Information and Timing in Repeated Bargaining*

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Abstract

We study a repeated version of the Myerson-Satterthwaite bilateral bargaining problem with serially correlated types. We show that efficient, unsubsidized, and individually rational trade is possible if and only if the expected dynamic virtual surplus from the efficient allocation is positive. Using this condition, we study the effects of discounting, frequency of interaction and persistence of private information on equilibrium outcomes. When types follow stationary Gaussian Markov processes, this yields an anti-folk theorem: Efficient and inidividually rational trade requires a subsidy equal to the first-best surplus for any degree of mean-reversion and any discount rate. For a "renewal model" we characterize in closed form the condition between the discount rate, frequency of interaction, and persistence needed for the expected virtual surplus to be positive. We also solve for the second-best mechanism in the Gaussian model.

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1 Introduction

In this paper we study a problem of repeated bilateral bargaining with imperfectly persistent values. We ask when it is possible to achieve efficient outcomes, how much a designer would need to subsidize a relationship to achieve efficiency, and what is the most the traders could hope for if the designer is unwilling to subsidize their trade.

We are motivated by the following observations. First, the negative results in Myerson and Satterthwaite (1983, henceforth MS) state that in a one-shot bargaining with two-sided

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private information it is not possible to achieve efficiency without subsidizing the trade on average.

Second, many real-life bargaining situations are not one-shot but rather repeated. For example, an online publisher negotiates many times each year the sale of impressions to potential advertisers in a so-called "guaranteed delivery" market, which is a name for transactions negotiated one-on-one without the use of real-time-bidding auctions (with remaining inventory being left to these auctions, which in our model represent the cost of the seller). In such a situation the publisher and the advertiser develop long-term relationships and hence can transfer some of the surplus between periods to help overcome the MS curse. Moreover, in such situations trade can be re-negotiated frequently, especially with the help of automated agents and protocols.

Third, since we want to model repeated bargaining of a flow of services of some asset or a flow of impressions in our advertising example, we think it is important to take into account that values and costs of the buyer and seller are likely to be serially correlated. As a result, unlike in the iid world, there is a difference between comparative statics in the frequency of negotiations and in the discount rate (the first affecting both persistence of types and the discount factor and the second only the latter).

In our model types are drawn independently across the buyer and the seller and follow a first-order Markov process. We should that not only the stationary distribution and the persistence parameter of the process matter, but also the nature of the process does. To this end, we first provide a general necessary and sufficient condition for the existence of a mechanism that implements an allocation rule in a way that is incentive compatible, budget balanced (ex-ante) and individually-rational at time 0 (we also discuss how these properties can be strengthened for positive results). The condition is that the expected dynamic virtual surplus from the efficient allocation be nonnegative. As a benchmark, in case of iid types, if the supports of the buyer's and seller's type distribution coincide, this reduces to the requirement that the discount factor, δ, be more than $1/2$. The intuition follows from MS: there, to achieve efficiency the mechanism designer needs to subsidize the trade by an amount equal to the expected social surplus. If in the current period the designer can tax future expected surplus (with the threat of breaking up trade if any of the players refuses to pay the tax), he can reduce the necessary subsidy. The expected future continuation surplus is $1/(1-\delta)$ of the current expected surplus, which is more than 1 when $\delta > 1/2$.

This observation suggests that sufficiently frequent negotiations should always allow the parties to obtain efficiency. We show that this is not true. Increasing the frequency of negotiations may help (but not always), but it is often not sufficient to resolve the MS problem: while more frequent periods mean that there is more future surplus (relative to current surplus) that can be used to subsidize current trade, types become more persistent across periods and that increases the information rents agents need to earn in order for the mechanism to be incentive compatible!

That leads to a second conjecture: for a fixed frequency of bargaining, reducing the discount rate sufficiently should allow us to achieve efficiency. This is true in some environments, but surprisingly even this conjecture is wrong in our baseline model, in which types follow an underlying stationary Gaussian Markov process (an Ohlstein-Uhlenbeck, or O-U, process) with time-zero types drawn from the stationary distribution. In contrast, if types follow a renewal process (in which types are constant over time intervals until a change arrives at which point they are re-drawn from the initial distribution, i.e., the process renews at Poisson arrivals), then this second conjecture holds. An additional surprise is that in the benchmark Gaussian model the amount of subsidy needed for efficient, individually rational trade, expressed as a fraction of the per-period expected surplus from trade, is independent of the discount rate or the persistence of the process—it is always equal to the expected surplus as in the static model of MS.

When the expected virtual surplus from the efficient allocation is positive so that efficient, unsubsidized, and individually rational trade is possible, it can be implemented by a mechanism roughly analogous to the iid case discussed. At time zero—before any reports or trade but after the players have oberved their initial types—players pay a tax to the mechanism designer. The tax payments are incentive compatible because failing to pay them would stop future trades that are in expectations sufficiently profitable even to the worst-off types. Then in each period the players trade efficiently with each obtaining the full surplus and the mechanism designer subsidizing the trade from the time-zero tax. This mechanism is efficient and ex ante budget balanced by construction. Furthermore, it is in fact ex post incentive compatible and individually rational in every period. We also argue that, alternatively, it is possible to balance the budget ex post period by period using the balanced transfers of Athey and Segal (2007), if we impose incentive compatibility in the sense of perfect Bayesain equilibrium and require individual rationality only in period 0.

Our impossibility result for the Gaussian model is surprising in light of the results in Athey and Segal (2007) who showed that with finitely many types there always exists an efficient mechanism if discount rate is sufficiently close to zero. Our model has a continuum of types (as in MS), but it is not causing the impossibility per se (as we show by considering other initial distributions for the Gaussian model, and noting that the initial distribution in the renewal case can be taken to be normal).

In the last part of the paper we ask what are the second-best mechanisms in the Gaussian case and in particular how the inefficiency of trade changes over time. We show that the bangfor-the-buck method used by MS to describe the second-best allocations can be also applied to the repeated bargaining problem. Although persistence does not affect the subsidies necessary for achieving the first-best, it does change the second-best outcomes. In particular, the faster is mean-reversion of our process, the smaller are the inefficiencies in future dates.

We solve for the second-best mechanism by maximizing the expected dynamic virtual surplus, which only incorporates local incentive compatibility constraints. While in the Gaussian model this this relaxed-problem approach results in an allocation rule that is indeed implementable, in the case of the renewal process it does not. That makes it hard (if not impossible) to characterize the second-best allocations for the renewal model.

2 The Model

Consider the following dynamic bargaining environment with two-sided private information: There are two agents, a buyer (B) and a seller (S). In each period $t = 0, 1, \ldots, T$, with $T \leq \infty$, they bargain over the allocation of an indivisible good and a numeraire (for example, the seller can provide a service for which the buyer will pay in cash). We denote the period-t allocation of the good by $x_t \in \{0, 1\}$, with $x_t = 0$ if the object stays with the seller, and $x_t = 1$ if the buyer receives the object. The buyer's value for the object in period t is denoted by θ_t, and the seller's cost is denoted ω_t.[1] Both θ_t and ω_t are assumed to be private information of the respective agents. We let p_t^i, $i \in \{B, S\}$, denote the amount of numeraire received by agent i in period t.

The agents are risk neutral and discount future payoffs at a common discount factor $\delta \in (0, 1)$. (When we compare problems with a different frequency of interactions, we explicitly write the discount factor as $\delta = e^{-r\Delta}$ for a common discount rate $r > 0$ and duration of a period Δ.) The payoffs for the buyer and seller are, respectively,

$$\frac{1-\delta}{1-\delta^T} \sum_{t=0}^{T} \delta^t (x_t \theta_t + p_t^B) \quad \text{and} \quad \frac{1-\delta}{1-\delta^T} \sum_{t=0}^{T} \delta^t (p_t^S - x_t \omega_t).$$

That is, all payoff-relevant consequences of bargaining are captured by the resulting allocations of the object and transfers of numeraire.

The buyer's value θ_t is a Markov process $(F, F(\cdot|\theta)_{\theta \in \Theta})$ on $\Theta = (\underline{\theta}, \bar{\theta}) \subset \mathbb{R}$, with $-\infty \leq \underline{\theta} < \bar{\theta} \leq \infty$, where F is the cumulative distribution function (cdf) of the period-0 value θ_0,

[1] One possible interpretation is that the seller owns a perfectly durable object, and in each period the agents bargain over who should have the control of it in the current period. In this interpretation ω_t is best viewed as the opportunity cost. Alternatively, we can think of a model where in each period the seller can produce one unit of a non-storable good or service at cost ω_t.

and $F(\cdot|\theta)$ denotes the cdf of the period-t value given value θ in period $t-1$. We assume that each $F(\cdot|\theta)$ has full support on Θ, and that the process is bounded in expectation in the sense that for all $\theta_0 \in \mathbb{R}$, $\mathbb{E}[\sum_{t=0}^{T} \delta^t |\theta_t| \mid \theta_0] < \infty$. (The latter ensures that the buyer's expected utility from allocations exists.) The seller's cost ω_t is a Markov process $(G, G(\cdot|\omega)_{\omega \in \Omega})$ on $\Omega = (\underline{\omega}, \bar{\omega}) \subset \mathbb{R}$, with $-\infty \leq \underline{\omega} < \bar{\omega} \leq \infty$, which satisfies assumptions analogous to those on the buyer's process. We assume throughout that the type processes are independent between agents. We consider particular processes in Sections 4 and 5.

We are interested in answering when formal or informal institutions or trading mechanisms could be established to achieve efficiency in the above bargaining environment. In order to do this, we first use tools of dynamic mechanism design to derive a necessary condition for a type-dependent allocation path to be implementable in a direct revelation mechanism where in each period the agents simply report their types to the mechanism, which determines the allocation and transfers as a function of the history of the reports. In a dynamic setting the most permissive results about implementation are achieved with the least amount of information disclosure (see, e.g., Myerson, 1986). Accordingly, we assume that the agents' reports and transfers are confidential so that the agents only observe the realized allocations and their own transfers. Since the outcomes of any institution or relational contract can be replicated in a truthful equilibrium of such a direct mechanism, our necessary condition puts a bound on what can be achieved in dynamic bilateral bargaining in general. (When we discuss sufficient conditions for implementation, we show how to construct mechanisms where reports and transfers can be made public.)

Formally, a (dynamic direct) mechanism is a sequence of mappings $M = \{x_t, p_t^B, p_t^S\}_{t=0}^{T}$, where $x_t : \Theta^t \times \Omega^t \to \{0,1\}$ is the period-t allocation rule, and $p_t^B, p_t^S : \Theta^t \times \Omega^t \to \mathbb{R}$ are the transfer rules.[2] A mechanism M induces a dynamic game of incomplete information between the agents as follows: Let h_t^i denote agent i's period-t private history consisting of his types in periods $\{0, \ldots, t\}$, and his reports, transfers, and allocations in periods $\{0, \ldots, t-1\}$. Then $H_t^B = \Theta^t \times \Theta^{t-1} \times \{0,1\}^{t-1} \times \mathbb{R}^{t-1}$ and $H_t^S = \Omega^t \times \Omega^{t-1} \times \{0,1\}^{t-1} \times \mathbb{R}^{t-1}$ are the sets of all such histories for the buyer and the seller, respectively. A reporting strategy for the buyer is a function $\sigma^B : \cup_t H_t^B \to \Theta$ that maps histories into a report about the current value. Similarly, a strategy for the seller is a function $\sigma^S : \cup_t H_t^S \to \Omega$. Thus, in period t, given histories h_t^B and h_t^S, the mechanism implements the allocation $x_t(\sigma^B(h_t^B), \sigma^S(h_t^S))$ and transfers $p_t^i(\sigma^B(h_t^B), \sigma^S(h_t^S))$, $i \in \{S, B\}$.

We say that the buyer's history h_t^B is truthful if all his messages have been truthful, i.e., if $h_t^B = ((\theta^{t-1}, \theta_t), \theta^{t-1}, x^{t-1}, p^{t-1})$ for some $\theta^{t-1} \in \Theta^{t-1}$, $\theta_t \in \Theta$, $x^{t-1} \in \{0,1\}^{t-1}$, and

[2] Restricting attention to deterministic mechanisms is without loss for our results.

$p^{t-1} \in \mathbb{R}^{t-1}$. The buyer's strategy σ^B is truthful if it prescribes truthful reporting at truthful histories, i.e., if $\sigma^B(h_t^B) = \theta_t$ for all truthful h_t^B. Truthful histories and strategies for the seller are defined analogously.

The following definitions are standard:

Definition 1 *A mechanism is* incentive compatible *(IC) if it has a Bayesian Nash equilibrium in truthful strategies.*[3]

Definition 2 *A mechanism is* individually rational in period 0 *(IR) if for all initial types $\theta_0 \in \Theta$ and $\omega_0 \in \Omega$, the expected equilibrium payoff from the mechanism is positive.*[4]

Definition 3 *A mechanism is* efficient *(E) if for all t, $\theta^t \in \Theta^t$ and $\omega^t \in \Omega^t$, the allocation satisfies $x_t\left(\theta^t, \omega^t\right) = 1$ iff $\theta_t \geq \omega_t$ and so can be written as a function of θ_t and ω_t only. We denote the efficient rule by x^*.*

Definition 4 *A mechanism is* ex ante budget balanced *(BB) if*

$$\mathbb{E}\left[\sum_{t=0}^{T} \delta^t \left(p_t^S\left(\theta^t, \omega^t\right) + p_t^B\left(\theta^t, \omega^t\right)\right)\right] \leq 0.$$

We note that our negative results are made stronger by the fact that our notion of incentive compatibility does not impose perfection. For the positive results we show how to construct mechanisms where truthtelling actually forms a Perfect Bayesian Equilibrium (PBE).

[3] Formally, the game induced by the mechanism has a Bayesian Nash equilibrium in truthful strategies, if for all initial types $\theta_0 \in \Theta$ and $\omega_0 \in \Omega$, and all strategies σ^B and σ^S,

$$\mathbb{E}\left[\sum_{t=1}^{T} \delta^t[\theta_t x_t(\theta^t, \omega^t) + p_t^B(\theta^t, \omega^t)] \mid \theta_0\right] \geq \mathbb{E}\left[\sum_{t=1}^{T} \delta^t[\theta_t x_t(\sigma^B(h_t^B), \omega^t) + p_t^B(\sigma^B(h_t^B), \omega^t)] \mid \theta_0\right],$$

and

$$\mathbb{E}\left[\sum_{t=1}^{T} \delta^t[p_t^S(\theta^t, \omega^t) - \omega_t x_t(\theta^t, \omega^t)] \mid \omega_0\right] \geq \mathbb{E}\left[\sum_{t=1}^{T} \delta^t[p_t^S(\theta^t, \sigma^S(h_t^S)) - \omega_t x_t(\theta^t, \sigma^S(h_t^S))] \mid \omega_0\right].$$

[4] While we perform analysis requiring only period-0 IR constraints to hold, we show how in case of positive results the mechanism can be implemented with a help of a bank in a way that IR constraints are satisfied in every period.

3 A Necessary and Sufficient Condition for Efficient Trade

The following lemma extends to the dynamic setting a part of the characterization of incentive compatible (IC) mechanisms by Myerson and Satterthwaite (1983). It is established by combining insights from the work on efficient efficient dynamic mechanisms by Athey and Segal (2007) and Bergemann and Välimäki (2010) with the techniques from Pavan, Segal and Toikka (2011).

Definition 5 *The type processes* $(F, F(\cdot|\theta)_{\theta\in\Theta})$ *and* $(G, G(\cdot|\omega)_{\omega\in\Omega})$ *are* regular *if they satisfy the following conditions:*

1. *F and each $F(\cdot|\theta)$ are absolutely continuous with densities f and $f(\cdot|\theta)$, respectively, that are strictly positive on Θ. The kernel $F(\cdot|\cdot)$ is continuously differentiable and there exists $c^B < \frac{1}{\delta}$ such that for all $(\theta', \theta) \in \Theta^2$,*

$$\left| \frac{\partial F(\theta'|\theta)/\partial\theta}{f(\theta'|\theta)} \right| < c^B.$$

2. *G and each $G(\cdot|\omega)$ are absolutely continuous with densities g and $g(\cdot|\omega)$, respectively, that are strictly positive on Ω. The kernel $G(\cdot|\cdot)$ is continuously differentiable and there exists $c^S < \frac{1}{\delta}$ such that for all $(\omega', \omega) \in \Omega^2$*

$$\left| \frac{\partial G(\omega'|\omega)/\partial\omega}{g(\omega'|\omega)} \right| < c^S.$$

Definition 6 *The type processes* $(F, F(\cdot|\theta)_{\theta\in\Theta})$ *and* $(G, G(\cdot|\omega)_{\omega\in\Omega})$ *satisfy* FOSD *if $F(\cdot|\theta') \leq F(\cdot|\theta)$ (i.e., $F(\cdot|\theta')$ first-order stochastically dominates $F(\cdot|\theta)$) for all $\theta' > \theta$ and $G(\cdot|\omega') \leq G(\cdot|\omega)$ for all $\omega' > \omega$.*

Regularity of the type processes is a sufficient condition for us to be able to use a dynamic envelope theorem argument to pin down the agents' equilibrium payoffs as a function of their first period types. FOSD allows us to deduce that these equilibrium payoffs are monotone so that the only relevant period-0 participation constraints are the ones for the lowest initial type of the buyer and the highest initial type of the seller.

Lemma 1 *Suppose that the type processes are regular and satisfy FOSD.*[5] *There exists an efficient, ex ante budget balanced, incentive compatible, and individually rational (E, BB, IC, IR) mechanism if and only if*

$$\frac{1-\delta}{1-\delta^T}\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t^*(\theta_t,\omega_t)\left(\theta_t - \frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t) - \omega_t - \frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right] \geq 0, \tag{1}$$

where

$$I_t(\theta^t) = \prod_{s=1}^{t} -\frac{\partial F(\theta_s|\theta_{s-1})/\partial\theta_{s-1}}{f(\theta_s|\theta_{s-1})} \quad \textit{and} \quad J_t(\omega^t) = \prod_{s=1}^{t} -\frac{\partial G(\omega_s|\omega_{s-1})/\partial\omega_{s-1}}{g(\omega_s|\omega_{s-1})},$$

respectively, are the impulse responses of the period-t types of the buyer and the seller.[6]

Proof. We first establish the necessity of (1). Suppose that M is an E, BB, IC, IR mechanism. Since M is IC and type processes are regular, the dynamic envelope theorem of Pavan, Segal, and Toikka (2011) implies that the buyer's equilibrium utility given period-0 type θ_0, denoted $U(\theta_0)$, is Lipschitz continuous in θ_0 with

$$U'(\theta_0) = \frac{1-\delta}{1-\delta^T}\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t^*(\theta^t,\omega^t)I_t(\theta^t) \mid \theta_0\right] \quad \text{a.e. } \theta_0. \tag{2}$$

An analogous expression obtains for the seller's equilibrium profit $\Pi(\omega_0)$. For bounded domains of types, we have

$$\begin{aligned}
0 &\leq_{\text{By } IR} U(\underline{\theta}) + \Pi(\bar{\omega}) \\
&= \mathbb{E}\left[U(\theta_0) - \int_{\underline{\theta}}^{\theta_0} U'(r)dr + \Pi(\omega_0) + \int_{\omega_0}^{\bar{\omega}} \Pi'(s)ds\right] \\
&= \mathbb{E}\left[U(\theta_0) - \frac{1-F(\theta_0)}{f(\theta_0)}U'(\theta_0) + \Pi(\omega_0) + \frac{G(\omega_0)}{g(\omega_0)}\Pi'(\omega_0)\right] \\
&\leq_{\text{By } BB} \frac{1-\delta}{1-\delta^T}\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t^*(\theta_t,\omega_t)\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t) - \frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right],
\end{aligned}$$

[5] We state the result for regular processes under FOSD to streamline the exposition. It can be generalized to non-regular processes by defining the impulse responses via a state representation of the type process as in Pavan, Segal and Toikka (2011). We use such a generalization in *Section* *** where we consider the renewal process whose transition distributions have atoms. Similarly, if the type sets Θ and Ω are bounded, it is possible to dispense with FOSD and derive an analogous inequality by starting from the worst-off initial types of the agents under the efficient allocation rule, which then need not be the lowest buyer type and the highest seller type.

[6] By convention, the product over the empty set equals one and thus $I_0(\theta_0) \equiv J_0(\theta_0) \equiv 1$.

where the first line follows by IR, the second by the fundamental theorem of calculus, the third by Fubini's theorem, and the last line by BB, the law of iterated expectations, and the envelope formula (2).

For unbounded domains, note that FOSD implies that $I_t(\theta^t) \geq 0$ for all t, θ^t, and hence U is an increasing continuous function. Thus $\lim_{\gamma\to\underline{\theta}} U(\gamma)$ is well-defined, and for all θ_0 we have

$$\lim_{\gamma\to\underline{\theta}} U(\gamma) = U(\theta_0) - \lim_{\gamma\to\underline{\theta}} \int_\gamma^{\theta_0} U'(r)dr = U(\theta_0) - \int_{\underline{\theta}}^{\theta_0} U'(r)dr,$$

where the last equality follows by the Monotone Convergence Theorem since $U' \geq 0$. An analogous expression obtains for $\lim_{\beta\to\bar{\omega}} \Pi(\beta)$. The rest of the reasoning is as before:

$$\begin{aligned} 0 &\leq_{\text{By } IR} \lim_{\gamma\to\underline{\theta}} U(\gamma) + \lim_{\beta\to\bar{\omega}} \Pi(\beta) \\ &\leq \frac{1-\delta}{1-\delta^T}\mathbb{E}\left[\sum_{t=0}^{T} \delta^t x_t^*(\theta_t,\omega_t)\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)} I_t(\theta^t) - \frac{G(\omega_0)}{g(\omega_0)} J_t(\omega^t)\right)\right], \end{aligned}$$

where the first inequality follows by IR, and the second by arguments identical to the case of bounded supports.

We then turn to sufficiency of (1). Note first that the mechanism $M = \{x^*, p^B, p^S\}$, where $p_t^B(\theta^t,\omega^t) = -\omega_t x_t^*(\theta_t,\omega_t)$ and $p_t^S(\theta^t,\omega^t) = \theta_t x_t^*(\theta_t,\omega_t)$ for all t, θ^t and ω^t, is E, IC, and IR. (This is the Team mechanism of Athey and Segal, 2007.) Indeed, E follows by construction; IC follows since the game effectively separates into a sequence of static problems as there are no linkages between periods in x^* or p^i, and in each period t, the payments are equal to the static VCG payments; IR follows since for each agent the payoff from truthtelling is equal to the social surplus (period by period), which is nonnegative. Furthermore, as in the necessity part of the proof, the envelope theorem and FOSD imply that the equilibrium payoffs $U(\theta_0)$ and $\Pi(\omega_0)$ are, respectively, nondecreasing and nonincreasing in type.

To conclude the proof, we show that if (1) is satisfied, then we can adjust the above transfers to satisfy BB while preserving IC and IR. Write $U(\underline{\theta}) = \lim_{\theta_0\to\underline{\theta}} U(\theta_0)$ and $\Pi(\bar{\omega}) =$

$\lim_{\omega_0 \to \bar{\omega}} \Pi(\omega_0)$.

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{t=0}^{T} \delta^t \left(p_t^S\left(\theta^t, \omega^t\right) + p_t^B\left(\theta^t, \omega^t\right)\right)\right] \\
= \; & \frac{1-\delta^T}{1-\delta} \mathbb{E}\left[U(\theta_0) + \Pi(\omega_0)\right] - \mathbb{E}\left[\sum_{t=0}^{T} \delta^t x_t^*(\theta_t, \omega_t)(\theta_t - \omega_t)\right] \\
= \; & \frac{1-\delta^T}{1-\delta}\left(U(\underline{\theta}) + \Pi(\bar{\omega})\right) - \mathbb{E}\left[\sum_{t=0}^{T} \delta^t x_t^*(\theta_t, \omega_t)\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)} I_t(\theta^t) - \frac{G(\omega_0)}{g(\omega_0)} J_t(\omega^t)\right)\right].
\end{aligned}
$$

Thus, if we change the first-period transfers to $\hat{p}_0^B(\theta_0, \omega_0) = p_0^B(\theta_0, \omega_0) - \frac{1-\delta^T}{1-\delta} U(\underline{\theta})$ and $\hat{p}_0^S(\theta_0, \omega_0) = p_0^S(\theta_0, \omega_0) - \frac{1-\delta^T}{1-\delta} \Pi(\bar{\omega})$, then (1) implies

$$
\mathbb{E}\left[\hat{p}_0^B(\theta_0, \omega_0) + \hat{p}_0^S(\theta_0, \omega_0) + \sum_{t=1}^{T} \delta^t \left(p_t^S\left(\theta^t, \omega^t\right) + p_t^B\left(\theta^t, \omega^t\right)\right)\right] \leq 0,
$$

i.e., the mechanism is BB. Furthermore, the change clearly preserves IC as we are just subtracting a constant. It also preserves IR since, by construction, we have $U(\underline{\theta}) = \Pi(\bar{\omega}) = 0$, so IR follows by monotonicity of U and Π. ■

For $T = 0$, the inequality in Lemma 1 is the familiar condition from Myerson and Satterthwaite (1983) that the virtual surplus (i.e., the expected trading surplus less information rents) needs to be positive. The dynamic setting adds additional surplus from future periods and additional information rents. The rents are additionally discounted by the impulse responses since today's types are imperfect signals of future types.

We note that the necessity part of Lemma 1 is made stronger by requiring IC only in the sense of Bayesian equilibrium, IR only in period 0, and BB only ex ante. The same is true of information disclosure: Making reports and transfers public would add additional incentive-compatibility constraints for the agents, which would only make it harder to achieve efficiency. Thus, for a given pair of type processes, we can show that efficient trade is impossible under any unsubsidized institutions based on voluntary exchange by simply showing that inequality (1) is not satisfied.

However, for sufficiency we may want to impose IC, IR, or BB in a stronger sense. To this end, we note first that for the m echanism constructed in the proof, truthtelling is actually an ex post equilibrium (not just within period, but also with respect to future types). In particular, there exists a Perfect Bayesian Equilibrium (PBE) in truthful strategies. This also implies that reports and transfers can be made public. As for participation constraints, the mechanism in the proof of Lemma 1 is clearly individually rational at the interim stage of every period t. (For periods $t > 0$ this is true even ex post.) Thus the inequality (1) is in fact sufficient for the existence of an efficient, ex ante budget balanced, ex post IC mechanism that is IR in every period.

Implementing an ex ante budget balanced mechanism requires access to third party financing. In environments where that is not available, budget balance needs to be imposed period by period. This can be achieved by using the "Balanced Team Transfers" of Athey and Segal (2007). The resulting mechanism can be shown to have a PBE in truthful strategies (Athey and Segal only consider a Bayesian Nash equilibrium). Furthermore, if inequality (1) is satisfied, then the mechanism is also IR in period 0.[7] Therefore, (1) is sufficient for the existence of an efficient, per period BB, perfect Bayesian IC mechanism that is IR in period 0.

Our analysis leaves open the question about the exact conditions needed for the existence of an efficient, per period BB, perfect Bayesian IC mechanism that is IR in every period. Athey and Segal (2007, 200?) show that such a mechanism exists in a class of Markov environments with finitely many types provided that the agents are sufficiently patient. We conjecture that the same is true in our model provided that inequality (1) is satisfied for some $\delta < 1$. However, it appears that (1) by itself is not sufficient for this.

We finish this section by considering the following useful benchmark.

Corollary 1 *Suppose types are iid over time, $\Theta = \Omega$, and $T = \infty$. Then there exists an E, BB, IC, IR mechanism if and only if $\delta \geq \frac{1}{2}$.*

Proof. In the i.i.d. case the impulse responses are zero for all $t > 0$. Let S denote the periodic ex ante expected surplus from the efficient allocation. Let R denote the sum of expected information rents in period 0. From MS we know that $R = 2S$ when $\Theta = \Omega$. Hence for $T = \infty$ condition (1) becomes

$$(1-\delta)(S - 2S) + \delta S \geq 0,$$

[7] To see this, note that for all efficient mechanisms we have

$$\begin{aligned}\frac{1-\delta^T}{1-\delta}\left(U(\underline{\theta}) + \Pi(\bar{\omega})\right) &= \mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t^*(\theta_t,\omega_t)\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t) - \frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right] \\ &\quad + \mathbb{E}\left[\sum_{t=0}^{T}\delta^t\left(p_t^S\left(\theta^t,\omega^t\right) + p_t^B\left(\theta^t,\omega^t\right)\right)\right].\end{aligned}$$

If the mechanism is exactly ex ante budget balanced, then the second line disappears and the sum of utilities for the worst-off types is positive whenever (1) holds. Hence, if need be, we can make a type-independent period-0 transfer between the agents to ensure period-0 IR.

which is satisfied if and only if $\delta \geq \frac{1}{2}$. ∎

The general remarks about the sufficiency direction of Lemma 1 apply equally well in the iid case. In particular, if we want the implementation and the environment to satisfy additional conditions, then the critical discount factor may be higher. For an analysis of that problem, see Athey and Miller (2007). We note here that if the values are drawn from a symmetric distribution ($G = F$), there is a simple implementation that is budget balanced and individually rational in every period: In the beginning of every period the seller is supposed to make the good a common property and the buyer is supposed to pay a fixed fee to him for that; otherwise the parties terminate the relationship. In this way they create a partnership, where each owns 50% of the good and we can use the results from Cramton, Gibbons, and Klemperer (1987) to establish the existence of a mechanism that dissolves the partnership efficiently without outside subsidies and is individually rational conditional on the initial transfer (of goods and money). If the distributions are symmetric, a simple ascending auction achieves that. The only additional constraint that needs to be checked then is that in every period, no matter what is the realized type, each player is willing to make the first transfer (of goods or money), given the threat that no future trade takes place otherwise. This will be satisfied for δ high enough. Returning to our motivating example, in practice the publisher and the advertiser can implement efficient trade by a long-term relational contract (that either party can break at any time) in which the status quo is that the seller will sell half of the traffic at a pre-specified, constant price and then the buyer and seller will negotiate among themselves additional trades (that can go either way). This works if types are iid. But what if types are serially correlated?[8]

4 Impossibility of Efficiency under Stationary Gaussian Processes

In this section we assume that $T = \infty$, and the buyer's and seller's types follow a symmetric AR(1) process induced by an underlying Ornstein-Uhlenbeck (OU) process, which is the unique stationary Gaussian Markov process. We provide the notation for the buyer types since the seller types behave analogously. Let the buyer's "latent type" process $\hat{\theta}_\tau$ be defined by

$$d\hat{\theta}_\tau = -\alpha\hat{\theta}_\tau d\tau + \sigma dB_\tau,$$

[8] We do not study the case of perfect persistence since that is equivalent to the static problem in MS.

where B_τ is the standard one-dimensional Brownian motion, and $\hat{\theta}_0 = \theta_0$ (with θ_0 drawn from the initial distribution $F = G$). We introduce the following notation to facilitate changes in the frequency of bargaining: we count time either in periods, t, with each period having length Δ, or in real time, τ, so that period t corresponds to real time $\tau = t\Delta$.

The solution to the stochastic differential equation for $\hat{\theta}_\tau$ is

$$\hat{\theta}_\tau = \hat{\theta}_0 e^{-\alpha\tau} + \sigma \int_0^\tau e^{-\alpha(\tau-s)} dB_s.$$

Thus we have

$$E[\hat{\theta}_\tau|\theta_0] = e^{-\alpha\tau}\theta_0, \quad \text{and} \quad Var[\hat{\theta}_\tau|\theta_0] = (1 - e^{-2\alpha\tau})\frac{\sigma^2}{2\alpha}.$$

The process has a stationary distribution $N\left(0, \frac{\sigma^2}{2\alpha}\right)$, which we denote by F_{st}. Let $q \equiv \sqrt{\frac{\sigma^2}{2\alpha}}$.

The latent types $\hat{\theta}_\tau$ induce the buyer's actual types by sampling the above OU process at fixed intervals of length Δ. Therefore, the buyer's type θ_t follows a linear AR(1) process

$$\theta_t = \gamma\theta_{t-1} + \varepsilon_t, \tag{3}$$

where $\gamma = e^{-\alpha\Delta}$ and $\varepsilon_t \sim N(0, \underbrace{(1 - e^{-2\alpha\Delta})\frac{\sigma^2}{2\alpha}}_{=(1-\gamma^2)q^2})$. That is, we assume that the type stays constant for the duration Δ and at the end of the period it switches to the position of the underlying OU process.[9]

This modeling choice allows us to perform the analysis in discrete time and let $\Delta \to 0$ represent frequent opportunities to bargain. More frequent trading opportunities have two effects: as Δ decreases the per-period discount factor, $\delta = e^{-r\Delta}$, tends to 1, but at the same time the correlation of types between adjacent periods increases as the persistence parameter, γ, tends to 1 and the variance of the innovation term ε tends to 0. Therefore, in a model with persistent types, more frequent transactions are not equivalent to a lower discount rate.[10]

We now analyze whether it is possible to implement the efficient allocation rule x^*. It is straightforward to verify that the Gaussian AR(1) process (3) is regular and satisfies FOSD, and hence Lemma 1 applies. In particular, the impulse responses simplify to $I_t(\theta^t) = J_t(\omega^t) = \gamma^t$ and the inequality (1) becomes

[9] An alternative model would be to let the type change over the duration of a period, which would require the computation of the expected discounted type over the duration of a period given the current type. In such a model, a more frequent bargaining would help not only resolve issues of information asymmetry but also improve total first-best surplus. To isolate the effects of incentives we prefer our model. .

[10] The difference between making r smaller and Δ smaller is reminiscent of the results in Abreu, Milgrom and Pearce (1991) but the tradeoff is different: in their model smaller Δ affects the informativeness of per-period noisy monitoring, while in our period it affects per-period serial correlations of types.

$$(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t x_t^*(\theta^t,\omega^t)\left(\theta_t-\omega_t-\frac{1-F(\theta_0)}{f(\theta_0)}\gamma^t-\frac{F(\omega_0)}{f(\omega_0)}\gamma^t\right)\right]\geq 0. \qquad (4)$$

We decompose the expression on the left-hand side of (4) into total surplus and information rents. Denote the total expected first-best surplus by

$$S=(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t x_t^*\left(\theta_t-\omega_t\right)\right],$$

where we omit the arguments of x_t^* to simplify notation. Letting Φ denote the cdf of the standard normal distribution, we can write the expected period-t information rent of the buyer as

$$\begin{aligned}R_t^B&=\gamma^t\mathbb{E}\left[x_t^*\frac{1-F(\theta_0)}{f(\theta_0)}\right]\\&=\gamma^t\mathbb{E}\left[\mathbb{E}[\mathbf{1}_{\{\theta_t-\omega_t\geq 0\}}\mid\theta_0,\omega_0]\frac{1-F(\theta_0)}{f(\theta_0)}\right]\\&=\gamma^t\mathbb{E}\left[\Phi\left(\frac{\gamma^t(\theta_0-\omega_0)}{q\sqrt{2(1-\gamma^{2t})}}\right)\frac{1-F(\theta_0)}{f(\theta_0)}\right],\end{aligned}$$

where the last step follows, since for $t>0$,

$$\theta_t\sim N\left(e^{-\alpha\Delta t}\theta_0,(1-e^{-2\alpha\Delta t})\frac{\sigma^2}{2\alpha}\right)\quad\text{and}\quad\omega_t\sim N\left(e^{-\alpha\Delta t}\omega_0,(1-e^{-2\alpha\Delta t})\frac{\sigma^2}{2\alpha}\right),$$

so by independence,

$$\theta_t-\omega_t\sim N\left(e^{-\alpha\Delta t}(\theta_0-\omega_0),(1-e^{-2\alpha\Delta t})\frac{\sigma^2}{\alpha}\right).$$

Similarly, the expected time-t information rent of the seller is

$$R_t^S=\gamma^t\mathbb{E}\left[x_t^*\frac{F(\omega_0)}{f(\omega_0)}\right]=\gamma^t\mathbb{E}\left[\Phi\left(\frac{\gamma^t(\theta_0-\omega_0)}{q\sqrt{2(1-\gamma^{2t})}}\right)\frac{F(\omega_0)}{f(\omega_0)}\right].$$

Thus the necessary condition (4) for the implementability of x^* can be re-stated as

$$S-(1-\delta)\sum_{t=0}^{\infty}\delta^t\left(R_t^B+R_t^S\right)\geq 0. \qquad (5)$$

That is, the total available expected surplus S needs to be enough to pay the expected discounted information rents of both agents.

4.1 Stationary case

Suppose $F = F_{st}$, that is, both types at time zero are drawn from the stationary distribution of the underlying OU process. Then the expected (first-best) total surplus is equal to the static first-best surplus under F_{st} given by

$$\begin{aligned} S_{st} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\theta}(\theta-\omega)\, f_{st}(\omega)\, d\omega f_{st}(\theta)\, d\theta \\ &=_{\text{by parts}} \int_{-\infty}^{\infty} F_{st}(\theta)\left(1-F_{st}(\theta)\right) d\theta \\ &= q\int_{-\infty}^{\infty}\Phi\left(\frac{\theta}{q}\right)\left(1-\Phi\left(\frac{\theta}{q}\right)\right) d\frac{\theta}{q} = \frac{q}{\sqrt{\pi}}, \end{aligned} \tag{6}$$

where we have used $F_{st}(\theta) = \Phi\left(\frac{\theta}{q}\right)$. (Recall that q is the standard deviation of the stationary distribution.)

In the appendix we establish the following technical lemma:

Lemma 2 (Technical Lemma) *For all* $z \in (0,1)$*,*

$$A(z) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} z\Phi\left(\frac{z(x-y)}{\sqrt{2(1-z^2)}}\right)\phi(x)\, dx\right)\Phi(y)dy = \frac{1}{\sqrt{\pi}}$$

This leads to our first main result:

Theorem 1 (Impossibility) *Suppose types follow the underlying OU process and time-zero types are drawn from the stationary distribution.*

1) For all strictly positive r*,* σ*,* Δ *and* α*, there does not exist a mechanism that is efficient, individually rational, ex-ante budget balanced and time-zero individually rational (E,IC,BB and IR).*

2) Moreover, the per-period subsidy, expressed as a fraction of total expected per-period surplus, needed to satisfy the necessary condition (1) *is independent of* α*,* σ, Δ *and* r*. It is equal to the expected per-period surplus as in the static problem.*

Proof. Using (6) we can simplify (5) to:

$$\begin{aligned}
&\frac{q}{\sqrt{\pi}} - (1-\delta)\sum_{t=0}^{\infty}(\gamma\delta)^t\,\mathbb{E}\left[\Phi\left(\frac{\gamma^t(\theta_0-\omega_0)}{q\sqrt{2(1-\gamma^{2t})}}\right)\left(\frac{1-F_{st}(\theta_0)}{f_{st}(\theta_0)}+\frac{F_{st}(\omega_0)}{f_{st}(\omega_0)}\right)\right] \\
=_{\text{change of variables}}&\frac{q}{\sqrt{\pi}} \\
&-(1-\delta)\sum_{t=0}^{\infty}(\gamma\delta)^t\,q\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\Phi\left(\frac{\gamma^t(x-y)}{\sqrt{2(1-\gamma^{2t})}}\right)\phi(x)\,dx\right)\Phi(y)dy \\
&-(1-\delta)\sum_{t=0}^{\infty}(\gamma\delta)^t\,q\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\Phi\left(\frac{\gamma^t(x-y)}{\sqrt{2(1-\gamma^{2t})}}\right)\phi(x)\,dx\right)(1-\Phi(y))\,dy \\
\geq\ & 0
\end{aligned}$$

By Lemma 2, letting $\gamma^t = z$ (note that $z \in (0,1)$ since $\alpha > 0$ and $\Delta > 0$), the expressions on the third and fourth line are equal to $\frac{q}{\sqrt{\pi}}$. Hence the inequality simplifies further to

$$\frac{q}{\sqrt{\pi}}(1-2) \geq 0$$

for every r, Δ and α. Thus the necessary condition cannot be satisfied.

Finally, the LHS of this expression is equal to $-\frac{q}{\sqrt{\pi}}$, which is equal to S_{st} for all parameters. So the dynamics does not make it easier to achieve first-best. We need the same subsidy per-period as we do in a static problem. ■

4.2 Varying the initial distribution

Above we assumed that the initial distribution was the stationary distribution of the OU process. Here we relax that assumption and assume instead that θ_0 and ω_0 are independent draws from a cdf F. The formulas for R_t^B and R_t^S still apply, but we need to re-evaluate the expectations of S and of the rents with respect to the time-zero types. Furthermore, the first best social surplus is now a function of r and Δ, say $S(r,\Delta)$, and we have $S(r,\Delta) \to S$ as $r \to 0$ (for all Δ) since for all initial distributions the process converges to the stationary distribution.

Proposition 1 *If* $\mathbb{E}\left[\frac{1-F(\theta_0)}{f(\theta_0)}\right], \mathbb{E}\left[\frac{F(\omega_0)}{f(\omega_0)}\right] < \infty$, *then for all* γ *there exists* $\bar{\delta}$ *such that for all* $\delta > \bar{\delta}$,

$$(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t x_t^*\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)}\gamma^t - \frac{F(\omega_0)}{f(\omega_0)}\gamma^t\right)\right] > 0.$$

Proof. Let $B = \mathbb{E}\left[\frac{1-F(\theta_0)}{f(\theta_0)}\right] + \mathbb{E}\left[\frac{F(\omega_0)}{f(\omega_0)}\right]$. The joint information rent satisfies

$$(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\gamma^t x_t^*\left(\frac{1-F(\theta_0)}{f(\theta_0)} + \frac{F(\omega_0)}{f(\omega_0)}\right)\right] \leq (1-\delta)\sum_{t=0}^{\infty}\delta^t\gamma^t B = \frac{1-\delta}{1-\delta\gamma}B$$

Thus, as $\delta \to 1$, we have

$$(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t x_t^*\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)}\gamma^t - \frac{F(\omega_0)}{f(\omega_0)}\gamma^t\right)\right] \geq S(\frac{-\log\delta}{\Delta},\Delta) - \frac{1-\delta}{1-\delta\gamma}B \to S > 0,$$

proving the claim. ■

Thus, truncating the tails of the initial distribution arbitrarily is sufficient to over-turn the anti-folk Theorem 1. In fact, for r sufficiently small we may even take the limit by sending $\Delta \to 0$:

Proposition 2 *If $\mathbb{E}\left[\frac{1-F(\theta_0)}{f(\theta_0)}\right], \mathbb{E}\left[\frac{F(\omega_0)}{f(\omega_0)}\right] < \infty$, then for all α there exists $\bar{r}$ such that for all $r < \bar{r}$,*

$$\lim_{\Delta\to 0}(1-e^{-r\Delta})\mathbb{E}\left[\sum_{t=0}^{\infty}e^{-r\Delta t}x_t^*\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)}e^{-\alpha\Delta t} - \frac{F(\omega_0)}{f(\omega_0)}e^{-\alpha\Delta t}\right)\right] > 0.$$

Proof. $B = \mathbb{E}\left[\frac{1-F(\theta_0)}{f(\theta_0)}\right] + \mathbb{E}\left[\frac{F(\omega_0)}{f(\omega_0)}\right]$. The joint limit information rent satisfies

$$\lim_{\Delta\to 0}(1-e^{-r\Delta})\sum_{t=0}^{\infty}e^{-(r+\alpha)\Delta t}\mathbb{E}\left[x_t^*\left(\frac{1-F(\theta_0)}{f(\theta_0)} + \frac{F(\omega_0)}{f(\omega_0)}\right)\right] \leq \lim_{\Delta\to 0}(1-e^{-r\Delta})\sum_{t=0}^{\infty}e^{-(r+\alpha)\Delta t}B = \frac{r}{r+\alpha}B.$$

Thus the limit virtual surplus converges to $S > 0$ as $r \to 0$.

■

Note that if F has bounded support, then $\mathbb{E}\left[\frac{1-F(\theta_0)}{f(\theta_0)}\right], \mathbb{E}\left[\frac{F(\omega_0)}{f(\omega_0)}\right] < \infty$. Therefore:

Corollary 2 *Suppose that types at time 0 are drawn from a distribution with bounded support and then follow the underlying OU process. Then for all α, σ, Δ there exists $\bar{r} > 0$ such that for all $r < \bar{r}$ the necessary condition (1) is satisfied at the efficient rule x^*.*

What if the initial distributions do not have bounded expected inverse hazard rates $\mathbb{E}\left[\frac{1-F(\theta_0)}{f(\theta_0)}\right]$ and $\mathbb{E}\left[\frac{F(\omega_0)}{f(\omega_0)}\right]$, but are not the stationary distribution either? This is an open question. What we can show is that if the initial distribution is an arbitrary normal distribution (but still symmetric between the players), then for small enough r the necessary condition cannot be satisfied (since we can show that as $r \to 0$, the variance of the initial distribution does not matter).

4.3 Two periods, normal distributions

To gain more intuition, suppose there are 2 periods $t \in \{0, 1\}$, no discounting, and that we draw the initial types from the standard normal distribution $N(0,1)$ in period 0, and in period 1 we have

$$\theta_1 = \gamma\theta_0 + \varepsilon,$$

where ε is again drawn from $N(0,1)$, and $\gamma \in (0,1)$. Then from the ex-ante perspective, θ_1 is distributed according to $N(0, 1+\gamma^2)$. Let $q = \sqrt{1+\gamma^2}$.

The total expected first-best surplus is

$$\begin{aligned} S &= \int_{-\infty}^{\infty} \Phi(\theta)(1-\Phi(\theta))\, d\theta + q\int_{-\infty}^{\infty} \Phi\left(\frac{\theta}{q}\right)\left(1-\Phi\left(\frac{\theta}{q}\right)\right) d\frac{\theta}{q} \\ &= \frac{1+\sqrt{1+\gamma^2}}{\sqrt{\pi}}. \end{aligned}$$

The first-period information rent is, by standard reasoning, $\frac{2}{\sqrt{\pi}}$ (so twice the surplus contribution from period zero). How about the additional dynamic information rent from the second period? We have

$$R_2^B = \gamma E\left[x_2^* \frac{\Phi(\omega_0)}{\phi(\omega_0)}\right] = \gamma E\left[\Phi\left(\frac{\gamma(\theta_0-\omega_0)}{\sqrt{2}}\right)\frac{\Phi(\omega_0)}{\phi(\omega_0)}\right].$$

By (13) we get

$$R_2^B = \gamma E\left[\Phi\left(\frac{\gamma(\theta_0-\omega_0)}{\sqrt{2}}\right)\frac{\Phi(\omega_0)}{\phi(\omega_0)}\right] = \gamma\int_{-\infty}^{+\infty}\left(1-\Phi\left(\omega_0\frac{\gamma}{\sqrt{\gamma^2+2}}\right)\right)\Phi(\omega_0)\, d\omega_0$$

By (14) we then get

$$R_2^B = \gamma\frac{\sqrt{1+\left(\frac{\gamma}{\sqrt{\gamma^2+2}}\right)^2}}{\frac{\gamma}{\sqrt{\gamma^2+2}}\sqrt{2\pi}} = \frac{\sqrt{\gamma^2+1}}{\sqrt{\pi}}.$$

By symmetry, the same is true for the seller's information rent. Thus, the expected second period information rents add up to exactly twice the expected second period total surplus.

5 Renewal process

We now turn to a different model of the evolution of types, which we call the *renewal model.*

The period-0 types are drawn from some distributions F and G, which are assumed to be absolutely continuous with strictly positive densities f and g on Θ and Ω, respectively. For simplicity, we assume that the supports coincide (i.e., $\Theta = \Omega$). There is an underlying continuous-time "latent type" process $\hat{\theta}_\tau$ that starts at $\hat{\theta}_0 = \theta_0$ (again, we describe the evolution of the buyer type; the seller type changes analogously). Horizon is infinite, $t \in \{1, ...\}$. In the discrete-time bargaining period t, the type of the buyer re-sets to the latent type, $\theta_t = \hat{\theta}_{t\Delta}$ and remains constant till the beginning of the next period.

The latent type evolves as follows: it stays constant in almost all τ, other than at discrete events that arrive with a constant Poisson arrival rate λ. Upon arrival the type changes discontinuously and is drawn from the initial distribution F independently of the history. As a result, given realized period-t type θ_t, the next period type θ_{t+1} is equal to θ_t with probability $\gamma = e^{-\lambda\Delta}$; with probability $(1-\gamma)$ it is drawn anew from F (renewed independently of the history). Note that this implies that θ_t follows an AR(1) process where the distribution of θ_{t+1} conditional on θ_t has an atom at θ_t. The stationary distribution of this process is F.

The above renewal process is not a regular process. However, we show in the appendix (TO BE ADDED) that Lemma 1 can be extended to this case using the results of Pavan, Segal, and Toikka (2011). The impulse responses take the form $I_t(\theta^t) = \mathbf{1}_{\{\theta_0=\theta_1=\cdots=\theta_t\}}$ and $J_t(\omega^t) = \mathbf{1}_{\{\omega_0=\omega_1=\cdots=\omega_t\}}$, and the inequality (1) becomes

$$(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t x_t^*\left(\theta_t - \omega_t - \frac{1-F(\theta_0)}{f(\theta_0)}\mathbf{1}_{\{\theta_t=\theta_0\}} + \frac{G(\omega_0)}{g(\omega_0)}\mathbf{1}_{\{\omega_t=\omega_0\}}\right)\right] \geq 0.^{11} \tag{7}$$

Again, it is useful to decompose this expression between the expected total surplus and information rents. Let the expected surplus be

$$S_t = \mathbb{E}\left[x_t^*(\theta_t - \omega_t)\right] = \mathbb{E}\left[\max\{0, \theta_t - \omega_t\}\right].$$

Since the processes for θ_t and ω_t are stationary, $S_t = S$ for all t, where S is the static first-best total surplus. In order to simplify notation, we let

$$\eta_0 \equiv \frac{1-F(\theta_0)}{f(\theta_0)}, \quad k_0 \equiv \frac{G(\omega_0)}{g(\omega_0)}.$$

[11] Since F and G are absolutely continuous, we may ignore the possibility of drawing the same type again upon renewal when evaluating the expectation. Thus we may ignore the intermediate types in the formulas for the impulse responses.

With this notation, the inequality (7) becomes:

$$S - (1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[x_t^* \left(\eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}} + k_0 \mathbf{1}_{\{\omega_t=\omega_0\}}\right)\right] \geq 0. \tag{8}$$

When is the expected surplus from efficient trading sufficient to cover the (dynamic) information rents of the players?

Theorem 2 *Suppose types follow the renewal process with arrival rate λ. There exists an E, BB, IC, IR mechanism if and only if $\delta \geq \frac{1}{2-\gamma}$. In the continuous-time limit, as $\Delta \to 0$, this condition is satisfied if and only if $\lambda \geq r$.*

Proof. Evaluate the expected information rent separately for each agent. Consider:

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[x_t^* \eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}}\right].$$

We first take the expectations with respect to $\{\theta_t, \omega_t\}$ for $t > 0$, keeping $\{\theta_0, \omega_0\}$ as given. The probability that $\{\theta_t = \theta_0\}$ is γ^t. Conditional on $\{\theta_t = \theta_0\}$, we split the histories into two possibilities: either $\omega_t = \omega_0$ (also with probability γ^t) or not. If $\omega_t \neq \omega_0$ then trade is efficient ($x_t^* = 1$) if $\omega_t < \theta_0$, and event with probability $G(\theta_0)$. Therefore we get:

a) If $\theta_0 - \omega_0 \geq 0$ then

$$\begin{aligned}\mathbb{E}\left[x_t^* \eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}} | \theta_0, \omega_0\right] &= \gamma^t(\gamma^t + \left(1-\gamma^t\right) G(\theta_0))\eta_0 \\ &= \gamma^{2t}\eta_0 + \gamma^t\left(1-\gamma^t\right) G(\theta_0)\eta_0\end{aligned}$$

b) If $\theta_0 - \omega_0 < 0$ then

$$\mathbb{E}\left[x_t^* \eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}} | \theta_0, \omega_0\right] = \gamma^t\left(1-\gamma^t\right) G(\theta_0)\eta_0.$$

Let $Q = \Pr(\theta_0 \geq \omega_0)$. Then overall, integrating also over $\{\theta_0, \omega_0\}$ we get (using the law of iterated expectations):

$$\begin{aligned}\mathbb{E}\left[x_t^* \eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}}\right] &= Q\mathbb{E}\left[x_t^* \eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}} | \theta_0 - \omega_0 \geq 0\right] + (1-Q)\mathbb{E}\left[x_t^* \eta_0 \mathbf{1}_{\{\theta_t=\theta_0\}} | \theta_0 - \omega_0 < 0\right] \\ &= \gamma^{2t} Q\mathbb{E}\left[\eta_0 | \theta_0 - \omega_0 \geq 0\right] + \gamma^t\left(1-\gamma^t\right)\mathbb{E}\left[G(\theta_0)\eta_0\right].\end{aligned}$$

We can further simplify this expression by noting that:

$$Q\mathbb{E}\left[\eta_0|\theta_0-\omega_0\geq 0\right]=\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\omega}}^{\theta}\frac{1-F(\theta)}{f(\theta)}g\left(\omega\right)d\omega f\left(\theta\right)d\theta$$
$$=\int_{\underline{\theta}}^{\bar{\theta}}G(\theta)(1-F(\theta))d\theta,$$

and

$$\mathbb{E}\left[G(\theta_0)\eta_0\right]=\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\omega}}^{\bar{\omega}}G(\theta)\frac{1-F(\theta)}{f(\theta)}g\left(\omega\right)d\omega f\left(\theta\right)d\theta$$
$$=\int_{\underline{\theta}}^{\bar{\theta}}G(\theta)(1-F(\theta))d\theta.$$

Summing up:

$$\mathbb{E}\left[x_t^*\eta_0\mathbf{1}_{\{\theta_t=\theta_0\}}\right]=\gamma^t\int_{\underline{\theta}}^{\bar{\theta}}G(\theta_0)(1-F(\theta_0))d\theta_0=\gamma^t\mathbb{E}\left[x_0^*\eta_0\right], \tag{9}$$

where the second equality follows from observing that

$$\mathbb{E}\left[x_0^*\eta_0\right]=QE\left[\eta_0|\theta_0-\omega_0\geq 0\right].$$

Analogous calculations for the seller yield

$$\mathbb{E}\left[x_t^*k_0\mathbf{1}_{\{\omega_t=\omega_0\}}\right]=\gamma^t\mathbb{E}\left[x_0^*k_0\right] \tag{10}$$

Putting together (8), (9) and (10), we have:

$$S-(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[x_t^*\left(\eta_0\mathbf{1}_{\{\theta_t=\theta_0\}}+k_0\mathbf{1}_{\{\omega_t=\omega_0\}}\right)\right]$$
$$=S-(1-\delta)\sum_{t=0}^{\infty}\delta^t\gamma^t\underbrace{\mathbb{E}\left[x_0^*\left(\eta_0+k_0\right)\right]}_{\equiv R}$$
$$=S-\frac{1-\delta}{1-\gamma\delta}R.$$

Now, note that R is the sum of information rents in the static problem and S is the total surplus in the static problem. From MS we know that $R=2S$. Hence we get that the necessary condition (1) is equivalent in the renewal model to:

$$S\left(1-2\frac{1-\delta}{1-\gamma\delta}\right)\geq 0 \tag{11}$$

which is satisfied if and only if $\delta \geq \frac{1}{2-\gamma}$. In the i.i.d. limit, $\gamma \to 0$, it corresponds to $\delta \geq \frac{1}{2}$; in the perfect persistence limit, $\gamma \to 1$, for any $\delta < 1$ it is violated (intuitively, when $\gamma = 1$ the problem is isomorphic to a static problem in MS).

Finally, recall that $\gamma = e^{-\lambda\Delta}$ and $\delta = e^{-r\Delta}$. Taking the limit of the term in (11) we get:

$$\lim_{\Delta\to 0}\left(1-2\frac{1-e^{-r\Delta}}{1-e^{-(r+\lambda)\Delta}}\right)=\frac{\lambda-r}{\lambda+r}$$

and that is positive if and only if $\lambda \geq r$. ■

Corollary 3 *In the renewal model, more frequent bargaining helps achieve efficiency (E,BB,IC and IR) but frequent bargaining may not be enough.*

Proof. In the proof of the Theorem 2, we showed that the necessary and sufficient condition for efficiency is:

$$1-2\frac{1-e^{-r\Delta}}{1-e^{-(r+\lambda)\Delta}}\geq 0.$$

The left-hand side is decreasing in Δ and hence it is easier to satisfy this condition when Δ is smaller. This observation and the Theorem imply that if $r > \lambda$ then for no Δ it is possible to achieve efficiency (with BB, IC and IR). ■

5.1 Asymmetric persistence

To finish this section, consider the asymmetric arrival case. In particular, suppose that the arrival rates for the latent processes are λ_B and λ_S respectively, so that $\gamma_B = e^{-\Delta\lambda_B}$ and $\gamma_S = e^{-\Delta\lambda_S}$. The calculations of information rents separately for each player are the same as in the proof above and after we put them together we get:

$$\begin{aligned}&S-(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[x_t^*\left(\eta_0\mathbf{1}_{\{\theta_t=\theta_0\}}+k_0\mathbf{1}_{\{\omega_t=\omega_0\}}\right)\right]\\&=S-(1-\delta)\sum_{t=0}^{\infty}\delta^t\gamma_B^t\mathbb{E}\left[x_0^*\eta_0\right]+(1-\delta)\sum_{t=0}^{\infty}\delta^t\gamma_S^t\mathbb{E}\left[x_0^*k_0\right].\end{aligned}$$

Simple algebra allows us to establish:

Corollary 4 *Suppose types follow the renewal process with arrival rates* λ_B *and* λ_S*. There exists an E, BB, IC, IR mechanism if and only if* $\delta \geq \frac{1}{1+\sqrt{(1-\gamma_B)(1-\gamma_S)}}$*. In the continuous time limit,* $\Delta \to 0$*, this condition is satisfied if and only if* $\sqrt{\lambda_B \lambda_S} \geq r$*.*

Proof. Note first that in the static model, $\mathbb{E}[x_0^* \eta_0] = \mathbb{E}[x_0^* k_0] = S$. (This holds for the pivot mechanism, and hence by revenue equivalence it holds for any efficient mechanism.) Therefore,

$$
\begin{aligned}
& S - (1-\delta) \sum_{t=0}^{\infty} \delta^t \gamma_B^t \mathbb{E}[x_0^* \eta_0] + (1-\delta) \sum_{t=0}^{\infty} \delta^t \gamma_S^t \mathbb{E}[x_0^* k_0] \\
= \quad & S \left(1 - \frac{1-\delta}{1-\delta\gamma_B} - \frac{1-\delta}{1-\delta\gamma_S} \right)
\end{aligned}
$$

This is positive iff

$$
\delta \geq \frac{1}{1+\sqrt{(1-\gamma_B)(1-\gamma_S)}}
$$

Letting $\Delta \to 0$ we get:

$$
\lim_{\Delta \to 0} \left(1 - \frac{1-e^{-r\Delta}}{1-e^{-(r+\lambda_B)\Delta}} - \frac{1-e^{-r\Delta}}{1-e^{-(r+\lambda_S)\Delta}} \right) = \frac{\lambda_B \lambda_S - r^2}{(r+\lambda_B)(r+\lambda_S)}
$$

and that is positive if and only if $\sqrt{\lambda_B \lambda_S} \geq r$. ■

6 Second Best

When efficiency is not implementable, we look for the IC, IR, BB mechanism with the highest expected total surplus.

Definition 7 *The mechanism* M *is* second best *if it maximizes* $\mathbb{E}\left[\sum_{t=0}^{T} \delta^t x_t(\theta^t, \omega^t)(\theta_t - \omega_t)\right]$ *subject to IC, IR, BB.*

In keeping with the approach in the previous sections, we are using the weak versions of incentive compatibility, individual rationality, and budget balance, but will comment on the possibility of satisfying stronger versions.

Lemma 3 *Suppose type processes are regular and satisfy FOSD. Suppose further that* $\frac{1-F(\theta_0)}{f(\theta_0)}$*,* $-\frac{G(\omega_0)}{g(\omega_0)}$*,* $I_t(\theta^t)$*, and* $-J_t(\omega^t)$ *are nonincreasing in* θ_0*,* ω_0*,* θ^t*, and* ω^t*, respectively, for all* t*. Then* x^{**} *is the allocation rule in a second best mechanism only if it solves*

$$
\max_x \mathbb{E}\left[\sum_{t=0}^{T} \delta^t x_t(\theta^t, \omega^t)(\theta_t - \omega_t) \right]
$$

subject to

$$\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t(\theta^t,\omega^t)\left(\theta_t-\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)-\omega_t-\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right]\geq 0. \qquad (12)$$

Proof. By inspection, the necessity part of the proof of Lemma 1 makes no reference to efficiency (other than using notation x^* for the allocation rule), and hence the same argument shows that all IC, IR, BB mechanisms satisfy

$$\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t(\theta^t,\omega^t)\left(\theta_t-\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)-\omega_t-\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right]\geq 0.$$

This implies that the maximization problem in Lemma 3 is a relaxed version of the one in the definition of second best. Hence it suffices to prove that there exists a solution x^{**} to the former such that $M=\{x^{**},p^B,p^S\}$ is IC, IR, BB for some p^B and p^S. Since the problem is linear in the allocation, there exists a Lagrange multiplier $\lambda\geq 0$ such that any solution x^{**} maximizes the Lagrangean

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t(\theta^t,\omega^t)\left(\theta_t-\omega_t\right)\right]\\
&+\lambda\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t(\theta^t,\omega^t)\left(\theta_t-\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)-\omega_t-\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right]\\
=\;&(1+\lambda)\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t(\theta^t,\omega^t)\left(\theta_t-\omega_t-\frac{\lambda}{1+\lambda}\left(\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)+\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right)\right]
\end{aligned}$$

If $\lambda=0$, then $x^{**}=x^*$ a.s. This implies that (1) is satisfied, and thus there exists an E, BB, IC, IR mechanism, and the second best coincides with the first best. So suppose $\lambda>0$. Note that the Lagrangean can be maximized pointwise, i.e., for all t, θ^t and ω^t,

$$x_t^{**}(\theta^t,\omega^t)\in\arg\max_{x_t}x_t\left[\theta_t-\omega_t-\frac{\lambda}{1+\lambda}\left(\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)+\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right].$$

Note that under the assumptions of the lemma, the objective function has increasing differences in (x,θ_s) and $(x,-\omega_s)$ for $0\leq s\leq t$. By Topkis' theorem, we may then take x_t^{**} to be nondecreasing in θ_s and nonincreasing in ω_s for all s. Thus the allocations of both agents are strongly monotone. As the type processes satisfy FOSD, the results of Pavan, Segal, and Toikka (2011) then imply that x_t^{**} is implementable in a (perfect Bayesian) IC mechanism. Furthermore, the mechanism can be taken to be exactly BB (even period by period) by using the balancing argument of Athey and Segal (2007). Finally, by exact BB we have

$$\frac{1-\delta^T}{1-\delta}\left(U(\underline{\theta})+\Pi(\bar{\omega})\right)=\mathbb{E}\left[\sum_{t=0}^{T}\delta^t x_t^{**}(\theta_t,\omega_t)\left(\theta_t-\omega_t-\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)-\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right)\right],$$

and thus the mechanism is IR (in period 0) by (12). ■

As is clear from the proof, the second best mechanism can be taken to be IC in the sense of a truthful PBE, and BB period by period, if IR is required only in period 0. Alternatively, if we only require ex ante BB, then it is possible to satisfy IR period by period. (Details to be added...)

Analogously to the static case, trade occurs in the second best mechanism only when the buyer's value exceeds the seller's cost by a sufficient margin. In the dynamic setting the margin is history-dependent with the agents trading in period t if

$$\theta_t-\omega_t\geq\frac{\lambda}{1+\lambda}\left(\frac{1-F(\theta_0)}{f(\theta_0)}I_t(\theta^t)+\frac{G(\omega_0)}{g(\omega_0)}J_t(\omega^t)\right),$$

where $\lambda\geq 0$ is the Lagrange multiplier for the maximization problem in Lemma 3, which is constant in t. However, at this level of generality it is difficult to characterize the second best allocation much further. (For example, the assumptions we have made so far allow for distortions to be nonmonotone in time.)

It can readily be verified that the stationary Gaussian model of 4 satisfies the assumptions of Lemma 3 (note that the impulse responses are simply $I_t(\theta^t)\equiv J_t(\theta^t)\equiv\gamma^t$). In this case the condition for trade simplifies to

$$\theta_t-\omega_t\geq\gamma^t\frac{\lambda}{1+\lambda}\left(\frac{1-F_{st}(\theta_0)}{f_{st}(\theta_0)}+\frac{F_{st}(\omega_0)}{f_{st}(\omega_0)}\right).$$

Recalling that $\gamma=e^{-\alpha\Delta}<1$, we see that the allocation converges to the first best allocation as $t\to\infty$. However, note that Theorem 1 implies that even for high δ, ex ante expected payoffs will be bounded away from first best.

The renewal model fails the condition of Lemma 3 as the impulse responses are not monotone. Indeed, it can be verified that following this method gives a candidate allocation rule that is not implementable. The problem is that the shape of the impulse response leads to a distortion to the allocation only in the case the type has not been renewed. However, such an allocation rule is not "monotone enough" to be implementable. As there is no known method that would allow us to analytically maximize with respect to all the incentive compatibility constraints (and not just the local ones incorporated in the virtual surpus), we leave the question about the second best in the renewal model for future work.

7 Concluding Remarks

8 Appendix

Proof of Lemma 2. We use two observations:

1) For all a and b,

$$\Phi\left(\frac{a}{\sqrt{b^2+1}}\right)=\int_{-\infty}^{\infty}\Phi\left(a+bx\right)d\Phi\left(x\right). \tag{13}$$

2) For $c>0$,

$$\int_{-\infty}^{\infty}\left(1-\Phi\left(yc\right)\right)\Phi(y)dy=\int_{-\infty}^{\infty}\left(1-\Phi\left(y\right)\right)\Phi(cy)dy=\frac{\sqrt{1+c^2}}{c\sqrt{2\pi}}. \tag{14}$$

They allow us to simplify:

$$\begin{aligned}
A\left(z\right) &=_{\text{by (13)}} \int_{-\infty}^{\infty} z\Phi\left(-y\frac{z}{\sqrt{2-z^2}}\right)\Phi(y)dy \\
&= \int_{-\infty}^{\infty} z\left(1-\Phi\left(y\frac{z}{\sqrt{2-z^2}}\right)\right)\Phi(y)dy \\
&=_{\text{by (14)}} \frac{z\sqrt{1+1/\left(\frac{z}{\sqrt{2-z^2}}\right)^2}}{\sqrt{2\pi}}=\frac{1}{\sqrt{\pi}}
\end{aligned}$$

■

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