RATIONALIZABILITY AND FINITE-ORDER IMPLICATIONS OF EQUILIBRIUM

JONATHAN WEINSTEIN AND MUHAMET YILDIZ

Abstract. Present economic theories assume common knowledge of the type structure after specifying the first or the second orders of beliefs. We analyze the set of equilibrium predictions that can be deduced from the knowledge of equilibrium and players' beliefs at finite orders. For generic finite-action games and the games with unidimensional action spaces and single-peaked preferences, we show that, if the space of underlying uncertainty is sufficiently rich, then these equilibrium predictions will be equivalent to the predictions that follow from rationalizability—at every instance in which the players' first-order beliefs are common knowledge. In particular, unless the game is dominance solvable, the equilibrium will be highly sensitive to high orders of beliefs, and present economic theories may be misleading.

Key words: higher-order uncertainty, rationalizability, incomplete information, equilibrium.

JEL Numbers: C72, C73.

"Game theory ... is deficient to the extent it assumes other features to be common knowledge, such as one player's probability assessment about another's preferences or information. I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumption will the theory approximate reality." Wilson (1987)

1. Introduction

Most present economic theories are based on equilibrium analysis of models that are closed after specifying the first and second-order beliefs, i.e., the beliefs about underlying uncertainty and the beliefs about other players' beliefs about underlying uncertainty. These models assume that the specified belief structure is common knowledge, i.e., conditional on the first and the second-order beliefs, all of the players' higher-order beliefs are common knowledge. There is generally no reason to believe that these assumptions are satisfied in the in the actual

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incomplete-information situation modeled. Hence, these theories will be misleading when the impact of higher-order beliefs on equilibrium behavior is large. There are examples that suggest that this might be the case. To overcome this fundamental deficiency, one may want to close the model at higher orders and hence weaken the common knowledge assumption by specifying more orders of beliefs. As Wilson (1987), one may hope that by specifying more and more orders of beliefs, the theory would approximate reality. We will demonstrate that this is not the case. For generic normal-form games,[1] actual predictive power of any closed model will be no more than that of rationalizability, no matter how many orders of beliefs are specified; any prediction that does not follow from rationalizability will be driven by the assumption made when the model is closed.

Consider a situation where players have incomplete information about some payoff-relevant parameter. Each player has a probability distribution about the parameter, which represents his first-order beliefs, a probability distribution about other players' first-order beliefs, which represents his second-order beliefs, and so on. Imagine a researcher who has computed an equilibrium of this game, where a type of a player is an infinite-hierarchy of his beliefs, and would like to make prediction about the action of a player i according to this equilibrium. Fix a type τ_i of player i as his actual type, and write $A_i^1(\tau_i)$ for the set of all actions that are played by some type of i whose first-order beliefs agree with τ_i. This set is the set of actions that the researcher cannot rule out if he only knows the first-order beliefs and assumes that the player plays according to the equilibrium. Similarly, write $A_i^k(\tau_i)$ for the set of actions that the researcher cannot rule out by looking at the first k orders of beliefs. Write $A_i^\infty(\tau_i)$ for the limit of these (decreasing) sets as k approaches infinity, i.e., the set of all actions that cannot be ruled out by the researcher by looking at (arbitrarily many) finite orders of beliefs.

This definition can be put another way. Consider two researchers who agree on the equilibrium played. One researcher is certain that player i is of type τ_i. The other (slightly suspicious) researcher is willing to agree with this assessment for the first k orders of beliefs but does not have any further assumption. The set $A_i^k(\tau_i)$ is precisely the set of actions that will not be ruled out by the second researcher.

In a model that is closed at order k, all higher-order beliefs are determined by the first k orders of beliefs and the assumption that is made when the model

[1]That is, in games where there are no ties and when the other players have more than one strategy, a player's payoffs from a strategy cannot be obtained from a mixture of other strategies. Clearly the complement of this set is of lower dimension, and hence has Lebesgue measure zero.

is closed. We wish to emphasize the sensitivity to the closing assumption. In a given equilibrium, the model predicts a unique action for each possible beliefs at orders 1 through k, namely the equilibrium action for the complete type implied by this beliefs and the closing assumption. But in the general model, every other action in $A_i^k(\tau_i)$ is played by a type whose first k orders of beliefs will be exactly as this type (but will fail the closing assumption.) Therefore, we cannot rule out any action in $A_i^k(\tau_i)$ without resorting to the assumption that is made at closing the model at order k.

In order to avoid a technical issue[2], in this version we will focus analyzing $A_i^k(\tau_i)$ for the case in which, according to τ_i, the players' first-order beliefs happen to be common knowledge. Of course, this means that we will be considering deviations from common knowledge at orders higher than k.

Our main result gives a lower bound for $A_i^k(\tau_i)$. Assume that our fixed equilibrium has full range, i.e., every action is played by some type.[3] For countable-action games, we show that $A_i^k(\tau_i)$ includes all actions which survive the first k iterations of eliminating all action which are never a *strict* best reply under τ_i. In particular, $A_i^\infty(\tau_i)$ includes all actions that survive iterated elimination of actions that cannot be a strict best reply. On the other hand, $A_i^k(\tau_i)$ is a subset of actions that survive the first k iterations of eliminating strictly dominated actions, and hence $A_i^\infty(\tau_i)$ is a subset of rationalizable actions. When there are no ties, these elimination procedures lead to the same outcome, and therefore $A_i^\infty(\tau_i)$ is precisely equal to the set of rationalizable outcomes. We also extend this characterization to games in which the action spaces are one-dimensional compact intervals and preferences are single-peaked with continuous best-reply functions. Of course, these games include many classical economic models, such as Cournot oligopoly, provision of public goods, etc.

To illustrate the main argument in the proof of the lower bound, we now explain why $A_i^1(\tau_i)$ includes all actions that survive the first round of elimination process. Let $\tilde{\tau}_i$ vary over the set of types that agree with τ_i at first order (i.e., concerning the underlying parameter) but may have any beliefs at higher orders (i.e., concerning the other players' type profile.) Our full range assumption implies that there are types $\tilde{\tau}_i$ with any beliefs whatsoever about other players' equilibrium action profile. Given any action a_i of i that is a strict best reply to his fixed belief about the parameter and some belief about the other players' actions, there is a type $\tilde{\tau}_i$ who has these beliefs in equilibrium,

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[2]There is a proof that would work for arbitrary games and types but suffers from possible non-measurability of certain function.

[3]This assumption will hold if the parameter space representing the underlying uncertainty is rich enough.

and therefore must play the strict best reply, a_i, in equilibrium. This argument will be formalized as part of an inductive proof of the main result.

For generic games, the above procedures are equivalent, and we have a characterization. Nevertheless, usually, when we fix a game tree, there will be ties in the normal-form representation. In that case, these procedures may differ significantly. While the set of rationalizable strategies remains large, our elimination procedure will lead to a subset of the result of iterated admissibility, i.e., iterated elimination of weakly dominated strategies . In perfect information games, iterated admissibility yields backwards induction, and hence our bounds will not be powerful. In such games, using our techniques one can find a characterization for sequential equilibria in terms of iterated admissibility. Of course, such a characterization will be a positive result.

We extend our result beyond the full-range assumption. To do this, given any subset of the range of equilibrium, we consider a new iterative process that starts with this subset. At each iteration we consider the set of all strict best responses against beliefs with supports in the previous set.[4] Now $A_i^k(\tau_i)$ will include all actions available at kth iteration. Notice that this is not an elimination procedure and the sets may easily grow, yielding a large lower bound. As an example, we consider Cournot oligopoly with a general inverse-demand function with usual regularity conditions and sufficiently many firms with identical constant (positive) marginal cost. We assume that demand depends on a real-valued parameter θ such that a firm's best reply is continuous and increasing with respect to θ. Consider again the confident researcher and his slightly skeptical friend. The former is confident that it is common knowledge that $\theta = \bar{\theta}$, while the latter is only willing to concede that it is common knowledge that $\left|\theta - \bar{\theta}\right| \leq \varepsilon$ and agrees with the kth-order mutual knowledge of $\theta = \bar{\theta}$. He is an arbitrarily generous skeptic; he is willing to concede the above for arbitrarily small $\varepsilon > 0$ and arbitrarily large finite k. We show that the skeptic nonetheless cannot rule out any output level that is not strictly dominated.

Our results have two implications. Firstly, while there are epistemic foundations for iterative admissibility (Brandenburger and Keisler (2002)) and rationalizability (Bernheim (1985), Pearce (1985)), the epistemic foundations for Nash equilibrium in general games are very weak (Aumann and Brandenburger (1995)). Despite this, equilibrium analysis is frequently used because it offers much sharper predictions. Our result shows that this predictive power is deceptive because it requires the full power of the (common-knowledge) assumption made in closing the model; if we weaken that assumption, however slightly, equilibrium is no more powerful than iterated admissibility.

[4]These sets are analyzed also by Milgrom and Roberts (1991).

Secondly, our result points to a close link between the equilibrium impact of higher-order uncertainty and higher-order reasoning. For generic games, assuming kth-order mutual knowledge of payoffs and that a fixed equilibrium (with full support) is played is equivalent to assuming kth-order mutual knowledge of rationality and common knowledge of payoffs.[5] This implies that when the equilibrium impact of high-order uncertainty is large, the impact of high-order failures of rationality is also large, suggesting that predictions will be highly unreliable without a very accurate knowledge of players' reasoning capacity.

2. Basic Definitions and Preliminary Results

Notation 1. Given any list $X_1, X_2, \ldots$ of sets, write $X_{-i} = \prod_{j \neq i} X_j$, $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots) \in X_{-i}$, and $(x_i, x_{-i}) = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots)$. Likewise, for any family of functions $f_j : X_j \to Y_j$, we define $f_{-i} : X_{-i} \to Y_{-i}$ by $f_{-i}(x_{-i}) = (f_j(x_j))_{j \neq i}$. Given any metric space (X, d), write $\Delta(X)$ for the space of probability distributions on X, suppressing the fixed σ-algebra on X which at least contains all open sets and singletons; when we use product spaces, we will always use the product σ-algebra. We will write δ_x for the probability distribution that puts probability 1 on $\{x\}$.

We consider a game with finite set of players $N = \{1, 2, \ldots, n\}$. The source of underlying uncertainty is a payoff-relevant parameter $\theta \in \Theta$ where (Θ, d) is a compact, complete and separable metric space, where d is a metric on set Θ. Each player i has action space A_i and utility function $u_i : \Theta \times A \to \mathbb{R}$ where $A = \prod_i A_i$. We endow the game with the universal type space of Brandenburger and Dekel (1993), a variant of an earlier construction by Mertens and Zamir (1985). Types are defined using the auxiliary sequence $\{X_k\}$ of sets defined inductively by $X_0 = \Theta$ and $X_k = [\Delta(X_{k-1})]^n \times X_{k-1}$ for each $k > 0$. We endow each X_k with the weak topology and the σ-algebra generated by this topology. A player i's first order beliefs (about the underlying uncertainty θ) are represented by a probability distribution τ_i^1 on X_0, second order beliefs (about all players' first order beliefs and the underlying uncertainty) are represented by a probability distribution τ_i^2 on X_1, etc. Therefore, a *type* τ_i of a player i is a member of $\prod_{k=1}^{\infty} \Delta(X_{k-1})$. Since a player's kth-order beliefs contain information about his lower-order beliefs, we need the usual coherence requirements. We write $\mathcal{T} = \prod_{i \in N} \mathcal{T}_i$ for the subset of $\left(\prod_{k=1}^{\infty} \Delta(X_{k-1})\right)^n$ in which it is common knowledge that the players' beliefs are coherent, i.e., the players know their own

[5] Of course, there is a close relationship between assumptions about rationality and payoff uncertainty. But this relationship is not straightforward. In non-generic games, A_i^{∞} differs from rationalizability, and the rationality assumption for iterative admissibility cannot be common knowledge (Brandenburger and Keisler (2002)), while payoffs can be common knowledge in our model.

beliefs and their marginals from different orders agree. We will use the variables $\tau_i, \tilde{\tau}_i \in \mathcal{T}_i$ as generic types of any player i and $\tau, \tilde{\tau} \in \mathcal{T}$ as generic type profiles. For every $\tau_i \in \mathcal{T}_i$, there exists a probability distribution κ_{τ_i} on $\Theta \times \mathcal{T}_{-i}$ such that

$$\tau_i^k = \delta_{\tau_i^{k-1}} \times \mathrm{marg}_{\Theta \times [\Delta(X_{k-2})]^{N \setminus \{i\}}} \kappa_{\tau_i}, \qquad (\forall k) \tag{2.1}$$

and $\tau_i^1 = \mathrm{marg}_\Theta \kappa_{\tau_i}$, where marg denotes the marginal distribution. Conversely, given any distribution κ_{τ_i} on $\Theta \times \mathcal{T}_{-i}$, we can define $\tau_i \in \mathcal{T}_i$ via (2.1). A *strategy* of a player i is any measurable function $s_i : \mathcal{T}_i \to A_i$. Given any type τ_i and any profile s_{-i} of strategies, we write $\pi(\cdot|\tau_i, s_{-i}) \in \Delta(\Theta \times A_{-i})$ for the joint distribution of the underlying uncertainty and the other players' actions induced by τ_i and s_{-i}. Formally, $\pi(\cdot|\tau_i, s_{-i}) = \kappa_{\tau_i} \circ \beta^{-1}$ where $\beta : (\theta, \tau_{-i}) \mapsto (\theta, s_{-i}(\tau_{-i}))$.

Definition 1 (Best Reply)**.** *For each $i \in N$ and for each belief $\pi \in \Delta(\Theta \times A_{-i})$, we write $BR_i(\pi)$ for the set of maximizers of $E_\pi[u_i(\theta, a_i, a_{-i})]$ over $a_i \in A_i$, where E_π is the expectation operator with respect to π. When $BR_i(\pi)$ is singleton, with slight abuse of notation, we write $BR_i(\pi)$ for its unique member. We sometimes write $BR_i(\theta^i, \sigma_{-i})$ for $BR_i(\delta_{\theta^i} \times \sigma_{-i})$.*

A strategy profile $s^* = (s_1^*, s_2^*, \ldots)$ is a *Bayesian Nash equilibrium* iff at each τ_i,

$$s_i^*(\tau_i) \in BR_i\left(\pi\left(\cdot|\tau_i, s_{-i}^*\right)\right).$$

An equilibrium s^* is said to have *full range* iff

$$s^*(\mathcal{T}) = A. \tag{(FR)}$$

The following assumption implies that every equilibrium s^* has full range.

Assumption 1 (Richness of Θ)**.** *Given any $i \in N$, any $\mu \in \Delta(A_{-i})$, and any a_i, there exists $\nu \in \Delta(\Theta)$, such that*

$$BR_i(\nu \times \mu) = \{a_i\}.$$

Lemma 1. *Under Assumption 1, every equilibrium s^* has full range.*

Proof. The proofs that are omitted in the text are in the appendix. □

Notation 2. For some arbitrary $(\theta^1, \ldots, \theta^n) \in \Theta^N$, write τ_i^{CK} for the type of a player i who is certain that it is common knowledge that each player j is certain that $\theta = \theta^j$.[6] For notational convenience we will suppress the dependence on $(\theta^1, \ldots, \theta^n)$.

[6] The assumption that the players' first-order beliefs are of this degenerate form is only for the sake of simplicity. As long as the players' first order beliefs are common knowledge, our results remain valid.

We are interested in how equilibrium is robust against the failure of common knowledge assumption in high orders. We now formalize our notion of robustness.

Equilibrium prediction with finite-order information of payoffs. Let us fix an equilibrium s^* and a type τ_i^{CK} of a player i, who believes that it is common knowledge that each player j is certain that $\theta = \theta^j$ for some $\left(\theta^1, \ldots, \theta^n\right)$. According to equilibrium, he will play $s_i^* \left(\tau_i^{CK}\right)$. Now imagine a researcher who only knows the first kth-order beliefs of player i and knows that equilibrium s^* is played. All the researcher can conclude from this information is that i will play one of the actions in

$$A_i^k \left[s^*, \tau_i^{CK}\right] = \left\{ s_i^* \left(\tau_i\right) | \tau_i^l = \left(\tau_i^{CK}\right)^l \quad \forall l \leq k \right\}.$$

Assuming plausibly that a researcher can verify only finite orders of a player's beliefs, all a researcher can ever know is that player i will play one of the actions in

$$A_i^\infty \left[s^*, \tau_i^{CK}\right] = \bigcap_{k=1}^{\infty} A_i^k \left[s^*, \tau_i^{CK}\right].$$

It turns out that the set $A_i^\infty \left[s^*, \tau_i^{CK}\right]$ is closely related to the set of rationalizable actions. To establish this close link, we will now define rationalizability and a strong version of iterative admissibility at instances in which the players' perceptions of underlying uncertainty is common knowledge.

Rationalizability and Iterated Admissibility. We will define the rationalizable actions for a type τ_i^{CK}. This set will be equal to the usual rationalizable actions of player i for the game in which it is common knowledge that each player j is certain that $\theta = \theta^j$. Define sets $S_j^k \left[\tau_i^{CK}\right]$, $j \in N$, $k = 0, 1, \ldots$, iteratively as follows. Set $S_j^0 \left[\tau_i^{CK}\right] = A_j$. For each $k > 0$, let Σ_{-j}^{k-1} be the set of all probability distributions on $S_{-j}^{k-1} \left[\tau_i^{CK}\right]$, i.e., the set of all possible beliefs of player j on other players' allowable actions that are not eliminated in the first $k-1$ rounds. Write

$$S_j^k \left[\tau_i^{CK}\right] = \bigcup_{\sigma_{-j} \in \Sigma_{-j}^{k-1}} BR_j \left(\theta^j, \sigma_{-j}\right)$$

for the set of all all actions a_j of j that are best reply against some of his beliefs in Σ_{-j}^{k-1}. The set of all rationalizable actions for player i (with type τ_i^{CK}) is

$$S_i^\infty \left[\tau_i^{CK}\right] = \bigcap_{k=0}^{\infty} S_i^k \left[\tau_i^{CK}\right].$$

Next we define the set of strategies that survive iterative elimination of strategies that are never strict best reply, denoted by $W^\infty \left[\tau_i^{CK}\right]$, similarly. We set

$W_j^0\left[\tau_i^{CK}\right] = A_j$,

$$W_j^k\left[\tau_i^{CK}\right] = \left\{a_j | BR_j\left(\theta^j, \sigma_{-j}\right) = \{a_j\} \text{ for some } \sigma_{-j} \in \Delta\left(W_{-j}^{k-1}\left[\tau_i^{CK}\right]\right)\right\},$$

and

$$W_i^\infty\left[\tau_i^{CK}\right] = \bigcap_{k=0}^{\infty} W_i^k\left[\tau_i^{CK}\right].$$

Notice that we eliminate a strategy if it is not a strict best-response to any belief on the remaining strategies of the other players. Clearly, this yields a smaller set than the result of iterative admissibility (i.e., iterative elimination of weakly dominant strategies).[7] In some games, iterative admissibility may yield strong predictions. For example, in finite perfect information games it leads to backward induction outcomes. Nevertheless, in generic normal-form games all these concepts are equivalent and usually have weak predictive power.

Lemma 2. *For finite-action games in normal form, if the payoffs are generic under* τ_i^{CK}*, then*

$$W^\infty\left[\tau_i^{CK}\right] = S^\infty\left[\tau_i^{CK}\right].$$

Our next result states that, knowing finitely many orders of a player's beliefs and the equilibrium, a researcher can predict that a rationalizable strategy will be played.

Proposition 1. *For any equilibrium* s^**, any player* i*, and any* τ_i^{CK} *as in Notation 2,*

$$A_i^k\left[s^*, \tau_i^{CK}\right] \subseteq S_i^k\left[\tau_i^{CK}\right] \qquad \forall k \geq 0;$$

in particular,

$$A_i^\infty\left[s^*, \tau_i^{CK}\right] \subseteq S_i^\infty\left[\tau_i^{CK}\right].$$

Proof. For $k = 0$, the proposition is true by definition. Assume that it is true for some $k-1 \geq 0$, i.e., for any i, $A_{-i}^{k-1}\left[s^*, \tau_i^{CK}\right] \subseteq S_{-i}^{k-1}\left[\tau_i^{CK}\right]$. Now take any τ_i with $\tau_i^l = \left(\tau_i^{CK}\right)^l$ for all $l \leq k$. Under τ_i, player i is certain that $\theta = \theta^i$ and $\tau_{-i}^l = \left(\tau_{-i}^{CK}\right)^l$ for each $l \leq k-1$. Hence, by induction hypothesis, he is certain that $s_{-i}^*(\tau_{-i}) \in S_{-i}^{k-1}\left[\tau_i^{CK}\right]$. Thus, $\pi\left(\cdot|\tau_i, s_{-i}^*\right) = \delta_{\theta^i} \times \mu$ for some $\mu \in \Delta\left(A_{-i}^{k-1}\left[s^*, \tau_i^{CK}\right]\right)$. Therefore,

$$s_i^*(\tau_i) \in BR_i\left(\pi\left(\cdot|\tau_i, s_{-i}^*\right)\right) \subseteq S_i^k\left[\tau_i^{CK}\right],$$

showing that $A_i^k\left[s^*, \tau_i^{CK}\right] \subseteq S_i^k\left[\tau_i^{CK}\right]$. □

[7]In particular, if we use non-reduced normal-form of an extensive-form game, many strategies will be outcome equivalent, in which case our procedure will eliminate all of these strategies. To avoid such over-elimination, we can use reduced-form, by representing all outcome-equivalent strategies by only one strategy.

Since the set of rationalizable strategies is typically very large, it is not surprising that, using the knowledge of finite orders of beliefs and equilibrium, a researcher can predict that a rationalizable action will be played. Nevertheless, we will illustrate next that, typically, this is all a researcher can deduce.

3. Countable-action games

We will now consider the countable-action games and show that equilibrium predictions with the knowledge of finite-order beliefs cannot be sharper than that of iterative elimination of strategies that are never strict best-response. Since the latter is equivalent to rationalizability for generic games, this will yield a characterization for these games. (A game is said to be a *countable-action game* iff A_i is countable or finite for each $i \in N$.)

Proposition 2. *For any countable-action game, any equilibrium s^* with full range, any $k \in \mathbb{N}$, $i \in N$, and any τ_i^{CK} as in Notation 2,*

$$W_i^k \left[\tau_i^{CK}\right] \subseteq A_i^k \left[s^*, \tau_i^{CK}\right];$$

in particular,

$$W_i^\infty \left[\tau_i^{CK}\right] \subseteq A_i^\infty \left[s^*, \tau_i^{CK}\right].$$

Proof. We will use mathematical induction. For $k = 0$, the statement is given by the full-range assumption. Assume that the proposition is true for some k, i.e.,

$$W_j^k \left[\tau_i^{CK}\right] \subseteq A_j^k \left[s^*, \tau_i^{CK}\right] \qquad (\forall j)$$

for some k. We will fix arbitrary i and show that

$$W_i^{k+1} \left[\tau_i^{CK}\right] \subseteq A_i^{k+1} \left[s^*, \tau_i^{CK}\right].$$

Take any $a_i \in W_i^{k+1} \left[\tau_i^{CK}\right]$. By definition, there exists a probability distribution σ_{-i} on $W_{-i}^k \left[\tau_i^{CK}\right]$ such that

$$BR_i \left(\theta^i, \sigma_{-i}\right) = \{a_i\}. \tag{3.1}$$

By induction hypothesis, for each $a_{-i} \in W_{-i}^k \left[\tau_i^{CK}\right]$, there also exists $\tau_{-i}[a_{-i}] \in \mathcal{T}_{-i}$ such that

$$s_{-i}^* \left(\tau_{-i}[a_{-i}]\right) = a_{-i} \tag{3.2}$$

and

$$\tau_{-i}^l [a_{-i}] = \left(\tau_{-i}^{CK}\right)^l \quad \forall l \leq k. \tag{3.3}$$

Now, let τ_i be the type of player i who is certain that $\theta = \theta^i$ and assigns probability $\sigma_{-i}(a_{-i})$ to each type $\tau_{-i}[a_{-i}]$, i.e.,

$$\kappa_{\tau_i} = \delta_{\theta^i} \times \left(\sum_{a_{-i} \in W^k_{-i}\left[\tau_i^{CK}\right]} \sigma_{-i}(a_{-i})\, \delta_{\tau_{-i}[a_{-i}]} \right).$$

Firstly, τ_i and τ_i^{CK} agree up to the $k+1$st order. To see this, note first that $\tau_i^1 = \delta_{\theta^i} = \left(\tau_i^{CK}\right)^1$. Moreover, for any l with $1 < l \leq k+1$, we have

$$\begin{aligned} \mathrm{marg}_{[\Delta(X_{l-2})]^{N\setminus\{i\}}} \kappa_{\tau_i} &= \sum \sigma_{-i}(a_{-i})\, \mathrm{marg}_{[\Delta(X_{l-2})]^{N\setminus\{i\}}} \delta_{\tau_{-i}[a_{-i}]} = \sum \sigma_{-i}(a_{-i})\, \delta_{\tau_{-i}^{l-1}[a_{-i}]} \\ &= \sum \sigma_{-i}(a_{-i})\, \delta_{\left(\tau_{-i}^{CK}\right)^{l-1}} = \delta_{\left(\tau_{-i}^{CK}\right)^{l-1}}, \end{aligned}$$

where the summations are taken over $a_{-i} \in W^k_{-i}\left[\tau_i^{CK}\right]$, and the third equality is due to (3.3). Hence, by (2.1),

$$\tau_i^l = \delta_{\theta^i} \times \delta_{\tau_i^{l-1}} \times \delta_{\left(\tau_{-i}^{CK}\right)^{l-1}} = \left(\tau_i^{CK}\right)^l.$$

But, under τ_i, $s^*_{-i}(\tau_{-i})$ is distributed with σ_{-i}, i.e., the joint distribution of θ and $s^*_{-i}(\tau_{-i})$ is

$$\pi\left(\cdot|\tau_i, s^*_{-i}\right) = \delta_{\theta^i} \times \sum \sigma_{-i}(a_{-i})\, \delta_{s^*_{-i}(\tau_{-i}[a_{-i}])} = \delta_{\theta^i} \times \sum \sigma_{-i}(a_{-i})\, \delta_{a_{-i}},$$

where the summations are taken over $a_{-i} \in W^k_{-i}\left[\tau_i^{CK}\right]$ and the second equality is by (3.2). Therefore,

$$s^*_i(\tau_i) \in BR_i\left(\pi\left(\cdot|\tau_i, s^*_{-i}\right)\right) = BR_i\left(\theta^i, \sigma_{-i}\right) = \{a_i\}.$$

This shows that $a_i = s^*_i(\tau_i) \in A_i^{k+1}\left[s^*, \tau_i^{CK}\right]$, and thus $W_i^{k+1}\left[\tau_i^{CK}\right] \subseteq A_i^{k+1}\left[s^*, \tau_i^{CK}\right]$. □

For generic finite-action games, Propositions 1 and 2 yield a characterization.

Proposition 3. *Under the notation and the assumptions of Proposition 2, if the payoffs are generic under τ_i^{CK}, then*

$$A_i^k\left[s^*, \tau_i^{CK}\right] = S_i^k\left[\tau_i^{CK}\right].$$

That is, in generic finite-action games, a researcher's predictions based on finite orders of players' beliefs and equilibrium will be equivalent to the predictions that follow from rationalizability. For non-generic games, we have weaker conclusions:

$$W_i^\infty\left[\tau_i^{CK}\right] \subseteq A_i^\infty\left[s^*, \tau_i^{CK}\right] \subseteq S_i^\infty\left[\tau_i^{CK}\right].$$

4. Nice games

We will now consider a class of "nice" games, which are widely used in economic theory, and show that $A_i^k\left[s^*,\tau_i^{CK}\right] = S_i^k\left[\tau_i^{CK}\right]$ for each k whenever equilibrium s^* has full range.

Definition 2. *A game is said to be* nice *iff for each* i, $A_i = [0,1]$ *and* $u_i(\theta,\cdot,a_{-i})$ *is a single-peaked function maximized at some* $BR_i(\theta,a_{-i})$ *where* $BR_i(\theta,a_{-i})$ *is continuous in* a_{-i}.[8]

Our first lemma presents simple but very useful properties of rationalizable strategies in nice games.

Lemma 3. *For any nice game and any* τ_i^{CK} *as in Notation 2, the following are true.*

(1) *Each* $S_i^k\left[\tau_i^{CK}\right]$ *is a closed interval in* A_i.
(2) *For each* $a_i^k \in S_i^k\left[\tau_i^{CK}\right]$, *there exists* $a_{-i}^{k-1} \in S_{-i}^{k-1}\left[\tau_i^{CK}\right]$ *such that*

$$a_i^k = \{BR_i\left(\theta^i, a_{-i}^{k-1}\right)\}.$$

Lemma 3.2 implies that a researcher's predictions about the equilibrium behavior with only finite-order information of beliefs will be equivalent to predictions that follow from rationalizability—as our next proposition establishes.

Proposition 4. *For any nice game, any equilibrium* s^* *with full range, for any* $k \in \mathbb{N}$, $i \in N$, *and* τ_i^{CK} *as in Notation 2,*

$$S_i^k\left[\tau_i^{CK}\right] = A_i^k\left[s^*,\tau_i^{CK}\right];$$

in particular,

$$S_i^\infty\left[\tau_i^{CK}\right] = A_i^\infty\left[s^*,\tau_i^{CK}\right].$$

Proof. Proposition 1 establishes already that $A_i^k\left[s^*,\tau_i^{CK}\right] \subseteq S_i^k\left[\tau_i^{CK}\right]$. We will now prove the reverse inequality using symmetric arguments to the proof of Proposition 2. Assume that $S_{-i}^{k-1}\left[\tau_i^{CK}\right] \subseteq A_{-i}^{k-1}\left[s^*,\tau_i^{CK}\right]$. Then, Lemma 3.2 states that each $a_i^k \in S_i^k\left[\tau_i^{CK}\right]$ is a unique best reply to some $a_{-i}^{k-1} = s_{-i}^*(\tau_{-i})$ where $\tau_{-i}^l = \left(\tau_{-i}^{CK}\right)^l$ for each $l \le k-1$. Now consider the type τ_i that puts point mass at $\left(\theta^i,\tau_{-i}\right)$. Then,

$$s_i^*(\tau_i) \in BR_i\left(\theta^i, s_{-i}^*(\tau_{-i})\right) = BR_i\left(\theta^i, a_{-i}^k\right) = \left\{a_i^k\right\}.$$

But, by definition of τ^{CK}, we have $\tau_i^l = \left(\tau_i^{CK}\right)^l$ for each $l \le k$, showing that $a_i^k \in S_i^k\left[\tau_i^{CK}\right]$, i.e., $S_i^k\left[\tau_i^{CK}\right] \subseteq A_i^k\left[s^*,\tau_i^{CK}\right]$. □

[8] This assumption is satisfied if u_i is strictly quasi-concave in a_i and continuous in a.

5. Extensions

We have so far assumed that the equilibrium at hand has full range, which allowed us to consider large changes. A researcher may be certain that it is common knowledge that the set of parameters are restricted to a small subset, or equivalently, the equilibrium considered may not vary much as the beliefs about the underlying uncertainty change. We will now present extensions of our result to such cases.

Local Rationalizability. Take any τ^{CK} and any $B \subset A$ such that $s^*\left(\tau^{CK}\right) \in B$. We Define sets $S_i^k\left[B;\tau_i^{CK}\right]$, $i \in N$, $k \in \mathbb{N}$, by setting

$$\begin{aligned} S^0\left[B;\tau_i^{CK}\right] &= B, \\ S_i^k\left[B;\tau_i^{CK}\right] &= \bigcup_{\sigma_{-i}\in\Delta\left(S_{-i}^{k-1}\left[B;\tau_i^{CK}\right]\right)} BR_i\left(\theta^i,\sigma_{-i}\right). \end{aligned}$$

Notice that this is the same procedure as iterated strict dominance, except that the initial set is restricted to a subset. Unlike iterated strict dominance, these sets can become larger as k increases. Hence we define the set of *locally rationalizable* strategies by

$$S^\infty\left[B;\tau_i^{CK}\right] = \bigcap_{k=0}^{\infty}\bigcup_{l=k}^{\infty} S^l\left[B;\tau_i^{CK}\right].$$

Notice that the set $S^\infty\left[B;\tau_i^{CK}\right]$ may be much larger than B. We define local version of W^∞, similarly, by setting $W^0\left[B;\tau_i^{CK}\right] = B$,

$$W_i^k\left[B;\tau_i^{CK}\right] = \left\{a_i \in A_i | BR_i\left(\theta^i,\sigma_{-i}\right) = \{a_i\} \text{ for some } \sigma_{-i} \in \Delta\left(W_{-i}^{k-1}\left[B;\tau_i^{CK}\right]\right)\right\},$$

and $W^\infty\left[B;\tau_i^{CK}\right] = \bigcap_{k=0}^{\infty}\bigcup_{l=k}^{\infty} W^l\left[B;\tau_i^{CK}\right]$. Notice that we consider all actions in our elimination process.

Proposition 5. *Let s^* be an equilibrium, $i \in N$ and τ_i^{CK} be as in Notation 2. If the game has countable actions, then*

$$W^\infty\left[s^*\left(\mathcal{T}\right);\tau_i^{CK}\right] \subseteq A^\infty\left[s^*,\tau_i^{CK}\right] \subseteq S^\infty\left[s^*\left(\mathcal{T}\right);\tau_i^{CK}\right].$$

If the game is nice or a generic finite-action game (under τ_i^{CK}), then for any $B \subseteq s^*\left(\mathcal{T}\right)$,

$$S^\infty\left[B;\tau_i^{CK}\right] \subseteq A^\infty\left[s^*,\tau_i^{CK}\right].$$

The last statement implies that, for nice and finite-action games, even the slight changes in very higher-order beliefs will have substantial impact on equilibrium behavior, unless the game is locally dominance-solvable, i.e., $S^\infty\left[B;\tau_i^{CK}\right] = \left\{s^*\left(\tau^{CK}\right)\right\}$ for some open neighborhood B of $s^*\left(\tau^{CK}\right)$. In our next section, we will show that many important games lack local dominance-solvability, and hence anything less than common knowledge of players' perception will lead to substantially different outcomes.

6. Applications

6.1. **Cournot Oligopoly.** First consider the linear case. When we have only a duopoly, the game is dominant-solvable, and hence Proposition 1 implies that higher-order beliefs have negligible impact on equilibrium whenever the players' first-order beliefs turn out to be common knowledge. Weinstein and Yildiz (2003) shows that this is in fact the case for entire type space, a fact that is also implied by a result of Nyarko (1996). On the other hand, when $n \geq 3$, any production level that is less than or equal to the monopoly production is rationalizable, and hence Proposition 4 implies that a researcher cannot rule out any such output level for a firm no matter how many orders of beliefs he specifies. We will now show a more disturbing fact. For fairly general oligopoly models we will show that when n is sufficiently large, any such outcome will be in $S_i^\infty \left[B; \tau_i^{CK}\right]$ for every neighborhood B of $s^*\left(\tau^{CK}\right)$. Therefore, by Proposition 5, even a slight doubt about the model in very high orders will lead a researcher to fail to rule out any outcome that is less than monopoly outcome as a firm's equilibrium output.

6.1.1. ***General Cournot Model.*** Consider n firms with identical constant marginal cost $c > 0$. Simultaneously, each firm i produces $q_i \in [0,1]$ at cost $q_i c$ and sell its output at price $P(Q;\theta)$ where $Q = \sum_i q_i$ is the total supply. (Here we impose capacity constraints in order to be consistent with our general model. We will focus on the cases in which these constraints are not binding.) For some fixed $\bar{\theta}$, we assume that Θ is a closed interval with $\bar{\theta} \in \Theta \neq \left\{\bar{\theta}\right\}$. We also assume that $P\left(0;\bar{\theta}\right) > 0$, $P\left(\cdot;\bar{\theta}\right)$ is strictly decreasing when it is is positive, and $\lim_{Q\to\infty} P\left(Q;\bar{\theta}\right) = 0$. Therefore, there exists a unique $\hat{Q}$ such that

$$P\left(\hat{Q};\bar{\theta}\right) = c.$$

We assume that, on $[0,\hat{Q}]$, $P\left(\cdot;\bar{\theta}\right)$ is continuously twice-differentiable and

$$P' + QP'' < 0.$$

It is well known that, under the assumptions of the model, (i) the profit function, $u\left(q, Q;\bar{\theta}\right) = q\left(P\left(q+Q\right) - c\right)$, is strictly concave in own output q; (ii) the unique best response $q^*\left(Q_{-i}\right)$ to others' aggregate production Q_{-i} is strictly decreasing on $[0,\hat{Q}]$ with slope bounded away from 0 (i.e., $\partial q^*/\partial Q_{-i} \leq \lambda$ for some $\lambda < 0$); (iii) equilibrium outcome at $\tau^{CK}\left(\bar{\theta},\ldots\bar{\theta}\right)$, $s^*\left(\tau^{CK}\left(\bar{\theta},\ldots\bar{\theta}\right)\right)$, is unique and symmetric (Okuguchi and Suzumura (1971)).

Lemma 4. *In the general Cournot model, for any equilibrium s^* and for τ^{CK} under which it is common knowledge that $\theta = \bar{\theta}$ for arbitrary $\bar{\theta}$, there exists $\bar{n} < \infty$ such that for any $n > \bar{n}$ and for any $B = \prod_{i \in N} \left[s^*\left(\tau^{CK}\right) - \epsilon, s^*\left(\tau^{CK}\right) + \epsilon\right] \subset A$ with $\epsilon > 0$, we have*

$$S_i^\infty \left[B; \tau^{CK}\right] = \left[0, q^M\right] \quad (\forall i \in N),$$

where $q^M \in [0,1]$ is the monopoly outcome under $P\left(\cdot; \bar{\theta}\right)$.

Proof. Let $\bar{n}$ be any integer greater than $1 + 1/|\lambda|$, where λ is as in (ii). Take any $n > \bar{n}$. By (iii), $B = [\underline{q}^0, \bar{q}^0]^n$ for some $\underline{q}^0, \bar{q}^0 \in [0,1]$ with $\underline{q}^0 < \bar{q}^0$. By (ii), for any $k > 0$, $S^k\left[B; \tau^{CK}\right] = [\underline{q}^k, \bar{q}^k]^n$, where

$$\bar{q}^k = q^*\left((n-1)\,\underline{q}^{k-1}\right) \text{ and } \underline{q}^k = q^*\left((n-1)\,\bar{q}^{k-1}\right).$$

Define $\underline{Q}^k \equiv (n-1)\,\underline{q}^k$, $\bar{Q}^k \equiv (n-1)\,\bar{q}^k$, and $Q^* = (n-1)\,q^*$, so that

$$\bar{Q}^k = Q^*\left(\underline{Q}^{k-1}\right) \text{ and } \underline{Q}^k = Q^*\left(\bar{Q}^{k-1}\right).$$

Since the slope of Q^*, $(n-1)\,\lambda$, is strictly less than 1, $\underline{Q}^k$ decreases with k and becomes 0 at some finite $\bar{k}$, and $\bar{Q}^k$ increases with k and takes value $Q^*(0) = (n-1)\,q^M$ at $\bar{k}+1$. That is, $S^k\left[B; \tau^{CK}\right] = \left[0, q^M\right]^n$ for each $k > \bar{k}$. Therefore, $S^\infty\left[B; \tau^{CK}\right] = \left[0, q^M\right]^n$. □

Together with Proposition 5, this lemma yields the following.

Proposition 6. *In the general Cournot model, assume that $\Theta = \left[\bar{\theta} - \varepsilon, \bar{\theta} + \varepsilon\right]$ for arbitrarily small $\varepsilon > 0$, and that the best-response function $q^*\left(Q_{-i}; \theta\right)$ is a continuous and strictly increasing function of θ at $\left(Q_{-i}, \bar{\theta}\right)$ where $Q_{-i} = (n-1)\,s_j^*\left(\tau^{CK}\right)$ is the others' aggregate output in equilibrium, and τ^{CK} is the type profile under which it is common knowledge that $\theta = \bar{\theta}$. Then,*

$$A_i^\infty\left[s^*, \tau_i^{CK}\right] = \left[0, q^M\right] \quad (\forall i \in N),$$

where $q^M \in [0,1]$ is the monopoly outcome under $P\left(\cdot; \bar{\theta}\right)$.

Proof. Firstly, by (i) above, we have a nice game. Moreover, by the assumption in the hypothesis, there exists $B \subset s^*(T)$ as in Lemma 4. Hence, Lemma 4 and Proposition 5 imply

$$\left[0, q^M\right] = S_i^\infty\left[B; \tau_i^{CK}\right] \subseteq A_i^\infty\left[s^*, \tau_i^{CK}\right] \subseteq \left[0, q^M\right],$$

yielding the desired equality. □

In Proposition 6, the assumption that $q^*\left(Q_{-i}; \theta\right)$ is responsive to θ guaranties that θ is a payoff-relevant parameter. Hence, our proposition can be spelled out as follows. Assume that it is actually common knowledge among the firms that a payoff-relevant parameter θ takes value $\bar{\theta}$. Imagine a researcher who does not know this but is certain that it is common knowledge that θ is in some arbitrarily give neighborhood of $\bar{\theta}$. Hence the researcher has a very slight uncertainty about the complete information case. Suppose in addition the researcher can learn the players' beliefs at arbitrary finite orders. If there are sufficiently many firms in the market, no matter how many orders of beliefs the researcher correctly specifies, he will not be able to rule out any output level that is not strictly dominated. In that case, any prediction based on equilibrium that is not implied by strict dominance will be invalid whenever we slightly deviate from the idealized complete information model.

Appendix A. Omitted proofs

Proof of Lemma 1. Take any i and any a_i. Take any $\gamma \in \Delta(\mathcal{T}_{-i})$, and let $\mu = \gamma \circ \left(s^*_{-i}\right)^{-1} \in \Delta(A_{-i})$. Let ν be as in Assumption 1. Define τ_i as the type such that $\kappa_{\tau_i} = \nu \times \gamma$. Notice that $\pi\left(\cdot|\tau_i, s^*_{-i}\right) = \kappa_{\tau_i} \circ \beta^{-1} = (\nu \times \gamma) \circ \beta^{-1} = \nu \times \left(\gamma \circ \left(s^*_{-i}\right)^{-1}\right) = \nu \times \mu$. Hence, $s^*(\tau_i) = BR_i\left(\pi\left(\cdot|\tau_i, s^*_{-i}\right)\right) = BR_i(\nu \times \mu) = a_i$. □

Proof of Lemma 3. We will use induction on k. For $k = 0$, part 1 is true by definition. Assume that part 1 is true for some $k-1$, i.e., $S^{k-1}_j\left[\tau^{CK}_i\right]$ is a closed subset of $A_j = [0,1]$ for each j. That is, $S^{k-1}_{-i}\left[\tau^{CK}_i\right]$ is a compact set. Hence, there exist $\underline{a}^k_i = \min_{a_{-i} \in S^{k-1}_{-i}\left[\tau^{CK}_i\right]} BR_i\left(\theta^i, a_{-i}\right)$ and $\bar{a}^k_i = \max_{a_{-i} \in S^{k-1}_{-i}\left[\tau^{CK}_i\right]} BR_i\left(\theta^i, a_{-i}\right)$. Since the preferences are single-peaked and all the action profiles outside of $S^{k-1}_{-i}\left[\tau^{CK}_i\right]$ are eliminated for $j \neq i$, $\bar{a}^k_i$ strictly dominates each $a_i > \bar{a}^k_i$, and $\underline{a}^k_i$ strictly dominates each $a_i < \underline{a}^k_i$. Hence, $S^k_i \subseteq \left[\underline{a}^k_i, \bar{a}^k_i\right]$. Moreover, since $S^{k-1}_{-i}\left[\tau^{CK}_i\right]$ convex and $BR_i\left(\theta^i, a_{-i}\right)$ is continuous in a_{-i}, $BR_i\left(\theta^i, S^{k-1}_{-i}\left[\tau^{CK}_i\right]\right)$ is connected. Thus, $\left[\underline{a}^k_i, \bar{a}^k_i\right] \supseteq BR_i\left(\theta^i, S^{k-1}_{-i}\left[\tau^{CK}_i\right]\right) \supseteq S^k_i$. Therefore, $S^k_i = BR_i\left(\theta^i, S^{k-1}_{-i}\left[\tau^{CK}_i\right]\right) = \left[\underline{a}^k_i, \bar{a}^k_i\right]$, a closed interval. Since $S^k_i = BR_i\left(\theta^i, S^{k-1}_{-i}\left[\tau^{CK}_i\right]\right)$, by definition, each $a^k_i \in S^k_i$ is the unique best response to some $a^{k-1}_{-i} \in S^{k-1}_{-i}\left[\tau^{CK}_i\right]$. □

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