Logical Differencing in Dyadic Network Formation Models with Nontransferable Utilities

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Abstract

This paper considers a semiparametric model of dyadic network formation under nontransferable utilities (NTU). NTU arises frequently in real-world social interactions that require bilateral consent, but by its nature induces additive non-separability. We show how unobserved individual heterogeneity in our model can be canceled out without additive separability, using a novel method we call logical differencing. The key idea is to construct events involving the intersection of two mutually exclusive restrictions on the unobserved heterogeneity, based on multivariate monotonicity. We provide a consistent estimator and analyze its performance via simulation, and apply our method to the Nyakatoke risk-sharing networks.

Keywords: dyadic network formation, semiparametric estimation, nontransferable utilities, additive nonseparability, differencing

JEL Codes: C14, C21, C25, D85

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1 Introduction

This paper considers a semiparametric model of dyadic network formation under *nontransferable utilities* (NTU), which arise naturally in the modeling of real-world social interactions that require bilateral consent. For instance, friendship is usually formed only when both individuals in question are willing to accept each other as a friend, or in other words, when both individuals derive sufficiently high utilities from establishing the friendship. It is often plausible that the two individuals may derive very different utilities from the friendship for a variety of reasons: for example, one of them may simply be more introvert than the other and derive lower utilities from the friendship. In addition, there may not be a feasible way to perfectly transfer utilities between the two individuals. Monetary payments may not be customary in many social contexts, and even in the presence of monetary or in-kind transfers, *utilities* may not be perfectly transferable through these feasible forms of transfers, say, when individuals have different marginal utilities with respect to these transfers.[1] Given the considerable academic and policy interest in understanding the underlying drivers of network formation,[2] it is not only theoretically interesting but also empirically relevant to incorporate NTU in the modeling of network formation.

This paper contributes to the line of econometric literature on network formation by introducing and incorporating *nontransferable utilities* into dyadic network formation models.[3] Previous work in this line of literature focuses primarily on case of *transferable utilities*, as represented in Graham (2017), which considers a parametric model with homophily effects and individual unobserved heterogeneity of the following form:

$$D_{ij} = \mathbb{1}\left\{w\left(X_i, X_j\right)' \beta_0 + A_i + A_j \geq U_{ij}\right\} \tag{1}$$

where D_{ij} is an observable binary variable that denotes the presence or absence of a link between individual i and j, $w\left(X_i, X_j\right)$ represents a (symmetric) vector of pairwise observable characteristics specific to ij generated by a known function w of the individual observable

[1] See surveys by Aumann (1967), Hart (1985) and McLean (2002) for discussions on the implications of NTU on link (bilateral relationship) and group formation from a micro-theoretical perspective.

[2] For example, the formation of friendship among U.S. high-school students has been studied by a long line of literature, such as Moody (2001), Currarini, Jackson, and Pin (2009, 2010), Boucher (2015), Currarini et al. (2016), Xu and Fan (2018) among others.

[3] It should be pointed out that the line of econometric literature on *strategic network formation* models, which primarily uses pairwise stability (Jackson and Wolinsky, 1996) as the solution concept for network formation, often builds NTU (along with link interdependence) into the econometric specification from scratch. See, for example, De Paula, Richards-Shubik, and Tamer (2018), Graham (2016), Leung (2015), Menzel (2015), Mele (2017a), Mele (2017b) and Ridder and Sheng (2017). This paper does not belong to that line of literature but instead contributes to the line of econometric literature on dyadic network formation models, which abstracts away from link interdependence but usually incorporates more flexible forms of unobserved individual heterogeneity.

characteristics X_i and X_j of i and j, while A_i and A_j stand for unobserved individual-specific degree heterogeneity and U_{ij} is some idiosyncratic utility shock. Model (1) essentially says that, if the (stochastic) *joint surplus* generated by a bilateral link $s_{ij} := w(X_i, X_j)' \beta_0 + A_i + A_j - U_{ij}$ exceeds the threshold zero, then the link between i and j is formed. The model implicitly assumes that the link surplus can be freely distributed among the two individuals i and j, and that bargaining efficiency is always achieved, so that the undirected link is formed if and only if the link surplus is positive. Given this specification, Graham (2017) provides consistent and asymptotically normal maximum-likelihood estimates for the homophily effect parameters β_0, assuming that the exogenous idiosyncratic pairwise shocks U_{ij} are independently and identically distributed with a logistic distribution. Recently, Candelaria (2016) and Toth (2017) provide semiparametric generalizations of Graham (2017), while Gao (2020) established nonparametric identification of a class of index models that further generalize (1).

This paper, however, generalizes Graham (2017) along a different direction, and seeks to incorporate the natural micro-theoretical feature of NTU into this class of network formation models. To illustrate[4], consider the following simple adaption of model (1) with two threshold-crossing conditions:

$$D_{ij} = \mathbb{1}\left\{w(X_i, X_j)' \beta_0 + A_i \geq U_{ij}\right\} \cdot \mathbb{1}\left\{w(X_i, X_j)' \beta_0 + A_j \geq U_{ji}\right\}, \tag{2}$$

where the unobserved individual heterogeneity A_i and A_j *separately* enter into two different threshold-crossing conditions. This formulation could be relevant to scenarios where A_i represents individual i's own intrinsic valuation of a generic friend: for a relatively shy or introvert person i, a lowerA_i implies that i is less willing to establish a friendship link, regardless of how sociable the counterparty is. For simplicity, suppose for now that $w(X_i, X_j) \equiv \mathbf{0}$ and $U_{ij} \sim_{iid} U_{ji} \sim F$. Focusing completely on the effects of A_i and A_j, it is clear that the TU model (1) implies that only the sum of "sociability", $A_i + A_j$, matters: the linking probability among pairs with $A_i = A_j = 1$ (two moderately social persons) should be exactly the same as the linking probability among pairs with $A_i = 2$ and $A_j = 0$ (one very social person and one very shy person), which might not be reasonable or realistic in social scenarios. In comparison, the linking probability among pairs with $A_i = 2$ and $A_j = 0$ is lower than the linking probability among pairs with $A_i = A_j = 1$ under the NTU model (2) with i.i.d. U_{ij}

[4]Starting from Section 2, we consider a more general specification than the illustrative model (1) introduced here.

and U_{ji} that follow any log-concave distribution[5]:

$$\begin{aligned} & \mathbb{E}\left[D_{ij}\middle| w\left(X_i, X_j\right) \equiv \mathbf{0}, A_i = 2, A_j = 0\right] \\ = & F\left(0\right) F\left(2\right) \\ < & F\left(1\right) F\left(1\right) \\ = & \mathbb{E}\left[D_{ij}\middle| w\left(X_i, X_j\right) \equiv \mathbf{0}, A_i = A_j = 1\right] \end{aligned}$$

This is intuitive given the observation that, under bilateral consent, the party with relatively lower utility is the pivotal one in link formation. Moreover, even though we maintain strict monotonicity in the unobservable characteristics A_i and A_j, the NTU setting can still effectively incorporate homophily effects on unobserved heterogeneity: given that $w\left(X_i, X_j\right) \equiv \mathbf{0}$ and $A_i + A_j = 2$, the linking probability is effectively decreasing in $\left|A_i - A_j\right|$ under log-concave F. Hence, by explicitly modeling NTU in dyadic network formation, we can accommodate more flexible or realistic patterns of conditional linking probabilities and homophily effects that are not present under the TU setting.

However, the NTU setting immediately induces a key technical complication: as can be seen explicitly in model (2), the observable indexes ($W_{ij}'\beta_0$ and $W_{ji}'\beta_0$) and the unobserved heterogeneity terms (A_i and A_j) are no longer additively separable from each other. In particular, notice that, even though the utility specification for each individual inside each of the two threshold-crossing conditions in model (2) remains completely linear and additive, the multiplication of the two (nonlinear) indicator functions directly destroys both linearity and additive separability, rendering inapplicable most previously developed econometric techniques that arithmetically "difference out" the "two-way fixed effects" A_i and A_j based on additive separability.[6]

Given this technical challenge, this paper proposes a new identification strategy termed *logical differencing*, which helps cancel out the unobserved heterogeneity terms, A_i and A_j, without requiring additive separability but leveraging the logical implications of *multivariate monotonicity* in model (2). The key idea is to construct an observable event involving the intersection of two mutually exclusive restrictions on the fixed effects A_i and A_j, which

[5]A distribution is log-concave if $F\left(x\right)^{\lambda} F\left(y\right)^{1-\lambda} \leq F\left(\lambda x + \left(1-\lambda\right) y\right)$. Many commonly used distributions, such as uniform, normal, exponential, logistic, chi-squared distributions, are log-concave. See Bagnoli and Bergstrom (2005) for more details on log-concave distributions from a microeconomic theoretical perspective.

[6]Equivalently, one could write model (2) in an alternative form as a "single" *composite* threshold-crossing condition:

$$D_{ij} = \mathbb{1}\left\{\min\left\{W_{ij}'\beta_0 + A_i - U_{ij}, W_{ji}'\beta_0 + A_j - U_{ji}\right\} \geq 0\right\},$$

where additive separability is again lost in this alternative formulation.

logically imply an event that can be represented without A_i or A_j. Specifically, in the context of the illustrative model (2) above, we start by considering the event where a given individual $\overline{i}$ is *more popular* than another individual $\overline{j}$ among a group of individuals k with observable characteristics $X_k = \overline{x}$ while $\overline{i}$ is simultaneously *less popular* than another individual $\overline{j}$ among a group of individuals with a certain realization of observable characteristics $\underline{x}$. This is the same as the conditioning event in Toth (2017) and analogous to the tetrad comparisons made in Candelaria (2016). However, instead of using arithmetic differencing to cancel out the unobserved heterogeneity $A_{\overline{i}}$ and $A_{\overline{j}}$ as in Candelaria (2016) and Toth (2017), we make the following logical deductions based on the monotonicity of the conditional popularity of $\overline{i}$ in $w\left(X_{\overline{i}}, \overline{x}\right)' \beta_0$ and $A_{\overline{i}}$. First, the event that $\overline{i}$ is *more popular* than another individual $\overline{j}$ among the group of individuals with $X_k = \overline{x}$ implies that either $w\left(X_{\overline{i}}, \overline{x}\right)' \beta_0 > w\left(X_{\overline{j}}, \overline{x}\right)' \beta_0$ or $A_{\overline{i}} > A_{\overline{j}}$, while the event that $\overline{i}$ is *less popular* than another individual $\overline{j}$ among a different group of individuals with $X_l = \underline{x}$ implies that either $w\left(X_{\overline{i}}, \underline{x}\right)' \beta_0 < w\left(X_{\overline{j}}, \underline{x}\right)' \beta_0$ or $A_{\overline{i}} < A_{\overline{j}}$. Second, when both events occur simultaneously, we can logically deduce that either $w\left(X_{\overline{i}}, \overline{x}\right)' \beta_0 > w\left(X_{\overline{j}}, \overline{x}\right)' \beta_0$ or $w\left(X_{\overline{i}}, \underline{x}\right)' \beta_0 < w\left(X_{\overline{j}}, \underline{x}\right)' \beta_0$ must have occurred, because $A_{\overline{i}} > A_{\overline{j}}$ and $A_{\overline{i}} < A_{\overline{j}}$ cannot simultaneously occur. Intuitively, the "switch" in the relative popularity of $\overline{i}$ and $\overline{j}$ among the two groups of individuals with characteristics $\overline{x}$ and $\underline{x}$ cannot be driven by individual unobserved heterogeneity $A_{\overline{i}}$ and $A_{\overline{j}}$, and hence when we indeed observe such a "switch", we obtain a restriction on the parametric indices $w\left(X_{\overline{i}}, \overline{x}\right)' \beta_0$, $w\left(X_{\overline{i}}, \overline{x}\right)' \beta_0$, $w\left(X_{\overline{i}}, \underline{x}\right)' \beta_0$, and $w\left(X_{\overline{j}}, \underline{x}\right)' \beta_0$, which helps identify β_0.

Based on this identification strategy we provide sufficient conditions for point identification of the parameter β_0 up to scale normalization as well as a consistent estimator for β_0. Our estimator has a two-step structure, with the first step being a standard nonparametric estimator of conditional linking probabilities, which we use to assert the occurrence of the conditioning event, while in the second step we use the identifying restriction on β_0 when the conditioning event occurs. The computation of the estimator essentially follows the same method proposed in (Gao and Li, 2020), with some adaptions to the network data setting. We analyze the finite-sample performance in a simulation study, and present an empirical illustration of our method using data from Nyakatoke on risk-sharing network collected by Joachim De Weerdt.

This paper belongs to the line of literature that studies dyadic network formation in a single large network setting, including Blitzstein and Diaconis (2011), Chatterjee, Diaconis, and Sly (2011), Yan and Xu (2013), Yan, Leng, and Zhu (2016), Graham (2017), Charbonneau (2017), Dzemski (2017), Jochmans (2017), Yan, Jiang, Fienberg, and Leng (2018), Candelaria (2016), Toth (2017) and Gao (2020). According to our knowledge Shi and Chen (2016) is the only previous paper that explicitly incorporates NTU into dyadic network formation models, but Shi and Chen (2016) considers a fully parametric model and establishes the consistency and asymptotic normality of the maximum likelihood estimators. In contrast, we consider a semiparametric model here where the functional form of the idiosyncratic shock distribution is left unrestricted.

This paper is also related to a line of research that utilizes the form of link formation models considered here in order to study structural social interaction models: for instance, Arduini et al. (2015), Auerbach (2016), Goldsmith-Pinkham and Imbens (2013), Hsieh and Lee (2016) and Johnsson and Moon (2017). In these papers, the social interaction models are the main focus of identification and estimation, while the link formation models are used mainly as a tool (a control function) to deal with network endogeneity or unobserved heterogeneity problems in the social interaction model. Even though some of the network formation models considered in this line of literature is consistent with the NTU setting, this line of literature is usually not primarily concerned with the full identification and estimation of the network formation model itself.

It should be pointed out that in this paper we do not consider link interdependence in network formation. See Graham (2015), Chandrasekhar (2016) and de Paula (2016) for reviews on the econometric literature on strategic network formation with link interdependence.

This paper is also a companion paper to Gao and Li (2020), which similarly leverages multivariate monotonicity in a multi-index structure under a panel multinomial choice setting, which incorporate rich individual-product specific unobserved heterogeneity in the form of an infinite-dimensional fixed effect that enters into individual's utility functions in an additively nonseparable way. The structural similarity between network data and panel data has long been noted in the econometric literature, but it should also be pointed out that the network structure considered in this paper is technically more complicated than the panel structure, as there are no direct ways in the network setting to make "intertemporal comparison" as in the panel setting that holds the fixed effects unchanged across two observable periods of time. It is precisely this additional complication induced by the network setting that requires the technique of logical differencing proposed in this paper.

The rest of the paper is organized as follows. In Section 2, we describe the general specifications of the dyadic network formation model we consider. Section 3 establishes identification of the parameter of interests in our model, and also provides a consistent tetrad estimator. Simulation results are reported in Section 4. We present an empirical illustration of our method using the risk-sharing data of Nyakatoke in Section 5. We conclude with Section 6. Proofs are available in the Appendix.

2 A Nonseparable Dyadic Network Formation Model

We consider the following dyadic network formation model:

$$\mathbb{E}\left[\left. D_{ij} \right| X_i, X_j, A_i, A_j\right] = \phi\left(w\left(X_i, X_j\right)' \beta_0, A_i, A_j\right) \tag{3}$$

where:

- $i \in \{1, ..., n\}$ denote a generic individual in a group of n individuals.
- X_i is a $\mathbb{R}^{d_x}$-valued vector of observable characteristics for individual i. This could include wealth, age, education and ethnicity of individual i.
- D_{ij} denotes a binary observable variable that indicates the presence or absence of an undirected and unweighted link link between two distinct individuals i and j: $D_{ij} = D_{ji}$ for all pairs of individuals ij, with $D_{ij} = 1$ indicating that ij are linked while $D_{ij} = 0$ indicating that ij are not linked.
- $w : \mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_w}$ is a known function that is *symmetric*[7] with respect to its two vector arguments.
- $\beta_0 \in \mathbb{R}^{d_\beta}$ is an unknown finite-dimensional parameter of interest. Assume $\beta_0 \neq \mathbf{0}$ so that we may normalize $\|\beta_0\| = 1$, i.e., $\beta_0 \in \mathbb{S}^{d_\beta - 1}$.
- A_i is an unobserved scalar-valued variable that represents unobserved individual heterogeneity.
- $\phi : \mathbb{R}^3 \to \mathbb{R}$ is an unknown measurable function that is symmetric with respect to its second and third arguments.

In addition, we impose the following two assumptions:

Assumption 1 (Monotonicity)**.** *ϕ is weakly increasing in each of its arguments.*

Assumption 1 is the key assumption on which our identification analysis is based, which requires that the conditional linking probability between individuals with characteristics (X_i, A_i) and (X_j, A_j) be monotone in a parametric index $\delta_{ij} := w\left(X_i, X_j\right)' \beta_0$ as well as the unobserved individual heterogeneity terms A_i and A_j. It should be noted that, given monotonicity, increasingness is without loss of generality as ϕ, β_0 and A_i, A_j are all unknown or unobservable. Also, Assumption 1 is only requiring that ϕ is monotonic in the index

[7]Our method can also be adapted to the case with *asymmetric* w. See Remark 1.

$w(X_i, X_j)' \beta_0$ as a whole, not individual components of $w(X_i, X_j)$. Therefore, we may include nonlinear or non-monotone functions $w(\cdot, \cdot)$ on the observable characteristics as long as Assumption 1 is maintained.

Next, we also impose a standard random sampling assumption:

Assumption 2 (Random Sampling). *(X_i, A_i) is i.i.d. across $i \in \{1, ..., n\}$.*

In particular, Assumption 2 allows arbitrary dependence structures between the observable characteristics X_i and the unobservable characteristic A_i.

Model (3) along with the specifications and the two assumptions introduced above encompass a large class of dyadic network formation models in the literature. For example, the standard dyadic network formation model (1) studied by Graham (2017) can be written as

$$\mathbb{E}[D_{ij} | X_i, X_j, A_i, A_j] = F\left(w(X_i, X_j)' \beta_0 + A_i + A_j\right)$$

where F is the CDF of the standard logistic distribution. For the semiparametric version considered by Candelaria (2016); Toth (2017); Gao (2020), we can simply take F to be some unknown CDF. In either case, the monotonicity of the CDF F and the additive structure of $w(X_i, X_j)' \beta_0 + A_i + A_j$ immediately imply Assumption 1.

However, our current model specification and assumptions further incorporate a larger class of dyadic network formation models with potentially nontransferable utilities. Specifically, consider the joint requirement of two threshold-crossing conditions,

$$D_{ij} = \mathbb{1}\left\{u\left(w(X_i, X_j)' \beta_0, A_i, A_j, \epsilon_{ij}\right) \geq 0\right\} \cdot \mathbb{1}\left\{u\left(w(X_j, X_i)' \beta_0, A_j, A_i, \epsilon_{ji}\right) \geq 0\right\} \quad (4)$$

where u is an unknown function that is not necessarily symmetric with respect to its second and third arguments (A_i, A_j), and $(\epsilon_{ij}, \epsilon_{ji})$ are idiosyncratic pairwise shocks that are i.i.d. across the each unordered ij pair with some unknown distribution. In particular, notice that model (2) is a special case of (4). Suppose we further impose the following two lower-level assumptions Assumption 1a and 1b:

Assumption (1a). *$(\epsilon_{ij}, \epsilon_{ji})$ are independent from (X_i, A_i, X_j, A_j).*

Assumption (1b). *u is weakly increasing in its first three arguments.*

Then the conditional linking probability

$$\begin{aligned}
&\mathbb{E}\left[D_{ij}\middle|\, X_i, X_j, A_i, A_j\right] \\
=&\int \mathbb{1}\left\{u\left(w\left(X_i, X_j\right)^{\prime}\beta_0, A_i, A_j, \epsilon_{ij}\right) \geq 0\right\} \cdot \mathbb{1}\left\{u\left(w\left(X_j, X_i\right)^{\prime}\beta_0, A_j, A_i, \epsilon_{ji}\right) \geq 0\right\} d\mathbb{P}\left(\epsilon_{ij}, \epsilon_{ji}\right) \\
=:&\ \phi\left(w\left(X_i, X_j\right)^{\prime}\beta_0, A_i, A_j\right). \qquad (5)
\end{aligned}$$

can be represented by model (3) with Assumption 1 satisfied.

In particular, notice that we do not require $\epsilon_{ij} \perp \epsilon_{ji}$. In fact, $\epsilon_{ij} \equiv \epsilon_{ji}$ is readily incorporated in our model. If u is furthermore assumed to be symmetric with respect to its second and third arguments (A_i and A_j), then our model degenerates to the case of transferable utilities,

$$D_{ij} = \mathbb{1}\left\{u\left(w\left(X_i, X_j\right)^{\prime}\beta_0, A_i, A_j, \epsilon_{ij}\right) \geq 0\right\},$$

where effectively only one threshold crossing condition will be determining the establishment of a given network link.

Remark 1 (Symmetry of $w\left(X_i, X_j\right)$). To explain the key idea of our identification strategy in a notation-economical way, we will be focusing on the case of symmetric w in the most of the following sections. However, it should be pointed out that our method can also be applied to the case where w is allowed to be asymmetric in (4), so that individual utilities based on observable characteristics can also be made asymmetric (nontransferable). In that case, model (4) need to be modified as

$$\mathbb{E}\left[D_{ij}\middle|\, X_i, X_j, A_i, A_j\right] = \phi\left(w\left(X_i, X_j\right)^{\prime}\beta_0, w\left(X_j, X_i\right)^{\prime}\beta_0, A_i, A_j\right), \qquad (6)$$

where $w\left(X_i, X_j\right)^{\prime}\beta_0$ may be different from $w\left(X_j, X_i\right)^{\prime}\beta_0$, but ϕ is symmetric with respect to its first two arguments $w\left(X_i, X_j\right)^{\prime}\beta_0, w\left(X_j, X_i\right)^{\prime}\beta_0$ *whenever* $A_i = A_j$. Moreover, Assumption 1 should also be understood as monotonicity with respect to all *four* arguments of ϕ. See Appendix (A.5) for more discussion on how our identification strategy can be adapted to accommodate asymmetric w under appropriate conditions.

3 Identification and Estimation

3.1 Identification via Logical Differencing

In this section, we explain the key idea of our identification strategy. We construct a mutually exclusive event to cancel out the unobservable heterogeneity A_i and A_j, which leads to an identifying restriction on β_0. We call this technique "*logical differencing*".

For each fixed individual $\bar{i}$, and each possible $\bar{x} \in \mathbb{R}^{d_x}$, define

$$\rho_{\bar{i}}(\bar{x}) := \mathbb{E}\left[D_{\bar{i}k} \middle| X_k = \bar{x} \right] \tag{7}$$

as the linking probability of this specific individual $\bar{i}$ with a group of individuals, individually indexed by k, with the same observable characteristics $X_k = \bar{x}$ (but potentially different fixed effects A_k). Clearly, $\rho_i(\bar{x})$ is directly identified from data in a single large network.

Suppose that individual $\bar{i}$ has observed characteristics $X_{\bar{i}} = x_{\bar{i}}$ and unobserved characteristics $A_{\bar{i}} = a_{\bar{i}}$. Then, by model (3) we have

$$\begin{aligned} \rho_{\bar{i}}(\bar{x}) &= \mathbb{E}\left[\mathbb{E}\left[D_{\bar{i}k} \middle| X_k = \bar{x}, A_k, X_{\bar{i}} = x_{\bar{i}}, A_{\bar{i}} = a_{\bar{i}} \right] \middle| X_k = \bar{x} \right] \\ &= \mathbb{E}\left[\phi\left(w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0, a_{\bar{i}}, A_k \right) \middle| X_k = \bar{x} \right] \\ &=: \psi_{\bar{x}}\left(w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0, a_{\bar{i}} \right), \end{aligned} \tag{8}$$

where the expectation in the second to last line is taken over A_k conditioning on $X_k = \bar{x}$. As we allow A_k and X_k to be arbitrarily correlated, the $\psi_{\bar{x}}$ function defined in the last line of (8) is dependent on $\bar{x}$. In the same time, notice that $\psi_{\bar{x}}$ does not depend on the identity of $\bar{i}$ beyond the values of $w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0$ and $a_{\bar{i}}$. By Assumption 1, $\psi_{\bar{x}}\left(w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0, a_{\bar{i}} \right)$ must be bivariate weakly increasing in the index $w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0$ and the unobserved heterogeneity scalar $a_{\bar{i}}$. We now show how to use bivariate monotonicity to obtain identifying restrictions on β_0.

Fixing two distinct individuals $\bar{i}$ and $\bar{j}$ in the population, we first consider the event that *individual* $\bar{i}$ is *strictly more popular than individual* $\bar{j}$ among the *group* of individuals with observed characteristics $X_k = \bar{x}$:

$$\rho_{\bar{i}}(\bar{x}) > \rho_{\bar{j}}(\bar{x}), \tag{9}$$

which is an event directly identifiable from observable data given (7). Even though event (9) is the same conditioning event as considered in Toth (2017) and analogous to the tetrad comparisons made in Candelaria (2016), we now exploit the following logical deduction based on the bivariate monotonicity of the conditional popularity of $\bar{i}$ in $w\left(X_{\bar{i}}, \bar{x}\right)' \beta_0$ and $A_{\bar{i}}$ without the assumption of additivity between them. Specifically, writing $\left(x_{\bar{i}}, a_{\bar{i}}\right)$ and $\left(x_{\bar{j}}, a_{\bar{j}}\right)$ as the observable and unobservable characteristics of individuals $\bar{i}$ and $\bar{j}$, by (8) we have

$$\begin{aligned} & \rho_{\bar{i}}(\bar{x}) > \rho_{\bar{j}}(\bar{x}). \\ \Leftrightarrow \quad & \psi_{\bar{x}}\left(w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0, a_{\bar{i}} \right) > \psi_{\bar{x}}\left(w\left(x_{\bar{j}}, \bar{x}\right)' \beta_0, a_{\bar{j}} \right) \\ \Rightarrow \quad & \left\{ w\left(x_{\bar{i}}, \bar{x}\right)' \beta_0 > w\left(x_{\bar{j}}, \bar{x}\right)' \beta_0 \right\} \text{ OR } \left\{ a_{\bar{i}} > a_{\bar{j}} \right\}, \end{aligned} \tag{10}$$

Note that the last line of equation (10) is a natural necessary (but not sufficient) condition for $\rho_{\bar{i}}(\overline{x}) > \rho_{\bar{j}}(\overline{x})$ under bivariate monotonicity.

Now, consider the event that *individual* $\bar{i}$ is *strictly less popular than individual* $\bar{j}$ among the *group* of individuals with observed characteristics $X_h = \overline{x}$, i.e.,

$$\rho_{\bar{i}}(\underline{x}) < \rho_{\bar{j}}(\underline{x}). \tag{11}$$

Then, by a similar argument to (10), we deduce

$$\rho_i(\underline{x}) < \rho_j(\underline{x}) \quad \Rightarrow \quad \left\{w(x_{\bar{i}}, \underline{x})' \beta_0 < w\left(x_{\bar{j}}, \underline{x}\right)' \beta_0\right\} \text{ OR } \left\{a_{\bar{i}} < a_{\bar{j}}\right\}. \tag{12}$$

Notice that the event $\left\{a_{\bar{i}} < a_{\bar{j}}\right\}$ in (12) is mutually exclusive with the event $\left\{a_{\bar{i}} > a_{\bar{j}}\right\}$ that shows up in (10).

Next, consider the event that the two events (9) and (11) described above *simultaneously happen*. Then, by (10), (12) and basic logical operations, we have

$$\begin{aligned}
& \left\{\rho_{\bar{i}}(\overline{x}) > \rho_{\bar{j}}(\overline{x})\right\} \text{ AND } \left\{\rho_{\bar{i}}(\underline{x}) < \rho_{\bar{j}}(\underline{x})\right\} \\
\Rightarrow \quad & \left(\left\{w(x_{\bar{i}}, \overline{x})' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \text{ OR } \left\{a_{\bar{i}} > a_{\bar{j}}\right\}\right) \\
& \text{AND } \left(\left\{w(x_{\bar{i}}, \underline{x})' \beta_0 < w\left(x_{\bar{j}}, \underline{x}\right)' \beta_0\right\} \text{ OR } \left\{a_{\bar{i}} < a_{\bar{j}}\right\}\right) \\
\Leftrightarrow \quad & \left(\left\{w(x_{\bar{i}}, \overline{x})' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \text{ AND } \left\{w(x_{\bar{i}}, \underline{x})' \beta_0 < w\left(x_{\bar{j}}, \underline{x}\right)' \beta_0\right\}\right) \\
& \text{OR } \left(\left\{w(x_{\bar{i}}, \overline{x})' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \text{ AND } \left\{a_{\bar{i}} < a_{\bar{j}}\right\}\right) \\
& \text{OR } \left(\left\{a_{\bar{i}} > a_{\bar{j}}\right\} \text{ AND } \left\{w(x_{\bar{i}}, \underline{x})' \beta_0 < w\left(x_{\bar{j}}, \underline{x}\right)' \beta_0\right\}\right) \\
& \text{OR } \left(\left\{a_{\bar{i}} > a_{\bar{j}}\right\} \text{ AND } \left\{a_{\bar{i}} < a_{\bar{j}}\right\}\right) \\
\Rightarrow \quad & \left(\left\{w(x_{\bar{i}}, \overline{x})' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \text{ AND } \left\{w(x_{\bar{i}}, \underline{x})' \beta_0 < w\left(x_{\bar{j}}, \underline{x}\right)' \beta_0\right\}\right) \\
& \text{OR } \left\{w(x_{\bar{i}}, \overline{x})' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \\
& \text{OR } \left\{w(x_{\bar{i}}, \underline{x})' \beta_0 < w\left(x_{\bar{j}}, \underline{x}\right)' \beta_0\right\} \\
\Leftrightarrow \quad & \left\{\left(w(x_{\bar{i}}, \overline{x}) - w\left(x_{\bar{j}}, \overline{x}\right)\right)' \beta_0 > 0\right\} \text{ OR } \left\{\left(w(x_{\bar{i}}, \underline{x}) - w\left(x_{\bar{j}}, \underline{x}\right)\right)' \beta_0 < 0\right\},
\end{aligned} \tag{13}$$

The derivations above exploit two simple logical properties: first,

$$\left\{a_{\bar{i}} > a_{\bar{j}}\right\} \text{ AND } \left\{a_{\bar{i}} < a_{\bar{j}}\right\} \quad = \quad \text{FALSE},$$

and second,

$$\left\{w\left(x_{\bar{i}}, \overline{x}\right)' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \text{ AND } \left\{a_{\bar{i}} < a_{\bar{j}}\right\} \quad \Rightarrow \quad \left\{w\left(x_{\bar{i}}, \overline{x}\right)' \beta_0 > w\left(x_{\bar{j}}, \overline{x}\right)' \beta_0\right\},$$

which uses only necessary but not sufficient condition, so that we can obtain an identifying restriction (13) on β_0 that does not involve $a_{\bar{i}}$ nor $a_{\bar{j}}$. These two forms of logical operations together enable us to "difference out" (or "cancel out") the unobserved heterogeneity terms $a_{\bar{i}}$ and $a_{\bar{j}}$.

In contrast with various forms of "*arithmetic differencing*" techniques proposed in the econometric literature (including Candelaria, 2016 and Toth, 2017 specific to the dyadic network formation literature), our proposed technique does *not* rely on additive separability between the parametric index $w\left(x_{\bar{i}}, \overline{x}\right)' \beta_0$ and the unobserved heterogeneity term $a_{\bar{i}}$. Instead, our identification strategy is based on multivariate monotonicity and utilizes logical operations rather than standard arithmetic to cancel out the unobserved heterogeneity terms. Hence we term our method "*logical differencing*".

The identifying arguments above are derived for a fixed pair of individuals $\bar{i}$ and $\bar{j}$, but clearly the arguments can be applied for any pair of individuals ij with observable characteristics x_i and x_j. Writing

$$\tau_{ij}\left(\overline{x}, \underline{x}\right) := \mathbb{1}\left\{\rho_i\left(\overline{x}\right) > \rho_j\left(\overline{x}\right)\right\} \cdot \mathbb{1}\left\{\rho_i\left(\underline{x}\right) < \rho_j\left(\underline{x}\right)\right\},$$
$$\lambda\left(\overline{x}, \underline{x}; x_i, x_j; \beta\right) := \mathbb{1}\left\{\left(w\left(x_i, \overline{x}\right) - w\left(x_j, \overline{x}\right)\right)' \beta_0 \leq 0\right\} \cdot \mathbb{1}\left\{\left(w\left(x_i, \underline{x}\right) < w\left(x_j, \underline{x}\right)\right)' \beta_0 \geq 0\right\},$$

for each $\beta \in \mathbb{S}^{m-1}$, we summarize the identifying arguments above by the following lemma.

Lemma 1 (Identifying Restriction)**.** *Under model* (3) *and Assumptions 1 and 2, we have.*

$$\tau_{ij}\left(\overline{x}, \underline{x}\right) = 1 \quad \Rightarrow \quad \lambda\left(\overline{x}, \underline{x}; x_i, x_j; \beta_0\right) = 0.$$

A simple (but clearly not unique) way to build a criterion function based on the above lemma is to define

$$Q\left(\beta\right) := \mathbb{E}_{ij,kl}\left[\tau_{ij}\left(X_k, X_l\right) \lambda_{ij}\left(X_k, X_l; X_i, X_j; \beta\right)\right], \tag{14}$$

where the expectation is $\mathbb{E}_{ij,kl}$ taken over random samples of ordered tetrads (i,j,k,l) from the population, and (X_i, X_j, X_k, X_l) denote the random variables corresponding to the observable characteristics of (i,j,k,l). According to Lemma 1, $Q(\beta_0) = 0$, which is always smaller than or equal to $Q(\beta) \geq 0 = Q(\beta_0)$ for any $\beta \neq \beta_0$ because $\tau_{ij} \geq 0$ and $\lambda_{ij} \geq 0$ by construction.

Observing that the scale of β_0 is never identified, we write

$$B_0 := \left\{\beta \in \mathbb{S}^{d_\beta - 1} : Q(\beta) = 0\right\}$$

to represent the normalized "identified set" relative to the criterion Q defined above. Lemma 1 implies that $\beta_0 \in B_0$, but in general there is no guarantee that B_0 is a singleton. The next subsection contains a set of sufficient conditions that guarantees $B_0 = \{\beta_0\}$.

3.2 Sufficient Conditions for Point Identification

We now present a set of sufficient conditions that guarantee point identification of β_0 on the unit sphere $\mathbb{S}^{m-1}$.

Assumption 3 (Full Directional Support)**.** *There exist a pair of $\overline{x}, \underline{x}$, both of which lie in the support of $Supp(X_i)$, such that $Supp(w(\overline{x}, X_i) - w(\underline{x}, X_i))$ contains all directions in $\mathbb{R}^m$.*

When $w(\overline{x}, \underline{x})$ is a component-wise Euclidean distance function, i.e., $w_h(\overline{x}, \underline{x}) = |\overline{x}_h - \underline{x}_h|$ where h indexes each coordinate of possibly vector valued $w(\cdot, \cdot)$ function, Assumption 3 is satisfied if $Supp(X_i)$ has nonempty interior[8], which is analogous to the standard assumption imposed for point identification on the unit sphere. When some components of $w(\overline{x}, X_j)$ have discrete range space, we need to require that at least one component of $w(\overline{x}, X_i) - w(\underline{x}, X_i)$ have full support on $\mathbb{R}$ and the coordinate of β_0 it corresponds to is nonzero, such that it creates enough variation in $w(\overline{x}, X_i) - w(\underline{x}, X_i)$ to guarantee Assumption 3 is satisfied.

Assumption 4 (Conditional Support of A_i)**.** *A_i is continuously distributed on the same support, conditional on $X_i = x_i$ for any $x_i \in \mathrm{supp}(X_i)$.*

Assumption 5 (Continuity of ϕ)**.** *ϕ is continuous with respect to the second and third arguments.*

Assumption 4 together with Assumption 2 implies that conditional on X_i and X_j, for two randomly sampled agents i, j, 0 is in the support of $|A_i - A_j|$. Assumption 5 then ensures

[8] When $Supp(X_i)$ has nonempty interior, there exist $\overline{x}, \underline{x} \in Supp(X_i)$ such that $\overline{x} >> \underline{x}$ in the point-wise sense and $\times_{h=1}^{d_x} [\underline{x}_h, \overline{x}_h] \subseteq int(Supp(X_i))$. In particular, $\frac{1}{2}(\overline{x} + \underline{x}) \in int(Supp(X_i))$ and thus $0 \in int(Supp(w(\overline{x}, X_i) - w(\underline{x}, X_i)))$. Consequently, one can construct a $\varepsilon-$ball around origin for $Supp(w(\overline{x}, X_i) - w(\underline{x}, X_i))$ by choosing X_i from $\times_{h=1}^{d_x} [\underline{x}_h, \overline{x}_h]$ and Assumption 3 is satisfied.

that $\tau_{ij}(\overline{x}, \underline{x}) = 1$ occurs with strictly positive probability, which is required for the point identification result.

Next, we lay out the lemma that will be used in the proof of point identification of β_0.

Lemma 2 (Tools for Point Identification)**.** *Under model* (3)*, Assumptions 1, 2, 3, 4, and 5, for each $\beta \in \mathbb{S}^{m-1} \backslash \beta_0$, there exist x_i, x_j, $\overline{x}$, and $\underline{x}$ all in the support of X_i such that*

$$\tau_{ij}(\overline{x}, \underline{x}) = 1, \tag{15}$$

$$\lambda_{ij}(\overline{x}, \underline{x}; x_i, x_j; \beta_0) = 0, \tag{16}$$

$$\lambda_{ij}(\overline{x}, \underline{x}; x_i, x_j; \beta) = 1. \tag{17}$$

We are now ready to present the point identification result.

Theorem 1 (Point Identification of β_0)**.** *Under model* (3) *and Assumptions 1, 2, 3, 4, and 5. Then β_0 is the unique minimizer of $Q(\beta)$ defined in 14 over the unit sphere $\mathbb{S}^{d_\beta - 1}$. Furthermore, for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\inf_{\beta \in \mathbb{S}^{d_\beta - 1} \backslash B(\beta_0, \epsilon)} Q(\beta) \geq Q(\beta_0) + \delta,$$

where $B(\beta_0, \epsilon) = \left\{\beta \in \mathbb{S}^{d_\beta - 1} : \|\beta - \beta_0\| \leq \epsilon\right\}$.

Remark 2 (Asymmetry of w, Continued)*.* In Appendix (A.5), we show how the identification arguments and assumptions above can be adapted to accommodate asymmetry of w. In short, the technique of logical differencing applies without changes, but the identifying restriction we obtained become weaker. In particular, when w is *antisymmetric* in the sense that $w(\overline{x}, \underline{x}) + w(\underline{x}, \overline{x}) \equiv 0$, the identifying restriction we obtained through logical differencing becomes trivial, and $B_0 = \mathbb{S}^{d_\beta - 1}$. However, with asymmetric but not antisymmetric w, it is still feasible to strengthen Assumption 3 so as to obtain point identification. See more discussions in Appendix (A.5).

3.3 Tetrad Estimation and Consistency

We now proceed to present a consistent estimator of β_0 in the framework of extremum estimation. We will first construct the sample criterion function and show how to estimate β_0 via a two-step estimation procedure. Then we will list one additional assumption before presenting the consistency result.

Define the sample analog of the population criterion $Q(\beta)$ in (14) by

$$\widehat{Q}_n(\beta) := \frac{(n-4)!}{n!} \sum_{1 \le i \ne j \ne k \ne l \le n} \mathbb{1}\{\hat{\rho}_i(X_k) > \hat{\rho}_j(X_k)\} \cdot \mathbb{1}\{\hat{\rho}_i(X_l) < \hat{\rho}_j(X_l)\} \cdot \left[\begin{array}{c} \mathbb{1}\left\{ w(X_i, X_k)' \beta \le w(X_j, X_k)' \beta \right\} \\ \cdot \mathbb{1}\left\{ w(X_i, X_l)' \beta \ge w(X_j, X_l)' \beta \right\} \end{array} \right], \tag{18}$$

where $\hat{\rho}_i(x)$ is a first-step nonparametric estimator of $\rho_i(x)$. The two-step tetrad estimator for β_0 is defined as

$$\widehat{\beta}_n := \arg \min_{\beta \in \mathbb{S}^{d_\beta - 1}} \widehat{Q}_n(\beta). \tag{19}$$

The first-step estimation of $\rho_i(x) := \mathbb{E}\left[D_{ik} \middle| i, X_k = x\right]$ function to is standard nonparametric regression problem. Computationally, one can fix each i in the sample, and regress D_{ik}, the indicator function for the link between i and k, on the basis functions chosen by the researcher evaluated at observable characteristics X_k for all $k \ne i$. There are many tools readily available to nonparametrically estimate $\rho_i(x)$ in the first stage. For example, one can use kernel, sieve, or neural networks. In Section 4, we use second order sieves with knot at the median to estimate $\rho_i(x)$ for the simulation study. Theoretical properties of our sieve estimator $\hat{\rho}_i(x)$ can be found in Chen (2007).

It is worth mentioning that we can smooth each component of $\tau_{ij}(\overline{x}, \underline{x})$ to achieve better numerical performance as long as the sign of the differences between $\rho_i(x)$ and $\rho_j(x)$ is preserved. Recall that

$$\tau_{ij}(\overline{x}, \underline{x}) := \mathbb{1}\{\rho_i(\overline{x}) > \rho_j(\overline{x})\} \cdot \mathbb{1}\{\rho_i(\underline{x}) < \rho_j(\underline{x})\}. \tag{20}$$

When $\rho_i(x)$ is close to $\rho_j(x)$, the estimation of $\tau_{ij}(\overline{x}, \underline{x})$ may be imprecise and sensitive to errors during data collection and analysis procedure. Therefore, we may wish to smooth both $\mathbb{1}\{\rho_i(\overline{x}) > \rho_j(\overline{x})\}$ and $\mathbb{1}\{\rho_i(\underline{x}) < \rho_j(\underline{x})\}$ such that the potential bias caused by the imprecise estimation at the boundary point of 0 is smaller. In practice, we can do so by applying a known smooth one-directional function H on $\rho_i(x) - \rho_j(x)$. A concrete example of H is the standard normal CDF, i.e. replace $\mathbb{1}\{\rho_i(\overline{x}) > \rho_j(\overline{x})\}$ with $2 \times \Phi\left[(\rho_i(\overline{x}) - \rho_j(\overline{x}))_+\right] - 1$ and replace $\mathbb{1}\{\rho_i(\underline{x}) < \rho_j(\underline{x})\}$ with $2 \times \Phi\left[(\rho_j(\underline{x}) - \rho_i(\underline{x}))_+\right] - 1$ in $\tau_{ij}(\overline{x}, \underline{x})$, where $(c)_+$ is the positive part of c, otherwise 0, and Φ is the CDF of standard normal $\mathcal{N}(0, 1)$. We use smoothed $\tau_{ij}(\overline{x}, \underline{x})$ in the simulation part. See Section 4 for details.

For the second step, we estimate β_0 by minimizing the sample criterion function $\widehat{Q}_n(\beta)$ over the unit sphere $\mathbb{S}^{d_\beta - 1}$ after plugging in the first stage estimator $\hat{\tau}_{ij}(\overline{x}, \underline{x})$. To exploit the topological characteristics of the parameter space $\mathbb{S}^{d_\beta - 1}$, i.e. compactness and convexity, we develop a new bisection-style nested rectangle algorithm that recursively shrinks and refines an adaptive grid on the angle space. The key novelty of the algorithm is that instead of working with the edges of the Euclidean parameter space $\mathbb{R}^{d_\beta}$, we deterministically "cut" the angle space in each dimension of $\mathbb{S}^{d_\beta-1}$ to search for the area that minimizes $\widehat{Q}_n(\beta)$. Additional measures are taken to ensure the search algorithm is conservative. Simulation and empirical results show that our algorithm performs reasonably well with a relatively small sample size. Gao and Li (2020) provides more details regarding the implementation in a panel multinomial choice setting.

For consistency, we impose the following assumption regarding the first-step nonparametric estimator $\hat{\rho}_i(\cdot)$ of the $\rho_i(\cdot)$ function.

Assumption 6 (Uniform Consistency for $\rho_i(\cdot)$)**.** *$\hat{\rho}_i(\cdot)$ is a uniformly consistent estimator of $\rho_i(\cdot)$ for each agent i.*

The usual kernel and sieve methods we mentioned above to estimate $\rho_i(x)$ have been proved to satisfy Assumption 6: see Bierens (1983) for results on kernel estimators and Chen (2007) on sieve estimators.

Lemma 3 (Uniform Convergence of $\widehat{Q}_n(\beta)$)**.** *Under model* (3) *and Assumptions 1, 2, 3, 4, 5, and 6, we have*

$$\sup_{\beta\in\mathbb{S}^{d_\beta-1}} \left|\widehat{Q}_n(\beta) - Q_n(\beta)\right| \xrightarrow{p} 0.$$

Finally, we state the consistency result of the tetrad estimator $\widehat{\beta}_n$.

Theorem 2. *Under model* (3) *and Assumptions 1, 2, 3, 4, 5, and 6, $\hat{\beta}_n$ is consistent for β_0, i.e.,*

$$\widehat{\beta}_n \xrightarrow{p} \beta_0.$$

4 Simulation

In this section, we conduct a simulation study to analyze the finite-sample performance of our two-step tetrad estimator. We start by specifying the data generating process (DGP) of the Monte Carlo simulations. Next, we show and discuss the performance of our 2-step estimation method under the baseline setup. Then, we vary the number of individuals N, the dimension of the pairwise observable characteristics $\overline{d}$, and the degree of correlation between X and A to further examine the robustness of our method. Finally, we show how the method performs when $w(X_i, X_j)$ is an asymmetric function of X_i and X_j, i.e. $w(X_i, X_j) \neq w(X_j, X_i)$.

4.1 Setup of Simulation Study

For each DGP configuration, we run $B = 100$ independent simulations of model 3 with the following network formation rule unknown to the econometrician for each agent pair (i, j)

$$D_{ij} = \mathbb{1}\left\{w\left(X_i, X_j\right)' \beta_0 + A_i > \epsilon_{ij}\right\} \cdot \mathbb{1}\left\{w\left(X_j, X_i\right)' \beta_0 + A_j > \epsilon_{ji}\right\}, \tag{21}$$

where the usual linear additivity is excluded by construction that D_{ij} equals the product of two indicator functions. In (21), D_{ij} equals one if i and j are connected, zero otherwise. X_i and X_j are $d_x \times 1$ vectors of observable characteristics of individual i and j, respectively. $w\left(X_i, X_j\right)$ is a known vector-valued function mapping (X_i, X_j) pairs to a $d_w \times 1$ vector. β_0 is a $d_\beta \times 1$ vector of structural parameter of interest. We maintain $d_x = d_w = d_\beta = \overline{d}$ in all our configurations. A_i represents the unobservable scalar valued fixed effect that is possibly correlated with X_i. ϵ_{ij} is the scalar valued iid random shocks independent of X and A.

In our baseline DGP configuration where we fix $N = 100$ and $\overline{d} = 3$, each coordinate of X_i is drawn independently across both individuals i and dimensions d from a uniform distribution on $[-0.5, 0.5]$. Then we compute $W_{ij}^{(d)}$, the d^{th} coordinate of $w\left(X_i, X_j\right)$ vector, as $W_{ij}^{(d)} = \left|X_i^{(d)} - X_j^{(d)}\right|$. Note that for the baseline setup we maintain the symmetry of W_{ij} in (X_i, X_j) pairs, i.e., $W_{ij} = W_{ji}$. Later on, we will relax this restriction and investigate the asymmetric case where $W_{ij} \neq W_{ji}$.

Next, we construct the unobserved heterogeneity A_i. To allow for the correlation between A_i and X_i, we draw iid sequence ξ_i independently from X_i from a uniform distribution on $[-0.5, 0.5]$ and let $A_i = corr \times X_i^{(1)} + (1 - corr) \times \xi_i$, where $corr$ controls the degree of correlation between X_i and A_i and is set to be 0.2. Later on, we will vary the correlation to see how robust our estimator is against correlation between A and X. As for the random utility shock ϵ_{ij}, we draw them from a uniform distribution on $[0, 1]$. Note that our estimation method does not require the knowledge of the distribution of A_i or ϵ_{ij}. We set the true $\beta_0 \in \mathbb{R}^{d_\beta}$ to be $(1, ..., 1)'$, and estimate the direction of β_0, represented by the normalized vector $\overline{\beta}_0 := \beta_0 / \|\beta_0\|$ on the unit sphere $\mathbb{S}^{d_\beta - 1}$ because the scale of β_0 is not identified. We shall maintain the specification of (ϵ, A, β_0) and the network formation rule (21) to be the same across all simulations.

Our method allows for asymmetry of W_{ij} in (X_i, X_j) pairs. To numerically show this, for the last coordinate $d = \overline{d}$ we compute $W_{ij}^{(\overline{d})}$ as $\left|2X_i^{(\overline{d})} - X_j^{(\overline{d})}\right| \times (2/3)$. The reason for multiplying $2/3$ is to make the size of $W_{ij}^{(\overline{d})}$ similar to other coordinates of W_{ij}. This way we generate asymmetry because $W_{ij}^{(\overline{d})} \neq W_{ji}^{(\overline{d})}$ unless $\left|X_i^{(d)}\right| = \left|X_j^{(d)}\right|$, which is a probability zero event under our DGP setting. For other dimensions $d = 1, ..., \overline{d} - 1$, we maintain the baseline assumption. As a robustness check, we also vary N and $\overline{d}$ under asymmetry to show how our method works.

To summarize, for each of the $B = 100$ simulations we randomly generate data on the characteristics of and the network structure among individuals. Then based on the observable $(X_i, W_{ij}, D_{ij})_{i,j\in\{1,\dots,N\}}$ matrix we construct our two-step estimator $\widehat{\beta}$ for the true parameter of interest $\overline{\beta}_0$. Specifically, we use a sieve estimator with 2nd-order spline with its knot at median for the first-stage nonparametric estimation of $\rho_i(\cdot)$. The spline is chosen to ensure a relatively small number of regressors in the nonparametric regression considering the small size of N. In the second stage, we adapt to the adaptive-gird search on the unit sphere algorithm developed in Gao and Li (2020) to calculate $\widehat{\beta}$ that minimizes the sample criterion function $\widehat{Q}(\beta)$ over the unit sphere. We refer interested readers to that paper for more technical details. It should be noted that constrained by computational power, when calculating the sample criterion $\widehat{Q}(\beta)$ for each $\beta \in \mathbb{S}^{\overline{d}-1}$ we randomly draw $M = 1000$ (i, j) pairs of individuals and vary across all possible (k, l) pairs excluding i or j. One can improve those results by increasing M when computational constraint is not present. Lastly, we compare our estimator $\widehat{\beta}$ with the true parameter value $\overline{\beta}_0$ based on several performance metrics including rMSE, mean norm deviations (MND), and maximum absolute deviation (MAB).

4.2 Results under Symmetric Pairwise Observable Characteristics

Baseline Results

For the baseline configuration, we fix number of individuals $N = 100$, dimension of W_{ij} $\overline{d} = 3$, number of (i, j) pairs used in evaluating $\widehat{Q}(\beta)$ $M = 1000$, and number of simulations $B = 100$. We define for each simulation round b the argmin set estimator $\widehat{B}_b$ as the set of points that achieve the minimum of $\widehat{Q}(\beta)$ over the unit sphere $\mathbb{S}^{\overline{d}-1}$. Under point identification, any element from $\widehat{B}_b$ is a consistent point estimator for $\overline{\beta}_0$. In particular, we further define ,for each simulation $b = 1, \dots, B$ and each dimension $d = 1, \dots, \overline{d}$ of β

$$\widehat{\beta}^l_{b,d} := \min \widehat{B}_{b,d}, \quad \widehat{\beta}^u_{b,d} := \max \widehat{B}_{b,d}, \quad \widehat{\beta}^m_{b,d} := \frac{1}{2}\left(\widehat{\beta}^l_{b,d} + \widehat{\beta}^u_{b,d}\right),$$

where $\widehat{\beta}^l_{b,d}$ is the minimum value along dimension d for simulation round b of the argmin set $\widehat{B}_b$, $\widehat{\beta}^u_{b,d}$ is the maximum value along dimension d for simulation round b of the argmin set $\widehat{B}_b$, and $\widehat{\beta}^m_{b,d}$ is the middle point along dimension d for simulation round b of the argmin set $\widehat{B}_b$. Note by construction for each simulation round b, the argmin set $\widehat{B}_b$ is a subset of the rectangle $\widehat{\Xi}_b := \times_{d=1}^{\overline{d}} \left[\widehat{\beta}^l_{b,d},\ \widehat{\beta}^u_{b,d}\right]$. We calculate the baseline performance using $\widehat{\beta}^l, \widehat{\beta}^u, \widehat{\beta}^m$

Table 1: Baseline Performance

		β_1	β_2	β_3
bias	$\frac{1}{B}\sum_b\left(\hat{\beta}^m_{b,d}-\overline{\beta}_{0,d}\right)$	-0.0021	0.0052	-0.0053
upper bias	$\frac{1}{B}\sum_b\left(\hat{\beta}^u_{b,d}-\overline{\beta}_{0,d}\right)$	0.0048	0.0118	-0.0002
lower bias	$\frac{1}{B}\sum_b\left(\hat{\beta}^l_{b,d}-\overline{\beta}_{0,d}\right)$	-0.0091	-0.0015	-0.0105
mean$(u-l)$	$\frac{1}{B}\sum_b\left(\hat{\beta}^u_{b,d}-\hat{\beta}^l_{b,d}\right)$	0.0138	0.0132	0.0103
root MSE	$\sqrt{\frac{1}{B}\sum_b\left\Vert\hat{\beta}^m_b-\overline{\beta}_0\right\Vert^2}$		0.0488	
mean norm deviations	$\frac{1}{B}\sum_b\left\Vert\hat{\beta}^m_b-\overline{\beta}_0\right\Vert$		0.0417	
max absolute deviations	$\max_d\left\vert\frac{1}{B}\sum_b\left(\hat{\beta}^m_{b,d}-\overline{\beta}_{0,d}\right)\right\vert$		0.0053	

respectively.

Below in Table 1 we report the performance of our estimators. In the first row of Table 1 we calculate the mean bias across $B = 100$ simulations using $\widehat{\beta}^m$ along each dimension $d = 1, ..., \overline{d}$. The result shows the estimation bias is very small across all dimensions with a magnitude between -0.0053 and 0.0052. Similar performance is observed using $\widehat{\beta}^u$ and $\widehat{\beta}^l$ as shown in row 2 and 3. We do not find any sign of persistent over/under- estimation of $\overline{\beta}_0$ across each dimension. Row 4 measures the average width of the rectangle $\widehat{\Xi}$ along each dimension. The size of $\widehat{\Xi}$ is very small, indicating a very tight area for the estimated set. In the second part of Table 1 we report rMSE, MND, and MAB, all of which are small in magnitude and provide evidence that our estimator work well in finite sample.

Results Varying N and $\overline{d}$

In this section we vary the number of individuals N and dimension of W_{ij} $\overline{d}$ to examine how robust our method is against these variations. We investigate the performance when $N = 50, 100, 200$ and $\overline{d} = 3, 4$, respectively. We maintain the symmetry in W_{ij} and other distributional assumptions as in baseline setup. M, the number of (i, j) pairs used to evaluate objective function, is set to be 1000 in all simulations. Note that one could make M larger for larger N to better capture the more information available from the increase in N. In this sense, our results are conservative below. Results are summarized in Table 2.

Table 2: Results Varying N and $\overline{d}$

$\overline{d}=3$	rMSE	MND	MAB	$\overline{d}=4$	rMSE	MND	MAB
$N=50$	0.0839	0.0724	0.0051	$N=50$	0.1119	0.1030	0.0091
$N=100$	0.0488	0.0417	0.0053	$N=100$	0.0692	0.0647	0.0038
$N=200$	0.0334	0.029	0.0043	$N=200$	0.0543	0.0523	0.0038

The left part of Table 2 shows the performance of our estimator when N changes and $\overline{d}$ is fixed at 3. When N increases, rMSE, MND and sum of absolute bias all show moderate decline in magnitude, indicating the performance is improving. Similar pattern is also observed for $\overline{d}=4$. This is intuitive because with more individuals in the sample, one can achieve more accurate estimation of $\rho_i(\cdot)$ and calculation of $\widehat{Q}(\beta)$. Moreover, we can see even with a relatively small sample size of $N=50$, the rMSE is 0.0839 when $\overline{d}=3$ and 0.1119 when $\overline{d}=4$, showing that our method is informative and accurate. When $N=200$, the performance is very good, with rMSE being as small as 0.0334 and 0.0543 for $\overline{d}=3$ and $\overline{d}=4$, respectively. When we fix N and compare between $\overline{d}=3$ and $\overline{d}=4$, it is clear the increase in $\overline{d}$ adversely affects the performance of our estimator, with rMSE and MND increasing for each N. Overall, Table 2 provides evidence that our method is able to estimate $\overline{\beta}_0$ accurately even with a small sample size.

Results Varying ***corr***

Correlation between observable characteristics X and unobservable fixed effect A is important in network formation models. We show how our estimator performs when the correlation between X and A varies. Recall that we construct A_i as

$$A_i = corr \times X_i^{(1)} + (1 - corr) \times \xi_i, \tag{22}$$

where ξ_i is iid uniform on $[-0.5, 0.5]$ and is independent of X_i. We set *corr* to be 0.2 in the baseline configuration. In Table 3, we vary *corr* from 0.20 to 0.90 while fixing $N=100$, $\overline{d}=3$, $M=1000$ and obtain the performance of our estimator among $B=100$ simulations when W_{ij} is symmetric.

It can be seen from Table 3 that even though increase in *corr* adversely affects the performance of our estimator, the magnitude of the impact is relatively small. For example,

Table 3: Results Varying *corr*

$corr$	rMSE	MND	MAB
0.20	0.0488	0.0417	0.0053
0.50	0.0489	0.0435	0.0186
0.75	0.0763	0.0690	0.0506
0.90	0.1010	0.0951	0.0743

Table 4: Results under Asymmetry

$\overline{d} = 3$	rMSE	MND	MAB	$\overline{d} = 4$	rMSE	MND	MAB
$N = 50$	0.1498	0.1403	0.0936	$N = 50$	0.2225	0.2124	0.1521
$N = 100$	0.1096	0.1028	0.0741	$N = 100$	0.1751	0.1695	0.1301
$N = 200$	0.0943	0.0893	0.0672	$N = 200$	0.1595	0.1555	0.1222

rMSE only increases from 0.0488 to 0.1010 when $corr$ increase dramatically from 0.2 to 0.9. Similar pattern is also observed using other performance metrics. Therefore, our estimator is robust against correlation between X and A.

4.3 Results under Asymmetric Pairwise Observable Characteristics

In this section, we investigate how our method works when W_{ij} is asymmetric. To introduce asymmetry, we construct $W_{ij}^{(\overline{d})} = \left|2X_i^{(\overline{d})} - X_j^{(\overline{d})}\right| \times (2/3)$ for each i, j pair. The reason for multiplying 2/3 is to make the size of $W_{ij}^{(\overline{d})}$ similar to other coordinates of W_{ij}. As discussed before, under our DGP $W_{ij}^{(\overline{d})} \neq W_{ji}^{(\overline{d})}$ unless $\left|X_i^{(\overline{d})}\right| = \left|X_j^{(\overline{d})}\right|$, which is a probability zero event. For $d = 1, ..., \overline{d} - 1$, we follow the configuration for $W_{ij}^{(d)}$ mentioned in section 4.1 for the asymmetric case. We maintain other distributional assumptions for X, A, ϵ and fix the number of (i, j) pairs M at 1000 for evaluation of $\widehat{Q}(\beta)$. Finally, we vary N and D under the asymmetric setting to show how our estimator performs. Table 4 summarizes the results.

From Table 4 one can see our method performs reasonably well when W_{ij} is asymmetric. First, when the number of individuals N increases, the overall performance is improved, with rMSE decreasing from 0.1498 to 0.0943 for $\overline{d} = 3$ and from 0.2225 to 0.1595 for $\overline{d} = 4$ when N increases from 50 to 200. This result is caused by the more information available in the sample and echos what we have seen for the symmetric W_{ij} case. When the dimension of W_{ij} $\overline{d}$ increases from 3 to 4, the performance is worse, with rMSE increasing from 0.0943 to 0.1595 for $N = 200$. It shows that more data (information) is required for accurate estimation when the dimension of $\overline{\beta}_0$ is larger. Second, when compared with the symmetric W_{ij} case, the overall performance under asymmetry in W_{ij} is worse, with rMSE being 0.1498 for asymmetric W_{ij} versus 0.0839 for symmetric W_{ij} when $N = 50$ and $\overline{d} = 3$. In Appendix A.5 we discuss the implications of asymmetric W_{ij}. It is shown there the identifying power of the objective function is in general "less restrictive" than the corresponding identifying restriction in Lemma 1. Therefore, one would expect larger bias than symmetric W_{ij} case, which is exactly what one observes in Table 4. Recall that we set total number of (i, j) pairs for the evaluation of objective function M to be 1000 for all simulations. Based on results in Table 4, when W_{ij} is asymmetric and computational power allows, we suggest one increases M to improve performance.

5 Empirical Illustration

As an empirical illustration, we estimate a network formation model under NTU with data of a small village network called Nyakatoke in Tanzania. Nyakatoke is a small Haya community of 119 households in 2000 located in the Kagera Region of Tanzania. We are interested in how important factors, such as wealth, distance, and blood or religious ties, are relative to each other in deciding the formation of risk-sharing links among local residents. The estimation results demonstrate that our proposed method produces estimates that are consistent with economic intuition.

5.1 Data Description

The risk-sharing data of Nyakatoke, collected by Joachim De Weerdt in 2000, cover all of the 119 households in the community. It includes the information about whether or not two households are linked in the insurance network. It also provides detailed information on total USD assets and religion of each household, as well as kinship and distance between households. See De Weerdt (2004); De Weerdt and Dercon (2006); De Weerdt and Fafchamps (2011) for more details of this dataset.

To define the dependent variable *link*, the interviewer asks each household the following question:

"*Can you give a list of people from inside or outside of Nyakatoke, who you can personally rely on for help and/or that can rely on you for help in cash, kind or labor?*"

The data contains three answers of "bilaterally mentioned", "unilaterally mentioned", and "not mentioned" between each pair of households. Considering the question is about whether one can rely on the other for help, we interpret both "bilaterally mentioned" and "unilaterally mentioned" as they are connected in this undirected network, meaning that *link* equals 1. We also run a robustness check by constructing a weighted network based on the answers, i.e. "bilaterally mentioned" means *link* equals 2, "unilaterally mentioned" means *link* equals 1, and "not mentioned" means *link* equals 0, and found that results are very similar.

We estimate the coefficients for wealth difference, distance and tie between households with our two-step estimator. *Wealth* is defined as the total assets in USD owned by each household in 2000, including livestocks, durables and land. *Distance* measures how far away two households are located in kilometers. *Tie* is a discrete variable, defined to be 3 if members of one household are parents, children and/or siblings of members of the other household, 2 if nephews, nieces, aunts, cousins, grandparents and grandchildren, 1 if any other blood relation applies or if two households share the same religion, and 0 if no blood religious tie exists. Following the literature we take natural log on *wealth* and *distance*, and we construct the *wealth difference* variable as the absolute difference in *wealth*.

Figure 1 illustrates the structure of the insurance network in Nyakatoke. Each node in the graph represents a household. The solid line between two nodes indicates they are connected, i.e., *link* equals 1. The size of each node is proportional to the USD wealth of each household. Each node is colored according to its rank in *wealth*: green for the top quartile, red for the second, yellow for the third and purple for the fourth quartile.

In the dataset there are 5 households that lack information on *wealth* and/or *distance*. We drop these observations, resulting in a sample size N of 114. The total number of ordered household pairs is 12,882. Summary statistics for the dependent and explanatory variables used in our analysis are presented in Table 5.

5.2 Methodology

To estimate β_0, we need to first estimate $\rho_i(x) := \mathbb{E}\left[D_{ik}\middle|i, W_{ik} = w\right]$ in order to construct $\tau_{ij}(\cdot)$. We use the second degree spline sieve with its knot at the median to estimate $\rho_i(w)$.

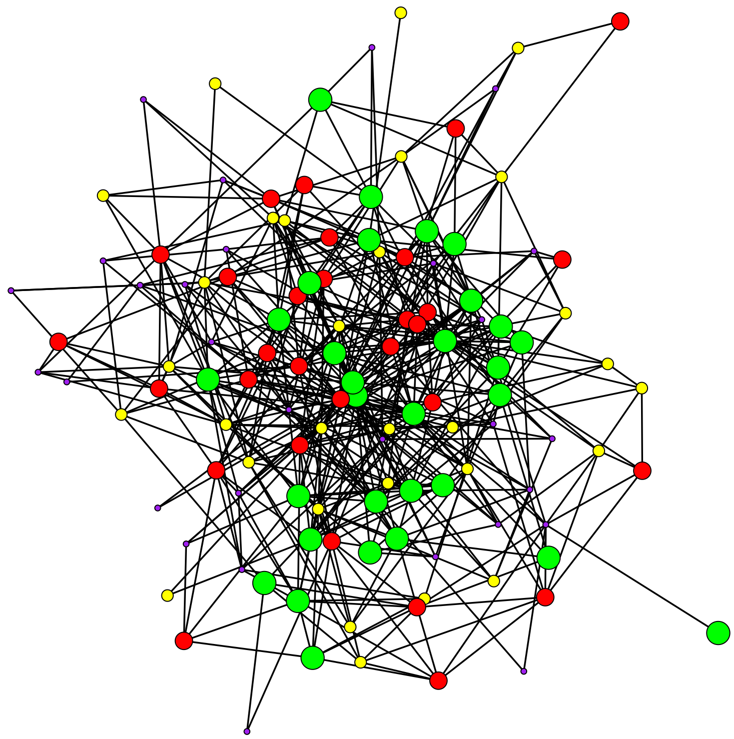

Figure 1: A Graphical Illustration of the Insurance Network of Nyakatoke

Table 5: Empirical Application: Summary Statistics

Variable	Obs	Mean	Std. Dev.	Min	Max
link	12,882	0.0732	0.2606	0	1
|(ln) wealth difference|	12,882	1.0365	0.8228	0.0004	5.8898
(ln) distance	12,882	6.0553	0.7092	2.6672	7.4603
tie	12,882	0.4260	0.6123	0.0000	3.0000

Table 6: Empirical Application: Estimation Results

Variable	$\hat{\beta}^m$	$\left[\hat{\beta}^l,\ \hat{\beta}^u\right]$
|(ln) wealth difference|	-0.1948	$[-0.1964,\ -0.1932]$
(ln) distance	-0.8036	$[-0.8043,\ -0.8029]$
tie	0.5619	$[0.5608,\ 0.5630]$

Specifically, for each household i in the data, we regress dependent variable *link* D_{ik} on each dimension of W_{ik}, W_{ik}^2, and $\left[\left(W_{ik} - \text{median}\left(W_{ik}\right)\right)_+\right]^2$ including constant for $k \neq i$. The reason why we could regress on basis functions constructed with W instead of X is because X affects D only through W. We obtain an estimator $\hat{\rho}_i\left(\cdot\right)$ evaluated at each realized $W_{ik} = w$ in the data for each household i. We also smooth each component of $\tau_{ij}\left(\cdot\right)$, i.e. $\mathbb{1}\left\{\rho_i\left(\overline{w}\right) > \rho_j\left(\overline{w}\right)\right\}$ and $\mathbb{1}\left\{\rho_i\left(\underline{w}\right) < \rho_j\left(\underline{w}\right)\right\}$ with normal CDF to improve the performance. In the second stage, we estimate β_0 with $\hat{\beta}$ that minimizes the sample criterion $\widehat{Q}\left(\beta\right)$ by adapting to the adaptive-grid search on the unit sphere algorithm developed in Gao and Li (2020). As shown in finite sample simulations, the method is able to converge fast to the area that contains true β_0.

5.3 Results and Discussion

Table 6 summarizes our estimation results. The column of $\hat{\beta}^m$ corresponds to the center of the estimated rectangle $\hat{\Xi}$. We will use it as the point estimator of the coefficients for each variable of $\mathbf{W}$ vector. $\left[\hat{\beta}^l,\ \hat{\beta}^u\right]$ corresponds to the upper and lower bound of $\hat{\Xi}$. While the scale of β_0 is unidentified, we can still compare the estimated coefficients with each other to obtain an idea about which variable affects the formation of the link more than the other.

The estimated coefficients for each variable conform well with economic intuition. Our method estimate the coefficient for *absolute wealth difference* to be negative in the range of $[-0.1964,\ -0.1932]$, which implies the more absolute difference in wealth between two households, the lower likelihood they are connected. The estimated set for *distance* is $[-0.8043,\ -0.8029]$. It is natural households rely more on neighbors for help than ones that live farther away. The estimated coefficient for *tie* falls in the positive range of $[0.5608,\ 0.5630]$, which is also consistent with economic intuition that one would depend on support from family when negative shock occurs.

It is worth mentioning the estimated set $\hat{\Xi}$ is very tight in each dimension, with a maximum width of 0.0032 for *tie.* Usually the discreteness could make the estimated set wide, but our algorithm is able to circumvent this issue by leveraging the large support in the two other continuous variables, i.e., wealth difference and distance. The relative magnitude and sign of coefficient for *tie* are estimated in line with expectation. The empirical results show that our proposed estimator is able to generate economically intuitive estimates under NTU.

6 Conclusion

This paper considers a semiparametric model of dyadic network formation under nontransferable utilities, a natural and realistic micro-theoretical feature that translates into the lack of additive separability in econometric modeling. We show how a new methodology called *logical differencing* can be leveraged to cancel out the two-way fixed effects, which correspond to unobserved individual heterogeneity, without relying on arithmetic additivity. The key idea is to exploit the logical implication of weak multivariate monotonicity and use the intersection of mutually exclusive events on the unobserved fixed effects. It would be interesting to explore whether and how the idea of *logical differencing*, or more generally the use of fundamental logical operations, can be applied to other econometric settings.

Simulation results show that our method performs reasonably well with a relatively small sample size, and robust to various configurations. The empirical illustration using the real network data of Nyakatoke reveals that our method is able to capture the essence of the network formation process by generating estimates that conform well with economic intuition.

This paper also reveals several further research questions regarding dyadic network formation models under the NTU setting. First, given the observation that the NTU setting can capture "homophily effects" with respect to the unobserved heterogeneity (under log-concave error distributions) while imposing monotonicity in the unobserved heterogeneity in the same time, it is interesting to investigate whether we can differentiate homophily effects generated by "intrinsic preference" from homophily effects generated by bilateral consent, NTU and log-concave errors. Second, admittedly the identifying restriction obtained in this paper becomes uninformative when we have *antisymmetric* pairwise observable characteristics. However, preliminary analysis based on an adaption of Gao (2020) to the NTU setting suggests that individual unobserved heterogeneity can be nonparametrically identified up to location and inter-quantile range normalizations. After the identification of individual unobserved heterogeneity terms (A_i), it becomes straightforward to identify the index parameter β_0 based on the observable characteristics, even in the presence of antisymmetric pairwise characteristics. However, consistent estimators of A_i and β_0 in a semiparametric framework based on identification strategy in Gao (2020) are still being developed. We thus leave these research questions to future work.

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Appendix

A Proofs

A.1 Proof of Lemma 2

Proof. For notational simplicity, we denote $\Delta(x; x_i, x_j)$ to be $w(x_i, x) - w(x_j, x)$ and d_β by m. It follows that

$$\lambda_{ij}(\overline{x}, \underline{x}; x_i, x_j; \beta) = \mathbb{1}\left\{\Delta(\overline{x}; x_i, x_j)'\beta \leq 0\right\}\mathbb{1}\left\{\Delta(\underline{x}; x_i, x_j)'\beta \geq 0\right\}. \tag{23}$$

Therefore, the event (16) is equivalent to $\left\{\Delta(\overline{x}; x_i, x_j)'\beta_0 > 0\right\} \cup \left\{\Delta(\underline{x}; x_i, x_j)'\beta_0 < 0\right\}$ and the event (17) is equivalent to $\left\{\Delta(\overline{x}; x_i, x_j)'\beta \leq 0\right\} \cap \left\{\Delta(\underline{x}; x_i, x_j)'\beta \geq 0\right\}$. By Assumption 3, there exist x_i and x_j in $Supp\,(X_i)$ such that $\Delta\,(X_k; x_i, x_j)$ has full directional support. Hence, given any β_0 and $\beta \neq \beta_0$ in $\mathbb{S}^{m-1}$, there exists some $\overline{x} \in Supp\,(X_i)$ such that

$$\Delta\,(\overline{x}; x_i, x_j)'\beta_0 > 0 \text{ AND } \Delta\,(\overline{x}; x_i, x_j)'\beta \leq 0,$$

and some $\underline{x} \in Supp\,(X_i)$ such that

$$\Delta\,(\underline{x}; x_i, x_j)'\beta_0 < 0 \text{ AND } \Delta\,(\underline{x}; x_i, x_j)'\beta \geq 0.$$

Hence, (16) and (17) hold simultaneously with strictly positive probability. Denote the set of $(x_i, x_j, \overline{x}, \underline{x})$ satisfying these restrictions by

$$\Xi := \left\{(x_i, x_j, \overline{x}, \underline{x}) \,\middle|\, \begin{array}{c} \Delta(\overline{x}; x_i, x_j)'\beta_0 > 0,\ \Delta(\underline{x}; x_i, x_j)'\beta_0 < 0, \\ \Delta(\overline{x}; x_i, x_j)'\beta \leq 0, \text{ and } \Delta(\underline{x}; x_i, x_j)'\beta \geq 0. \end{array}\right\}. \tag{24}$$

Note that Ξ_{ij} occurs with strictly positive probability.

For such a combination of x_i, x_j, $\overline{x}$, and $\underline{x}$, we show next the event (15) holds with strictly positive probability. According to the fact that $\left\{\Delta(\overline{x}; x_i, x_j)'\beta_0 > 0\right\}$ holds for x_i, x_j, $\overline{x}$, and $\underline{x}$, under Assumption 5 there exists some $\epsilon_1 > 0$ such that $\rho_i(\overline{x}) > \rho_j(\overline{x})$ whenever $|A_i - A_j| \leq \epsilon_1$. This is true because when the difference between A_i and A_j is small enough, the relative magnitude of $\rho_i\,(\overline{x})$ compared to $\rho_j\,(\overline{x})$ will be solely determined by whether $\Delta(\overline{x}; x_i, x_j)'\beta_0 > 0$ or not according to (7). Similarly, there exists some $\epsilon_2 > 0$ such that $\rho_i(\underline{x}) < \rho_j(\underline{x})$ whenever $|A_i - A_j| \leq \epsilon_2$. Thus, there exists some $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ such that

$$\begin{aligned}\mathbb{P}\left\{\tau_{ij}\left(\overline{x},\underline{x}\right)=1\right\} &\geq \mathbb{P}\left\{\left|A_i-A_j\right|\leq\epsilon,\left(x_i,x_j,\overline{x},\underline{x}\right)\in\Xi\right\}\\ &= \mathbb{P}\left\{\left|A_i-A_j\right|\leq\epsilon\right|\left(x_i,x_j,\overline{x},\underline{x}\right)\in\Xi\right\}\mathbb{P}\left\{\left(x_i,x_j,\overline{x},\underline{x}\right)\in\Xi\right\}\\ &> 0, \end{aligned} \tag{25}$$

where the first inequality holds by $\{|A_i - A_j| \leq \epsilon, (x_i, x_j, \overline{x}, \underline{x}) \in \Xi\}$ is sufficient for $\{\tau_{ij}(\overline{x}, \underline{x}) = 1\}$ and the last inequality holds by Assumption 4.

Therefore, we conclude the three events (15), (16), and (17), hold simultaneously with strictly positive probability for some x_i, x_j, $\overline{x}$, and $\underline{x}$ all in the support of X. □

A.2 Proof of Theorem 1

Proof. By Lemma 1, we have $\beta_0 \in \arg\min_{\beta\in\mathbb{S}^{m-1}} Q(\beta)$ because $Q(\beta_0) = 0 \leq Q(\beta)$ by the construction of the population criterion $Q(\cdot)$. Furthermore, we have β_0 is the unique minimizer of $Q(\beta)$ because for any $\beta \neq \beta_0$, we have

$$\begin{aligned}Q(\beta) &= \mathbb{E}\left[\lambda_{ij}\left(\overline{x},\underline{x};x_i,x_j;\beta\right)\tau_{ij}\left(\overline{x},\underline{x}\right)\right]\\ &= \mathbb{P}\left\{\left\{\lambda_{ij}\left(\overline{x},\underline{x};x_i,x_j;\beta\right)=1\right\}\cap\left\{\tau_{ij}\left(\overline{x},\underline{x}\right)=1\right\}\right\}>0,\end{aligned} \tag{26}$$

where the first equality holds by (14) and the last inequality holds by Lemma 2.

Next, we show that $\mathbb{S}^{m-1}$ is a compact set and $Q(\beta)$ is continuous on $\mathbb{S}^{m-1}$, which together with the uniqueness of β_0 shown in (26) guarantee the identification result holds by Newey and McFadden (1994). The former claim is true by the definition of $\mathbb{S}^{m-1}$. To prove the continuity of $Q(\beta)$, define

$$g_{ij}(z,\beta) := \lambda_{ij}\left(\overline{x},\underline{x};x_i,x_j;\beta\right)\tau_{ij}\left(\overline{x},\underline{x}\right) \tag{27}$$

and let z denote $(\overline{x}, \underline{x}; x_i, x_j)$. Following Newey and McFadden (1994), the sufficient condition for the continuity of $Q(\beta)$ is

(i) $\mathbb{P}\{g_{ij}(z,\beta)$ is continuous at $\beta = \beta^*\} = 1$ for every $\beta^* \in \mathbb{S}^{m-1}$, and

(ii) $\mathbb{E}\sup_{\beta\in\mathbb{S}^{m-1}} |g_{ij}(z,\beta)| < \infty$.

Part (i) is true because $\lambda_{ij}(\overline{x}, \underline{x}; x_i, x_j; \beta)$ is a binary function of $z = (\overline{x}, \underline{x}; x_i, x_j)$ and the change in value from 0 to 1 or from 1 to 0 only occurs when $d(\overline{x}; x_i, x_j)'\beta = 0$ or $d(\underline{x}; x_i, x_j)'\beta = 0$. Under Assumption 3, these two events have zero probability of happening. Thus, part (i) is verified. For part (ii), note that by construction $g_{ij}(z,\beta) \in \{0,1\}$ is a bounded function of β for all z. Therefore,

$$\mathbb{E} \sup_{\beta \in \mathbb{S}^{m-1}} |g_{ij}(z, \beta)| \leq 1 < \infty. \tag{28}$$

Hence we have for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{\beta \in \mathbb{S}^{m-1} \backslash B(\beta_0, \epsilon)} Q(\beta) \geq Q(\beta_0) + \delta, \tag{29}$$

where $B(\beta_0, \epsilon) := \left\{ \beta \in \mathbb{S}^{m-1} : \|\beta - \beta_0\| \leq \epsilon \right\}$. □

A.3 Proof of Lemma 3

Proof. Define the infeasible criterion $\widetilde{Q}_n(\beta)$ as

$$\begin{aligned} \widetilde{Q}_n(\beta) := \frac{(n-4)!}{n!} \sum_{1 \leq i \neq j \neq k \neq l \leq n} & \mathbb{1}\{\rho_i(X_k) > \rho_j(X_k)\} \cdot \mathbb{1}\{\rho_i(X_l) < \rho_j(X_l)\} \\ & \times \left[\begin{array}{c} \mathbb{1}\left\{ d(X_k; X_i, X_j)' \beta \leq 0 \right\} \\ \times \mathbb{1}\left\{ d(X_l; X_i, X_j)' \beta \geq 0 \right\} \end{array} \right]. \end{aligned} \tag{30}$$

By triangular inequality, we have

$$\sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - Q(\beta) \right| \leq \sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - \widetilde{Q}(\beta) \right| + \sup_{\beta \in \mathbb{S}^{m-1}} \left| \widetilde{Q}(\beta) - Q(\beta) \right|. \tag{31}$$

According to the decomposition (31), we divide our proof into two steps.

Step I. $\sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - \widetilde{Q}(\beta) \right| \xrightarrow{p} 0$.

By the fact that $\lambda_{ij}(X_k, X_l; X_i, X_j; \beta)$ is either 0 or 1 for any $\beta \in \mathbb{S}^{m-1}$, we have

$$\begin{aligned} & \sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - \widetilde{Q}(\beta) \right| \\ = & \frac{(n-4)!}{n!} \sum_{1 \leq i \neq j \neq k \neq l \leq n} \sup_{\beta \in \mathbb{S}^{m-1}} |\lambda_{ij}(X_k, X_l; X_i, X_j; \beta)| \\ & \times \left| \begin{array}{c} \mathbb{1}\{\rho_i(X_k) > \rho_j(X_k)\} \cdot \mathbb{1}\{\rho_i(X_l) < \rho_j(X_l)\} \\ -\mathbb{1}\{\hat{\rho}_i(X_k) > \hat{\rho}_j(X_k)\} \cdot \mathbb{1}\{\hat{\rho}_i(X_l) < \hat{\rho}_j(X_l)\} \end{array} \right| \\ \leq & \frac{(n-4)!}{n!} \sum_{1 \leq i \neq j \neq k \neq l \leq n} \left| \begin{array}{c} \mathbb{1}\{\rho_i(X_k) > \rho_j(X_k)\} \cdot \mathbb{1}\{\rho_i(X_l) < \rho_j(X_l)\} \\ -\mathbb{1}\{\hat{\rho}_i(X_k) > \hat{\rho}_j(X_k)\} \cdot \mathbb{1}\{\hat{\rho}_i(X_l) < \hat{\rho}_j(X_l)\} \end{array} \right| \\ \leq & \frac{(n-4)!}{n!} \sum_{1 \leq i \neq j \neq k \neq l \leq n} \left[\begin{array}{c} |\mathbb{1}\{\rho_i(X_k) > \rho_j(X_k)\} - \mathbb{1}\{\hat{\rho}_i(X_k) > \hat{\rho}_j(X_k)\}| \\ + |\mathbb{1}\{\rho_i(X_l) < \rho_j(X_l)\} - \mathbb{1}\{\hat{\rho}_i(X_l) < \hat{\rho}_j(X_l)\}| \end{array} \right], \end{aligned} \tag{32}$$

where the first inequality uses $|\lambda_{ij}(X_k, X_l; X_i, X_j; \beta)|$ is bounded from above by 1 and the last inequality uses the fact that whenever the LHS of the last inequality equals 1, the RHS must always equals 1.

It follows that

$$\begin{aligned}
&\mathbb{E} \sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - \widetilde{Q}(\beta) \right| \\
&\leq \mathbb{E} \left| \mathbb{1}\{\rho_i(X_k) > \rho_j(X_k)\} - \mathbb{1}\{\hat{\rho}_i(X_k) > \hat{\rho}_j(X_k)\} \right| \\
&\quad + \mathbb{E} \left| \mathbb{1}\{\rho_i(X_l) < \rho_j(X_l)\} - \mathbb{1}\{\hat{\rho}_i(X_l) < \hat{\rho}_j(X_l)\} \right|
\end{aligned} \tag{33}$$

By Assumption 6, we obtain

$$\mathbb{E} \sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - \widetilde{Q}(\beta) \right| \to 0 \tag{34}$$

using Dominated Convergence Theorem.

Finally, by Markov inequality, we have

$$\sup_{\beta \in \mathbb{S}^{m-1}} \left| \widehat{Q}_n(\beta) - \widetilde{Q}(\beta) \right| \xrightarrow{p} 0. \tag{35}$$

Step II. $\sup_{\beta \in \mathbb{S}^{m-1}} \left| \widetilde{Q}_n(\beta) - Q(\beta) \right| \xrightarrow{p} 0.$

For this part of the proof, we adapt to section 9.5 of Toth (2017) and use existing results from the U-process literature. We have $\left\{ \widetilde{Q}_n(\beta) - Q(\beta) : \beta \in \mathbb{S}^{m-1} \right\}$ is a centered U-process of order 4. We follow the arguments from the seminal papers Nolan and Pollard (1987) and Sherman (1994). For a systematic understanding of U-statistics, we refer the readers to Serfling (2009).

First, we show $\left\{ \widetilde{Q}_n(\beta) - Q(\beta) : \beta \in \mathbb{S}^{m-1} \right\}$ is Euclidean for the constant envelope of 1 (See Definition 8 in Nolan and Pollard (1987)). To see why, first note that the unsymmetrized kernel of $\widetilde{Q}_n(\beta) - Q(\beta)$ for any $\beta \in \mathbb{S}^{m-1}$ is defined to be

$$\begin{aligned}
kernel := {} & \lambda_{ij}(X_k, X_l; X_i, X_j; \beta)\, \mathbb{1}\{\rho_i(X_k) > \rho_j(X_k)\} \\
& \times \cdot \mathbb{1}\{\rho_i(X_l) < \rho_j(X_l)\} \\
& - \mathbb{E}\Big[\tau_{ij}(X_k, X_l) \cdot \lambda_{ij}(X_k, X_l; X_i, X_j; \beta)\Big].
\end{aligned} \tag{36}$$

The kernel defined in (36) belongs to a Euclidean class if and only if the function class of $\lambda_{ij}(X_k, X_l; X_i, X_j; \beta)$ indexed by β is Euclidean because the property is closed under finite addition, multiplication and linear operations, see Nolan and Pollard (1987). By (23), we have

$$\lambda_{ij}\left(X_k, X_l; X_i, X_j; \beta\right) = \mathbb{1}\left\{d(X_k; X_i, X_j)'\beta \leq 0\right\} \mathbb{1}\left\{d(X_l; X_i, X_j)'\beta \geq 0\right\}. \tag{37}$$

Note that the function class of $\lambda_{ij}\left(X_k, X_l; X_i, X_j; \beta\right)$ indexed by β is Euclidean if and only if the function class of $d(X_k; X_i, X_j)'\beta$ is Euclidean, again by closure under finite multiplication and indicator functions.

Define the function class $\mathscr{G}$ of $g\left(X; Y, Z\right) := d(X; Y, Z)'\beta$ to be

$$\mathscr{G} := \left\{d(X; Y, Z)'\beta \middle| \beta \in \mathbb{S}^{m-1}\right\}. \tag{38}$$

We have $\mathscr{G}$ forms a finite dimensional vector space of functions as long as $m < \infty$. By Lemma 18 of Nolan and Pollard (1987), the collection of all sets of the form $\{g \geq 0\}$ or $\{g \leq 0\}$ or $\{g > 0\}$ or $\{g < 0\}$ for any $g \in \mathscr{G}$ is a polynomial class, which implies $\{\text{graph}\left(g\right) : g \in \mathscr{G}\}$ is a polynomial class of sets because any class of subsets of $\mathbb{R}$ is a polynomial class. From this result and Lemma 19 of Nolan and Pollard (1987), we have $\mathscr{G}$ is Euclidean. Therefore, the kernel defined in (36) indeed belongs to a Euclidean class, and according to Corollary 7 in Sherman (1994), we have

$$\sup_{\beta \in \mathbb{S}^{m-1}} \left|\widetilde{Q}_n\left(\beta\right) - Q\left(\beta\right)\right| \xrightarrow{p} 0. \tag{39}$$

Combining (35) and (39), we have

$$\sup_{\beta \in \mathbb{S}^{m-1}} \left|\widehat{Q}_n\left(\beta\right) - Q\left(\beta\right)\right| \xrightarrow{p} 0. \tag{40}$$

□

A.4 Proof of Theorem 2

Proof. We aim to prove, for any $\epsilon > 0$, $\mathbb{P}\left(\|\widehat{\beta}_n - \beta_0\|\right) > \epsilon \to 0$. According to the proof in Theorem 1, we have for any $\epsilon > 0$, there exists $\delta > 0$ such that $\inf_{\beta \in \mathbb{S}^{m-1} \backslash B_m(\beta_0, \epsilon)} Q\left(\beta\right) \geq Q\left(\beta_0\right) + \delta$, where $B_m(\beta_0, \epsilon) = \left\{\beta \in \mathbb{S}^{m-1} : \|\beta - \beta_0\| \leq \epsilon\right\}$. It follows that there exist $\delta > 0$ such that

$$\mathbb{P}\left(\|\widehat{\beta}_n - \beta_0\| > \epsilon\right) = \mathbb{P}\left(\widehat{\beta}_n \in \mathbb{S}^{m-1} \backslash B_m\left(\beta_0, \epsilon\right)\right) \leq \mathbb{P}\left(Q\left(\widehat{\beta}_n\right) \geq Q\left(\beta_0\right) + \delta\right). \tag{41}$$

By construction of $\widehat{\beta}_n$, we have $\widehat{Q}_n(\widehat{\beta}_n) - \widehat{Q}_n(\beta_0) \leq 0$. Therefore,

$$
\begin{aligned}
&\mathbb{P}\left(Q(\widehat{\beta}_n) \geq Q(\beta_0) + \delta\right) \\
&= \mathbb{P}\Big(Q(\widehat{\beta}_n) - \widehat{Q}_n(\widehat{\beta}_n) + \widehat{Q}_n(\widehat{\beta}_n) - \widehat{Q}_n(\beta_0) + \widehat{Q}_n(\beta_0) - Q(\beta_0) \geq \delta\Big) \\
&\leq \mathbb{P}\Big(Q(\widehat{\beta}_n) - \widehat{Q}_n(\widehat{\beta}_n) + 0 + \widehat{Q}_n(\beta_0) - Q(\beta_0) \geq \delta\Big).
\end{aligned} \tag{42}
$$

It follows that

$$
\mathbb{P}\left(Q(\widehat{\beta}_n) \geq Q(\beta_0) + \delta\right) \leq \mathbb{P}\left(\sup_{\beta \in \mathbb{S}^{m-1}} \left|\widehat{Q}_n(\beta) - Q(\beta)\right| \geq \delta/2\right). \tag{43}
$$

By Lemma 3, we have for any $\delta > 0$

$$
\mathbb{P}\left(\sup_{\beta \in \mathbb{S}^{m-1}} \left|\widehat{Q}_n(\beta) - Q(\beta)\right| \geq \delta/2\right)\Big) \to 0 \text{ as } n \to \infty. \tag{44}
$$

Therefore, we have for any $\epsilon > 0$

$$
\mathbb{P}\left(\|\widehat{\beta}_n - \beta_0\| > \epsilon\right) \leq \mathbb{P}\left(\sup_{\beta \in \mathbb{S}^{m-1}} \left|\widehat{Q}_n(\beta) - Q(\beta)\right| \geq \delta/2\right) \to 0 \text{ as } n \to \infty. \tag{45}
$$

□

A.5 Asymmetry of Pairwise Observable Characteristics

So far we have been focusing on the case with symmetric pairwise observable characteristics, i.e.,

$$
w(X_i, X_j) \equiv w(X_j, X_i).
$$

In this section, we briefly discuss how our method can be adapted to accommodate asymmetric pairwise observable characteristics.

As in Remark 1, consider the adapted model (6):

$$
\mathbb{E}\left[D_{ij} \middle| X_i, X_j, A_i, A_j\right] = \phi\left(w(X_i, X_j)' \beta_0, w(X_j, X_i)' \beta_0, A_i, A_j\right) \tag{46}
$$

where w needs not be symmetric with respect to its two vector arguments and $\phi : \mathbb{R}^4 \to \mathbb{R}$ is required to be monotone in all its four arguments.

The technique of logical differencing still applies in the exactly same way as before. Specifically, the event $\left\{\rho_{\bar{i}}(\overline{x}) > \rho_{\bar{j}}(\overline{x})\right\}$ implies that

$$
\left\{w(X_{\bar{i}}, \overline{x})' \beta_0 > w\left(X_{\bar{j}}, \overline{x}\right)' \beta_0\right\} \text{ OR } \left\{w(\overline{x}, X_{\bar{i}})' \beta_0 > w\left(\overline{x}, X_{\bar{j}}\right)' \beta_0\right\} \text{ OR } \left\{A_{\bar{i}} > A_{\bar{j}}\right\},
$$

while the event $\left\{\rho_{\bar{i}}\left(\underline{x}\right) < \rho_{\bar{j}}\left(\underline{x}\right)\right\}$ implies that

$$\left\{w\left(X_{\bar{i}},\underline{x}\right)'\beta_0 < w\left(X_{\bar{j}},\underline{x}\right)'\beta_0\right\} \text{ OR } \left\{w\left(\underline{x},X_{\bar{i}}\right)'\beta_0 < w\left(\underline{x},X_{\bar{j}}\right)'\beta_0\right\} \text{ OR } \left\{A_{\bar{i}} < A_{\bar{j}}\right\}.$$

The joint occurrence of $\left\{\rho_{\bar{i}}\left(\overline{x}\right) > \rho_{\bar{j}}\left(\overline{x}\right)\right\}$ and $\left\{\rho_{\bar{i}}\left(\underline{x}\right) < \rho_{\bar{j}}\left(\underline{x}\right)\right\}$ now implies that

$$\begin{aligned} &\left\{w\left(X_{\bar{i}},\overline{x}\right)'\beta_0 > w\left(X_{\bar{j}},\overline{x}\right)'\beta_0\right\} \text{ OR } \left\{w\left(\overline{x},X_{\bar{i}}\right)'\beta_0 > w\left(\overline{x},X_{\bar{j}}\right)'\beta_0\right\} \\ \text{OR } &\left\{w\left(X_{\bar{i}},\underline{x}\right)'\beta_0 < w\left(X_{\bar{j}},\underline{x}\right)'\beta_0\right\} \text{ OR } \left\{w\left(\underline{x},X_{\bar{i}}\right)'\beta_0 < w\left(\underline{x},X_{\bar{j}}\right)'\beta_0\right\}, \end{aligned} \tag{47}$$

which is in general "less restrictive" than the corresponding identifying restriction in Lemma 1.

In particular, in the extreme case where w is *antisymmetric* in the sense of

$$w\left(X_i,X_j\right) \equiv -w\left(X_j,X_i\right),$$

the identifying restriction on the RHS of

$$\left\{w\left(X_{\bar{i}},\overline{x}\right)'\beta_0 > w\left(X_{\bar{j}},\overline{x}\right)'\beta_0\right\} \text{ OR } \left\{w\left(\overline{x},X_{\bar{i}}\right)'\beta_0 > w\left(\overline{x},X_{\bar{j}}\right)'\beta_0\right\}$$

becomes

$$\left\{w\left(X_{\bar{i}},\overline{x}\right)'\beta_0 \neq w\left(X_{\bar{j}},\overline{x}\right)'\beta_0\right\},$$

which can be generically true and thus becomes (almost) trivial.

Correspondingly, Assumption 3 needs to be strengthened for point identification:

Assumption (3a). *There exist a pair of* $\overline{x},\underline{x}$*, both of which lie in the support of* $Supp\left(X_i\right)$*, such that*

$$Supp\left(w\left(\overline{x},X_i\right) - w\left(\underline{x},X_i\right)\right) \cap Supp\left(w\left(X_i,\overline{x}\right) - w\left(X_i,\underline{x}\right)\right)$$

contains all directions in $\mathbb{R}^m$*.*

Clearly, the case of antisymmetric w is ruled out by Assumption 3a. Assumption A.5 ensures that, for any $\beta \neq \beta_0$, there exist in-support x_i and x_j such that

$$\begin{aligned} &\left\{w\left(x_i,X_k\right)'\beta_0 > w\left(x_j,X_k\right)'\beta_0\right\} \text{ AND } \left\{w\left(x_i,X_l\right)'\beta_0 < w\left(x_j,X_l\right)'\beta_0\right\} \\ \text{AND} &\left\{w\left(X_k,x_i\right)'\beta_0 > w\left(X_k,x_j\right)'\beta_0\right\} \text{ AND } \left\{w\left(X_l,x_i\right)'\beta_0 < w\left(X_l,x_j\right)'\beta_0\right\} \end{aligned} \tag{48}$$

and

$$\begin{aligned} & \left\{w\left(x_i, X_k\right)' \beta \leq w\left(x_j, X_k\right)' \beta\right\} \text{ AND } \left\{w\left(x_i, X_l\right)' \beta \geq w\left(x_j, X_l\right)' \beta\right\} \\ \text{AND} & \left\{w\left(X_k, x_i\right)' \beta \leq w\left(X_k, x_j\right)' \beta\right\} \text{ AND } \left\{w\left(X_l, x_i\right)' \beta \geq w\left(X_l, x_j\right)' \beta\right\} \end{aligned} \tag{49}$$

occur simultaneously with strictly positive probability. (48) and (49) are sufficient for $\left\{\rho_{\bar{i}}\left(\overline{x}\right) > \rho_{\bar{j}}\left(\overline{x}\right)\right\}$ and $\left\{\rho_{\bar{i}}\left(\underline{x}\right) < \rho_{\bar{j}}\left(\underline{x}\right)\right\}$ to occur simultaneously under the maintained assumption on the support of A_i. It thus can guarantee point identification of β_0.

The estimator can be correspondingly adapted in an obvious manner.