns. However, we found that the convergence to the optimum portfolio rule is slow when the volatility of asset returns is high. Therefore, it might be safer to use only multinomial asset returns.

3.4 Consumption and wealth distribution

What does the stationary consumption and wealth distribution look like in the recursive competitive equilibrium? Although the wealth distribution is not a relevant state variable for describing the recursive equilibrium, it has drawn the attention of many researchers since the time of Pareto (1896).

Since by Theorem 2.1 consumption is proportional to wealth, we only need to look at the wealth distribution. The answer crucially depends on whether consumers are infinitely lived or not. Consider a consumer who starts with initial wealth w_{ini}. If consumers are infinitely lived, letting $\sigma_s = 1 - a_s^{-\frac{1}{\gamma}}$ be the saving rate in state s, the consumer's wealth at the beginning of period T, w_T, satisfies

$$\log w_T = \log w_{\mathrm{ini}} + \sum_{t=0}^{T-1} \log[\sigma_{s_t} R_{t+1}(\theta_{s_t})] =: \log w_{\mathrm{ini}} + \sum_{t=0}^{T-1} X_t.$$

Conditioning on the realization of states $s_1, s_2, \dots$ and the returns on public assets, the random variables $\{X_t\}$ are independent across time and i.i.d. across individuals. Hence by the Lindeberg-Lévy central limit theorem, as $T \to \infty$ log wealth will behave as $\log w_T \sim \log w_{\mathrm{ini}} + Z$, where Z is a normal random (whose mean and variance depend on the realization of states and the returns on public assets but are of order $O(T)$) which is i.i.d. across individuals. Since the term corresponding to the initial wealth is negligible compared to the term corresponding to wealth growth, the cross-sectional wealth distribution becomes lognormal. This argument is precisely the same as the original formulation of Gibrat (1931).

Things drastically change when consumers die, which is more realistic. If we assume that consumers die with probability $\delta > 0$ between periods, the stationary age distribution is geometric with mean $1/\delta$. Let ν_δ be a geometric random variable with mean $1/\delta$ which describes the age of a consumer. Then, the consumer's wealth at the beginning of period T, w_T, satisfies

$$\begin{aligned}\log w_T &= \log w_{\text{ini}} + \sum_{t=1}^{\nu_\delta} \log[\sigma_{s_{T-t}} R_{T-t+1}(\theta_{s_{T-t}})] \\ &=: \log w_{\text{ini}} + \sum_{t=1}^{\nu_\delta} X_{T-t},\end{aligned}$$

assuming $\nu_\delta \le T$. When the death probability δ is small, by Theorem A.1 the growth of log wealth $\log w_T - \log w_{\text{ini}} = \sum_{t=1}^{\nu_\delta} X_{T-t}$ is approximately Laplace, or the ratio between the current and initial wealth is approximately double Pareto (Champernowne, 1953). In particular, if the initial wealth distribution is lognormal, then the cross-sectional wealth distribution becomes double Pareto-lognormal.[8]

In Toda and Walsh (2011) we fitted the lognormal and the double Pareto-lognormal distributions to quarterly household consumption data in U.S., and found that (i) the lognormal distribution is rejected against the double Pareto-lognormal distribution in 97 out of 98 quarters by the likelihood ratio test at significance level 0.05, and (ii) the double Pareto-lognormal distribution is not rejected in 77 out of 96 quarters (79% of the time) by the Kolmogorov test at significance level 0.05. These findings imply that the consumption distribution can be well explained by the present general equilibrium model with heterogeneous agents, but only when we take death into account.

4 Asset pricing

4.1 Asset pricing and risk-premium

The following proposition gives a formula for pricing any asset.

Proposition 4.1. *Let $\mathcal{F}_t = \sigma(\mathbf{R}_t, \dots, \mathbf{R}_0, s_t, \dots, s_0)$ be the σ-algebra generated by all past values of asset returns and states and $\{D_t\}_{t=0}^\infty$ be an $\mathcal{F}_t$-adapted stochastic process (*i.e.*, D_t is $\mathcal{F}_t$-measurable). Suppose that the distribution of D_{t+1} conditional on $\mathcal{F}_t$ depends only on s_t. If $s_t = s$, then the time t price of an asset that pays $D = D_{t+1}$ at period $t+1$ (and zero in any other periods) is given by*

$$P_s(D) = \frac{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} D \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]}, \tag{4.1}$$

where a_s satisfies (2.2a) *and θ_s is the optimal portfolio in Theorem 2.1. In particular, the price of the state s' Arrow security $P_{ss'}$ and the gross risk-free rate R_s satisfy*

$$P_{ss'} = \frac{\pi_{ss'} a_{s'} \mathrm{E}\left[R(\theta_s)^{-\gamma} \mid s, s'\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]} \tag{4.2a}$$

$$\frac{1}{R_s} = \frac{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]}. \tag{4.2b}$$

[8] See Appendix A for the definition of these distributions.

Proof. By Theorem 2.1, when $\gamma \neq 1$ the optimal consumption rule is $c(w, s) = a_s^{-\frac{1}{\gamma}} w$, where a_s satisfies (2.2a). This is true even for $\gamma = 1$ (with $a_s = \frac{1}{1-\beta}$). By the Euler equation, the optimal consumption rule, and (2.2a), we obtain

$$
\begin{aligned}
P_s(D) &= \mathrm{E}\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} D \,\middle|\, s\right] = \mathrm{E}\left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} D \,\middle|\, s\right] \\
&= \mathrm{E}\left[\beta \frac{a_{s_{t+1}}}{a_s} [(1 - a_s^{-\frac{1}{\gamma}}) R(\theta_s)]^{-\gamma} D \,\middle|\, s\right] \\
&= \frac{1}{(a_s^{\frac{1}{\gamma}} - 1)^\gamma} \mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} D \,\middle|\, s\right] = \frac{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} D \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]}.
\end{aligned}
$$

By setting $D = 1\{s_{t+1} = s'\}$ or $D = 1$ we obtain the price of the state s' Arrow security or the price of the risk-free bond. $\square$

Proposition 4.2. *The risk premium of the optimal portfolio* $\mathrm{E}[R(\theta) \mid s] - R_s$ *is positive.*

Proof. Take any portfolio rule $\{\theta_t\}$ adapted to $\mathcal{F}_t$ and consider the consumption rule $c(w, s) = a_s^{-\frac{1}{\gamma}} w$ found in Theorem 2.1. Then the expected utility under this consumption/portfolio rule is

$$
\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{(a_{s_t}^{-\frac{1}{\gamma}} w_t)^{1-\gamma}}{1-\gamma} = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t a_{s_t}^{1-\frac{1}{\gamma}} \frac{w_0^{1-\gamma}}{1-\gamma} \prod_{\tau=0}^{t-1} (1 - a_{s_\tau}^{-\frac{1}{\gamma}})^{1-\gamma} R_{\tau+1}(\theta_\tau)^{1-\gamma}.
$$

Therefore, if we consider two portfolio rules $\{\theta_t\}$ and $\{\theta_t'\}$, conditioning on $\mathcal{F}_\tau$ it turns out that θ is worse than θ' if we have

$$
\mathrm{E}\left[u(R_{t+1}(\theta_t)) \mid s_t\right] \leq \mathrm{E}\left[u(R_{t+1}(\theta_t')) \mid s_t\right]
$$

for all t and s_t with at least one strict inequality, where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Now suppose that the risk-premium $\mathrm{E}[R(\theta_s) \mid s] - R_s$ were less than or equal to zero in state s. Take $\theta_t = \theta_{s_t}$ (the optimal portfolio in Theorem 2.1) and θ_t' consisting entirely of the risk-free asset if $s_t = s$ and $\theta_t' = \theta_t$ otherwise. Letting R_s be the gross risk-free rate in state s, the return on θ_t' is R_s if $s_t = s$ and $R(\theta_s)$ otherwise. Since u is increasing and strictly concave, by Jensen's inequality we obtain

$$
\mathrm{E}\left[u(R(\theta_s)) \mid s\right] < u(\mathrm{E}\left[R(\theta_s) \mid s\right]) \leq u(R_s),
$$

which implies that the risk-free asset dominates the optimal portfolio in state s. Since θ_t and θ_t' are identical in states $s_t \neq s$, we have a contradiction. $\square$

The reason why we proved that the risk-premium on the optimal portfolio is positive is because it is not trivial. In fact, if an asset pays more in a bad state than in a good state, the asset may well have a negative risk-premium, such as a home or auto insurance.

The following theorem provides an explicit formula for the risk-premium.

Theorem 4.3. *Let R^j be the return of asset j, R_s be the gross risk-free rate in state s, and θ_s be the optimal portfolio in state s. Then*

$$
\mathrm{E}\left[R^j \,\middle|\, s\right] - R_s = -\frac{\mathrm{Cov}\left[a_{s'} R(\theta_s)^{-\gamma}, R^j \,\middle|\, s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} \mid s\right]}. \tag{4.3}
$$

Proof. Dividing both sides of (4.1) by $P_s(D)$ and substituting asset j in place of D, we obtain

$$1 = \frac{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} R^j \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]}.$$

Using $\mathrm{E}[XY \mid s] = \mathrm{Cov}[X, Y \mid s] + \mathrm{E}[X \mid s]\,\mathrm{E}[Y \mid s]$ for $X = a_{s'} R(\theta_s)^{-\gamma}$ and $Y = R^j$, we obtain

$$1 = \frac{\mathrm{Cov}\left[a_{s'} R(\theta_s)^{-\gamma}, R^j \mid s\right] + \mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} \mid s\right] \mathrm{E}\left[R^j \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{1-\gamma} \mid s\right]}.$$

Using (4.2b) and rearranging terms, we obtain (4.3). □

4.2 Relevance of market incompleteness for asset pricing

In a recent paper Krueger and Lustig (2010) showed that under some assumptions the absence of insurance markets for idiosyncratic labor income risk has no effect on the premium for aggregate risk. The assumptions are (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk, (iv) a constant capital share of income, and (v) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income. In this section I show that the irrelevance result of Krueger and Lustig depends on the constant capital share of income and the implicit (and rather stringent) assumption that the growth rate of labor income is the product of idiosyncratic and aggregate components.

The crucial assumption of Krueger and Lustig (2010) is given by Equation (2) on page 7,

$$\eta_t(s^t) = (1 - \alpha(z_t))\eta(y_t, z_t)e_t(z^t),$$

where y is an individual state, z is an aggregate state, $s = (y, z)$ is the joint state, e_t is the aggregate endowment, α is the aggregate labor income share, $\eta(y_t, z_t)$ is the idiosyncratic labor income shock, and η_t is the individual labor income. Suppose in our baseline model of Sections 2 and 3 there are only two assets, public and private, whose return is $\mathbf{R}_t = (R_t^1, R_t^2)$, where asset 1 is public and asset 2 is private. Corresponding to the assumption of Krueger and Lustig, suppose that the return on the private asset is proportional to the return of the public asset, so $R^2 = \eta R^1$, where η is a random variable independent from R^1 and independent across individuals conditional on the aggregate state s. Then the return on a portfolio $\theta = (\theta^1, \theta^2)$ is given by

$$R_t(\theta) = R_t^1\theta^1 + R_t^2\theta^2 = (\theta^1 + \theta^2\eta)R_t^1.$$

Substituting this into the formula for the risk-premium (4.3), we obtain

$$\begin{aligned}
\mathrm{E}\left[R^j \mid s\right] - R_s &= -\frac{\mathrm{Cov}\left[a_{s'} R(\theta_s)^{-\gamma}, R^j \mid s\right]}{\mathrm{E}\left[a_{s'} R(\theta_s)^{-\gamma} \mid s\right]} \\
&= -\frac{\mathrm{E}\left[(\theta^1 + \theta^2\eta)^{-\gamma} \mid s\right] \mathrm{Cov}\left[a_{s'} (R^1)^{-\gamma}, R^j \mid s\right]}{\mathrm{E}\left[(\theta^1 + \theta^2\eta)^{-\gamma} \mid s\right] \mathrm{E}\left[a_{s'} (R^1)^{-\gamma} \mid s\right]} \\
&= -\frac{\mathrm{Cov}\left[a_{s'} (R^1)^{-\gamma}, R^j \mid s\right]}{\mathrm{E}\left[a_{s'} (R^1)^{-\gamma} \mid s\right]}. \qquad (4.4)
\end{aligned}$$

Although the right-most expression in (4.4) does not directly depend on the idiosyncratic shock η, $\{a_s\}_{s=1}^S$ indirectly depends on η through (2.2). Therefore, even in the special case that the return on the private asset is the product of idiosyncratic and aggregate components, in general the idiosyncratic shock has a nontrivial impact on the equity premium. If the evolution to the next period's state is independent from asset returns, *i.e.*, if s_{t+1} and $\mathbf{R}_{t+1}$ are independent conditional on s_t, then (4.4) becomes

$$\mathrm{E}\left[R^j \mid s\right] - R_s = -\frac{\mathrm{Cov}\left[(R^1)^{-\gamma}, R^j \mid s\right]}{\mathrm{E}\left[(R^1)^{-\gamma} \mid s\right]},$$

which does not depend on the idiosyncratic shock η.

There are two reasons why market incompleteness is relevant for the equity premium. First, the return on the private asset (or the growth rate of labor income, in the terminology of Krueger and Lustig) is not necessarily the product of idiosyncratic and aggregate components, as we saw above. In fact, using a model similar to the one in Sections 2 and 3 (with log utility and i.i.d. asset returns), Krebs and Wilson (2004) obtain an equity premium of 1% with incomplete markets as opposed to 0.11% with complete markets. Second, the asset pricing implications of a production economy and an endowment economy are different, as argued convincingly in Akdeniz and Dechert (2007). In the current model the next period's resources are produced through the stochastic saving technologies, and the allocation of resources between these technologies are not fixed but optimally chosen by individuals. This is why a_s in (2.2) is affected by the idiosyncratic shock and (4.4) does not by itself imply the irrelevance of incomplete markets.

5 Firms

In this section we introduce firms to the baseline model of Sections 2 and 3. As before the state of the economy is described by a finite state Markov chain on $\{1, 2, \ldots, S\}$ with transition probabilities $\Pr(s \to s') = \pi_{ss'}$. In the economy there is an "all purpose" good which can either be consumed or invested as physical or human capital. The supply side of the economy consists of J firms indexed by $j = 1, 2, \ldots, J$. Firm j has a constant returns to scale neoclassical production function $F_{js}(K, H)$ in state s, where K, H denote the input of the efficiency unit of physical and human capital, respectively. The presence of the subscript s implies that the production function may depend on the state of the economy. We assume all standard assumptions for F_{js}, namely, that F_{js} is twice continuously differentiable, increasing and strictly concave in both arguments, $\frac{\partial}{\partial K} F_{js}(0, H) = \frac{\partial}{\partial H} F_{js}(K, 0) = \infty$, and $\frac{\partial}{\partial K} F_{js}(\infty, H) = \frac{\partial}{\partial H} F_{js}(K, \infty) = 0$.

At each period, each firm rents physical and human capital from consumers. Thus a firm's decision problem is

$$\max_{K,H \geq 0} [F_{js_t}(K, H) - r_{jt}K - r_{0t}H],$$

a static problem, where r_{jt} denotes the rental rate of physical capital firm j faces at period t and r_{0t} is the wage rate.

We keep all assumptions on consumers introduced in Section 3. In particular, each consumer dies with probability $\delta \geq 0$ each period and is replaced by a newborn consumer whose initial wealth is drawn from an initial distribution Ψ_0. We assume that a newborn consumer's wealth cannot be used for production at the period the consumer is born.

If a consumer rents physical capital k_t^j to firm j at period t and invests x_t^j, then the physical capital at the beginning of the next period will be

$$k_{t+1}^j = z_{t+1}^j[(1-\delta_t^j)k_t^j + x_t^j],$$

where δ_t^j is the depreciation rate of physical capital used by firm j, and z_{t+1}^j denotes the shock to the efficiency unit of physical capital to firm j that occurs between periods t and $t+1$. This shock may include capital obsolescence or capital-augmenting technological change. If a consumer has human capital k_t^0 at period t and invests x_t^0, then the human capital at the beginning of the next period will be

$$k_{t+1}^0 = z_{t+1}^0[(1-\delta_t^0)k_t^0 + x_t^0],$$

where δ_t^0 is the depreciation rate and z_{t+1}^0 denotes the shock to the efficiency unit of human capital that occurs between periods t and $t+1$, both assumed to be i.i.d. across individuals. Here we allow disinvestment of human capital ($x_t^0 < 0$), which can be interpreted as cutting work hours or switching to a less demanding job. We assume that the distribution of the random vector $\mathbf{z}_{t+1} = (z_{t+1}^0, z_{t+1}^1, \dots, z_{t+1}^J)$ depends only on the current state s_t and is time homogeneous. We impose a similar assumption on the depreciation rates. Because the physical capital of each firm evolves stochastically, the rental rate r_{jt} may differ across firms. However, since human capital is not firm-specific but individual-specific, the wage (per efficiency unit of human capital) r_{0t} must be common across all firms.

Finally, there is an arbitrary number of assets in zero net supply. Since by the nature of the model there will be no trade in assets in zero net supply, in what follows we shall ignore these assets.

Each consumer maximizes his expected utility (3.1) subject to the constraints

$$c_t + \sum_{j=0}^J x_t^j = \sum_{j=0}^J r_{jt}k_t^j, \tag{5.1a}$$

$$k_{t+1}^j = z_{t+1}^j[(1-\delta_t^j)k_t^j + x_t^j],\ j = 0, 1, \dots, J. \tag{5.1b}$$

(5.1a) is the budget constraint: the left-hand side is the sum of consumption and investment, which must be equal to the right-hand side, the income from all sources. (5.1b) is the equation of motion for the physical capital invested in each firm or the human capital. Note that our formulation in (5.1) is more general than that in Krebs (2006): Equation (2) on (Krebs, 2006, p. 510) only allows for depreciation after production, but through $\mathbf{z}_{t+1}$ we allow for factor obsolescence or factor-augmenting technological change.

5.1 Existence of equilibrium

Let $\mathcal{I}$ be the set of agents. Given the initial distribution of physical and human capital $((k_{i0}^j)_{j=0}^J)_{i\in\mathcal{I}}$, a sequential equilibrium is defined by a sequence of quantities

$$\left\{(c_{it}, (k_{i,t+1}^j)_{j=0}^J)_{i\in\mathcal{I}}, (K_{t+1}^j, H_{t+1}^j)_{j=1}^J\right\}_{t=0}^\infty$$

and prices $\left\{(r_{jt})_{j=0}^{J}\right\}_{t=1}^{\infty}$ such that (i) consumers and firms optimize, and (ii) markets clear. The following Theorem 5.1 shows that the case with firms is similar to the baseline model studied in Sections 2 and 3. However, since shocks occur after investment but before production takes place, agents face more uncertainty. As the following theorem shows, a recursive competitive equilibrium exists if there are a continuum of agents, but the equilibrium is no longer the same as that of the single agent economy.

Theorem 5.1. *Let everything be as above and suppose that the set of agents $\mathcal{I}$ is a continuum with mass 1. Then there exists a recursive competitive equilibrium which is similar to the one with linear stochastic savings technology described in Section 3. However, the equilibrium generally differs from the one in the single agent economy.*

Proof.

Step 1. If the distribution of the rental rate of physical and human capital $(r_{jt})_{j=0}^{J}$ depends only on the previous state s_{t-1}, then the individual decision problem reduces to an optimal consumption/portfolio problem studied in Section 2. In particular, every consumer has a common saving rate and a common portfolio of investment in the physical capital of J firms and human capital, which depend only on the aggregate state s.

Let $k_t'^j = (1-\delta_t^j)k_t^j + x_t^j$ be the amount of physical capital allocated to firm j (if $j \geq 1$) or the amount of human capital (if $j = 0$) *after* production and investment. Since the investment x_t^j is unrestricted, so is $k_t'^j$. Adding total capital after depreciation $\sum_{j=0}^{J}(1-\delta_t^j)k_t^j$ to the budget constraint (5.1a), we obtain

$$c_t + \sum_j k_t'^j = w_t := \sum_j (1 + r_{jt} - \delta_t^j)k_t^j, \tag{5.2}$$

where w_t is the wealth of the consumer including the production in period t.

Let $\Delta^J = \left\{\theta \in \mathbb{R}_+^{J+1} \,\middle|\, \sum_{j=0}^{J}\theta_j = 1\right\}$ be the usual simplex. Define the "portfolio share" at period t, $\theta_t \in \Delta^J$, by $k_t'^j = \theta_t^j(w_t - c_t)$ for $j = 0, 1, \dots, J$. Using (5.1), the physical or human capital at the beginning of period $t+1$ becomes

$$k_{t+1}^j = z_{t+1}^j \theta_t^j (w_t - c_t). \tag{5.3}$$

By (5.2) and (5.3), the consumer's wealth in period $t+1$ is

$$w_{t+1} = \sum_j (1 + r_{j,t+1} - \delta_{t+1}^j)k_{t+1}^j = R_{t+1}(\theta_t)(w_t - c), \tag{5.4}$$

where $\theta_t = (\theta_t^0, \dots, \theta_t^J) \in \Delta^J$ and the return on the portfolio is

$$R_{t+1}(\theta_t) = \sum_j (1 + r_{j,t+1} - \delta_{t+1}^j) z_{t+1}^j \theta_t^j. \tag{5.5}$$

The budget constraint (5.4) has the same form as the budget constraint in Section 2. Since by assumption the distribution of $(r_{j,t+1})_{j=0}^{J}$ depends only on the current state s_t, so does the distribution of $R_{t+1}(\theta_t)$. Thus we have reduced the individual decision problem to an optimal consumption/portfolio problem studied in Section 2. Hence by Theorem 2.1 every consumer invests the wealth in the physical capital of J firms and human capital in a common proportion which depends only on the current state s_t.

Step 2. If the optimal portfolio rule depends only on the current state, then the distribution of the rental rates $(r_{jt})_{j=0}^J$ depends only on the previous state s_{t-1}.

Now suppose that there is a continuum of agents with preferences (3.1), each starting with some initial wealth. Let $\Theta_s \in \Delta^J$ be the common portfolio share found in Step 1, given $\{r_{jt}\}$. Let K_t^j, H_t^j be the amount of physical and human capital employed by firm j at period t and $H_t = \sum_j H_t^j$ the total human capital in the economy. Noting that consumers die with probability δ and adding (5.3) across all individuals in the economy, for $j \geq 1$ we obtain

$$K_{t+1}^j = z_{t+1}^j \Theta_{s_t}^j \sigma_{s_t}((1-\delta)W_t + \delta\bar{W}), \tag{5.6}$$

where σ_s is the common saving rate of consumers in state s (thus $c_t = (1 - \sigma_{s_t})w_t$), W_t is the aggregate wealth in period t, and $\bar{W}$ is the average wealth a newborn agent starts with. For human capital ($j = 0$), since the human capital shock z_{t+1}^0 is conditionally i.i.d. across individuals, by the law of large numbers we obtain

$$H_{t+1} = K_{t+1}^0 = \mathrm{E}\left[z_{t+1}^0 \,\middle|\, I_{t+1}\right] \Theta_{s_t}^0 \sigma_{s_t}((1-\delta)W_t + \delta\bar{W}), \tag{5.7}$$

where $I_{t+1} = (s_t, s_{t+1}, (z_{t+1})_{j=1}^J)$ is the time $t+1$ aggregate information. Letting $\phi_t^j = H_t^j/H_t$ be the fraction of human capital employed by firm j, by firm optimization, (5.6), (5.7), and the homogeneity of production functions, the rental rates satisfy

$$r_{jt} = \frac{\partial}{\partial K} F_{js_t}(K_t^j, H_t^j) = \frac{\partial}{\partial K} F_{js_t}(z_t^j \Theta_{s_{t-1}}^j, \mathrm{E}\left[z_t^0 \,\middle|\, I_t\right] \Theta_{s_{t-1}}^0 \phi_t^j), \tag{5.8a}$$

$$r_{0t} = \frac{\partial}{\partial H} F_{js_t}(K_t^j, H_t^j) = \frac{\partial}{\partial H} F_{js_t}(z_t^j \Theta_{s_{t-1}}^j, \mathrm{E}\left[z_t^0 \,\middle|\, I_t\right] \Theta_{s_{t-1}}^0 \phi_t^j). \tag{5.8b}$$

By the standard assumptions on the production function, (5.8b) can be solved for ϕ_t^j as a function of r_{0t}, s_t, $\mathbf{z}_t$, s_{t-1}, $\Theta_{s_{t-1}}^j$, and $\Theta_{s_{t-1}}^0$. Using $\sum_j \phi_t^j = 1$, we can solve for r_{0t} as a function of s_t, $\mathbf{z}_t$, s_{t-1}, and $\Theta_{s_{t-1}}$. Hence ϕ_t^j is also a function of s_t, $\mathbf{z}_t$, s_{t-1}, and $\Theta_{s_{t-1}}$, and so is r_{jt} by (5.8a). Write this dependency as

$$r_{jt} = r_j(s_t, \mathbf{z}_t | s_{t-1}, \Theta_{s_{t-1}}),\ j = 0, 1, \dots, J. \tag{5.9}$$

Clearly the distribution of r_{jt} depends only on s_{t-1}. Although r_{jt} is a random variable, it is a deterministic function r_j of other random variables.

Step 3. Existence of a recursive equilibrium with no financial assets.

By Steps 1 and 2, the distribution of the rental rates depends only on the previous state if and only if the optimal portfolio depends only on the current state. To construct a recursive equilibrium, these two conditions need to hold simultaneously. Using (5.5) and (5.9), define the return on the individual portfolio $\theta_t \in \Delta^J$ when every other agent is choosing $\Theta_t \in \Delta^J$ by

$$R_{t+1}(\theta_t, \Theta_t) = \sum_j (1 + r_j(s_{t+1}, \mathbf{z}_{t+1} | s_t, \Theta_t) - \delta_{t+1}^j) z_{t+1}^j \theta_t^j. \tag{5.10}$$

By the analogy of Theorem 2.1 and Theorem 3.1, we can construct a recursive equilibrium if the following holds:

$$a_s^{\frac{1}{\gamma}} = 1 + \left(\beta(1-\delta)\,\mathrm{E}\left[a_{s'} R(\theta_s,\Theta_s)^{1-\gamma} \,\middle|\, s\right]\right)^{\frac{1}{\gamma}}, \tag{5.11a}$$

$$\theta_s = \operatorname*{arg\,max}_{\theta\in\Delta^J} \frac{1}{1-\gamma}\,\mathrm{E}\left[a_{s'} R(\theta,\Theta_s)^{1-\gamma} \,\middle|\, s\right], \tag{5.11b}$$

$$\theta_s = \Theta_s, \tag{5.11c}$$

where a_s is the coefficient of the value function: $V(w,s) = a_s \frac{w^{1-\gamma}}{1-\gamma}$ (see Theorem 2.1) and $R(\theta,\Theta)$ is given by (5.10). Conditions (5.11a) and (5.11b) are parallel to (2.2), which together form the individual optimality condition. If $\gamma = 1$, (5.11a) remains the same but $\frac{1}{1-\gamma}R(\theta,\Theta)^{1-\gamma}$ in (5.11b) should be replaced by $\log R(\theta,\Theta)$. Since by (5.10) $R(\theta,\Theta)$ is linear in θ, $\frac{1}{1-\gamma}R(\theta,\Theta)^{1-\gamma}$ and $\log R(\theta,\Theta)$ are strictly concave in θ, so θ_s in (5.11b) is unique. The new condition (5.11c) means that everybody should choose the same portfolio in equilibrium.

The proof that (5.11) has a solution is similar to the proof of Theorem 2.1 and therefore we only provide a sketch below. We set $x_s = a_s^{\frac{1}{\gamma}}$, $\mathbf{x} = (x_1,\dots,x_S)$, and define the mapping $T : \mathbb{R}_+^S \times (\Delta^J)^S \to \mathbb{R}_+^S \times (\Delta^J)^S$, where $(\mathbf{x}',\Theta_1',\dots,\Theta_S') = T(\mathbf{x},\Theta_1,\dots,\Theta_S)$, x_s' the right-hand side of (5.11a) with $\theta_s = \Theta_s$, and Θ_s' is the right-hand side of (5.11b). The domain $\mathbb{R}_+^S \times (\Delta^J)^S$ of the mapping T can be changed to a compact set $[0,M]^S \times (\Delta^J)^S$ by assuming that condition (2.1) holds for all Θ, where $R(\theta)$ is replaced by $R(\theta,\Theta)$ and minimization/maximization is over θ. Then by Berge's maximum theorem (Berge, 1959, p. 116), T is continuous. Hence the Brouwer fixed point theorem applies and (5.11) has a solution.

Step 4. Existence of a recursive equilibrium with an arbitrary number of assets with zero net supply.

The general case with assets with zero net supply is similar to the proof of Theorem 3.1.

Step 5. The equilibrium with a continuum of agents generally differs from the one with a single agent.

The single agent problem corresponds to replacing $\mathrm{E}\left[z_t^0 \,\middle|\, I_t\right]$ in (5.8) by just z_t^0 because the law of large numbers does not apply. Hence unless z_t^0 is common across all consumers, *i.e.*, unless there is no idiosyncratic component in the shock corresponding to human capital obsolescence or human capital-augmenting technological change, the equilibrium with a continuum of agents generally differs from the one with a single agent. □

Although the case with firms (Theorem 5.1) is quite similar to the case without firms but with only linear savings technology (Theorem 2.1 and Theorem 3.1), there are a few differences. First, because the return on the physical and human capital are endogenous (they depend on the equilibrium portfolio through (5.9) and (5.10)), not exogenous as in the case without firms in Section 2, we have more endogenous variables to pin down. This is why we apply the Brouwer fixed point theorem instead of constructing a fixed point through a monotone self map as in Theorem 2.1. However, if z_t^0 is common across all consumers, the multiple agent problem reduces to the single problem (see Theorem 5.2 below), in which case the proof of Theorem 2.1 directly applies.

Second, because the multiple agent problem differs from the single agent problem, there is some risk sharing in equilibrium. Although agents face exactly the same idiosyncratic shocks in the market (continuum of agents) economy and the autarky (single agent) economy, the agents in the market economy can insure against the wage shock through pooling their human capital (the term with $\mathrm{E}\left[z_t^0 \,\middle|\, I_t\right]$ in (5.8)), but not against the shock to the amount of individual human capital itself. In the autarky economy, on the other hand, the agent cannot insure against either. This conclusion differs from that of Krebs (2006), where the single agent and multiple agent problems are exactly the same and hence there is no risk sharing.

5.2 Generic constrained inefficiency of equilibrium

Because the equilibrium with linear savings technology as in Sections 2 and 3 is identical to a single agent (planning) problem, the equilibrium was constrained efficient (Theorem 3.1). This is not necessarily the case when there are firms, as we show in Theorem 5.2 below.

Theorem 5.2. *The equilibrium with firms is generically constrained inefficient. However, if the human capital shock z_t^0 is common across all consumers (*i.e.*, the only idiosyncratic shock is in human capital depreciation), then the equilibrium is constrained efficient.*

Proof.

Step 1. The equilibrium is constrained efficient only if

$$\theta_s = \operatorname*{arg\,max}_{\theta \in \Delta^J} \frac{1}{1-\gamma} \mathrm{E}\left[a_{s'} R(\theta,\theta)^{1-\gamma} \,\middle|\, s\right] \tag{5.12}$$

for all s, where $\{a_s\}_{s=1}^S$ is given by (5.11a).

If θ_s does not satisfy (5.12) for some s, then

$$\frac{1}{1-\gamma} \mathrm{E}\left[a_{s'} R(\theta_s^*,\theta_s^*)^{1-\gamma} \,\middle|\, s\right] > \frac{1}{1-\gamma} \mathrm{E}\left[a_{s'} R(\theta_s,\theta_s)^{1-\gamma} \,\middle|\, s\right] \tag{5.13}$$

for $\theta_s^* \in \Delta^J$ that attains the maximum of the right-hand side of (5.12). Suppose that at some node in state s all agents choose the portfolio θ_s^* instead of θ_s, but stick to the equilibrium portfolio thereafter. Since the value function is given by $V(w,s) = a_s \frac{w^{1-\gamma}}{1-\gamma}$ and $R(\theta,\theta)$ is the return on portfolio θ, if a typical agent has wealth w *after* consumption, by (5.13) his discounted future utility becomes

$$\frac{\beta(1-\delta)w^{1-\gamma}}{1-\gamma} \mathrm{E}\left[a_{s'} R(\theta_s^*,\theta_s^*)^{1-\gamma} \,\middle|\, s\right] > \frac{\beta(1-\delta)w^{1-\gamma}}{1-\gamma} \mathrm{E}\left[a_{s'} R(\theta_s,\theta_s)^{1-\gamma} \,\middle|\, s\right],$$

so everybody is better of by simultaneously switching to the portfolio θ_s^*. Therefore the equilibrium is constrained inefficient.

Step 2. The equilibrium is generically constrained inefficient.

By the equilibrium condition (5.11b), we have

$$\mathrm{E}\left[a_{s'}R(\theta_s,\theta_s)^{-\gamma}\frac{\partial}{\partial\theta}R(\theta_s,\theta_s)\,\middle|\,s\right]=\lambda_s\mathbf{1}, \tag{5.14}$$

where λ_s is the Lagrange multiplier for the portfolio constraint $\sum_{j=0}^{J}\theta_s^j=1$. If the equilibrium is constrained efficient, by (5.12) we have

$$\mathrm{E}\left[a_{s'}R(\theta_s,\theta_s)^{-\gamma}\left(\frac{\partial}{\partial\theta}R(\theta_s,\theta_s)+\frac{\partial}{\partial\Theta}R(\theta_s,\theta_s)\right)\,\middle|\,s\right]=\mu_s\mathbf{1}, \tag{5.15}$$

where μ_s is the Lagrange multiplier. Hence by (5.14) and (5.15) we obtain

$$\mathrm{E}\left[a_{s'}R(\theta_s,\theta_s)^{-\gamma}\frac{\partial}{\partial\Theta}R(\theta_s,\theta_s)\,\middle|\,s\right]=(\mu_s-\lambda_s)\mathbf{1}. \tag{5.16}$$

Since $\{a_s,\theta_s,\lambda_s\}_{s=1}^{S}$ (in total $(J+3)S$ unknowns) are determined by (5.11a), (5.14), and $\sum_{j=0}^{J}\theta_s^j=1$ (in total $(J+3)S$ equations), their solution generically does not satisfy (5.16) (additional JS equations, since μ_s can be chosen freely). Hence the equilibrium is generically constrained inefficient.

Step 3. The equilibrium is constrained efficient if the human capital shock z_t^0 is common across all consumers.

See Appendix B. The crucial assumption in proving that the equilibrium is constrained efficient is that there is no idiosyncratic component in the shock for human capital obsolescence or human capital-augmenting technological change, that is, $\mathrm{E}\left[z_t^0\,\middle|\,I_t\right]=z_t^0$ for all agents. Under this assumption we can replace $\mathrm{E}\left[z_t^0\,\middle|\,I_t\right]$ appearing in the definition of market rental rates (5.8) by just z_t^0, and using the constant returns to scale property of the production functions we can show that the multiple agent market economy is identical to an autarky economy. □

6 Further generalizations

In this section I further generalize the baseline model of Sections 2 and 3 by introducing endogenous labor supply or bequest. In both cases the generalized model mathematically reduces to the baseline model.

6.1 Endogenous labor supply

Introducing labor-leisure decision into the model with firms in Section 5 is straightforward. Suppose that agents have the utility function

$$\mathrm{E}_0\sum_{t=0}^{\infty}[\beta(1-\delta)]^t u(c_t,l_t,s_t),$$

where $l_t\in[0,1]$ is leisure and the period utility function $u(c,l,s)$ is assumed to take the form

$$u(c,l)=\frac{c^{1-\gamma}}{1-\gamma}v(l,s),$$

where v is continuous in l.[9] We interpret that by enjoying leisure l, the consumer is utilizing a fraction $1 - l$ of his or her human capital. Then the budget constraint (5.1a) (omitting individual subscripts) becomes

$$c_t + \sum_{j=0}^{J} x_t^j = r_{0t}(1 - l_t)k_t^0 + \sum_{j=1}^{J} r_{jt}k_t^j,$$

that is, the consumer is paid for the human capital actually used. The equation of motion (5.1b) remains the same if we assume that human capital depreciates whether it is used for production or not. If we assume that only human capital that is actually used for production depreciates, then the equation of motion (5.1b) becomes

$$k_{t+1}^0 = z_{t+1}^0[(1 - \delta_t^0)(1 - l_t)k_t^0 + l_t k_t^0 + x_t^0].$$

In either case, the analysis of Section 5 remains the same, with the only difference being that we have another endogenous variable, l, and therefore the maximization in (2.2b) will be also with respect to $l \in [0, 1]$. The leisure l_t will depend only on the current state s_t. In particular, if there is only one state (in which case the shocks to physical and human capital are i.i.d. across time), the labor supply will be constant (perfectly inelastic).

6.2 Bequest

In the baseline model of Section 3 personal wealth was destroyed upon death and there was no bequest, but what if there is? To model this situation suppose that when an agent enters each period (including $t = 0$) he dies with probability $\delta > 0$, and in the event of death he bequeaths his entire wealth.[10] The agent's utility function is

$$\mathrm{E}_0 \sum_{t=0}^{\infty} [\beta(1 - \delta)]^t [(1 - \delta)u(c_t) + \delta b(w_t)],$$

where c_t is period t consumption (if alive), w_t is the personal wealth at the beginning of period t, u is the period utility function from consumption and b is the bequest function (utility from bequest).

Using this preference we can write down the Bellman equation similar to (2.4). If we choose $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $b(w) = b\frac{w^{1-\gamma}}{1-\gamma}$, where $b > 0$ is a constant, by repeating the argument of Theorem 2.1 we can show that the value function has the form $V(w, s) = a_s \frac{w^{1-\gamma}}{1-\gamma}$ and the propensity to consume out of wealth is constant (which depends on the state of the economy s). Therefore, the case with bequest is mathematically similar to the case without bequest. However, the wealth distribution will now depend on how the bequest is split between heirs or how it is taxed, and will no longer be described by a simple double Pareto distribution as in Section 3.4.

[9] Typically $v(l, s)$ is increasing if $\gamma \le 1$ and decreasing if $\gamma > 1$, but monotonicity is unnecessary for obtaining the mathematical results. In fact, common sense suggests that people are happier when occupied ($l < 1$) than having nothing to do ($l = 1$), so it is neither necessarily realistic to assume the monotonicity with respect to leisure. We allow state-dependent utility by including s in the argument of v.

[10] Since the entire wealth includes human capital, this assumption is unrealistic. However, the analysis is almost identical even if an agent can bequeath only his financial wealth.

7 Concluding remarks

A Laplace and related distributions

The *Laplace* distribution has a density

$$f_{\mathrm{L}}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta}\mathrm{e}^{-\alpha|x-m|}, & (x \geq m) \\ \frac{\alpha\beta}{\alpha+\beta}\mathrm{e}^{-\beta|x-m|}, & (x < m) \end{cases}$$

where m is the mode and $\alpha, \beta > 0$ are scale parameters. It is called *symmetric* if $\alpha = \beta$. See Kotz et al. (2001) for an exhaustive survey of the Laplace distribution.

If X is Laplace distributed, then $Y = \exp(X)$ is said to be *double Pareto* distributed.[11] The density of the double Pareto distribution is

$$f_{\mathrm{dP}}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta}\frac{1}{M}\left(\frac{x}{M}\right)^{-\alpha-1}, & (x \geq M) \\ \frac{\alpha\beta}{\alpha+\beta}\frac{1}{M}\left(\frac{x}{M}\right)^{\beta-1}, & (0 \leq x < M) \end{cases}$$

where $M > 0$ is the mode and $\alpha, \beta > 0$ are power law exponents. The (discrete) double Pareto distribution first appeared in Champernowne (1953) and has been shown to fit the conditional income distribution (Toda, 2011b,a).

Perhaps the most important property of the Laplace distribution is that it is the only limit distribution of geometric sums: if $X_1, X_2, \dots$ are i.i.d. with finite variance, then the properly scaled geometric sum $\sum_{j=1}^{\nu_p} X_j$ (where ν_p is a geometric random variable with mean $1/p$) converges in distribution to a Laplace distribution as $p \to 0$. As the following Theorem A.1 taken from Toda (2012) shows, the i.i.d. assumption can be weakened to requiring only independence. This theorem, which is a counterpart of the Lindeberg-Feller central limit theorem for geometric sums, shows that it is a robust property that the limit of a geometric sum is a Laplace distribution.

Theorem A.1. *Let $\{X_j\}$ be a sequence of independent but not identically distributed (i.n.i.d) random variables such that $\mathrm{E}[X_j] = 0$ and $\mathrm{Var}[X_j] = \sigma_j^2$, and ν_p be a geometric random variable independent of X_j's with mean $1/p$. Suppose that*

1. $\lim_{n\to\infty} n^{-\alpha}\sigma_n^2 = 0$ *for some* $0 < \alpha < 1$ *and* $\sigma^2 := \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \sigma_j^2 > 0$ *exists, and*

2. *for all* $\epsilon > 0$ *we have*

$$\lim_{p\to 0} \sum_{j=1}^{\infty} (1-p)^{j-1} p \,\mathrm{E}\left[X_j^2 \left\{|X_j| \geq \epsilon p^{-\frac{1}{2}}\right\}\right] = 0.$$

Then, as $p \to 0$ the geometric sum $p^{\frac{1}{2}} \sum_{j=1}^{\nu_p} X_j$ converges in distribution to a symmetric Laplace distribution with mean 0 and variance σ^2.

[11] Hence the Laplace and the double Pareto distributions have the same relation as the normal and the lognormal distributions.

If X_1, X_2 are independent normal and Laplace random variables, their convolution $Y = X_1 + X_2$ is said to be normal-Laplace distributed (Reed, 2001, 2003; Reed and Jorgensen, 2004; Reed and Wu, 2008). The normal-Laplace distribution has four parameters, a location parameter μ and three scale parameters $\sigma, \alpha, \beta > 0$, with probability density function

$$f_{\mathrm{NL}}(x) = \frac{\alpha\beta}{\alpha+\beta}\left[\mathrm{e}^{\frac{\alpha^2\sigma^2}{2}-\alpha(x-\mu)}\Phi\left(\tfrac{x-\mu}{\sigma}-\alpha\sigma\right)+\mathrm{e}^{\frac{\beta^2\sigma^2}{2}+\beta(x-\mu)}\Phi\left(-\tfrac{x-\mu}{\sigma}-\beta\sigma\right)\right],$$

where Φ is the cumulative distribution function of the standard normal distribution. If X is normal-Laplace distributed, then $Y = \exp(X)$ is said to be double Pareto-lognormally distributed. It is clear from the above density that the Laplace and the normal distributions are special cases of the normal-Laplace distribution by letting $\sigma \to 0$ and $\alpha = \beta \to \infty$, respectively.

B Proof of Theorem 5.2

Instead of the optimization problem in the market economy, consider the autarky problem. Then the budget constraint (5.1a) is replaced by the resource constraint

$$c_t + \sum_{j=0}^{J} x_t^j = \sum_{j=1}^{J} F_{js_t}(k_t^j, \phi_t^j k_t^0),$$

where ϕ_t^j denotes the fraction of human capital allocated to production with technology j. Then (5.2) is accordingly replaced by

$$c_t + \sum_j k_t'^j = w_t := \sum_{j=1}^{J}[F_{js_t}(k_t^j, \phi_t^j k_t^0) + (1-\delta_t^j)k_t^j + (1-\delta_t^0)\phi_t^j k_t^0],$$

and the budget constraint (5.4) becomes

$$\begin{aligned} w_{t+1} = \sum_{j=1}^{J}[&F_{js_{t+1}}(z_{t+1}^j\theta_t^j, z_{t+1}^0\theta_t^0\phi_{t+1}^j) \\ &+ z_{t+1}^j(1-\delta_{t+1}^j) + z_{t+1}^0(1-\delta_{t+1}^0)\theta_t^0\phi_{t+1}^j]w_t. \end{aligned} \quad \text{(B.1)}$$

Since the allocation of human capital ϕ_{t+1}^j can be chosen after observing time $t+1$ shocks, the agent will choose it so as to maximize the right-hand side of (B.1). Hence (5.4) holds with

$$\begin{aligned} R_{t+1}(\theta_t) = \max_{\phi_{t+1}\in\Delta^{J-1}} \sum_{j=1}^{J}[&F_{js_{t+1}}(z_{t+1}^j\theta_t^j, z_{t+1}^0\theta_t^0\phi_{t+1}^j) \\ &+ z_{t+1}^j(1-\delta_{t+1}^j)\theta_t^j + z_{t+1}^0(1-\delta_{t+1}^0)\theta_t^0\phi_{t+1}^j]. \end{aligned} \quad \text{(B.2)}$$

The budget constraint (B.1) (maximized with respect to ϕ_{t+1}) has precisely the same form as the budget constraint in Section 2. The only difference is that $R_{t+1}(\theta)$ is linear in θ in Section 2, but not necessarily so in (B.1). However, since in the proof of Theorem 2.1 we have nowhere used the linearity of $R_{t+1}(\theta)$, the proof goes through without modifications under conditions similar to (2.1). Therefore the optimal portfolio depends only on the current state, which we denote by θ_s.

Define the rental rate of physical and human capital at period $t+1$ by

$$r_{j,t+1} = \frac{\partial}{\partial K} F_{js_{t+1}}(\theta^j_{s_t}, \theta^0_{s_t}\phi^j_{t+1}), \tag{B.3a}$$

$$r_{0,t+1} = \frac{\partial}{\partial H} F_{js_{t+1}}(\theta^j_{s_t}, \theta^0_{s_t}\phi^j_{t+1}), \tag{B.3b}$$

where $\phi_{t+1} \in \Delta^{J-1}$ is the maximizer of (B.2). The right-hand side of (B.3b) does not depend on j by considering the first-order condition of the maximization (B.2). Let $R_{t+1}(\theta)$ be the return on portfolio in the autarky economy defined by (B.2) and $R_{t+1}(\theta, \theta_{s_t})$ be the return on portfolio in the market economy defined by

$$R_{t+1}(\theta, \theta_{s_t}) = \sum_{j=0}^{J} (1 + r_{j,t+1} - \delta^j_{t+1}) z^j_{t+1} \theta^j, \tag{B.4}$$

where $r_{j,t+1}$ is given by (B.3) (see also (5.10)). Since production functions exhibit constant returns to scale, by (B.2)–(B.4) we obtain

$$\begin{aligned}
&R_{t+1}(\theta_{s_t}, \theta_{s_t}) \\
&= \sum_{j=1}^{J} [(1 + r_{j,t+1} - \delta^j_{t+1}) z^j_{t+1} \theta^j_{s_t} + (1 + r_{0,t+1} - \delta^0_{t+1}) z^0_{t+1} \theta^0_{s_t} \phi^j_{t+1}] \\
&= \sum_{j=1}^{J} [F_{js_{t+1}}(z^j_{t+1}\theta^j_{s_t}, z^0_{t+1}\theta^0_{s_t}\phi^j_{t+1}) + z^j_{t+1}(1 - \delta^j_{t+1}) + z^0_{t+1}(1 - \delta^0_{t+1})\theta^0_{s_t}\phi^j_{t+1}] \\
&= R_{t+1}(\theta_{s_t}).
\end{aligned}$$

Furthermore, by a straightforward calculation we obtain

$$\frac{\partial}{\partial \theta} R_{t+1}(\theta_{s_t}, \theta_{s_t}) = \frac{\partial}{\partial \theta} R_{t+1}(\theta_{s_t}).$$

Therefore θ_s (the optimal portfolio in autarky) satisfies (5.11), that is, θ_s is the equilibrium portfolio in the market economy. Since the equilibrium portfolio is also optimal in autarky, by the same argument as in the proof of Theorem 3.1, the equilibrium is constrained efficient.

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