

# MORAL HAZARD AND THE OPTIMALITY OF DEBT

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**ABSTRACT.** Why are debt securities so common? I show that debt securities minimize the welfare losses from the moral hazards of excessive risk-taking and lax effort. For any security design, the variance of the security payoff is a statistic that summarizes these welfare losses. Debt securities have the least variance, among all limited liability securities with the same expected value. The optimality of debt is exact in my benchmark model, and holds approximately in a wide range of models. I study both static and dynamic security design problems, and show that these two types of problems are equivalent. The models I develop are motivated by moral hazard in mortgage lending, where securitization may have induced lax screening of potential borrowers and lending to excessively risky borrowers. My results also apply to corporate finance and other principal-agent problems.

## 1. INTRODUCTION

Debt contracts are widespread, even though debt encourages excessive risk taking. A leading example of this phenomenon occurs in residential mortgage securitization. Prior to the recent financial crisis, mortgage lenders sold debt securities, backed by mortgage loans, to outside investors. The issuance of these securities may have weakened the incentives of mortgage lenders to lend prudently. Even before the financial crisis, the incentive problems associated with debt were well-known. Why did mortgage lenders sell debt securities? Would some other security design, such as equity, have created better incentives?

In this paper, I show that debt is the optimal security design in a model in which both reduced effort and excessive risk-taking are possible. In the model, the seller of the security (e.g. the mortgage lender) can alter the probability distribution of outcomes in arbitrary ways. I show that, to minimize the welfare losses arising from this moral hazard, the security's payout must be designed to minimize variance. Debt securities are optimal because, among all limited-liability securities

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*Key words and phrases.* security design, moral hazard, optimal contracts.

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with the same expected value, they have the least variance. This theory explains why mortgage lenders found it optimal to sell debt securities, even though debt encourages excessive risk taking.

In my model, the seller can choose any probability distribution for the value of the assets (e.g. the mortgage loans) backing the security, which allows for both reduced effort and risk-shifting. Effort refers to actions that change the mean asset value, and risk-shifting refers to actions that change other moments of the distribution. The security is the portion of the asset value received by the outside investors. If the seller retains a levered equity claim, she<sup>1</sup> has sold a debt security.

There are gains from trade, meaning that the outside investors value the security more than the seller does, holding the distribution of outcomes fixed. Both the outside investors and the seller are risk-neutral. There is a “zero-cost” distribution, which the seller will choose if she has no stake in the outcome. If the seller chooses any other probability distribution, she incurs a cost. In my benchmark model, the cost to the seller of choosing a probability distribution  $p$  is proportional to the Kullback-Leibler divergence<sup>2</sup> of  $p$  from the zero-cost distribution. Under these assumptions, the optimal limited liability security design is a debt contract.

In this model, the combined effects of reduced effort and risk-shifting can be summarized by one statistic, the variance of the security payoff. The gains from trade are proportional to the mean security payoff. I show that debt securities maximize a mean-variance tradeoff, which means that debt optimally balances the problems of reduced effort and risk-shifting against the gains from trade. This illustrates a key distinction between my model and the existing security design literature: I argue that debt is optimal precisely because both effort and risk-shifting are possible.

The classic paper of Jensen and Meckling [1976] argues that debt securities are good at providing incentives for effort, but create incentives for risk-shifting, while equity securities avoid risk-shifting problems, but provide weak incentives for effort. A natural conjecture, based on these intuitions, is that when both risk-shifting problems and effort incentives are important, the optimal security will be “in between” debt and equity. This paper shows, contrary to this intuition, that a debt security is optimal.

The argument of Jensen and Meckling [1976] that debt is best for inducing effort relies on a restriction to monotone security designs. The “live-or-die” result of Innes [1990] (see the appendix, section §A, Figure A.1) shows that when the seller can supply effort to improve the mean of the distribution, it is efficient to give the seller all of the asset value when the asset value is high, and nothing otherwise.<sup>3</sup> I show that when both risk-shifting and effort are important, debt is optimal.

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<sup>1</sup>Throughout the paper, I will use she/her to refer to the seller and he/his to the buyer of the security. No association of the agents to particular genders is intended.

<sup>2</sup>The KL divergence is also known as relative entropy. It is defined in section §2.

<sup>3</sup>This result assumes a monotone likelihood ratio property in effort.

Debt is “in between” the live-or-die security, which induces optimal effort, and equity, which avoids risk-shifting. I formalize this idea in section §4.

I also show that, with cost functions other than the KL divergence, the variance of the security design approximates the costs of reduced effort and risk-shifting. In these models, debt securities are approximately optimal. I then demonstrate that this mean-variance tradeoff applies to models in which the seller can choose only a few parameters, rather than the entire probability distribution (“parametric” models). Finally, I show that this mean-variance tradeoff applies to continuous-time models of moral hazard, in which the seller can dynamically choose to exert effort. Taken together, these results show that debt is optimal or approximately optimal in a wide range of models.

With more general cost functions, debt securities are approximately optimal. The approximation I use applies when the moral hazard and gains from trade are small relative to scale of the assets. Debt is first-order optimal, meaning that when this approximation is accurate, debt securities are a detail-free way to achieve nearly the same utility as the optimal security design. A mixture of debt and equity is second-order optimal. This can be interpreted as a “pecking order,” in which the security design grows more complex as the size of both the moral hazard problem and gains from trade grow, relative to the scale of the assets.

The literature on security design with moral hazard typically studies parametric models of moral hazard. One example of a parametric model is when the seller controls the mean and variance of a log-normal distribution. In these models of moral hazard, the design of the optimal contract depends on the set of controls available to the seller. I show that regardless of the controls available to the seller, the mean-variance tradeoff provides an approximate lower bound on the security designer’s utility, and debt maximizes this lower bound.<sup>4</sup> This lower bound applies to problems in which the standard “first-order approach” fails,<sup>5</sup> and to most single-parameter moral hazard problems. Because this lower bound does not depend on the actions available to the agent, it is similar in spirit to the robust contracting models of Carroll [2013] and Chassang [2013].

I also provide a micro-foundation for the security design problem with the KL divergence cost function. I show that a continuous-time moral hazard problem, similar to Holmström and Milgrom [1987], is equivalent to the static moral hazard problem. The equivalence of the static and dynamic problems provides an intuitive explanation for how the seller can create any probability distribution of outcomes. The key distinction between the dynamic models I discuss and the principal-agent models found in Holmström and Milgrom [1987] is limited liability. In Holmström and Milgrom [1987], linear contracts for the seller (agent) are optimal, because they induce the seller to take the same action each period. In my model, because of limited liability, the only way to always

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<sup>4</sup>The result that debt is approximately optimal in parametric models is a generalization of Bose et al. [2011].

<sup>5</sup>For a discussion of the “first-order” approach to principal-agent problems, see Jewitt [1988], Mirrlees [1999].

implement the efficient action is to offer the seller a very large share of the asset value. However, offering the seller a large share of the asset value limits the gains from trade. It is preferable to pay the seller nothing in the worst states of the world, and then at some point offer a linear payoff. Even though this design does not induce the seller to take the efficient action in every state, it achieves a much larger level of gains from trade. This design for the retained tranche, levered equity, corresponds to selling a debt security.

The models I present are motivated by the securitization of mortgage loans, and other asset-backed securities.<sup>6</sup> In this setting, the seller sells a security, backed by those loans, to outside investors. I refer these investors as a single agent, the “buyer.” The buyer bears some or all of the risk associated with these loans.<sup>7</sup> If the buyer does not observe the quality of the loans, there is a moral hazard problem, because the seller might not carefully screen borrowers when making loans. To mitigate this problem, the seller can retain exposure to the performance of the loans. One question that naturally arises is what the shape of this retained exposure should be. This is a security design problem, called “tranche retention” or “risk retention.” Tranche retention is the subject of new regulations after the recent financial crisis, in both the U.S. and E.U. (Fender and Mitchell [2009], Geithner [2011]). The issue of tranche retention by the seller is distinct from the question of how the buyer’s claims might subsequently be divided into tranches. This paper is silent about the latter issue.

In my model, the seller acts to maximize the value of the portion of the asset that she does not sell to the buyer. The most obvious interpretation of this assumption is that the seller retains the tranche not sold to the buyer, as is common practice in asset securitizations (Begley and Purnanandam [2013], Gorton [2008], Gorton and Metrick [2012]). Alternatively, imagine that there are two kinds of buyers, informed and uninformed, and that informed buyers require higher returns on their investments. The seller sells the security to the uninformed buyer, who cannot observe the seller’s actions, and the remaining portion of the cash flows to the informed buyer, who can observe the seller’s actions. This market structure will also cause the seller take actions that maximize the value of the “retained” tranche, even though she is selling that tranche to a third party. Two recent papers provide evidence that the size of the first loss tranche in residential and commercial mortgage-backed securities deals affected the subsequent default rates on those deals, consistent with the assumption that the design of the retained tranche altered the incentives of the originator (Begley and Purnanandam [2013] and Ashcraft et al. [2014]).

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<sup>6</sup>In this context, the asset in question (the mortgage loan) is also a debt, between the homeowner and the lender. The issues of effort and risk-shifting emphasized in this paper may also be relevant to the relationship between the homeowner and mortgage lender.

<sup>7</sup>This is a very stylized account of securitization. In particular, I model the originator of the mortgage loans and the sponsor of the securitization (usually an investment bank) as a single entity. See Gorton and Metrick [2012] for an in-depth description of the process.

In my benchmark model, debt is optimal regardless of whether the moral hazard occurs before or after the security is sold. With mortgage origination, it is natural to assume that the mortgage loans are made before the security is sold, and that this is where the bulk of the moral hazard exists. An example of the opposite timing is the so-called “Bowie bonds,” in which the musician David Bowie sold debt securities backed by song royalties to the insurance company Prudential Financial Inc. (Sylva [1999]). One possible concern in such a deal is that the actions taken by Bowie to promote himself and his music might be altered by the transaction, because of his reduced incentive to maximize the value of the song royalties. In that deal, David Bowie retained the rights to the song royalties, as long as the interest on the bonds was paid, mitigating this moral hazard problem. My models apply to this version of the tranche retention problem as well.

My models also apply to many other settings. One possible application in finance is the design of collateralized debt obligations, in which the security designer could choose a wide range of assets to put into the CDO. The models can be applied to corporate finance, when considering the capital structure of firms, and to principal-agent problems more generally.

I focus on one particular aspect of these problems: the flexible nature of the seller’s moral hazard problem. When a mortgage originator makes loans, she can screen borrowers along many different, unobservable dimensions. The benchmark model in this paper takes this idea to one extreme, allowing the seller to create any probability distribution of outcomes, subject to a cost. This approach to moral hazard problems was introduced by Holmström and Milgrom [1987]. It is conceptually similar to the notion of flexible information acquisition, emphasized in Yang [2012].<sup>8</sup> In contrast, much of literature on security design with moral hazard allows the seller to control only one or two parameters of the probability distribution. These papers do not find that debt is optimal.<sup>9</sup>

Several of these models are examples of the parametric models discussed previously. Closest to this paper is Hellwig [2009], who has a two-parameter model with continuous choices for risk-shifting and effort, and finds that a mix of debt and equity are optimal. In his model, risk-shifting is costless for the agent. Fender and Mitchell [2009] have a model of screening and tranche retention,

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<sup>8</sup>One key distinction between the approach of this paper and Holmström and Milgrom [1987], and the approach in Yang [2012], concerns the interpretation of the costs. In this paper, the cost of choosing a probability distribution should be interpreted as a cost associated with the actions required to cause that distribution to occur (underwriting or not underwriting mortgage loans, for example). In the rational inattention literature, which Yang [2012] builds on, gathering or processing information (as opposed to taking actions) is costly. This distinction is blurred in the rational inattention micro-foundation in the appendix, section §D. There is also a methodological relationship between the two approaches (see the appendix, section E.1 and section E.10).

<sup>9</sup>In Acharya et al. [2012], bank managers can both shift risk and pursue private benefits, but do this by choosing amongst three possible investments. In Biais and Casamatta [1999], there are three possible states and two levels of effort and risk-shifting. Edmans and Liu [2011], who argue that is efficient for the agent (not the principal) to hold debt claims, also have a binary project choice.

which is a single-parameter model. The model of Gorton and Pennacchi [1995] is also a single-parameter model that fits into my framework.

Innes [1990] advocates a moral-hazard theory of debt, but debt is optimal only when the seller controls a single parameter, and the security is constrained to be monotone. If the security does not need to be monotone, or if the seller controls both the mean and variance of a log-normal distribution, the optimal contract is not debt.<sup>10</sup> In the corporate finance setting, one argument for monotonicity is that a manager can borrow from a third party, claim higher profits, and then repay the borrowed money from the extra contract payments. In addition to the accounting and legal barriers to this kind of “secret borrowing,” the third party might find it difficult to force repayment.<sup>11</sup> Moreover, in the context of asset-backed securities, where cash flows are more easily verified, secret borrowing is even less plausible.<sup>12</sup>

Two strands of the existing security design literature have focused on adverse selection as a justification for debt. In these models, either the seller<sup>13</sup> or buyer<sup>14</sup> is endowed with, or can acquire, information about the asset’s value before trading the security. In the context of mortgage origination, there is empirical evidence for lax screening by originators who intended to securitize their mortgage loans, which suggests that moral hazard is a relevant issue.<sup>15</sup> There are also mechanisms to mitigate adverse selection by the seller, such as the inability to retain loans and random selection of loans into securitization (Keys et al. [2010]). The evidence of Jiang et al. [2013] shows that, for one particular lender, there was actually favorable selection of loans into securitization, due to delays in the securitization process that allowed additional information about loan quality to be revealed. For these reasons, I focus on moral hazard as an explanation for debt, while noting that moral hazard and buyer-side adverse selection are mutually compatible explanations for

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<sup>10</sup>For brevity, I have omitted this result from the paper. It is available upon request.

<sup>11</sup>This argument against secret borrowing was suggested to me by Oliver Hart.

<sup>12</sup>Another argument in favor of monotonicity from corporate finance concerns the possibility of the buyer (principal, outside shareholders) sabotaging the project. In the context of securitization, the buyer exerts minimal control over the securitization trust and sabotage is not a significant concern.

<sup>13</sup>See Nachman and Noe [1994], DeMarzo and Duffie [1999], and DeMarzo [2005].

<sup>14</sup>See Gorton and Pennacchi [1990], Dang et al. [2011], and Yang [2012]. The buyer might also anticipate adverse selection by a third-party in the future.

<sup>15</sup>See Demiroglu and James [2012], Elul [2011], Jiang et al. [2013], Keys et al. [2010], Krainer and Laderman [2014], Mian and Sufi [2009], Nadauld and Sherlund [2013], Purnanandam [2011], Rajan et al. [2010]. Some of this evidence is disputed (see Bubb and Kaufman [2014]). Some of this evidence is consistent with information asymmetries, but cannot distinguish between moral hazard and adverse selection.

debt.<sup>16</sup> Several other recent papers also model moral hazard and securitization, but focus on the intertemporal aspect of security design.<sup>17</sup>

In corporate finance, there are many theories to explain the prevalence of debt. I argue that most of these theories are not applicable to the tranche retention problem in securitization, which motivates my search for an alternative explanation. For example, there are usually no tax benefits to retaining one type of tranche over another. The cash flows underlying the security are readily verifiable, so arguments based on costly state verification (Townsend [1979], Gale and Hellwig [1985]) are not applicable. The servicer of the asset-backed security, who maintains the assets post-contracting, has limited ability to pursue non-pecuniary benefits or expand the scale of the assets. Explanations based on control or limiting investment (Aghion and Bolton [1992], Jensen [1986], Hart and Moore [1994]) are therefore not relevant for this context. Other explanations include standardization or regulatory treatment, but these explanations do not account for the wide prevalence of debt.

I begin in section §2 by explaining the benchmark security design problem, whose structure is used throughout the paper. I then show in section §3 that for a particular cost function, debt is optimal, and explain how this relates to a mean-variance tradeoff. Next, I show in section §4 that this mean-variance tradeoff holds approximately for a wider class of cost functions. I will then discuss higher-order approximations, and in section §5 discuss parametric models. Finally, in section §6 and section §7, I provide micro-foundations for the non-parametric models, from a continuous time model.

## 2. MODEL FRAMEWORK

In this section, I introduce the security design framework that I will discuss throughout the paper. The problem is close to Innes [1990] and other papers in the security design literature. There is a risk-neutral agent, called the “seller,” who owns an asset in the first period. In the second period, one of  $N + 1$  possible states, indexed by  $i \in \Omega = \{0, 1, \dots, N\}$ , occurs.<sup>18</sup> In each of these states, the seller’s asset has an undiscounted value of  $v_i$ . I assume that  $v_0 = 0$ ,  $v_i$  is non-decreasing in  $i$ , and that  $v_N > v_0$ .

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<sup>16</sup>In earlier versions of this paper, I showed that the moral hazard model I develop can be combined with the model of Yang [2012], and the the optimal security design is a debt. In contrast, combining a parametric model of moral hazard with the model of Yang [2012] would not generally result in debt as the optimal security design. I view buyer-side adverse selection theories of debt as being focused on why the senior (AAA) securities were designed as debt, whereas this paper focuses on the shape of the tranche retained by the seller. Following a different approach, Vanasco [2013] combines a seller-side adverse selection model with moral hazard.

<sup>17</sup>See Hartman-Glaser et al. [2012], and Malamud et al. [2013].

<sup>18</sup>Using a discrete outcome space simplifies the exposition, but is not necessary for the main results.

The seller discounts second period payoffs to the first period with a discount factor  $\beta_s$ . There is a second risk-neutral agent, the “buyer,” who discounts second period payoffs to the first period with a larger discount factor,  $\beta_b > \beta_s$ . Because the buyer values second period cash flows more than the seller, there are “gains from trade” if the seller gives the buyer a second period claim in exchange for a first period payment. I will refer to the parameter  $\kappa = \frac{\beta_b - \beta_s}{\beta_s}$  as the gains from trade.

I assume there is limited liability, so that in each state the seller can credibly promise to pay at most the value of the asset. I also assume that the seller must offer the buyer a security, meaning that the second period payment to the buyer must be weakly positive. In this sense, the seller must offer the buyer an “asset-backed security.” When the asset takes on value  $v_i$  at time one, the security pays  $s_i \in [0, v_i]$  to the buyer. Following the conventions of the literature, I will say that the security is a debt security if  $s_i = \min(v_i, \bar{v})$  for some  $\bar{v} \in (0, v_N)$ .

I also assume that the seller designs the security and makes a “take-it-or-leave-it” offer to the buyer at price  $K$ . If the buyer rejects the offer, the seller retains the entire asset. In this case, it is as if the seller had offered the “nothing” security at a zero price. I have given all the bargaining power to the seller, which simplifies the exposition and does not alter the main results.

I will briefly discuss several possible timing conventions for the sequence of decisions by the seller during the first period. In that period, the seller designs the security, sells it to the buyer (assuming the buyer accepts), and takes actions that will create or modify the assets backing the security. The timing convention refers to the order in which these three steps occur. In the first timing convention, the “shelf registration” convention (using the terminology of DeMarzo and Duffie [1999]), the security is designed before the assets are created, but sold afterward. In the second timing convention, the “origination” convention, the security is designed and sold after the assets are created. In the third timing convention, the “principal-agent” convention, the security is designed and sold before the seller takes her actions. In this last convention, it is natural to assume that the asset exists before the security is designed, but its payoffs are modified by the seller’s actions after the security is traded.

There are asset securitization examples for each of these timing conventions. For some asset classes, such as first-lien mortgages, the security design is standardized, and the “shelf registration” timing convention is appropriate. For more unusual assets, the security design varies deal-by-deal, and the “origination” timing convention is appropriate. In some cases, such as the Bowie bonds mentioned previously, maintaining incentives post-securitization is important, and the principal-agent timing convention applies.

Most of the results in this paper hold regardless of the timing convention. Intuitively, because the model is a “pure” moral hazard model, there is no information to signal. This contrasts with models based on adverse selection by the seller, such as DeMarzo and Duffie [1999], in which

the timing of events is crucial. For a more detailed description of the timing conventions and equilibrium concept, see section §B in the appendix.

The moral hazard problem occurs when the seller creates or modifies the asset. During this process, the seller will take a variety of actions, and these actions will alter the probability distribution of second period asset values over the sample space. Following Holmström and Milgrom [1987], I model the seller as directly choosing a probability distribution,  $p$ , over the sample space  $\Omega$ , subject to a cost  $\psi(p)$ . I will consider two versions of the model: one in which any probability distribution  $p$  can be chosen—which I will call “non-parametric”—and one in which  $p$  must belong to a parametric family of distributions.

I will make several assumptions about the cost function  $\psi(p)$ . First, I assume that there is a unique probability distribution,  $q$ , with full support over  $\Omega$ , that minimizes the cost. Second, because I will not consider participation constraints for the seller, I assume without loss of generality that  $\psi(q) = 0$ . I also assume that  $\psi(p)$  is strictly convex and smooth. Below, I will impose additional assumptions on the cost function, but first will describe the moral hazard and security design problems.

The moral hazard occurs because the seller cares only about maximizing the discounted value of her retained tranche. When the value of the asset is  $v_i$ , the discounted value of the retained tranche is

$$\eta_i = \beta_s(v_i - s_i).$$

Because of that assumption that  $v_0 = 0$ , and limited liability, it is always the case that  $\eta_0 = s_0 = 0$ . Denote the probability that the asset is worth  $v_i$  as  $p^i$ , under probability distribution  $p$ . The moral hazard sub-problem of the seller can be written as

$$(2.1) \quad \phi(\eta) = \sup_{p \in M} \left\{ \sum_{i>0} \eta_i p^i - \psi(p) \right\},$$

where  $M$  is the set of feasible probability distributions and  $\phi(\eta)$  is the indirect utility function. In the non-parametric case, when  $M$  is the set of all probability distributions on the sample space, the moral hazard problem has a unique optimal  $p$  for each  $\eta$ . In the parametric case, there may be multiple  $p \in M$  that achieve the same optimal utility for the seller. In that case, I assume that the seller chooses from this set of optimal  $p$  to maximize the buyer’s utility.<sup>19</sup>

The buyer cannot observe  $p$  directly, but can infer the seller’s choice of  $p$  from the design of the seller’s retained tranche  $\eta$ . At the security design stage, the buyer’s valuation of a security is determined by both the structure of the security  $s$  and the buyer’s inference about which probability

<sup>19</sup>This assumption is made for expository purposes and does not affect any of the paper’s results.

distribution the seller will choose,  $p(\eta)$ . Without loss of generality, I will define the units of the seller's and buyer's payoffs so that  $\beta_s \sum_i v_i p^i(\beta_s v_i) = 1$ . If the seller retains the entire asset, and takes actions in the moral hazard problem accordingly, the discounted asset value is one.<sup>20</sup>

Let  $s_i(\eta)$  be the security design corresponding to retained tranche  $\eta$ . The security design problem can be written as

$$(2.2) \quad U(\eta^*) = \max_{\eta} \left\{ \beta_b \sum_{i>0} p^i(\eta) s_i(\eta) + \phi(\eta) \right\}$$

subject to the limited liability constraint that  $\eta_i \in [0, \beta_s v_i]$ . From the seller's perspective, when she is designing the security, she internalizes the effect that her subsequent choice of  $p$  will have on the buyer's valuation, because that valuation determines the price at which she can sell the security. The security serves as a commitment device for the seller, providing an incentive for her to choose a favorable  $p$ . This commitment is costly, however, because allocating more of the available asset value to the retained tranche necessarily reduces the payout of the security, reducing the gains from trade.

The structure of this model generalizes Innes [1990]. In that paper, the model is parametric, with a single parameter, "effort." The convex cost of effort can be rewritten to depend on  $p$ , fitting into the framework described above. The probability distribution that would result from zero effort corresponds to  $q$ . More complex parametric models (such as the ones in Fender and Mitchell [2009] and Hellwig [2009]) can also fit into this framework.

From the earlier assumptions about the cost function, it follows that the cost function can be written as a divergence<sup>21</sup> between  $p$  and the zero-cost distribution,  $q$ , defined for all  $p, q \in M$  :

$$\psi(p) = D(p||q).$$

In section §3, I begin the paper by focusing on a particular divergence, the Kullback-Leibler divergence. The KL divergence, also called relative entropy, is defined as

$$D_{KL}(p||q) = \sum_{i \in \Omega} p^i \ln\left(\frac{p^i}{q^i}\right).$$

The KL divergence has the assumed convexity and differentiability properties, and also guarantees that the  $p$  chosen by the seller will be mutually absolutely continuous with respect to  $q$ . The KL

<sup>20</sup>I use this convention to ensure that the units correspond to an empirically observable quantity— the value of the assets, if those assets are retained by the seller. This convention is useful when calibrating the model (see the appendix, section §C).

<sup>21</sup>A "divergence" is similar to a distance, except that there is no requirement that it be symmetric between  $p$  and  $q$ , or that it satisfy the triangle inequality.

divergence has been used in a variety of economic models, notably Hansen and Sargent [2008], who use it to describe the set of models a robust decision maker considers. It also has many applications in econometrics, statistics, and information theory, and the connection between the security design problem and these topics will be discussed later in the paper. I will show that when the cost function is proportional to the KL divergence, debt is the optimal security design.

I will also discuss more general class of divergences, known as the “ $f$ -divergences” (Ali and Silvey [1966], Csiszár [1967]). This class of divergences, which include the KL divergence, can be written as

$$(2.3) \quad D_f(p||q) = \sum_{i \in \Omega} q^i f\left(\frac{p^i}{q^i}\right),$$

where  $f(u)$  is a convex function on  $\mathbb{R}^+$ , with  $f(1) = 0$  and  $f(u) \geq 0$ . I will limit my discussion to smooth  $f$ -functions, for mathematical convenience, and use the normalization that  $f''(1) = 1$ . Commonly used  $f$ -divergences include the Hellinger distance  $f(u) = 4(\frac{1}{2} - \sqrt{u} + \frac{1}{2}u)$ , the  $\chi^2$ -divergence  $f(u) = \frac{1}{2}(u^2 + 1 - 2u)$ , and the KL divergence  $f(u) = u \ln u - u + 1$ .

The  $f$ -divergences are themselves part of a larger class of divergences, the “invariant divergences.” In section §4 and section §5, I analyze the case of an arbitrary invariant divergence cost function, for non-parametric and parametric models, respectively. Invariant divergences are defined by their invariance with respect to sufficient statistics (Chentsov [1982], Amari and Nagaoka [2007]).<sup>22</sup> I will show that with invariant divergence cost functions, debt is approximately optimal.

The KL divergence, and the broader class of invariant divergences, are interesting because they are closely related to ideas from information theory. In the appendix, section §D, I illustrate this in a model based on rational inattention (Sims [2003]), in which the cost function is related to the KL divergence. The KL divergence cost function can also be micro-founded from a dynamic moral hazard problem. In section §6, I show that a continuous time problem is equivalent to the static moral hazard problem with the KL divergence cost function. In section §7, I extend this analysis to a more general class of continuous time problems and show that they are related, in a certain sense, to static moral hazard problems with invariant divergence cost functions.

In this section, I have introduced the framework that I will use throughout the paper. In the next section, I analyze the benchmark model, in which the cost function is the KL divergence.

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<sup>22</sup>The exact definition of an invariant divergence is rather technical, and not required to comprehend the results in this paper: one can interpret the results as applying to all  $f$ -divergences, with only a small loss of generality. I therefore refer interested readers to the aforementioned sources for the definition of an invariant divergence.

### 3. THE BENCHMARK MODEL

In this section, I discuss the non-parametric version of the model, in which the set  $M$  of feasible probability distributions is the set of all probability distributions on  $\Omega$ . I assume that the cost function is proportional to the KL divergence between  $p$  and  $q$ ,

$$\psi(p) = \theta D_{KL}(p||q).$$

where  $\theta$  is a positive constant that controls how costly it is for the seller to have  $p$  deviate from  $q$ . Larger values of  $\theta$  make deviations more costly, and therefore reduce the moral hazard. The KL divergence cost function guarantees an interior solution to equation (2.1), for all  $\eta$ . It follows that the indirect utility function  $\phi(\eta)$  is the convex conjugate (Fenchel-Legendre transform) of  $\psi(p)$ .

This observation suggests that the retained tranche  $\eta$  can also be thought of as a set of dual coordinates for the probability distribution  $p$ , in the sense of Amari and Nagaoka [2007]. As shown by Holmström and Milgrom [1987], for each  $\eta$  there is a unique  $p$  that the seller will choose, and for each  $p$  there is a unique  $\eta$  that will cause the seller to choose  $p$ . The notion of  $\eta$  as a set of probability coordinates will be emphasized later in the paper. In this section, several results related to this duality will allow me to characterize the optimal security design.

Because each  $\eta$  results in a unique  $p$ , and the function  $p(\eta)$  is differentiable and interior, the “first-order approach” for the security design problem is valid. The Lagrangian for the security design problem (taking into account the limited liability constraints) is

$$(3.1) \quad \mathcal{L}(\eta, \lambda, \omega) = \beta_b \sum_{i>0} p^i(\eta) s_i(\eta) + \phi(\eta) + \kappa \sum_{i>0} \lambda^i \eta_i + \kappa \sum_{i>0} \omega^i (\beta_s v_i - \eta_i),$$

where I have scaled the multipliers  $\lambda$  and  $\omega$  to simplify later equations. To derive the optimal security design, I will use several properties related to this duality. Using the envelope theorem, it follows from the moral hazard problem that

$$\frac{\partial \phi(\eta)}{\partial \eta_i} \Big|_{\eta=\eta^*} = p^i(\eta^*).$$

Intuitively, because the seller is maximizing her welfare over  $p$ , small changes in the security design affect her utility only through their direct impact on the cash flows received, and the indirect effects of the changes in  $p(\eta)$  can be ignored. With this observation, we can write the first-order condition (FOC) for the Lagrangian:

$$(3.2) \quad \kappa(p^i(\eta^*) - \lambda^i + \omega^i) - \beta_b \sum_{j>0} \frac{\partial p^i(\eta)}{\partial \eta_j} \Big|_{\eta=\eta^*} s_j(\eta^*) = 0.$$

This equation states that there are two welfare-relevant components of any infinitesimal security design change. First, there is a direct effect on the gains from trade, proportional to the constant  $\kappa$ . Second, there is a change in the buyer's valuation of the security, due to his anticipation of the seller's changing incentives. To analyze this equation further, I will use several additional properties related to the duality of  $\eta$  and  $p$ .

The first-order condition of the moral hazard problem is, for  $i > 0$ ,

$$\eta_i = \frac{\partial \psi(p)}{\partial p^i} \Big|_{p=p^*(\eta)},$$

where I use the notation that  $p \in \mathbb{R}_N^+$  (not  $\mathbb{R}_{N+1}^+$ ) and  $p^0 = 1 - \sum_{i>0} p^i$ . We can differentiate both sides by  $\eta_j$  and see that

$$\delta_i^j = \sum_{k>0} \frac{\partial \psi(p)}{\partial p^i \partial p^k} \Big|_{p=p(\eta^*)} \frac{\partial p^k(\eta)}{\partial \eta_j} \Big|_{\eta=\eta^*},$$

where  $\delta_i^j$  is the Kronecker delta. In matrix terms, the inverse of the Hessian of the cost function is the matrix  $[\frac{\partial p^k(\eta)}{\partial \eta_j} \Big|_{\eta=\eta^*}]_{kj}$ .<sup>23</sup> Using this fact, we can rewrite our first-order condition as

$$(3.3) \quad \beta_b s_j(\eta^*) = \kappa \sum_{i>0} [p^i(\eta^*) - \lambda^i + \omega^i] \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \Big|_{p=p(\eta^*)}.$$

This expression for the optimal security holds for any convex cost function that guarantees an interior solution to the moral hazard problem, not just the KL divergence. I will return to it when I consider more general cost functions. The equation demonstrates the close connection between the shape of the optimal security and the Hessian of the cost function, recognizing that  $p(\eta^*)$  is itself endogenous and determined in part by the cost function.

When the cost function is proportional to the KL divergence, its Hessian is proportional to the Fisher information matrix:

$$\frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \Big|_{p=p(\eta^*)} = \theta g_{ij}(p(\eta^*)),$$

where  $g_{ij}(p)$  is the  $(i, j)$  element of the Fisher information matrix, evaluated at  $p$ . To understand the security design first-order condition, consider the  $N = 2$  case (three second-period states, including the  $v_0 = 0$  state), for which the Fisher information matrix is

$$[g_{ij}(p)] = \begin{bmatrix} \frac{1}{p^1} & 0 \\ 0 & \frac{1}{p^2} \end{bmatrix} + \begin{bmatrix} \frac{1}{p^0} & \frac{1}{p^0} \\ \frac{1}{p^0} & \frac{1}{p^0} \end{bmatrix}.$$

<sup>23</sup>By assumption, the Hessian of the cost function is positive-definite, and therefore invertible.

The inverse Fisher information matrix, which I will denote  $[g^{ij}(p)]$ , can be written in the  $N = 2$  case as

$$[g_{ij}(p)]^{-1} = [g^{ij}(p)] = \begin{bmatrix} p^1 & 0 \\ 0 & p^2 \end{bmatrix} - \begin{bmatrix} p^1 \\ p^2 \end{bmatrix} \begin{bmatrix} p^1 & p^2 \end{bmatrix}.$$

The inverse Fisher information matrix can be used to compute the variance. For any security design  $s_i$ , the variance of  $s_i$  under probability distribution  $p$  is

$$V^P[s_i] = \sum_i \sum_j s_i s_j g^{ij}(p).$$

In words, in the non-parametric model, the Cramér-Rao bound is an equality. This observation can be used to connect the first order condition of the security design problem with moral hazard (equation (3.2) above) to a different security design problem. I will first discuss the connection between these two security design problems, and then demonstrate that debt is optimal.

Suppose that the asset value has a fixed probability distribution,  $\bar{p}$ . Consider the set of limited liability securities, and ask the following question: of all securities with the same expected value under  $\bar{p}$ , which security has the least variance in its payout? Formally, the problem is

$$\min_{s_i} V^{\bar{p}}[s_i],$$

subject to the limited liability constraints and that  $E^{\bar{p}}[s_i] = C$ , for some constant  $C$ . Here,  $V^{\bar{p}}[\cdot]$  and  $E^{\bar{p}}[\cdot]$  denote the variance and mean, respectively, under the probability distribution  $\bar{p}$ . The first-order condition of this problem is

$$\hat{\kappa}(\bar{p}^i - \lambda^i + \omega^i) - \beta_b \sum_{j>0} g^{ij}(\bar{p}) s_j = 0,$$

where  $\hat{\kappa}$  is the multiplier on the constraint that  $E^{\bar{p}}[s_i] = C$ . This equation is exactly the same as the first-order condition in the moral hazard problem, if  $\bar{p} = p(\eta^*)$  and  $\hat{\kappa} = \kappa$ .

The optimal contract in the security design problem with moral hazard problem is the one that minimizes the variance of its payout, among all limited liability securities with the same expected value. Intuitively, because debt securities are “flat” wherever possible, they minimize variance<sup>24</sup>, and are therefore optimal. Examining the equity, live-or-die, and debt securities shown in the appendix, Figure A.1, it is clear why the debt security minimizes the variance of the payout, among all limited-liability securities. The proof of proposition 1 below shows both that the variance-minimizing security is a debt contract, and that this is optimal in the security design problem with moral hazard.

<sup>24</sup>It seems likely that the variance-minimizing property of debt has been pointed out by previous authors, but I am not aware of any papers mentioning this fact.

**Proposition 1.** *In the non-parametric model, with the cost function proportional to the Kullback-Leibler divergence, the optimal security design is a debt contract,*

$$s_j(\eta^*) = \min(v_j, \bar{v}),$$

for some  $\bar{v} > 0$ . If the highest possible asset value is sufficiently large ( $v_N > \sum_i q^i v_i + \frac{\kappa}{\beta_b} \theta$ ), then  $\bar{v} < v_N$ .

*Proof.* See appendix, section E.1. □

The result in proposition 1 shows that debt is optimal, for any full-support zero-cost distribution  $q$ .<sup>25</sup> The condition that  $v_N$  be “high enough” is weak. If it was not satisfied for some sample space  $\Omega$  and zero-cost distribution  $q$ , one could include a new highest value  $v_{N+1}$  in  $\Omega$ , occurring with vanishingly small probability under  $q$ , such that the condition was satisfied. Intuitively, the sample space must contain high enough values to observe the “flat” part of the debt security.

The optimality of debt runs contrary to the themes of the existing literature, which emphasize that risk-shifting undoes the optimality of debt (Acharya et al. [2012], Biais and Casamatta [1999], Fender and Mitchell [2009], Hellwig [2009], Jensen and Meckling [1976]). The intuition for the optimality of debt comes from its mean-variance optimality. Higher mean values of the security (under a fixed distribution) are desirable because they drive the gains from trade. Higher variance in the security payout is problematic for two reasons. First, variance in the security payout reduces the seller’s incentive to increase the mean value of the asset, relative to the zero-variance (“sell nothing”) security. Second, variance in the security payout gives the seller an incentive to inefficiently shift cash flows towards her retained tranche, holding the mean asset value constant. Every security is subject to these two effects, although which of these effects is most prevalent depends on the security design. For equity securities, there is no risk-shifting, and the variance of the security design summarizes reduced effort. For the live-or-die security design, effort is high, possibly excessive, and there is also large amounts of risk-shifting. For debt securities, both reduced effort and risk-shifting occur. Under the KL divergence cost function, at the margin, for any security design, these two effects are exactly summarized by one statistic: the variance of the security payout. Debt securities are optimal because they are mean-variance optimal, minimizing the combined losses due to reduced effort and risk-shifting.

A different way to view this intuition is to consider small perturbations to the retained tranche,  $\eta_i = \eta_i^* + \tau \varepsilon_i$ , where for small enough  $\tau$ , the retained tranche still satisfies limited liability. Changing the retained tranche will cause the seller to change her behavior in the moral hazard problem. If the perturbation  $\varepsilon$  provides incentives that go “in the same direction” as the existing incentives

<sup>25</sup>The level of the debt,  $\bar{v}$ , depends on  $q$  indirectly, through the endogenous probability distribution  $p(\eta^*)$ .

$\eta$ , the seller will move the probability distribution  $p$  further away from  $q$ , and increase  $D_{KL}(p||q)$ . For small perturbations, the amount by which  $D_{KL}(p||q)$  will change is proportional to the covariance of  $\eta$  and  $\varepsilon$  under the endogenous probability distribution  $p(\eta^*)$ . The perturbation  $\varepsilon$  might also have a benefit, in that it causes the seller to increase the expected value of the asset under the endogenous probability distribution  $p(\eta)$ . The extent to which  $E^{p(\eta)}[\beta_s v_i]$  changes, for small perturbations, is  $Cov^{p(\eta^*)}[\beta_s v_i, \varepsilon_i]$ . Intuitively, if the seller's incentives become more aligned with the value of the asset, the seller will act to increase the asset value. The combined effects of the change in moral hazard costs and the change in asset value is  $Cov^{p(\eta^*)}[\beta_s s_i, \varepsilon_i]$ , or the extent the perturbation is aligned with the security design. This is also the opposite of the change in the variance of the security design, which explains why the variance of the security design summarizes the moral hazard problems.

Several of the assumptions in the benchmark model can be relaxed without altering the debt security result of proposition 1.<sup>26</sup> The lowest possible value,  $v_0$ , can be greater than zero. The seller and the buyer can Nash bargain over the price of the security, and the result will hold as long as the seller has some bargaining power. The buyer can be risk-averse, with any increasing, differentiable utility function. The probability distributions can be thought of as risk-neutral, rather than physical, probabilities. Finally, although the model is set up as a security design problem, similar results could be obtained in principal-agent and investor-entrepreneur contexts.

One additional aspect of these results worth emphasizing is the issue of redundant states. In the setup of the model, there can be two states,  $i$  and  $j$ , such that  $v_i = v_j$ . It is a result of proposition 1 that the security payoff will be identical in these states. One interpretation of this result is that randomized contracts are not optimal. Another interpretation relates to currency choice.<sup>27</sup> If we assume that the cost function is a pecuniary cost, it follows that the optimal security is debt in the currency of the cost function. For example, if a mortgage originator creates loans in the United States, and sells the loans to a euro zone bank, the model predicts that the security design will be a dollar-denominated debt, not a euro-denominated (or yen-denominated, ...) debt. Similarly, the model predicts that Danish mortgage securities would be denominated in Danish krone, consistent with observed practice (Frankel et al. [2004]).

A third way of interpreting this result relates to the practice of "pooling" multiple assets together into a single securitization structure. We can think of the redundant states  $v_i$  and  $v_j$  as two states in which the combined value of all the assets in the securitization pool is the same, but the composition of the asset value is different. It is a result of the model that the optimal security designs pays equally in those two states. Intuitively, while the security design could reward the seller for

<sup>26</sup>These assumptions are useful later in the paper, for both parametric models and non-parametric models without the KL divergence cost function.

<sup>27</sup>This interpretation was suggested to me by Roger Myerson.

increasing the value of some assets in the pool relative to other assets, it is not efficient to do so in the presence of this form of moral hazard.

The optimal security described in proposition 1 has an interesting comparative static. Define the “put option value” of a debt contract as the discounted difference between its maximum payoff  $\bar{v}$  and its expected value. For the optimal security design<sup>28</sup>,

$$(3.4) \quad P.O.V. = \beta_b \bar{v}(\eta^*) - \beta_b \sum_{i>0} p^i(\eta^*) s_i(\eta^*) = \kappa \theta.$$

When the constant  $\theta$  is large, meaning that it is costly for the seller to change the distribution, the put option will have a high value. Similarly, when the gains from trade,  $\kappa$ , are high, the put option will have a high value. For all distributions  $q$ , a higher put option value translates into a higher “strike” of the option,  $\bar{v}$ , although the exact mapping depends on the distribution  $q$  and the sample space  $\Omega$ . Restated, when the agents know that the moral hazard is small, or that the gains from trade are large, they will use a large amount of debt, resulting in a riskier debt security.<sup>29</sup>

Before generalizing this theory, I will briefly discuss the ways in which the model could be applied to data. Suppose that we had a reasonable estimate of  $\theta$ . To compute the riskiness of the optimal debt security, as measured by the “put option value,” the only additional information required is the gains from trade  $\kappa$ . There are several empirical papers that attempt to estimate this parameter directly (see the appendix, section §C). Armed with estimates of  $\theta$  and  $\kappa$ , one could compare actual security designs against the model’s predictions.

This approach would require an estimate of  $\theta$ . I will discuss two distinct approaches, and describe them in more detail in the appendix, section §C. The first approach could be thought of as an “experimental” approach. If an experimenter randomly assigned different sellers one of two retained tranches,  $\eta$  and  $\hat{\eta}$ , and observed the resulting endogenous probability distributions  $p(\eta)$  and  $p(\hat{\eta})$ , the experimenter could estimate the parameter  $\theta$  (assuming that the cost function was the KL divergence).<sup>30</sup> Unfortunately, because empirical work necessarily observes ex-post outcomes, rather than ex-ante probabilities, and the outcomes of different pools of loans are not independent, this approach is difficult to implement credibly. The second approach is to infer  $\theta$  from the design of securitizations (although one cannot then test whether these designs were optimal). This approach is somewhat more successful, insofar as it can be carried out using estimates from the empirical literature, but far from conclusive. For details, see the appendix, section §C.

<sup>28</sup>See lemma 3 in the appendix.

<sup>29</sup>The model also has interesting comparative statics for the zero-effort distribution  $q$ . I have omitted them for the sake of brevity.

<sup>30</sup>In fact, the experimenter could even test the assumption of the KL divergence cost function.

In the next section, I consider alternative cost functions, and show that the mean-variance intuition for the optimality of debt is a general phenomena.

#### 4. THE NON-PARAMETRIC MODEL WITH INVARIANT DIVERGENCES

In this section, I analyze the general case of invariant divergence cost functions. First, I will show that among a special subset of invariant divergences known as  $f$ -divergences, the Kullback-Leibler divergence is the only divergence that always results in debt as the optimal security design. Second, I will return to the general case of invariant divergence cost functions, and show that debt is approximately optimal for all invariant divergence cost functions. Third, I will discuss higher-order approximations, under which the optimal security design is a mixture of debt and equity.

I begin by assuming that the cost function is proportional to a smooth  $f$ -divergence (defined earlier, see equation (2.3)):

$$\psi(p) = \theta D_f(p||q).$$

These divergences are analytically tractable because they are decomposable, meaning that the cost of choosing some  $p^i$  is not affected by value of  $p^j$ ,  $j \neq i$ , except through the adding-up constraint on probability distributions. In some cases, such as the Hellinger distance or KL divergence, the seller's choice of  $p$  is guaranteed to be interior, but this is not true for all  $f$ -divergences.

Whenever the solution is interior, the optimal security design is characterized by equation (3.3). The Hessians of the  $f$ -divergences have a simple structure. Using this structure, I find the following result:

**Proposition 2.** *In the non-parametric model, with an  $f$ -divergence cost function, if the optimal security design is debt for all sample spaces  $\Omega$  and zero-cost probability distributions  $q$ , then that  $f$ -divergence is the Kullback-Leibler divergence.*

*Proof.* See appendix, section E.3. □

The statement of proposition 2 shows that the KL divergence is special, in the sense that it is the only smooth  $f$ -divergence that always results in debt as the optimal security design. The structure of the proof uses the “flat” part of the debt security to derive this result. I first show that to have a flat optimal contract, when the limited liability constraints don't bind, the  $f(u)$  function must be equal to  $u \ln u - u + 1$  for the values of  $u$  that (endogenously) correspond to the the flat part of the contract. I then show that for debt to be optimal for all sample spaces and zero-cost distributions,  $f(u) = u \ln u - u + 1$  for its entire domain.

What does this negative result imply about the mean-variance intuition discussed previously? I interpret proposition 2 as showing that the marginal losses due to reduced effort and risk-shifting

are not perfectly summarized by the variance of the security payout, unless the cost function is the KL divergence. However, the intuition that security payout variance is costly, because it reduces effort and increases risk-shifting, still holds. This leads me to investigate whether the utility in the security design problem can be approximated as a mean-variance tradeoff, and therefore whether debt is “approximately optimal.”

The approximation I consider is a first-order expansion of the utility function in the security design problem. I approximate the utility of using an arbitrary security design  $s$ , relative to selling nothing, to first order in  $\theta^{-1}$  and  $\kappa$ . When  $\theta^{-1}$  is small, and therefore  $\theta$  is large, it is difficult for the seller to change  $p$ . When  $\kappa$  is small, the gains from trade are low. One possible justification for this approximation is that the time period in question is short. Another possible justification is that legal recourse or other forms of monitoring deter the seller from taking extreme actions, explaining why  $\theta$  is high. I take this approximation around the limit point  $\theta^{-1} = \kappa = 0$ . This approximation applies when  $\theta^{-1}$  and  $\kappa$  are small but positive. The limit point itself is degenerate; because there is no moral hazard and no gains from trade, the security design does not matter. Using this approximation, I will show that debt securities achieve approximately the same utility as the optimal security design, for any invariant divergence cost function. Moreover, only debt contracts have this property, and it arises through the same mean-variance intuition discussed in the previous section.

The relevance of the approximation will depend on whether  $\theta^{-1}$  and  $\kappa$  are small enough, relative to the higher order terms of the utility function, for those terms to be negligible. This is a question that can only be answered in the context of a particular application. In the appendix, section §C, I discuss a calibration of the model relevant to mortgage origination, for which the approximation is accurate. It is important to distinguish between the relevance of the approximation and the economic importance of the problem. In the calibration for mortgage origination, debt achieves almost the same utility as the optimal security design, while the utility difference between the best debt contract and selling nothing is about 0.73% of the total asset value. A single mortgage securitization could involve billions of dollars in loans, and a single investment bank could sponsor many such deals each year. In this calibration, even though the moral hazard and gains from trade are small enough that debt is nearly optimal, the private gains for those involved in securitization are very large.<sup>31</sup>

With the KL divergence cost function, debt is the optimal security design because the KL divergence’s Hessian, with respect to  $p$ , is the Fisher information matrix. There is a broader connection

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<sup>31</sup>I emphasize that the private gains are large, because I do not model foreclosure or other externalities that may be relevant for social welfare.

between all invariant divergences with continuous second derivatives (which include all smooth  $f$ -divergences) and the Fisher information matrix. At the point  $q$ , the Hessian for all such divergences is proportional to the Fisher information matrix (Chentsov [1982]):

$$\frac{\partial^2 D(p||q)}{\partial p^i \partial p^j} \Big|_{p=q} = c \cdot g_{ij}(q),$$

where  $c$  is a positive constant. I will assume the cost function is proportional to an invariant divergence, with the previously mentioned convexity and differentiability properties,

$$\psi(p) = \theta c^{-1} D(p||q),$$

ensuring that the Hessian of the cost function is the positive constant  $\theta$  times the Fisher matrix. When  $\theta$  is large, the seller will endogenously choose a  $p$  that is close to  $q$ . In the neighborhood of  $q$ , the cost function “looks like” the KL divergence, in the sense of having the Fisher matrix as its Hessian. Putting these two ideas together, it is not surprising that the optimal security design resembles a debt contract.

I consider a first-order asymptotic expansion of the security design problem utility,  $U(s; \theta^{-1}, \kappa)$ , around the point  $\theta^{-1} = \kappa = 0$ . This approximation holds  $\beta_s$  fixed as it changes  $\kappa$ , implicitly changing  $\beta_b$ . I consider the utility of an arbitrary security  $s$ , relative to the “sell-nothing” security<sup>32</sup>, and derive the following result:

**Proposition 3.** *In the non-parametric model, with a smooth, convex, invariant divergence cost function, the difference in utilities achieved by an arbitrary security  $s$  and the sell-nothing security is*

$$(4.1) \quad U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \kappa E^q[\beta_s s_i] - \frac{1}{2} \theta^{-1} V^q[\beta_s s_i] + O(\theta^{-2} + \theta^{-1} \kappa).$$

*Proof.* See appendix, section E.4. □

The intuition of the security design problem as a mean-variance optimization problem holds (approximately) for all invariant divergences, not just the KL divergence. The proof argues that when  $\theta$  is large,  $p$  will be close to  $q$ , and then uses the aforementioned fact that the Hessian of the invariant divergence, at point  $q$ , is the Fisher information matrix at point  $q$ .

Two related results show that as both the moral hazard and gains from trade become small, the optimal contract converges to debt, and a debt contract is first-order optimal.

<sup>32</sup>There are two natural benchmark security designs, selling nothing and selling everything. The choice of benchmark does not affect the optimal security design. I use selling nothing as the benchmark, because it clarifies why debt is approximately optimal (the mean-variance tradeoff).

**Corollary 1.** Define  $s_{debt}$  as the unique solution (which is a debt security) to the mean-variance security design problem suggested by equation (4.1), ignoring the  $O(\theta^{-2} + \theta^{-1}\kappa)$  terms:

$$s_{debt}(\theta^{-1}, \kappa) = \arg \max_s \kappa E^q[\beta_s s_i] - \frac{1}{2} \theta^{-1} V^q[\beta_s s_i],$$

subject to the limited liability constraints. Under the conditions of proposition 3, defining  $s^*(\theta^{-1}, \kappa)$  as the optimal contract given  $\theta^{-1}$  and  $\kappa$ ,

- (1)  $\lim_{\theta^{-1} \rightarrow 0^+} s^*(\theta^{-1}, \bar{\kappa}\theta^{-1}) = s_{debt}(1, \bar{\kappa})$  and
- (2)  $U(s^*(\theta^{-1}, \kappa); \theta^{-1}) - U(s_{debt}(\theta^{-1}, \kappa); \theta^{-1}, \kappa) = O(\theta^{-2} + \theta^{-1}\kappa)$ .

*Proof.* See appendix, section E.5. □

The accuracy of the approximation that both the moral hazard and gains from trade are small will vary by application. The generality of proposition 3, which holds for all sample spaces, zero-cost distributions, and invariant divergences, suggests that as long as the moral hazard is not too large, the agents can neglect the details of the cost function. In this case, the mean-variance intuition for security design holds, and a debt security is approximately optimal.

The result of corollary 1 shows that when the gains from trade and moral hazard are small, but not zero, debt is approximately optimal in a way that other security designs are not. In Figure 4.1, I illustrate this idea. I plot the utility of the optimal debt contract, optimal equity contract, and the optimal contract, relative to selling everything, for different values of  $\theta$ , with  $\kappa = \bar{\kappa}\theta^{-1}$ .<sup>33</sup> As  $\theta$  becomes large, all security designs converge to the same utility. For intermediate values of  $\theta$ , the optimal debt contract achieves nearly the same utility as the optimal contract, which is what the approximation results show. For low values of  $\theta$ , the high-stakes case, the gap between the optimal debt contract and optimal contract grows.

These results hold for any divergence whose Hessian, at the point  $p = q$ , is proportional to the Fisher information matrix. The earlier discussion of security design with KL divergence cost functions emphasized the connection between debt securities and the Fisher information matrix. In light of that discussion, the mean-variance results presented here are not surprising, conditional on the assumption that the Hessian of the cost function is the Fisher information matrix. The less intuitive aspect of the result is the connection between that assumption and the invariant divergences, which are defined by certain desirable information-theoretic properties (Chentsov's theorem).

I next turn to the question of higher order approximations. The mean-variance characterization of the utility function, up to order  $\theta^{-1}$ , naturally raises the question of what the  $O(\theta^{-2} + \kappa\theta^{-1})$

<sup>33</sup>The results of proposition 3 reference the utility of a security relative to selling nothing, instead of selling everything. In Figure 4.1, I plot utilities relative to selling everything for visual clarity.

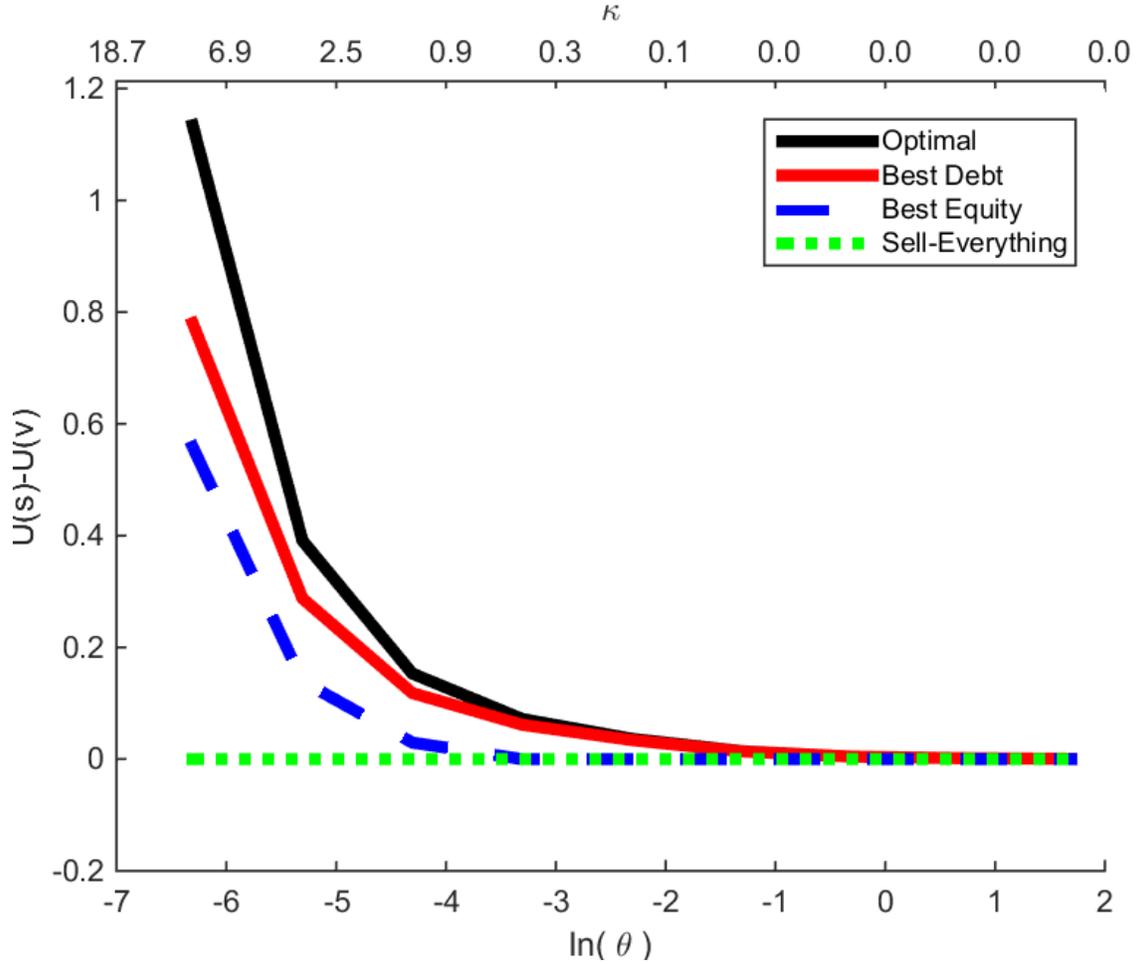


FIGURE 4.1. THE UTILITY OF VARIOUS SECURITY DESIGNS. This figure compares the utility of several security designs (debt, equity, and the optimal security design) relative to the utility of selling everything, for different values of  $\theta$ . The bottom x-axis is the value of  $\ln(\theta)$ , the top x-axis is the value of  $\kappa$ , and the y-axis is the difference in security design utility between the security (debt, equity, etc.) and selling everything. For each  $\theta$  and corresponding  $\kappa$ , the optimal debt security, equity security, and the optimal security are determined. Then, the utility of using each of the four securities designs, given  $\theta$  and  $\kappa$ , is computed. The cost function is a  $\alpha$ -divergence, with  $\alpha = -7$ , which was chosen because the optimal security design is easy to characterize (it is a mix of debt and equity) and it is sufficiently different from the KL divergence to ensure that debt is not always optimal. The gains from trade,  $\kappa$ , vary as  $\theta$  changes, with  $\kappa = \bar{\kappa}\theta^{-1}$ ,  $\bar{\kappa} = 0.0171$ . This parameter was chosen to be consistent with the calibration in the appendix, section §C. The discounting parameter for the seller is  $\beta_s = 0.5$ . The zero-cost distribution  $q$  is a discretized, truncated gamma distribution with mean 2, 0.3 standard-deviation, and an upper bound of 8. The outcome space  $v$  is a set of 401 evenly-spaced values ranging from zero to 8. The utilities are plotted for nine different values of  $\theta$ , ranging from  $2 \exp(-7)$  to  $2 \exp(1)$ , and linearly interpolated between those values.

terms look like. Fortunately, the theorem of Chentsov [1982] also characterizes the third derivative of all invariant divergences. Using this theorem, I derive the following lemma:

**Lemma 1.** *For every invariant divergence with continuous third derivatives, there exists a real number  $\alpha$  such that*

$$\frac{\partial^3 D_f(p||q)}{\partial p^i \partial p^j \partial p^k} \Big|_{p=q} = c \left( \frac{3 + \alpha}{2} \right) \partial_i g_{jk}(p) \Big|_{p=q},$$

where  $c$  is a positive constant such that  $c \cdot g_{ij}(q) = \frac{\partial^2 D_f(p||q)}{\partial p^i \partial p^j} \Big|_{p=q}$ .

*Proof.* See appendix, section E.6. □

The parameter  $\alpha$  controls how rapidly the curvature of the Hessian of the cost function changes, as  $p$  moves away from  $q$ . For the KL divergence,  $\alpha = -1$ , while the Hellinger distance and  $\chi^2$ -divergence have  $\alpha$  values of zero and negative three, respectively. Unsurprisingly, the parameter  $\alpha$  influences the design of the second-order optimal security.

**Proposition 4.** *Under the assumptions of proposition 5, the difference in utilities between an arbitrary security  $s$  and selling nothing is*

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) &= \kappa E^q[\beta_s s_i] - \frac{1}{2} \theta^{-1} V^q[\beta_s s_i] + \\ &\quad \kappa \theta^{-1} Cov^q[\beta_s s_i, \eta_i(s)] - \frac{3 + \alpha}{12} \theta^{-2} K_3^q[\eta_i(s), \beta_s s_i, \beta_s v_i] - \\ &\quad \frac{3 + \alpha}{6} \theta^{-2} K_3^q[\beta_s s_i, \beta_s s_i, \beta_s s_i] + O(\theta^{-3} + \kappa \theta^{-2}), \end{aligned}$$

where  $Cov^q[X, Y]$  is the covariance under probability distribution  $q$  and  $K_3^q[X, Y, Z]$  is the third cross-cumulant under  $q$ .

*Proof.* See appendix, section E.4. □

The covariance term in proposition 4 represents the interaction between the gains from trade and the moral hazard, while the third cross-cumulant terms can be thought of as accounting for the difference between  $p(\eta)$ , the endogenous probability distribution, and  $q$ . The economic intuition behind this result is harder to grasp than the mean-variance intuition, but fortunately the optimal security design helps clarify matters. Taking the first-order condition of this expression, I derive the following corollary:

**Corollary 2.** *Define  $s_{debt-eq}$  as the optimal security design for the problem suggested by proposition 4, ignoring the  $O(\theta^{-3} + \kappa \theta^{-2})$  terms. Assuming that  $\alpha < (1 + \frac{2}{\kappa})$ , the second-order optimal security design, for some constant  $\bar{v} > 0$ , is*

$$s_{debt-eq,i}(\theta^{-1}, \kappa) = \begin{cases} v_i & \text{if } v_i < \bar{v} \\ \max[-\frac{\kappa(1+\alpha)}{2+\kappa(1-\alpha)}(v_i - \bar{v}) + \bar{v}, 0] & \text{if } v_i \geq \bar{v}. \end{cases}$$

Under the conditions of proposition 4, defining  $s^*(\theta^{-1}, \kappa)$  as the optimal contract given  $\theta^{-1}$  and  $\kappa$ ,

$$U(s^*(\theta^{-1}, \kappa); \theta^{-1}, \kappa) - U(s_{debt-eq}(\theta^{-1}, \kappa); \theta^{-1}, \kappa) = O(\theta^{-3} + \kappa\theta^{-2}).$$

*Proof.* See appendix, section E.7. □

The higher-order optimal security design can be thought of as a mixture of debt and equity (at least when  $\alpha \leq -1$ ), whose slope is determined by the gains from trade parameter  $\kappa$  and the change of curvature parameter  $\alpha$ . For the KL divergence,  $\alpha = -1$ , the contract is always a debt contract, regardless of  $\kappa$ . As the moral hazard and gains from trade become small, in the limit considered earlier, the optimal contract converges to a debt contract, regardless of the parameter  $\alpha$ . The restriction that  $\alpha < (1 + \frac{2}{\kappa})$  is without loss of generality in the limit as  $\kappa \rightarrow 0$ . It also worth noting that for any  $\alpha > -1$ , the optimal contract is non-monotonic. For any invariant divergence cost function with such an  $\alpha$ , imposing the restriction that the security design be monotonic results in a debt contract as the (second-order) optimal contract. Viewed in this light, it is not surprising that many authors in the security design literature find that debt is optimal when assuming monotonicity.

One way to interpret this result is to think of  $\alpha$  as controlling the balance between effort and risk shifting. When  $\alpha$  is large and negative, the second-order optimal security design resembles an equity contract, because concerns about risk-shifting dominate concerns about effort. When  $\alpha$  is large and positive, the second-order optimal security design approaches a “live-or-die” contract, suggesting that concerns about effort dominate concerns about risk-shifting. The live-or-die, debt, and equity security designs exist on a continuum, indexed by  $\alpha$ , that formalizes what it means for debt to be “in between” the live-or-die and equity contracts. In Figure A.2, in the appendix, I illustrate the different second-order optimal security designs associated with varying values of  $\alpha$ .

The results for first-order and second-order optimal security designs can be summarized as a type of “pecking order” theory. When the moral hazard and gains from trade are small, the agents can use debt contracts. As the stakes grow larger, so that both the moral hazard and gains from trade are bigger concerns, the agents can use a mix of debt and equity. For very large stakes, the security design will depend on the precise nature of the moral hazard problem.

Having analyzed non-parametric models, I next turn to parametric models. I will show that the mean-variance intuition and the approximate optimality of debt also applies for these models.

## 5. PARAMETRIC MODELS WITH INVARIANT DIVERGENCES

The moral hazard models of Innes [1990], Hellwig [2009], and Fender and Mitchell [2009] can all be thought of as restriction the seller's choice of probability distributions to a parametric model. In this section, I discuss parametric moral hazard problem with invariant divergence cost functions.

I assume that the set of feasible probability distributions,  $M_\xi$ , is a curved exponential family, smoothly embedded into the set of all distributions on  $\Omega$ .<sup>34</sup> The parameters,  $\xi$ , are coordinates for the curved exponential family. I will rewrite the parametric moral hazard sub-problem as

$$(5.1) \quad \phi(\eta; M_\xi, \theta^{-1}) = \max_{p \in M_\xi} \left\{ \sum_{i>0} p^i \eta_i - \psi(p; \theta^{-1}) \right\}.$$

I assume the seller controls at least one parameter that alters the mean asset value, so the problem is not trivial. As noted earlier, in the parametric model there is no guarantee that there is a unique  $p$  which maximizes equation (5.1). I assume for all retained tranches  $\eta$ , for each optimal  $p$ , the parameters  $\xi$  corresponding to that  $p$  are interior. As in the previous section, I will assume that

$$\psi(p) = \theta c^{-1} D(p||q)$$

where  $D(p||q)$  is a invariant divergence and  $c$  is defined as in Chentsov's theorem, and continue to assume that  $D$  is smooth and convex in  $p$ . I also assume that  $q \in M_\xi$ , meaning that the "zero-cost" distribution is feasible. Let  $U(s; M_\xi, \theta^{-1})$  be the utility in the security design problem,

$$U(s; M_\xi, \theta^{-1}, \kappa) = \beta_s(1 + \kappa) \sum_{i>0} p^i(\eta(s); M_\xi, \theta^{-1}) s_i + \phi(\eta(s); M_\xi, \theta^{-1}),$$

where  $p(\eta; M_\xi, \theta^{-1})$  is the endogenous probability the seller would choose given  $\eta$ ,  $M_\xi$ , and  $\theta^{-1}$ . I have written  $p(\eta; M_\xi, \theta^{-1})$  as a function, assuming an arbitrary rule for choosing between different optimal  $p$ , in the event that there are multiple maximizers of equation (5.1). Unlike the non-parametric models considered previously, the mapping between securities and probability distributions is many-to-one. Moreover, even under the approximation considered in the previous section, the optimal contract does not approach debt. For example, in the Innes [1990] problem, the optimal contract in this limit is still a "live-or-die" contract. Nevertheless, I use the approximation to derive an analog to proposition 3 for parametric models:

**Proposition 5.** *In the parametric model, with a smooth, convex, invariant divergence cost function, the difference in utilities achieved by an arbitrary security  $s$  and the sell-nothing security, for sufficiently small  $\theta^{-1}$  and  $\kappa$ , is bounded below:*

<sup>34</sup>Because the set of all probability distributions on a discrete outcome space  $\Omega$  is an exponential family, this is almost without loss of generality. The key restriction is that the sub-manifold  $M_\xi$  is smoothly embedded.

$$U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) \geq \kappa E^q[\beta_s s_i] - \frac{1}{2} \theta^{-1} V^q[\beta_s s_i] + O(\theta^{-2} + \kappa \theta^{-1}; M_\xi).$$

The lower bound, to first order, does not depend on  $M_\xi$ , only on  $q$ . There exist parameters  $\xi$  and corresponding set of feasible probability distributions  $M_\xi$  such that the lower bound is tight. The notation  $O(\theta^{-2} + \kappa \theta^{-1}; M_\xi)$  indicates terms of order  $\theta^{-2}$ ,  $\kappa \theta^{-1}$ , or lower that may depend on  $M_\xi$ .

*Proof.* See appendix, section E.8. □

In the parametric model, there is still a mean-variance tradeoff between the gains from trade and the losses due to both reduced effort and risk-shifting. However, because the model is parametric, there are ways that the security can vary that do not alter the incentives of the seller, because there is no action she can take to respond to this variation. The seller's limited ability to respond to changed incentives explains why the result in proposition 5 is a lower bound.<sup>35</sup>

This lower bound applies to a large class of problems in the existing security design literature. For problems with a single choice variable, such as Innes [1990], the restriction to invariant divergence cost functions is almost without loss of generality, because all  $f$ -divergences are invariant, and the associated  $f$ -function can be any smooth, convex function. The lower bound even applies to problems where the standard “first-order-approach” is not valid (Jewitt [1988], Mirrlees [1999]). Although in general parametric moral hazard sub-problems do not have unique or everywhere-differentiable solutions, in the neighborhood of  $\theta^{-1} \rightarrow 0^+$  the policy function  $p(\eta; M_\xi, \theta^{-1})$  becomes unique and differentiable for all smooth manifolds  $M_\xi$ . The lower bound can therefore be thought of as a tractable approach to approximating problems that are otherwise difficult to analyze.

The lower bound is tight, in the sense that there exists an  $M_\xi$  such that the utility difference is equal to the mean-variance objective, up to order  $\theta^{-1}$ . The example that illustrates this provides an interesting economic intuition. Take the security  $s$  and corresponding retained tranche  $\eta$  as given. Suppose that  $M_\xi$  is an exponential family,

$$p^i(\xi) = q^i \exp(\xi_1 \eta_i + \xi_2 s_i - A(\xi)),$$

where  $\eta$  and  $s$  are the sufficient statistics of the distribution<sup>36</sup>, and  $A(\xi)$  is the log-partition function that ensures  $p^i(\xi)$  is a probability distribution for all  $\xi$ . The zero-cost distribution,  $q^i$ , corresponds to  $\xi_1 = \xi_2 = 0$ . This example is a “worst-case scenario” for the agents, over the set of possible

<sup>35</sup>The proof uses the monotonicity property of the Fisher information metric, and is closely related to the Cramér-Rao bound.

<sup>36</sup>Informally, the sufficient statistics capture all of the information in a data sample. In this case, if an observer saw the value of both the retained tranche and security, there would be no additional information that would be useful when trying to infer the seller's actions.

actions, holding the security design fixed. It is the worst case because the exponential family can also be expressed as a function of its dual coordinates,  $\tau$ , with  $\tau_1(\xi) = \sum_i p^i(\xi)\eta_i$  and  $\tau_2(\xi) = \sum_i p^i(\xi)s_i$  (Amari and Nagaoka [2007]). In effect, the seller separately controls the value of her retained tranche and the security. The seller will use only the action that increases the value of her retained tranche ( $\tau_1$ ) and will not use the action that increases the value of the security ( $\tau_2$ ). Relative to the case where the seller retained the entire asset, this situation exhibits both reduced effort and risk-shifting.

The lower bound in this example is tight, and there is no “costless variation,” because the restriction of the seller’s actions to this particular parametric family did not alter the probability distribution  $p(\eta)$  that she would choose, relative to the non-parametric model. The proof of proposition 5 relies on the fact that the security  $s$  is a sufficient statistic of the distribution, and that sufficient statistics (as estimators) attain the Cramér-Rao bound.

It is interesting to note the contrast between the worst-case scenarios for debt securities and equity securities. For debt securities, the worst-case scenario is that the seller controls an option-like payoff, which increases both the mean and variance of the asset value. For equity securities, the worst-case scenario is that the seller controls the mean outcome.

The security design utility in the parametric problem, under the small stakes approximation, exhibits a “monotonicity” with respect to the set of feasible distributions  $M_\xi$ .<sup>37</sup> If  $M_\xi \subseteq \hat{M}_\xi$ , then for all  $s$ , and sufficiently small  $\theta^{-1}$  and  $\kappa$ ,

$$U(s; M_\xi, \theta^{-1}, \kappa) \geq U(s; \hat{M}_\xi, \theta^{-1}, \kappa).$$

Earlier, I assumed that the seller could influence the mean of the probability distribution. It follows that giving the seller additional actions can only, in the small stakes limit, expand the scope for risk-shifting, and therefore reduce the utility that can be achieved in the security design problem. Eventually, as the set of feasible probability distributions approaches the entire probability simplex, the problem converges to the non-parametric case, and the lower bound is tight for all securities. In this case, as in the previous section, debt is optimal in the small stakes limit.

The lower bound in proposition 3 (the mean and variance terms) does not depend on the parameters  $\xi$ , which suggests an interpretation of debt as a robust security design. The robustness of debt is complementary to the result of Carroll [2013]. The key difference between this model and Carroll [2013] is the cost of each potential probability distribution,  $p$ . In Carroll [2013], each probability  $p$  could have different, arbitrary costs. When there is no structure on the cost of potential actions, risk-shifting concerns dominate concerns about reduced effort, and it is crucial that the buyer and seller’s payoffs be exactly aligned. In this case, the optimal security is equity. By

<sup>37</sup>See appendix section E.8, which proves this.

contrast, the invariant divergence cost assumption in this model imposes a structure on the cost that any probability distribution  $p$  would have, if that  $p$  were feasible. With the cost structure I have imposed, both reduced effort and risk-shifting are potentially important, the mean-variance intuition applies, and debt is approximately optimal. Together, these theories can explain the prevalence of both debt and equity securities. Debt securities are used when concerns about reduced effort and risk-shifting are both relevant. If concerns about risk-shifting dominate, equity securities are used.

In the appendix, section §D, I provide a micro-foundation for a version of the parametric model, using a model of rational inattention. The rational inattention model motivates my study of parametric models with the KL divergence cost function. The exact solution for the model cannot be characterized analytically, whereas the approximately optimal debt contract is simple to describe.

In the next two sections of the paper, I will discuss continuous time models of effort. I show that the optimality of debt, and the intuitions about mean-variance tradeoffs, apply in continuous time as well. These sections provide a micro-foundation for the non-parametric static models introduced earlier.

## 6. DYNAMIC MORAL HAZARD

In this section, I will analyze a continuous time effort problem. This problem is closely connected to the static models discussed previously. The role of this section is to explain how an agent could “choose a distribution,” and show that the mean-variance intuition and optimality of debt discussed previously apply in dynamic models.

I will study models in which the seller controls the drift of a Brownian motion. The contracting models I discuss are similar to those found in Holmström and Milgrom [1987], Schaeffler and Sung [1993], and DeMarzo and Sannikov [2006], among others. The models can be thought of as the continuous time limit of repeated effort models<sup>38</sup>, in which the seller has an opportunity each period to improve the value of the asset. Two recent papers are particularly relevant. The models I discuss are a special case of Cvitanić et al. [2009]. The contribution of this paper is to note that, in the special case I discuss, debt securities are optimal, and that the reason debt securities are optimal is that the mean-variance intuitions discussed earlier still hold. To demonstrate this, I use the results of Bierkens and Kappen [2012], who study a single-agent control problem (effectively, the seller’s moral hazard problem) and show that it is equivalent to a relative entropy minimization problem.

The timing in these models follows the principal-agent convention discussed earlier. At time zero, the seller and buyer trade a security. Between times zero and one, the seller will apply effort

<sup>38</sup>See Biais et al. [2007], Hellwig and Schmidt [2002], Sadzik and Stachetti [2013] for analysis of the relationship between discrete and continuous time models.

(or not) to change the value of the asset. At time one, the asset value is determined and the security payoffs occur.

Between times zero and one, the seller controls the drift of a Brownian motion. Define  $W$  as a Brownian motion on the canonical probability space,  $(\Omega, \mathcal{F}, \tilde{P})$ , and let  $\mathcal{F}_t^W$  be the standard augmented filtration generated by  $W$ . Denote the asset value at time  $t$  as  $V_t$ , and let  $\mathcal{F}_t^V$  be the filtration generated by  $V$ . The seller observes the history of both  $W_t$  and  $V_t$  at each time, whereas the buyer observes (or can contract on) only the history of  $V_t$ . This information asymmetry is what creates the moral hazard problem.

The initial value,  $V_0 > 0$ , is known to both the buyer and the seller. The asset value evolves as

$$dV_t = b(V_t, t)dt + u_t\sigma(V_t, t)dt + \sigma(V_t, t)dW_t,$$

where  $b(V_t, t)$  and  $\sigma(V_t, t) > 0$  satisfy standard conditions to ensure that, conditional on  $u_t = 0$  for all  $t$ , there is a unique, everywhere-positive solution to this SDE.<sup>39</sup> The seller's control,  $u_t$ , should be thought of as effort. Let  $\mathcal{U}$  be the set of admissible effort strategies, which I will define shortly. Effort is costly. There is a flow cost of effort, a general form of which is  $g(t, V_t, u_t)$ . The function  $g(\cdot)$  is weakly positive, twice-differentiable, and strictly convex in effort. For all  $t$  and  $V_t$ ,  $g(t, V_t, 0) = 0$ . Effort always improves the expected value of the asset. For all  $t$  and  $V_t$ , the expectation of  $V_1$  is increasing in  $u_s$ , for all  $s > t$ .<sup>40</sup>

The retained tranche,  $\eta(V)$ , is an  $\mathcal{F}_1^V$ -measurable random variable that can depend on the entire path of the asset value. I continue to assume limited liability, meaning that  $\eta(V) \in [0, \beta_s V_1]$  for all paths  $V$ . The seller's indirect utility function can be written as

$$\phi_{CT}(\eta) = \sup_{\{u_t\} \in \mathcal{U}} \phi_{CT}(\eta; \{u_t\}) = \sup_{\{u_t\} \in \mathcal{U}} \{E^{\tilde{P}}[\eta(V)] - E^{\tilde{P}}[\int_0^1 g(t, V_t, u_t)dt]\},$$

where  $E^{\tilde{P}}$  denotes the expectation at time zero under the physical probability measure. Alternatively, if the buyer and seller were risk-averse, but shared a common risk-neutral measure  $\tilde{P}$ , the problem would be identical. The key assumption in that case would be that the problem is small, in the sense that the outcome of this particular asset and security does not alter the common risk-neutral measure. To guarantee that utility is finite, I will make some additional assumptions, which I will explain shortly.

The security design problem is almost identical to the security design problem in the static models. The seller internalizes the effect that the security design will have on the price that the

<sup>39</sup>The following conditions are sufficient. For all  $V \in \mathbb{R}^+$  and  $t \in [0, 1]$ ,  $\sigma(V, t) > 0$  and  $|b(V, t)| + |\sigma(V, t)| \leq C(1 + |V|)$  for some positive constant  $C$ . For all  $t \in [0, 1]$ ,  $V, V' \in \mathbb{R}^+$ ,  $|b(V, t) - b(V', t)| + |\sigma(V, t) - \sigma(V', t)| \leq D|V - V'|$ , for some positive constant  $D$ . For all  $t \in [0, 1]$ ,  $\lim_{v \rightarrow 0^+} \sigma(v, t) = 0$ , and  $\lim_{v \rightarrow 0^+} b(t, v) \geq 0$ .

<sup>40</sup>This assumption places restrictions on the functions  $b(V_t, t)$  and  $\sigma(V_t, t)$ .

buyer is willing to pay, and solves

$$U_{CT}(s^*) = \sup_{s \in S} \{ \beta_b E^{\tilde{P}}[s(V)] + \phi_{CT}(\eta(s)) \},$$

where  $S$  is the set of  $\mathcal{F}_1^V$ -measurable limited liability security designs. Here, as in the static models,  $\eta(V) = \beta_s(V_1 - s(V))$ .

This model, which is a version of the one discussed by Schaeffler and Sung [1993] and Cvitanic et al. [2009], uses the “strong” formulation of the moral hazard problem. There is a second approach to the moral hazard problem, known as the “weak” formulation, discussed by those authors. The weak formulation is closely related to the static problems discussed earlier, and is equivalent to the strong formulation for the purpose of determining the optimal security design.

In the weak formulation, let  $X_t$  be a stochastic process that evolves as

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t,$$

where  $B$  is a Brownian motion on the probability space  $(\Omega, \mathcal{F}, Q)$ , with standard augmented filtration  $\mathcal{F}_t^B$ , and the  $b(\cdot)$  and  $\sigma(\cdot)$  are identical to the functions discussed above. Consider an  $\mathcal{F}_t^B$ -adapted control strategy  $u_t$  such that the stochastic exponential,  $Z_t = \exp(\int_0^t u_s dB_s - \frac{1}{2} \int_0^t u_s^2 ds)$ , is a martingale. For such a strategy, Girsanov’s theorem holds, and we can define a measure,  $P$ , that is absolutely continuous with respect to  $Q$ , such that  $\frac{dP}{dQ} = Z_1$ . Under the measure  $P$ ,  $B_t^P = B_t - \int_0^t u_s ds$  is a Brownian motion, and the process  $X_t$  evolves as

$$dX_t = b(X_t, t)dt + u_t \sigma(X_t, t)dt + \sigma(X_t, t)dB_t^P.$$

Under this measure  $P$ , the stochastic process  $X$  has the same law as the asset price  $V$  does under measure  $\tilde{P}$  in the strong formulation of the problem. Because the assumed effort strategies are  $\mathcal{F}_t^B$ -adapted, they can be written as functionals  $u_t = u(X, t)$ . For any such control strategy, it follows that

$$\phi_{CT}(\eta; \{u_t\}) = E^P[\eta(X)] - E^P[\int_0^1 g(t, X_t, u(X, t))dt],$$

meaning that the indirect utility in the strong formulation is the same as the indirect utility in the weak formulation. Here, I interpret  $X$  as the asset price, and require that  $\eta$  be  $\mathcal{F}_1^X = \mathcal{F}_1^B$ -measurable. By the same logic,  $E^{\tilde{P}}[s(V)] = E^P[s(X)]$  under these effort strategies.

I define the set of admissible strategies  $\mathcal{U}$  as the set of  $\mathcal{F}_t^B$ -adapted, square-integrable controls such that  $E[Z_1^4] < \infty$  (similar to the assumptions of Cvitanic et al. [2009]). Defining the set  $\mathcal{U}$  in this way ensures that Girsanov’s theorem can be applied. It follows that

$$\phi_{CT}(\eta) = \sup_{\{u_t\} \in \mathcal{U}} \{ E^P[\eta(X)] - E^P[\int_0^1 g(t, X_t, u_t)dt] \},$$

meaning that the optimal strategies in the weak and strong formulations, assuming they exist and are unique, are identical.

The restriction that  $u_t$  be  $\mathcal{F}_t^B$ -adapted has economic meaning. The filtrations generated by  $B_t$  and  $X_t$  are identical, and  $X_t$  is interpreted as the asset price. This restriction states that the seller's control strategy does not depend on the seller's private information ( $B^P$ ) but only the seller and buyer's common knowledge,  $X$  (and  $t$ ). Under the form of the flow cost function I have assumed, the optimal strategy will satisfy this restriction, even if it is not imposed (Cvitanic et al. [2009]).

The weak formulation can be rewritten to emphasize the idea of "choosing a distribution." The expected values of the security and the retained tranche depend only on the Radon-Nikodym derivative  $\frac{dP}{dQ}$ , and not directly on the control strategy.

Define the set  $M$  as the set of measures  $P$  that are absolutely continuous with respect to  $Q$ , and for whom  $E^Q[(\frac{dP}{dQ})^4] < \infty$ . Bierkens and Kappen [2012] (and the sources cited therein) show that any  $P \in M$  can be created through some control strategy  $\{u_t\}$ . I combine results found in Cvitanic et al. [2009] and Bierkens and Kappen [2012] into the following lemma:

**Lemma 2.** *For any effort strategy  $u(X, t) \in \mathcal{U}$ , the stochastic exponential  $Z_t = \exp(\int_0^t u(X, s)dB_s - \frac{1}{2} \int_0^t u(X, s)^2 ds)$  is an everywhere-positive martingale, and the measure defined by  $\frac{dP}{dQ} = Z_1$  is a measure in  $M$ . Conversely, for any measure  $P \in M$ , there exists an effort strategy  $u(X, t) \in \mathcal{U}$  such that  $\frac{dP}{dQ} = \exp(\int_0^1 u(X, s)dB_s - \frac{1}{2} \int_0^1 u(X, s)^2 ds)$ . The effort strategy  $u(X, t)$  is unique up to an evanescence.*

*Proof.* See appendix, section E.11. □

We can define a divergence,

$$D_g(P||Q) = \inf_{\{u_t\} \in \mathcal{U}} E^P[\int_0^1 g(t, X_t, u_t)dt],$$

subject to the constraint that  $\frac{dP}{dQ} = \exp(\int_0^1 u_s dB_s - \frac{1}{2} \int_0^1 u_s^2 ds)$ . By the uniqueness result in lemma 2, all strategies  $\{u_t\} \in \mathcal{U}$  that satisfy this constraint are identical for our purposes. Note that, because  $g(t, X_t, u_t) = 0$  if and only if  $u_t = 0$ , and is otherwise positive,  $D_g(P||Q)$  satisfies the definition of a divergence.

The moral hazard can be written as

$$\phi_{CT}(\eta) = \sup_{P \in M} \{E^Q[\frac{dP}{dQ}\eta(X)] - D_g(P||Q)\}.$$

I have now rewritten the continuous time moral hazard problem as a static problem, in which the seller chooses a probability measure subject to a cost that is described by a divergence. In light of the results for static models, two questions immediately arise. First, is there a  $g(\cdot)$  function

such that  $D_g(P||Q)$  is the Kullback-Leibler divergence? Second, are there  $g(\cdot)$  functions such that  $D_g(P||Q)$  is an invariant divergence?

The answer to the first question comes from the work of Bierkens and Kappen [2012] and the sources cited therein, who show that quadratic costs functions,  $g(t, X_t, u_t) = \frac{\theta}{2}u_t^2$ , lead to the KL divergence.<sup>41</sup> In the remainder of this section, I will analyze quadratic cost models, and show that debt is the optimal security design. In the next section, I will explore the question of whether there are  $g(\cdot)$  functions that lead to invariant divergences.

As mentioned earlier, some additional assumptions are necessary to ensure that utility in the moral hazard problem is finite. With quadratic costs, it is sufficient to assume that  $E^Q[\exp(4\theta^{-1}X_1)] < \infty$  (see Cvitanić et al. [2009]). This should be thought of as a restriction on the functions  $b(\cdot)$  and  $\sigma(\cdot)$  discussed previously. For a given  $P \in M$ , the KL divergence can be defined as

$$D_{KL}(P||Q) = E^P[\ln(\frac{dP}{dQ})].$$

For any  $u(X, t) \in \mathcal{U}$  such that  $\frac{dP}{dQ} = \exp(\int_0^1 u_s dB_s - \frac{1}{2} \int_0^1 u_s^2 ds)$ , it follows that

$$\begin{aligned} D_{KL}(P||Q) &= E^P[\int_0^1 u(X, s) dB_s - \frac{1}{2} \int_0^1 u(X, s)^2 ds] \\ &= E^P[\int_0^1 u(X, s) dB_s^P + \frac{1}{2} \int_0^1 u(X, s)^2 ds] \\ &= E^P[\frac{1}{2} \int_0^1 u(X, s)^2 ds]. \end{aligned}$$

For quadratic cost functions  $g(t, X_t, u_t) = \frac{\theta}{2}u_t^2$ , we have  $D_g(P||Q) = \theta D_{KL}(P||Q)$ . The moral hazard problem can be rewritten as

$$\phi_{CT}(\eta) = \sup_{P \in M} \{E^P[\eta(X)] - \theta D_{KL}(P||Q)\},$$

and the security design problem as

$$U_{CT}(s^*) = \sup_{s \in S} \{\beta_b E^{P^*(\eta(s))}[s(X)] + \phi_{CT}(\eta(s))\},$$

where  $P^*(\eta)$  is the measure chosen in the moral hazard problem. The model is a continuous sample space version of the non-parametric model discussed earlier. It is not surprising that debt is the optimal security design.

**Proposition 6.** *In the continuous time model, with the quadratic cost function, the optimal security design is a debt contract,*

<sup>41</sup>Note that this formulation rules out time discounting of the effort costs. One possible justification for this assumption is that the time period in question is short.

$$s_j(\eta^*) = \min(v_j, \bar{v}),$$

for some  $\bar{v} > 0$ .

*Proof.* See appendix, section E.10. The result is a special case of Cvitanić et al. [2009].  $\square$

Debt is the optimal security design in the continuous time model for same reasons it is optimal in the non-parametric model: it minimizes the variance of the security payout, holding fixed the mean value. The mean-variance intuition arises from concerns about reduced effort and risk-shifting, balanced against the gains from trade. Risk-shifting is possible in this model because of the seller can selectively alter her effort level, based on the path that the asset value has taken up to the current time. The surprising result of proposition 6 is that this sort of risk-shifting is equivalent to the risk-shifting in the static problem discussed earlier.

The intersection of these results with Holmström and Milgrom [1987] is intuitive. In the principal-agent framework, when the asset value Ito process is an arithmetic Brownian motion and the flow cost function is quadratic, without limited liability, a constant security for the principal is optimal. With limited liability, in the security design framework, optimal security simply reduces the constant payoff where necessary, and debt is optimal.

The debt security design may or may not be renegotiation-proof. Suppose that at some point, say time 0.5, the seller can offer the buyer a restructured security. Assume that at this time, there are no gains from trade.<sup>42</sup> If the current asset value is low enough, the debt security provides little incentive for the seller to continue putting in effort in the future. In this state, the buyer might agree to “write down” the debt security, even though he cannot receive any additional payments from the seller, because the buyer’s gains from increased effort by the seller could more than offset the loss of potential cash flows. In this model, write-downs can be Pareto-efficient if the time-zero expected value of the debt,  $E^P[s(X)]$ , is greater than  $\theta$ .<sup>43</sup> Write-downs will never be Pareto-efficient when  $\kappa$  and  $\theta^{-1}$  are both small, but could occur if both the gains from trade at time zero and the moral hazard were large.

One possible extension of this model concerns the timing of the seller’s actions and payment. In the context of mortgage origination, it might be reasonable to assume that the mortgage originator (seller) controls the drift of  $V_t$  from time zero to time one, when she is making the mortgage loans. However, the mortgage security might be constrained to pay both the originator and security buyers at time  $T > 1$ , and the drift of  $V_t$  between times one and  $T$  is not controlled by the originator. It turns out that this version of the model is related to the parametric problems discussed earlier. We

<sup>42</sup>Otherwise, if the asset value has increased, the seller will “lever up” and sell more debt to the buyer.

<sup>43</sup>Proof available upon request.

can still think of the seller as controlling the time  $T$  Radon-Nikodym derivative,  $\frac{dP}{dQ}$ , but subject to the constraint that the derivative be  $\mathcal{F}_1^B$ -measurable. One consequence of the invariance property of the KL divergence is that, if  $\frac{dP}{dQ}$  is  $\mathcal{F}_1^B$ -measurable, then the KL divergence between  $P$  and  $Q$ , projected onto  $\mathcal{F}_1^B$ , is the same as the KL divergence evaluated at time  $T$ . Similar to the parametric models discussed earlier, the seller cannot choose any distribution over paths from time zero to time  $T$ , but instead only a subset of all such distributions. Using arguments from that section, one can show that, under the approximation used earlier, a mean-variance tradeoff provides a lower bound on the utility any security design can achieve, and that debt maximizes this lower bound.

In the final section of the paper, I will discuss the second question mentioned earlier. Are there cost functions  $g(\cdot)$  for which  $D_g(P||Q)$  is an invariant divergence?

## 7. A MEAN-VARIANCE APPROXIMATION FOR CONTINUOUS TIME MODELS

For the static models discussed earlier, invariant divergence cost functions lead to models in which debt was approximately optimal. I will not directly answer the question of whether there functions  $g(\cdot)$  such that  $D_g(P||Q)$  is invariant. Instead, I will show that for all  $g(t, X_t, u_t) = \theta\psi(u_t)$ , where  $\psi(u_t)$  is a convex function, debt is approximately optimal. The approximations used in this section are identical to the ones discussed previously, in section §4. I consider problems in which both the moral hazard and gains from trade are small, relative to the scale of the assets. I show that the utility of arbitrary security designs can be characterized, to first-order, by a mean-variance tradeoff.

In this section, I will continue to analyze the models introduced in the previous section, with several small modifications. I will assume that the control is bounded,  $|u_t| \leq \bar{u}$ . This automatically ensures that all  $\mathcal{F}_t^B$ -adapted processes  $u_t$  are in  $\mathcal{U}$ , and simplifies the discussion of integrability conditions. I assume  $\psi$  satisfies the conditions required for  $g$  in the previous section, and in addition that for all  $|u| \leq \bar{u}$ ,  $\psi''(u) \in [K_1, K_2]$  for some positive constants  $0 < K_1 < 1 < K_2$ . I also normalize  $\psi''(0) = 1$ . I assume that, for bounded control strategies  $|u_t| \leq \bar{u}$ , the asset value has a finite fourth moment. That is,  $E^P[X_1^4] < \infty$  for all  $P$  for created by a bounded control strategy.

There is a sense in which any convex cost function  $\psi(u_t)$  resembles the quadratic cost function, as  $u_t$  becomes close to zero, because their second derivatives are the same. Similarly, in static models, all invariant divergences resemble the KL divergence, because their Hessian is the Fisher information matrix. I apply this idea to the divergences  $D_\psi$  induced by the convex cost functions  $\psi$ .

First, I define the Fisher information. Consider a model with a finite number of parameters,  $\tau$ , and associated probability distribution  $p(\omega; \tau)$ , over sample space  $\Omega$ . The Fisher information

matrix defined as

$$I_{ij} = E^{p(\omega; \tau)} \left[ \left( \frac{\partial \ln p(\omega; \tau)}{\partial \tau_i} \right) \left( \frac{\partial \ln p(\omega; \tau)}{\partial \tau_j} \right) \right].$$

I will define the Fisher information for the continuous time model, which has an infinite number of parameters, in an analogous way. Let  $\frac{dP}{dQ}(\gamma, \tau) = \exp(\int_0^1 (\gamma u_s + \tau v_s) dB_s - \frac{1}{2} \int_0^1 (\gamma u_s + \tau v_s)^2 ds)$ , where  $\gamma$  and  $\tau$  parametrize a perturbation in the direction defined by the square-integrable, predictable processes  $u_s$  and  $v_s$ .<sup>44</sup> The Fisher information, in the directions defined by  $u_s$  and  $v_s$ , is defined as

$$I(u, v) = E^{P(\gamma, \tau)} \left[ \left( \frac{\partial}{\partial \gamma} \ln \left( \frac{dP}{dQ}(\gamma, \tau) \right) \right) \left( \frac{\partial}{\partial \tau} \ln \left( \frac{dP}{dQ}(\gamma, \tau) \right) \right) \right]_{\gamma=\tau=0}.$$

Plugging in the definition of  $\frac{dP}{dQ}(\gamma, \tau)$ , and using the Ito isometry,  $I(u, v) = E^Q[\int_0^1 u_s v_s ds]$ . Next, consider the second variation of  $D_\psi(P(\gamma, \tau) || Q)$ ,

$$\begin{aligned} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \gamma} D_\psi(P(\gamma, \tau) || Q) |_{\gamma=\tau=0} &= \theta \frac{\partial}{\partial \tau} \frac{\partial}{\partial \gamma} E^{P(\gamma, \tau)} \left[ \int_0^1 \psi(\gamma u_s + \tau v_s) ds \right] |_{\gamma=\tau=0} \\ &= \theta E^Q \left[ \int_0^1 \psi''(0) u_s v_s ds \right]. \end{aligned}$$

It follows that  $\frac{\partial}{\partial \tau} \frac{\partial}{\partial \gamma} D_\psi(P(\gamma, \tau) || Q) |_{\gamma=\tau=0} = \theta I(u, v)$ .<sup>45</sup> For any cost function  $\psi$ , the second variation in the directions  $u_s$  and  $v_s$  is equal to the Fisher information in those directions. The divergences  $D_\psi$  resemble, locally, the KL divergence, in exactly the same way that all invariant divergences resemble the KL divergence.

I consider the same approximation discussed earlier, in which both  $\theta^{-1}$  and  $\kappa$  are small.<sup>46</sup> As the cost of effort rises, the seller will choose to respond less and less to the incentives provided by the retained tranche. Regardless of the cost function  $\psi$ , the divergence  $D_\psi(P || Q)$  will approach  $D_{KL}(P || Q)$ , and debt will be approximately optimal. To make this argument rigorous, I use Malliavin calculus in a manner similar to Monoyios [2013] to prove the following theorem:

**Proposition 7.** *For any limited liability security design  $s$ , the difference in utilities achieved by an arbitrary security  $s$  and the sell-nothing security is*

$$U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \kappa E^Q[\beta_s s] - \theta^{-1} \frac{1}{2} V^Q[\beta_s s] + O(\theta^{-2} + \theta^{-1} \kappa).$$

*Proof.* See appendix, section E.12. □

In the continuous time effort problem with an arbitrary convex cost function, debt securities are first-order optimal. The same mean-variance intuition that I discussed in static models applies to

<sup>44</sup>These perturbations are the Cameron-Martin directions used in Malliavin calculus.

<sup>45</sup>I am omitting several technicalities from this discussion. These technicalities are addressed in the proof of proposition 7.

<sup>46</sup>The limit I consider is related to the “large firm limit” discussed by Sannikov [2013].

continuous time models. The variance of the security payoff is again a summary statistic for the problems of reduced effort and risk shifting associated with the moral hazard problem.

## 8. CONCLUSION

In this paper, I offer a new explanation for the prevalence of debt contracts. Debt arises as the solution to a moral hazard problem with both effort and risk-shifting. Debt is exactly optimal in the non-parametric model, when the cost function is the KL divergence, for which I provide a continuous-time micro-foundation. Debt is approximately optimal for all invariant divergence cost functions, again because it is the mean-variance optimal contract. For parametric models, such as the rational inattention model, debt maximizes an approximate lower bound on the level of utility. In all of these results, debt is desirable because it minimizes the variance of the security payout, balancing the need to provide incentives for effort, minimize risk-shifting, and maximize trade. Taken together, these results offer a new explanation for why debt is observed in a wide range of settings.

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## APPENDIX A. ADDITIONAL FIGURES

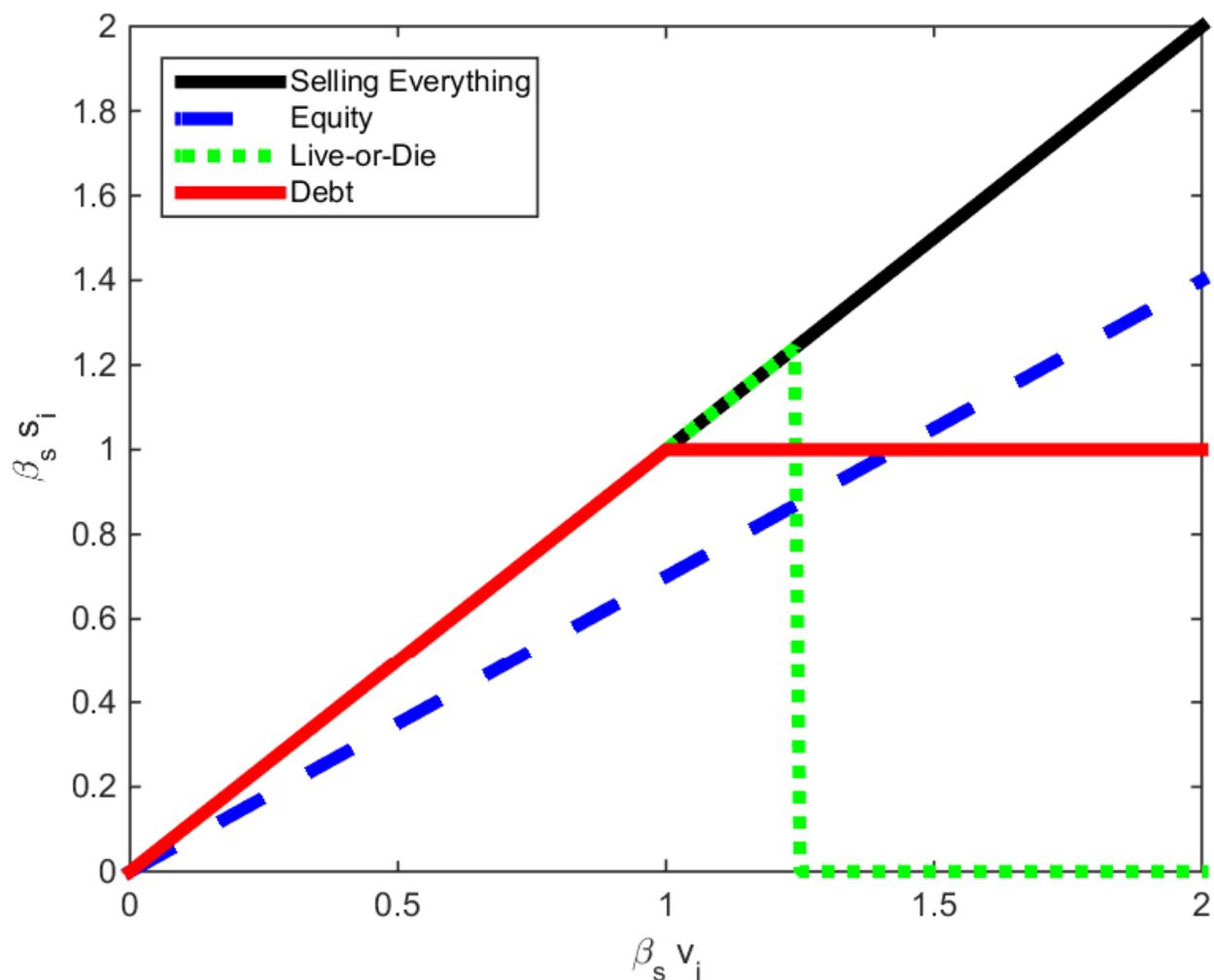


FIGURE A.1. POSSIBLE SECURITY DESIGNS. This figure illustrates several possible security designs: a debt security, an equity security, and the “live-or-die” security of Innes [1990]. The x-axis, labeled  $\beta_s v_i$ , is the discounted value of the asset, and the y-axis, labeled  $\beta_s s_i$ , is the discounted value of the security. The level of debt, the cutoff point for the live-or-die, and the fraction of equity are chosen for illustrative purposes. The discount factor for the seller is  $\beta_s = 0.5$ . The outcome space  $v_i$  is a set of 401 evenly-spaced values ranging from zero to 8. The x-axis is truncated to make the chart clearer.

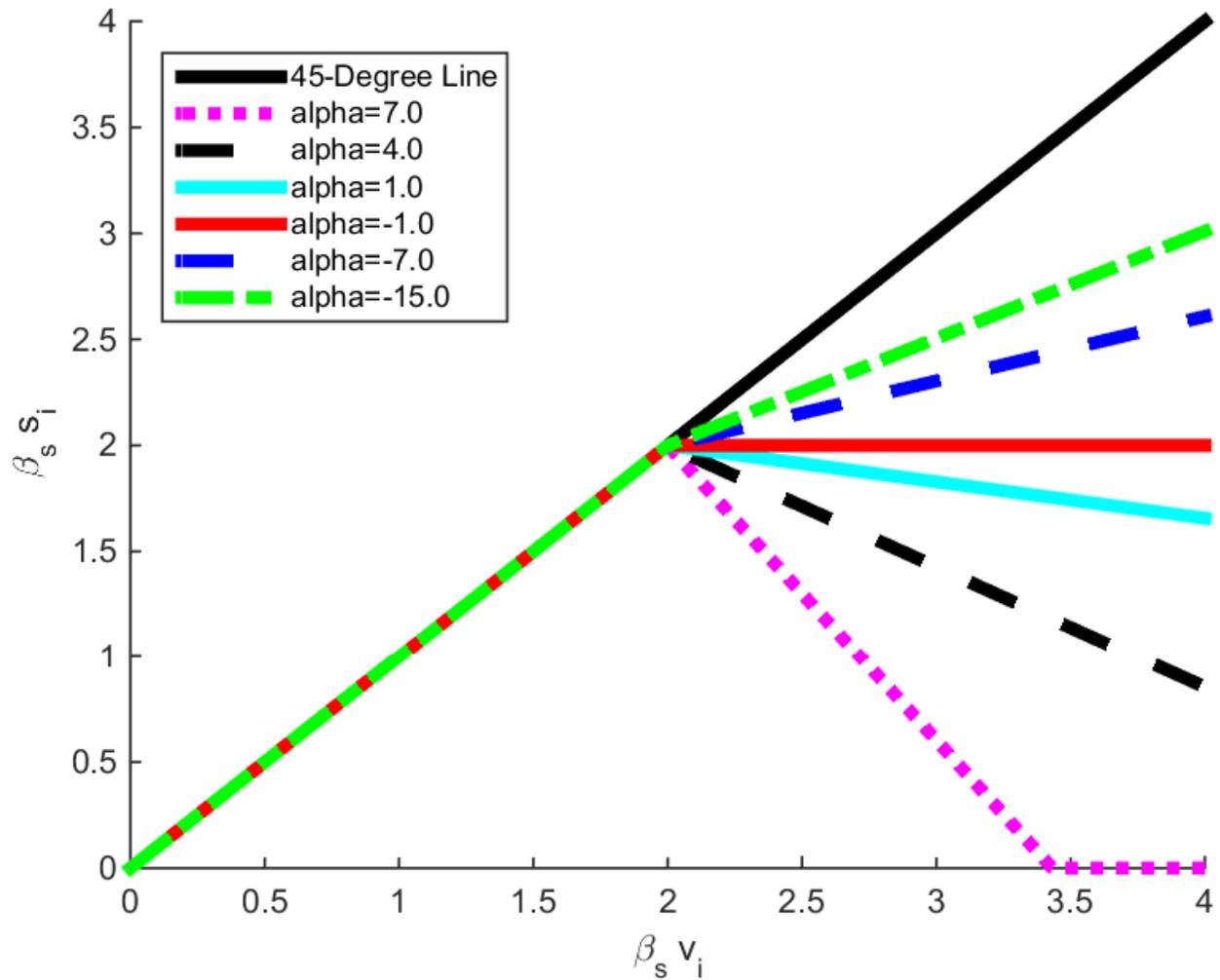


FIGURE A.2. SECOND-ORDER OPTIMAL SECURITY DESIGNS. This figure shows the second-order optimal security designs, for various values of the curvature parameter  $\alpha$ . The x-axis, labeled  $\beta_s v_i$ , is the discounted value of the asset, and the y-axis, labeled  $\beta_s s_i$ , is the discounted value of the security. These securities are plotted with the same  $\bar{v}$  for each  $\alpha$  (not an optimal  $\bar{v}$ ). The value of  $\kappa$  used to generate this figure is roughly 0.17, which was chosen to ensure that the slopes of the contracts would be visually distinct (and not because it is economically reasonable). The outcome space  $v$  is a set of 401 evenly-spaced values ranging from zero to 8.

## APPENDIX B. TIMING CONVENTIONS

TABLE 1. Timing Conventions During Period One

<i>Principal-Agent Timing</i>	<i>Origination Timing</i>	<i>Shelf Registration Timing</i>
(1) Security Designed	(1) Actions Taken	(1) Security Designed
(2) Security Traded	(2) Security Designed	(2) Actions Taken
(3) Actions Taken	(3) Security Traded	(3) Security Traded

The principal-agent timing convention is the simplest convention to analyze. In any sub-game perfect equilibrium, the seller takes actions that maximize the value of her retained tranche, because the price that she receives for the security has already been set. The buyer anticipates this, forming beliefs about the distribution of outcomes based on the design of the security. The buyer's beliefs affect the price that he is willing to pay for the security, and the seller internalizes this when designing the security. Multiple equilibria are possible if the seller's optimal actions for a particular retained tranche are not unique, or if there are multiple security designs that maximize the seller's utility. The moral hazard, in this timing convention, can occur either because the buyer is unaware of the seller's actions, or because he can observe those actions but is powerless to enforce any consequences based on them.

In the other two timing conventions, I use sequential equilibrium as the equilibrium concept and assume that the buyer does not observe the seller's actions. The buyer, because he is unaware of the actions the seller has taken, can accept or reject the security based only on its design and the price. In equilibrium, the seller will always take actions that maximize the value of her retained tranche— otherwise, she could profitably deviate by offering the same security at the same price while choosing different actions. The buyer will form beliefs about the seller's actions based on the design of the security, and accept or reject the seller's offer accordingly. It follows that the only sequential equilibria in these timing conventions are the sub-game perfect equilibria of the principal-agent timing convention. Intuitively, the model is a “pure moral hazard” model, meaning that the seller has no private information aside from knowledge of her own actions, and there is no signaling in the model.

## APPENDIX C. CALIBRATION

I will discuss the experimental approach first. This approach is consistent in spirit with the empirical literature on moral hazard in mortgage lending (Keys et al. [2010], Purnanandam [2011],

others). In that literature, the quasi-experiment compares no securitization ( $\eta_i = \beta_s v_i$ ) with securitization. If we assume securitization uses the optimal security design  $\eta^*$ , then  $\theta$  can be approximated as

$$\theta^{-1} \approx E^{p(\beta_s v)}[v_i] \cdot \frac{E^{p(\beta_s v)}[v_i] - E^{p^*}[v_i]}{Cov^{p^*}(v_i, s_i^*)}.$$

This formula illustrates the difficulties of calibrating the model using the empirical work on moral hazard in mortgage lending. For the purposes of the model, what matters is the loss in expected value due to securitization, relative to the risk taken on by the buyers, ex-ante. The empirical literature estimates ex-post differences, and the magnitude of these differences varies substantially, depending on whether the data sample is from before or during the crash in home prices. Converting this an ex-ante difference would require assigning beliefs to the buyer and seller about the likelihood of a crash. Estimating the ex-ante covariance, which can be understood as a measure of the quantity of “skin in the game,” is even more fraught. For these reasons, I have not pursued this calibration strategy further.

The second calibration strategy, which is somewhat more promising, is to use the design of mortgage securities to infer  $\theta$ . Essentially, by (crudely) estimating the other terms in the “put option value” equation (equation (3.4)), and assuming the model is correct, we can infer what the security designers thought the moral hazard was. Rearranging that equation,

$$\underbrace{\frac{\beta_b \bar{v} - \beta_b E^{p^*}[s_i]}{\beta_b E^{p^*}[s_i]}}_{Spread} \underbrace{\frac{E^{p^*}[s_i]}{E^{p^*}[v_i]}}_{Share} \left(1 - \underbrace{\frac{E^{p(\beta_s v)}[v_i] - E^{p^*}[v_i]}{E^{p(\beta_s(v))}[v_i]}}_{Moral Hazard}\right) \kappa^{-1} = \theta.$$

The spread term should be thought of as reflecting the initial spread between the assets purchased by the buyer and the discount rate, under the assumption that the bonds will not default. Using a 90/10 weighting on the initial AAA and BBB 06-2 ABX coupons reported in Gorton [2008], I estimate this as 34 basis points per year. In a different setting (CLOs), the work of Nadauld and Weisbach [2012] estimates the cost of capital advantage due to securitization at 17 basis points per year. The “share” term is the ratio of the initial market value of the security to the initial market value of the assets. Begley and Purnanandam [2013] document that the value of the non-equity tranches was roughly 99% of the principal value in their sample of residential mortgage securitizations. Similarly, the moral hazard term is likely to be small. The estimates of Keys et al. [2010], whose interpretation is disputed by Bubb and Kaufman [2014], imply that pre-crisis, securitized mortgage loans defaulted at a 3% higher rate<sup>47</sup> than loans held in portfolio. Assuming a 50% recovery rate, and using this as an estimate of the ex-ante expected difference in asset value, this suggests that the moral hazard term is roughly 1.5%, and therefore negligible

<sup>47</sup>After about one year, ~11% of securitized loans were in default, compared to ~8% of loans held in portfolio.

in this calibration. Combining all of these estimates, I find  $\theta$  of 2 is consistent with the empirical literature on securitization. This calibration assumed that the security design problem with the KL divergence was being solved. However, this formula also holds (approximately) under invariant divergences, conditional on the assumption that  $\theta^{-1}$  and  $\kappa$  are small enough.

The value of  $\theta = 2$  can be compared with the results of Figure 4.1. Under the assumptions used to generate that figure, which are described in its caption, I find that with  $\theta = 2$  and  $\kappa = 0.85$  (17 basis points per year times 5 years), debt would be achieve 99.96% of gains achieved by the optimal contract, relative to selling everything (and an even larger fraction of the gains relative to selling nothing). Under these parameters, the utility difference between the best debt security and selling nothing would be roughly 0.73% of the total asset value. While that might seem like an economically small gain, for a single deal described in Gorton [2008], SAIL 2005-6, the private gains of securitization would be roughly \$16.4mm. In contrast, the utility difference between the best equity security and selling nothing is about 0.56% of the total asset value. The private cost of using the optimal equity contract, instead of the optimal debt contract, would be roughly \$4mm for this particular securitization deal.

The numbers discussed in this calculation depend on the assumptions used in Figure 4.1, some of which are ad hoc. Nevertheless, they illustrate the general point that it is simultaneously possible for debt to be approximately optimal, and for the private gains of securitization to be large.

#### APPENDIX D. CONTRACTING WITH A RATIONALLY INATTENTIVE SELLER

In this section, I consider the problem of contracting with a rationally inattentive seller. I will show that this model is very close to the parametric security design model discussed in section §5, and that the lower bound result in proposition 3 applies.

One possible economic motivation for this type of problem comes from mortgage origination. Suppose that there is a set of  $A$  alternative groups of borrowers that the seller (a mortgage originator) could lend to. These groups could be borrowers from different zip codes, with different credit scores, etc., and a single borrower might belong to multiple groups. The seller can choose only one of these groups to lend to, and create the asset by making loans to this group. I assume that the seller can freely observe to which groups a potential borrower belongs, but is uncertain about which group she should lend to. I also assume that, under the seller's and buyer's common prior, lending to each alternative group will result in an asset with the same expected value and the same variance of asset value.

The buyer observes which group the seller chose to lend to. The buyer can also observe (in the second period) the ex-post returns of each group, not just the group the seller picked. The set of possible states of the world is  $X$ , where each state  $x \in X$  corresponds to a set of asset values for

each of the alternative groups, if that group had been chosen by the seller. There are  $N = |A| \cdot |X|$  possible combinations of states and choices by the seller. Let  $i$  be an index of these state-choice pairs,  $i \in \{0, \dots, N - 1\}$ , and let  $v_i$  denote the value of the asset the seller actually created in state-choice  $i$ . I will assume that  $v_0 = 0$ .

The value of the security that the seller offers to the buyer can depend on  $i$ , the state-choice pair. In this sense, the security can be “benchmarked,” and the payoff can depend not only on the ex-post value of the asset the seller created, but also on the ex-post value of the other alternatives she might have chosen. However, limited liability still applies, based on the value of the asset the seller actually created. I will denote the value of the security in state-choice  $i$  as  $s_i$ , and require  $s_i \in [0, v_i]$ .

The moral hazard in this problem comes from the seller’s ability to acquire information about which states  $x \in X$  are mostly likely, before choosing amongst the  $A$  alternatives. The quantity and nature of the information the seller acquires is not contractible, and this is what creates the moral hazard problem. The seller could choose an information structure to maximize the expected value of the assets she creates, but will instead choose an information structure to maximize the value of the retained tranche  $\eta$ . I will model the choice and cost of different information structures using the rational inattention framework of Sims [2003] and the results about discrete choice and rational inattention found in Matějka and McKay [2011].

As in previous sections, I will assume that the seller has a lower discount factor than the buyer, creating gains from trade. For expository purposes, I will use the “principal-agent” timing convention, but the optimal security design does not depend on the timing.<sup>48</sup> The timing of the model is as follows. During the first period:

- (1) The seller designs the security  $s_i$ , subject to limited liability.
- (2) The seller makes a take-it-or-leave-it offer to the buyer, at price  $K$ .
- (3) The buyer accepts or rejects the offer.
- (4) The seller chooses a signal structure, unobservable to the buyer.
- (5) The seller receives a signal about the likelihood of each state in  $X$ .
- (6) The seller chooses an alternative  $a \in A$ .

The seller’s choice of information structure and action is similar to a version of the parametric models discussed earlier. Let  $g(x)$  denote the common prior over the possible states  $x \in X$ . Following the standard simplification result used in rational inattention problems, I will think of the seller as choosing the conditional probability distribution of alternatives  $a \in A$ , for each state  $x$ .

<sup>48</sup>In the “origination” and “shelf registration” timing conventions, one can show that equilibria with pooling across signal realizations are ruled out in the optimal signal structure. In effect, endogenous information acquisition leads to pure moral hazard problems, in situations where exogenous signal structures would lead to adverse selection.

This model is parametric, because the seller cannot create any probability distribution over the  $N$  state-choice combinations. Instead, she is constrained to create only those probability distributions over the  $N$  state-choices that have the marginal distribution over states  $x \in X$  equal to  $g(x)$ . Economically, the mortgage originator does not control the amount that each group of mortgage borrowers will repay (the state). The best the mortgage originator can do is learn about which states are likely, and choose which group of borrowers to lend to accordingly.

The cost of the signal structure is described by mutual information, which can be rewritten as a KL divergence. The seller's moral hazard problem is

$$\phi_{RI}(\eta; \theta^{-1}) = \max_{p \in M_{RI}} \left\{ \sum_i p^i \eta_i - \theta D_{KL}(p || q(p)) \right\},$$

where  $M_{RI}$  denotes the set of probability distributions over state-choices with the marginal distribution over states equal to  $g(x)$ , and  $q(p)$  is the joint distribution between states and choices, when choices are independent of states and the marginal distribution of choices in  $q$  is the same as the marginal distribution in  $p$ . The constant of proportionality  $\theta$  is now interpreted as the cost per unit of information that the seller acquires. I will continue to assume that the solution is interior and unique. This assumption simplifies the proof and discussion, but can be relaxed (Matějka and McKay [2011] provide more primitive conditions under which the assumption would hold).

The key difference between this moral hazard problem and the parametric problem discussed previously is that the  $q$  distribution is endogenous, not fixed. This endogeneity is a property of the mutual information cost function. If the seller were to choose  $p$  so that her action were independent of the state, there would be no information cost, regardless of the marginal distribution of actions. Economically, if the mortgage originator decided to randomly choose a group of borrowers to give loans to, independent of economic fundamentals, this would require no costly information gathering, regardless of how the mortgage originator decided to randomize.

The security design problem is almost identical to the parametric moral hazard model. For an arbitrary security  $s$ ,

$$U_{RI}(s; \theta^{-1}, \kappa) = \beta_s(1 + \kappa) \sum_{i>0} p^i(\eta(s); \theta^{-1}) s_i + \phi_{RI}(\eta(s); \theta^{-1}),$$

where  $p(\eta; \theta^{-1})$  is the probability distribution over state-choices that the seller will choose, given retained tranche  $\eta$  and cost of information  $\theta$ . Despite the aforementioned complication that  $q$  is endogenous, the lower bound result of proposition 5 applies.

**Proposition 8.** *In the rational inattention model, the difference in utilities achieved by an arbitrary security  $s$  and the sell-nothing security, for sufficiently small  $\theta^{-1}$  and  $\kappa$ , is bounded below:*

$$U_{RI}(s; \theta^{-1}, \kappa) - U_{RI}(0; \theta^{-1}, \kappa) \geq \kappa E^{\bar{p}}[\beta_s s_i] - \frac{1}{2} \theta^{-1} V^{\bar{p}}[\beta_s s_i] + O(\theta^{-2} + \kappa \theta^{-1}),$$

where  $\bar{p}$  is the endogenous probability distribution that the seller would choose, if they could not gather information and the security was  $s$ .

*Proof.* See appendix, section E.9. □

The statement of proposition 8 discusses the endogenous probability distribution  $\bar{p}$ , which is the probability distribution the seller would choose if they could only choose the unconditional distribution of actions, and the actions were independent of the state. The key intuition is that different security designs can change the seller's incentives to pick one alternative over another, even if the seller could not gather any information at all. This could be true because the security treats different alternatives differently, or because the security's valuation depends on third or higher moments of the value distribution (only the mean and variance of each alternative under the prior are assumed to be identical). Debt maximizes the lower bound in proposition 8 only when small, symmetric perturbations to the debt security do not alter the unconditional distribution of actions.<sup>49</sup> This would hold true if the prior distribution of values were identical for each alternative.

In general, rational inattention problems will not result in debt as the optimal contract, because the seller does not control the state  $x \in X$ . If there are some states that offer higher payoffs than others, regardless of the alternative chosen, it is not efficient for the seller to receive higher cash flows in those "good" states. Additionally, the security design influences the unconditional probability that a particular alternative will be chosen. A debt contract, whose payoff does not vary by alternative chosen, only by outcome, may result in a sub-optimal choice of unconditional probabilities for different actions.

The lower bound is tight for symmetric contracts under two conditions.<sup>50</sup> The first is the exchangeable prior discussed by Matějka and McKay [2011]. When the prior is exchangeable, there is no reason for the security payoff to vary by alternative chosen, because each alternative is ex-ante equivalent. The second condition is that the states are symmetric, and there is no such thing as a "good state" or "bad state," only good and bad alternatives conditional on the state. One justification for this second condition is complete markets (see the discrete-outcome, complete asset market described in He [1990]). Under these two conditions, the lower bound is tight, and a debt security is approximately optimal.<sup>51</sup>

<sup>49</sup>By symmetric perturbations, I mean those perturbations that, like the debt contract, depend only on the value of the outcome, and not the alternative chosen.

<sup>50</sup>These conditions are sufficient, not necessary.

<sup>51</sup>In fact, under these two conditions, a debt security is optimal, not just approximately optimal.

Contracting problems with rationally inattentive agents are challenging to solve exactly, both analytically and computationally. The lower bound result of proposition 8 shows that, when the seller's ability to gather information is weak, debt contracts are a tractable, detail-free way of guaranteeing a minimum utility level.

## APPENDIX E. PROOFS

**E.1. Proof of proposition 1.** To prove that debt securities are optimal, I first re-characterize the security design first-order condition.

**Lemma 3.** *In the non-parametric model, with the cost function proportional to the Kullback-Leibler divergence, the optimal security design satisfies the following condition for all values of  $j$ :*

$$\beta_b s_j(\eta^*) = \beta_b \sum_{i>0} p^i(\eta^*) s_i(\eta^*) + \kappa \theta \left(1 - \frac{\lambda^j}{p^j(\eta^*)} + \frac{\omega^j}{p^j(\eta^*)}\right).$$

*Proof.* See appendix, section E.2. □

To characterize the optimal security, I analyze the equation of lemma 3 and the complementary slackness conditions for the multipliers, as in Yang [2012]. The complementary slackness conditions on the multipliers require that if  $\lambda^j > 0$ ,  $s_j = v_j$ , and that if  $\omega^j > 0$ ,  $s_j = 0$ . Define the endogenous positive constant

$$\bar{v} = \sum_{i>0} p^i(\eta^*) s_i(\eta^*) + \frac{\kappa}{\beta_b} \theta.$$

Consider the case where  $\lambda^j > 0$ . In this case, we must have  $\omega^j = 0$  and  $s_j = v_j$ . Using lemma 3, in this case  $v_j = \bar{v} - \frac{\kappa}{\beta_b} \theta \frac{\lambda^j}{p^j} < \bar{v}$ . Next, consider the case where  $\omega^j > 0$ . In this case, we would have  $\lambda^j = 0$ ,  $s_j = 0$ , and therefore  $0 = \bar{v} + \frac{\kappa}{\beta_b} \theta \frac{\omega^j}{p^j}$ , a contradiction. Finally, consider the case where  $\lambda^j = \omega^j = 0$ . In this case,  $s_j = \bar{v}$ . Combining these three cases, we see that if  $v_j < \bar{v}$ , we must have  $\lambda^j > 0$  and  $s_j = v_j$ , and if  $v_j \geq \bar{v}$ , we must have  $s_j = \bar{v}$ . Therefore,  $s_j$  is a debt contract.

Assume that  $v_N > \sum_i q^i v_i + \frac{\kappa}{\beta_b} \theta$ , and suppose that  $\bar{v} \geq v_N$ . The contract would be a sell-everything contract, and the seller would choose the minimum cost distribution,  $p^i(\eta^*) = q^i$ . Therefore,  $\bar{v} = \sum_i q^i v_i + \frac{\beta_b - \beta_s}{\beta_b \beta_s} \theta$ , a contradiction.

The argument above shows that any security satisfying the first-order condition of the Lagrangian is a debt contract. The existence of an optimal contract is guaranteed by the existence of a maximum of  $U(\eta)$  on the closed and bounded set of  $\mathbb{R}^N$  corresponding to limited liability securities, due to the continuity of  $U(\eta)$ . Because a maximum exists and  $U(\eta)$  is differentiable, the

first-order condition for the Lagrangian is necessary for optimality. Therefore the optimal security must be a debt security.

**E.2. Proof of lemma 3.** The security design equation, specialized to the Fisher metric Hessian, is

$$\beta_b s_j(\eta^*) = \sum_{i>0} \theta \kappa [p^i(\eta^*) - \lambda^i + \omega^i] g_{ij}(p(\eta^*)),$$

where

$$g_{ij}(p(\eta^*)) = \left( \frac{\delta_{ij}}{p^i(\eta^*)} + \frac{1}{p^0(\eta^*)} \right).$$

Taking expected values,

$$\begin{aligned} \beta_b \sum_{j>0} p^j(\eta^*) s_j(\eta^*) &= \theta \kappa \sum_i [p^i(\eta^*) - \lambda^i + \omega^i] + \\ &\theta \kappa \sum_i [p^i(\eta^*) - \lambda^i + \omega^i] \frac{1 - p^0(\eta^*)}{p^0(\eta^*)}. \end{aligned}$$

Simplifying,

$$\beta_b \sum_{j>0} p^j(\eta^*) s_j(\eta^*) = \frac{\theta \kappa}{p^0(\eta^*)} \sum_i [p^i(\eta^*) - \lambda^i + \omega^i].$$

Plugging this into the security design equation,

$$\beta_b s_j(\eta^*) = \beta_b \sum_{i>0} p^i(\eta^*) s_i(\eta^*) + \theta \kappa \sum_{i>0} [p^i(\eta^*) - \lambda^i + \omega^i] \frac{\delta_{ij}}{p^i(\eta^*)}.$$

which is equivalent to the desired result.

**E.3. Proof of proposition 2.** For any  $f$ -divergence, the Hessian is

$$\frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \Big|_{p=p(\eta^*)} = \theta \left[ \frac{\delta_{ij}}{q^i} f''\left(\frac{p^i(\eta^*)}{q^i}\right) + \frac{1}{q^0} f''\left(\frac{p^0(\eta^*)}{q^0}\right) \right].$$

To prove proposition 2, I suppose that there is some  $f$ -divergence such that, for every sample space  $\Omega$  and zero-cost distribution  $q$ , the optimal security design is debt. I will show that this  $f$ -divergence must be the KL divergence.

Take some  $\Omega$  and  $q$  such that the optimal security design is a debt, with at least two distinct indices  $j$  and  $j'$  such that  $s_j = s_{j'}$  and  $0 < s_j < \min(v_j, v_{j'})$ , and such that the solution to the moral hazard problem under the optimal contract is interior. The indices  $j$  and  $j'$  correspond to the “flat” part of the a debt contract. I construct an  $\Omega$  and  $q$  with this property below, showing

that such examples exist. Because the solution to the moral hazard problem is interior, from the security design equation,

$$s_j(\eta^*) = \sum_{i>0} \theta(\beta_s^{-1} - \beta_b^{-1}) [p^i(\eta^*) - \lambda^i + \omega^i] \left[ \frac{\delta_{ij}}{q^i} f''\left(\frac{p^i(\eta^*)}{q^i}\right) + \frac{1}{q^0} f''\left(\frac{p^0(\eta^*)}{q^0}\right) \right].$$

For the indices  $j$  and  $j'$  associated with the flat part of the debt contract, neither multiplier binds, and therefore

$$\frac{p^j(\eta^*)}{q^j} f''\left(\frac{p^j(\eta^*)}{q^j}\right) = \frac{p^{j'}(\eta^*)}{q^{j'}} f''\left(\frac{p^{j'}(\eta^*)}{q^{j'}}\right).$$

If the optimal security is a debt for some  $\Omega$  and  $q$ , this property must hold for all  $j$  and  $j'$  associated with the flat part of that debt contract. Below, I state a claim. I will first prove the rest of the proposition, assuming this claim, and then prove the claim.

*Claim 1.* For all pairs  $u_1 \in (0, 1]$  and  $u_2 \in [1, \infty)$ , there exist  $\Omega$  and  $q$  such that, under the optimal debt contract, there are indices  $j$  and  $j'$ , associated with the flat part of the debt contract, such that

$$\frac{p^j(\eta^*)}{q^j} = u_1$$

and

$$\frac{p^{j'}(\eta^*)}{q^{j'}} = u_2.$$

Assume the above claim is true. If the optimal security is a debt, then the optimal debt contract (the best contract in the class of debt contracts) is the optimal security (in the space of all securities). Because debt is optimal for all  $\Omega$  and  $q$ , including the ones constructed in the above claim, it follows that for any  $u_1$  and  $u_2$ , we must have

$$u_1 f''(u_1) = u_2 f''(u_2).$$

When  $u_1 = 1$ ,  $u f''(u) = 1$  by the normalization assumption, and therefore for all  $u \in (0, \infty)$ ,

$$u f''(u) = 1.$$

Solving the differential equation, using the normalization assumptions that  $f'(1) = 0$  and  $f(1) = 0$ ,

$$f(u) = u \ln u - u + 1,$$

proving the proposition. To complete the proof, I prove the above claim.

E.3.1. *Proof of Claim.* First, note the first-order condition for the moral hazard problem, assuming an interior solution, requires that

$$f'\left(\frac{p^i(\eta^*)}{q^i}\right) - f'\left(\frac{p^0(\eta^*)}{q^0}\right) = \theta^{-1}\eta_i = \theta^{-1}\beta_s(v_i - s_i).$$

For the flat region for some  $j$ ,

$$f'\left(\frac{p^j(\eta^*)}{q^j}\right) - f'\left(\frac{p^0(\eta^*)}{q^0}\right) = \theta^{-1}\beta_s(v_j - \bar{v}).$$

Let  $\Omega = \{0, v_1, v_2\}$ . I will construct, under the optimal debt security,

$$\frac{p^1(\eta^*)}{q^1} = u_1,$$

$$\frac{p^2(\eta^*)}{q^2} = u_2,$$

for given  $u_1 \in (0, 1]$  and  $u_2 \in [1, \infty)$ . Assume for now that the optimal debt contract has  $\bar{v} < v_1$  as its maximum value. By the first-order conditions, we have (using the normalization condition)

$$f'(u_1) - f'(u_0) = \theta^{-1}\beta_s(v_1 - \bar{v}),$$

where  $u_0 = \frac{p^0(\eta^*)}{q^0}$ . It follows that

$$f'(u_2) - f'(u_1) = \theta^{-1}\beta_s(v_2 - v_1).$$

Because  $f(u)$  is convex, this equation can be solved to pin down  $v_2$  given  $u_2$  and  $u_1$ , and  $v_1$ , noting that  $v_2 \geq v_1$  by the convexity and normalization of  $f$ , and  $u_2 \geq u_1$ . Next, I will use the adding up constraints to choose a  $q$ . We must have

$$q^0 + q^1 + q^2 = 1$$

and

$$q^0 u_0 + q^1 u_1 + q^2 u_2 = 1.$$

Putting these together,

$$u_0(q^1, q^2) = \frac{1 - q^1 u_1 - q^2 u_2}{1 - q^1 - q^2}.$$

The solution will be interior if  $u_0(q^1, q^2) > 0$ . Finally, I define the optimal debt security. Denote the class of retained tranches associated with debt securities as  $\eta(\bar{v})$ . The security design FOC with respect to  $\bar{v}$  is

$$\left(1 - \frac{\beta_b}{\beta_s}\right) \sum_{i>0} p^i(\eta^*) \frac{\partial \eta_i(\bar{v})}{\partial \bar{v}} + \beta_b \sum_{i,j>0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \Big|_{\eta=\eta^*} s_j(\eta^*) \frac{\partial \eta_i(\bar{v})}{\partial \bar{v}} = 0.$$

Define  $\bar{v}_{full}(q^1, q^2)$  as the solution to this equation, given  $q^1$  and  $q^2$ . The Hessian of  $\phi$  is the inverse of the Hessian of  $\psi$ . Using the Sherman-Morrison formula,

$$\frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \Big|_{\eta=\eta^*} = \theta^{-1} \frac{\delta_{ij} q^i}{f''(u_i)} - \theta^{-1} \frac{q^i q^j}{\sum_i \frac{q^i}{f''(u_i)}}.$$

Assuming that  $v_1 > \bar{v}$ , this simplifies to

$$(\beta_b^{-1} - \beta_s^{-1})(1 - u_0 q^0) + \beta_b \bar{v} \sum_{i,j>0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \Big|_{\eta=\eta^*} = 0.$$

Define  $\bar{v}(q^1, q^2)$  as the solution to this equation, which differs from  $\bar{v}_{full}(q^1, q^2)$  due to the assumption that  $\bar{v} < v_1$  and associated simplifications. Summing,

$$\sum_{i,j>0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \Big|_{\eta=\eta^*} = \theta^{-1} \sum_{i>0} \frac{q^i}{f''(u_i)} - \theta^{-1} \frac{\sum_{i,j>0} \frac{q^i q^j}{f''(u_i) f''(u_j)}}{\sum_i \frac{q^i}{f''(u_i)}}.$$

Because  $\phi$  is convex, we can think of this as

$$\sum_{i,j>0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \Big|_{\eta=\eta^*} = h(q^1, q^2),$$

where  $h(q^1, q^2)$  is a positive-valued function. Therefore,

$$\bar{v}(q^1, q^2) = (\beta_s^{-1} - \beta_b^{-1}) \theta \frac{q^1 u_1 + q^2 u_2}{h(q^1, q^2)}.$$

We can define  $v_1$  from

$$v_1 = \bar{v}(q^1, q^2) + \theta \beta_s^{-1} (f'(u_1) - f'(u_0(q^1, q^2))),$$

which is greater than  $\bar{v}$  by the convexity and normalization of  $f$ , as long as  $u_0 < u_1$ , which can be maintained by choice of  $q_1$  and  $q_2$ . It remains to be shown that  $\bar{v}(q^1, q^2)$  is the globally optimal contract, considering also contracts with  $\bar{v} > v_1$ . Note that in the limit as  $q^1 \rightarrow 0$ ,

$$\lim_{q^1 \rightarrow 0^+} \bar{v}_{full}(q^1, q^2) = \lim_{q^1 \rightarrow 0^+} \bar{v}(q^1, q^2),$$

because whether  $v_1 > \bar{v}$  or not does not alter the utility from the security design, since  $q^1$  is small. Moreover, in this limit,  $u_0(q^1, q^2)$  can be made arbitrarily small through a choice of  $q^2$ , preserving the requirement that  $u_0(q^1, q^2) \in (0, u_1)$ . It follows that there exist  $q^1$  and  $q^2$  such that  $u_0(q^1, q^2) < u_1$ ,  $0 < \bar{v}(q^1, q^2) < v_1 < v_2$ , and the desired result

$$\frac{p^1(\eta^*)}{q^1} = u_1,$$

$$\frac{p^2(\eta^*)}{q^2} = u_2,$$

holds.

**E.4. Proof of proposition 3 and proposition 4.** This section proves both proposition 4 and proposition 3, which is a special case. In this proof, I am using the summation convention. For example,  $p^i s_i$  is a summation,  $\sum_{i>0} p^i s_i$ . We define the utility generated by a particular contract as

$$U(s; \theta^{-1}) = \beta_b p^i(\eta(s)) s_i + \phi(\eta(s)).$$

The first-order condition for the moral hazard problem is

$$\eta_i - \partial_i \psi(p(\eta)) = 0.$$

In the neighborhood of  $\theta \rightarrow \infty$ ,

$$\lim_{\theta \rightarrow \infty} p(\eta) = q,$$

because otherwise the seller would receive unbounded negative utility. The solution to the moral hazard problem is guaranteed to be interior in this neighborhood, due to the assumption that  $q$  has full support. Expanding the function  $\partial_i \psi(p(\eta))$  around  $p = q$ ,

$$\eta_i - \partial^i \psi(q) - (p^j - q^j) \partial_i \partial_j \psi(q) - \frac{1}{2} (p^j - q^j) (p^k - q^k) \partial_i \partial_j \partial_k \psi(p^*) = 0,$$

where  $p^* = q + a^*(p - q)$  for some  $a^* \in (0, 1)$ . Because  $\psi(p)$  is invariant, we can simplify this to

$$p^j - q^j = \theta^{-1} g^{ij}(q) \eta_i - \frac{1}{2} g^{ij}(q) (p^l - q^l) (p^k - q^k) h_{ikl}(p^*),$$

where  $h_{ikl}(p^*) = \theta^{-1} c \partial_i \partial_j \partial_k \psi(p^*)$ . It follows that

$$p^j - q^j = O(\theta^{-1}).$$

Returning to the first-order condition, and expanding it up to order  $\theta^{-2}$ ,

$$\theta^{-1}\eta_i g^{il}(q) - \frac{1}{2}(p^j - q^j)(p^k - q^k)h_{ijk}(q)g^{il}(q) + O(\theta^{-3}) = (p^l - q^l).$$

Plugging this equation into itself,

$$\theta^{-1}\eta_i g^{il}(q) - \frac{1}{2}\theta^{-2}\eta_m \eta_n g^{jm}(q)g^{kn}(q)h_{ijk}(q)g^{il}(q) + O(\theta^{-3}) = (p^l - q^l).$$

For ease of notation, define

$$h^{lmn}(q) = g^{jm}(q)g^{kn}(q)h_{ijk}(q)g^{il}(q).$$

It also follows that to second order, the cost function can be approximated as

$$\psi(p) = \frac{\theta}{2}(p^i - q^i)(p^j - q^j)g_{ij}(q) + \frac{\theta}{6}(p^i - q^i)(p^j - q^j)(p^k - q^k)h_{ijk}(q) + O(\theta^{-3}),$$

Using the first-order condition,

$$\begin{aligned} \psi(p) &= \frac{1}{2}(p^i - q^i)\eta_i - \\ &\quad \frac{1}{4}\theta^{-1}(p^i - q^i)\eta_m \eta_n g^{jm}(q)g^{kn}(q)h_{ijk}(q) + \\ &\quad \frac{\theta}{6}(p^i - q^i)(p^j - q^j)(p^k - q^k)h_{ijk}(q) + O(\theta^{-3}), \end{aligned}$$

and

$$\begin{aligned} \psi(p) &= \frac{1}{2}\theta^{-1}\eta_j \eta_i g^{ij}(q) - \\ &\quad \frac{1}{4}\theta^{-2}\eta_m \eta_n \eta_i h^{ilm}(q) - \\ &\quad \frac{1}{4}\theta^{-2}\eta_m \eta_n \eta_i h^{ilm}(q) + \\ &\quad \frac{\theta^{-2}}{6}\eta_m \eta_n \eta_i h^{ilm}(q) + O(\theta^{-3}). \end{aligned}$$

Simplifying,

$$\psi(p) = \frac{1}{2}\theta^{-1}\eta_j \eta_i g^{ij}(q) - \frac{1}{3}\theta^{-2}\eta_m \eta_n \eta_i h^{ilm}(q) + O(\theta^{-3}).$$

The utility given by an arbitrary security is

$$U(s; \theta^{-1}, \kappa) = \beta_s(1 + \kappa)p^i(\eta(s))v_i - \kappa p^i(\eta(s))\eta_i(s) - \psi(p(\eta(s))),$$

which under these approximations is

$$\begin{aligned} U(s; \theta^{-1}, \kappa) &= \beta_s(1 + \kappa)(p^i - q^i)v_i + \beta_s(1 + \kappa)q^i v_i - \\ &\quad \kappa(p^i - q^i)\eta_i - \kappa q^i \eta_i - \\ &\quad \frac{\theta}{2}(p^i - q^i)(p^j - q^j)g_{ij}(q) - \frac{\theta}{6}(p^i - q^i)(p^j - q^j)(p^k - q^k)h_{ijk}(q) + O(\theta^{-3} + \kappa\theta^{-2}). \end{aligned}$$

and the expression can be rewritten as

$$\begin{aligned} U(s; \theta^{-1}, \kappa) &= \beta_s(1 + \kappa)\theta^{-1}\eta_i g^{il}(q)v_l - \frac{\theta^{-2}}{2}\beta_s\eta_j\eta_k v_i h^{ijk}(q) + \\ &\quad + \beta_s(1 + \kappa)q^i v_i - \\ &\quad \kappa\theta^{-1}\eta_i g^{il}(q)\eta_l - \kappa q^i \eta_i - \\ \text{(E.1)} \quad &\quad \frac{\theta^{-1}}{2}\eta_i\eta_j g^{ij}(q) + \frac{\theta^{-2}}{3}\eta_i\eta_j\eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa\theta^{-2}). \end{aligned}$$

For the zero security,

$$U(0; \theta^{-1}, \kappa) = \frac{1}{2}\beta_s^2\theta^{-1}v_i g^{il}(q)v_l - \frac{1}{6}\theta^{-2}\beta_s^3 v_j v_k v_i h^{ijk}(q) + \beta_s q^i v_i + O(\theta^{-3} + \kappa\theta^{-2}).$$

Taking the difference,

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) &= \beta_s(1 + \kappa)\theta^{-1}\eta_i g^{il}(q)v_l - \frac{\theta^{-2}}{2}\beta_s\eta_j\eta_k v_i h^{ijk}(q) + \\ &\quad + \beta_s\kappa q^i s_i - \frac{1}{2}\beta_s^2\theta^{-1}v_i g^{il}(q)v_l + \frac{1}{6}\theta^{-2}\beta_s^3 v_j v_k v_i h^{ijk}(q) - \\ &\quad \kappa\theta^{-1}\eta_i g^{il}(q)\eta_l - \\ &\quad \frac{\theta^{-1}}{2}\eta_i\eta_j g^{ij}(q) + \frac{\theta^{-2}}{3}\eta_i\eta_j\eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa\theta^{-2}). \end{aligned}$$

Substituting out (most) of the  $\eta$  terms for  $s$  and  $v$  terms,

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) &= -\frac{1}{2}\beta_s^2\theta^{-1}s_i g^{il}(q)s_l + \beta_s\kappa\theta^{-1}n_i g^{il}(q)s_l + \kappa\beta_s q^i s_i \\ &\quad + \frac{1}{6}\theta^{-2}\beta_s^2\eta_j s_k v_i h^{ijk}(q) + \frac{1}{3}\theta^{-2}\beta_s^3 s_j s_k s_i h^{ijk}(q) + O(\theta^{-3}). \end{aligned}$$

This expression is equivalent to the statement of the theorem, except that it remains to show that

$$s_i g^{il}(q) s_l = \text{Var}^q[s_i],$$

$$v_i g^{il}(q) s_l = \text{Cov}^q[v_i, s_i],$$

and

$$x_i y_j z_k h^{ijk}(q) = \left(\frac{3 + \alpha}{2}\right) K_3^q(x_j, y_k, z_i).$$

The first two follow from the definition of the non-parametric Fisher information metric (see also theorem 2.7 in Amari and Nagaoka [2007]). Applying lemma 1,

$$x_i y_j z_k h^{ijk}(q) = \left(\frac{3 + \alpha}{2}\right) x_i g^{il}(q) y_j g^{jm}(q) z_k g^{kn}(q) \partial_l g_{mn}(p)|_{p=q}.$$

This can also be written in terms of the derivative of the inverse Fisher metric,

$$x_i y_j z_k h^{ijk}(q) = -\left(\frac{3 + \alpha}{2}\right) x_i g^{il}(q) \partial_l (y_j z_k g^{jk}(p))|_{p=q}.$$

Because of the non-parametric nature of the problem,

$$y_j z_k g^{jk}(p) = E^p[y \cdot z] - E^p[y] E^p[z].$$

Differentiating,

$$\partial_l (y_j z_k g^{jk}(p))|_{p=q} = y_l \cdot z_l - y_l E^q[z] - z_l E^q[y].$$

Therefore,

$$\begin{aligned} -\frac{2}{3 + \alpha} x_i y_j z_k h^{ijk}(q) &= x_i g^{il}(q) \partial_l (y_j z_k g^{jk}(p))|_{p=q} \\ &= E^q[x \cdot y \cdot z] - E^q[x \cdot y] E^q[z] - E^q[x \cdot z] E^q[y] - \\ &\quad E^q[y \cdot z] E^q[x] + 2E^q[x] E^q[y] E^q[z], \end{aligned}$$

which is the definition of the third cross-cumulant.

**E.5. Proof of corollary 1.** As argued in the proof of proposition 3, in the limit as  $\theta \rightarrow \infty$ , the solution to the moral hazard problem is interior. It follows that the security design equation equation (3.3), for an invariant divergence, can be written (again, using the summation notation) as

$$\beta_s s_j^*(\theta^{-1}, \bar{\kappa} \theta^{-1}) = \frac{\bar{\kappa}}{1 + \bar{\kappa} \theta^{-1}} [p^i(\theta^{-1}) - \lambda^i(\theta^{-1}) + \omega^i(\theta^{-1})] \partial_i \partial_j D_f(p(\theta) || q).$$

Here, I am writing the endogenous probability distributions and multipliers as functions of  $\theta^{-1}$ , to simplify by discussion of limits. Defining  $\lim_{\theta^{-1} \rightarrow 0^+} \lambda^i(\theta^{-1}) = \lambda_{opt}^i$  and  $\lim_{\theta^{-1} \rightarrow 0^+} \omega^i(\theta^{-1}) = \omega_{opt}^i$ ,

$$\lim_{\theta^{-1} \rightarrow 0^+} s_j^*(\theta^{-1}, \bar{\kappa}\theta^{-1}) = \frac{\bar{\kappa}}{\beta_s} [q^i - \lambda_{opt}^i + \omega_{opt}^i] g_{ij}(q).$$

It follows that  $\lim_{\theta^{-1} \rightarrow 0^+} s_j^*(\theta^{-1}, \bar{\kappa}\theta^{-1})$  is a debt security, by the proof of proposition 1. Denote  $s_j^*$  as this limit. The definition of  $s_{debt}(1, \bar{\kappa})$  is that it is the maximizer of

$$\bar{\kappa} E^q[\beta_s s_i] - \frac{1}{2} V^q[\beta_s s_i],$$

over the set of limited liability securities, which is unique by the positive-definiteness of the Fisher information metric. If  $s_j^* \neq s_{debt}(1, \bar{\kappa})$ , then

$$\bar{\kappa} E^q[\beta_s s_i^*] - \frac{1}{2} V^q[\beta_s s_i^*] < \bar{\kappa} E^q[\beta_s s_{debt,i}(1, \bar{\kappa})] - \frac{1}{2} V^q[\beta_s s_{debt,i}(1, \bar{\kappa})].$$

However, because  $s^*(\theta^{-1}, \bar{\kappa}\theta^{-1})$  is the optimal security, we must have

$$U(s^*(\theta^{-1}, \bar{\kappa}\theta^{-1}); \theta^{-1}) - U(0; \theta^{-1}, \bar{\kappa}\theta^{-1}) \geq U(s_{debt}(1, \bar{\kappa}); \theta^{-1}, \bar{\kappa}\theta^{-1}) - U(0; \theta^{-1}, \bar{\kappa}\theta^{-1})$$

for all  $\theta$ . It follows that

$$\begin{aligned} \lim_{\theta^{-1} \rightarrow 0^+} \theta \{U(s^*(\theta^{-1}, \bar{\kappa}\theta^{-1}); \theta^{-1}, \bar{\kappa}\theta^{-1}) - U(0; \theta^{-1}, \bar{\kappa}\theta^{-1})\} \\ \geq \theta \{U(s_{debt}(1, \bar{\kappa}); \theta^{-1}, \bar{\kappa}\theta^{-1}) - U(0; \theta^{-1}, \bar{\kappa}\theta^{-1})\} \end{aligned}$$

which would require, by proposition 3, that

$$\bar{\kappa} E^q[\beta_s s_i^*] - \frac{1}{2} V^q[\beta_s s_i^*] \geq \bar{\kappa} E^q[\beta_s s_{debt,i}(1, \bar{\kappa})] - \frac{1}{2} V^q[\beta_s s_{debt,i}(1, \bar{\kappa})],$$

contradicting the assertion that  $s_j^* \neq s_{debt}(1, \bar{\kappa})$ . Therefore,  $s_j^* = s_{debt}(1, \bar{\kappa})$ . Defining  $S$  as the set of limited liability securities, the second part of the corollary follows from the fact that

$$U(s^*(\theta^{-1}, \kappa); \theta^{-1}) - U(0; \theta^{-1}, \kappa) \leq \sup_{s \in S} \{\kappa E^q[\beta_s s_i^*] - \frac{1}{2} \theta^{-1} V^q[\beta_s s_i^*]\} + \sup_{s \in S} \{O(\theta^{-2} + \kappa\theta^{-1})\},$$

and therefore

$$U(s^*(\theta^{-1}, \kappa); \theta^{-1}) - U(s_{debt}(\theta^{-1}, \kappa); \theta^{-1}, \kappa) \leq \sup_{s \in S} \{O(\theta^{-2} + \kappa\theta^{-1})\}.$$

Because the space of limited liability securities is compact and the utility function is continuous, it follows that

$$U(s^*(\theta^{-1}, \kappa); \theta^{-1}) - U(s_{debt}(\theta^{-1}, \kappa); \theta^{-1}, \kappa) = O(\theta^{-2} + \kappa\theta^{-1}).$$

**E.6. Proof of lemma 1.** The proof of this lemma uses Chentsov's theorem and several results from Amari and Nagaoka [2007]. We have, for any monotone divergence,

$$\frac{\partial^3 D(p||q)}{\partial p^i \partial p^j \partial p^k} \Big|_{p=q} = ch_{ijk}(q) = c\partial_i g_{jk}(p) \Big|_{p=q} + \Gamma_{jk,i}^{(\alpha)},$$

where  $\Gamma_{jk,i}^{(\alpha)}$  are the connection coefficients of the  $\alpha$ -connection in the m-flat coordinate system. Using results in Amari and Nagaoka [2007], p. 33 and 36,

$$\Gamma_{jk,i}^{(\alpha)} = \Gamma_{jk,i}^{(-1)} - \frac{1+\alpha}{2} T_{ijk} = -\frac{1+\alpha}{2} T_{ijk}$$

where  $T_{ijk}$  is a covariant symmetric tensor of degree three. Repeating the argument for the Riemannian connection,

$$\Gamma_{ij,k}^{(0)} = -\frac{1}{2} T_{ijk} = \Gamma_{ik,j}^{(0)}$$

and

$$c\partial_i g_{jk} = \Gamma_{ij,k}^{(0)} + \Gamma_{ik,j}^{(0)} = T_{ijk}.$$

It follows that

$$h_{ijk}(q) = \left(\frac{3+\alpha}{2}\right) \partial_i g_{jk}(p) \Big|_{p=q}.$$

**E.7. Proof of corollary 2.** From section E.4, the utility for a particular security design can be written as

$$\begin{aligned} U(s; \theta^{-1}, \kappa) &= \beta_s(1+\kappa)\theta^{-1}\eta_i g^{il}(q)v_l - \frac{\theta^{-2}}{2}\beta_s\eta_j\eta_k v_i h^{ijk}(q) + \\ &+ \beta_s(1+\kappa)q^i v_i - \\ &\kappa\theta^{-1}\eta_i g^{il}(q)\eta_l - \kappa q^i \eta_i - \\ &\frac{\theta^{-1}}{2}\eta_i \eta_j g^{ij}(q) + \frac{\theta^{-2}}{3}\eta_i \eta_j \eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa\theta^{-2}). \end{aligned}$$

Taking the FOC with respect to  $\eta$ ,

$$\begin{aligned}\kappa(\lambda^i - \omega^i) &= \beta_s(1 + \kappa)\theta^{-1}g^{il}(q)v_l - \theta^{-2}\beta_s\eta_k v_j h^{ijk}(q) - \\ &\quad 2\kappa\theta^{-1}g^{il}(q)\eta_l - \kappa q^i - \\ &\quad \theta^{-1}\eta_j g^{ij}(q) + \theta^{-2}\eta_j \eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa\theta^{-2}).\end{aligned}$$

where  $\lambda^i$  and  $\omega^i$  are the scaled limited liability multipliers. Recall the lemma,

$$h_{ijk}(q) = \left(\frac{3 + \alpha}{2}\right)\partial_i g_{jk}(p)|_{p=q}.$$

Define a new probability distribution,

$$\hat{p}(\eta) = q + \left(\frac{3 + \alpha}{2}\right)(p(\eta) - q).$$

I will use  $\hat{p}$  and  $p$  instead of  $\hat{p}(\eta)$  and  $p(\eta)$  to keep the notation compact. Taylor expanding,

$$\begin{aligned}g^{ij}(\hat{p}) &= g^{ij}(q) - g^{ik}(q)(\partial_l g_{km}(p)|_{p=q})g^{mj}(q)(\hat{p}^l - q^l) + O(\theta^{-2}) \\ &= g^{ij}(q) - g^{ik}(q)h_{lkm}(q)g^{mh}(q)(p^l - q^l) + O(\theta^{-2}).\end{aligned}$$

The approximate first-order condition in the moral hazard problem was

$$\theta^{-1}\eta_i g^{il}(q) + O(\theta^{-2}) = (p^l(\eta) - q^l).$$

Putting these two together,

$$g^{ij}(\hat{p}) = g^{ij}(q) - \theta^{-1}h^{ijk}(q)\eta_k + O(\theta^{-2}).$$

Using these two results,

$$\begin{aligned}\kappa(\lambda^i - \omega^i) &= \beta_s\theta^{-1}g^{il}(\hat{p})v_l + \\ &\quad \kappa\theta^{-1}g^{il}(q)(\beta_s s_l - \eta_l) - \kappa q^i - \\ &\quad \theta^{-1}\eta_j g^{ij}(\hat{p}) + O(\theta^{-3} + \kappa\theta^{-2}).\end{aligned}$$

$$\begin{aligned}\kappa(\lambda^i - \omega^i) &= \beta_s\theta^{-1}g^{il}(\hat{p})s_l + \\ &\quad \kappa\theta^{-1}g^{il}(q)(\beta_s s_l - \eta_l) - \kappa q^i + O(\theta^{-3} + \kappa\theta^{-2}).\end{aligned}$$

$$\kappa(\hat{p}^l - q^l) = \left(\frac{3 + \alpha}{2}\right)\kappa\theta^{-1}\eta_i g^{il}(q) + O(\kappa\theta^{-2})$$

$$\begin{aligned} \kappa(\lambda^i - \omega^i) &= \beta_s\theta^{-1}g^{il}(\hat{p})s_l + \kappa\theta^{-1}g^{il}(q)\beta_s s_l \\ &\quad - \kappa\hat{p}^i + \left(\frac{\alpha + 1}{2}\right)\kappa\theta^{-1}\eta_i g^{il}(q) + O(\theta^{-3} + \kappa\theta^{-2}). \end{aligned}$$

Note that

$$\kappa\theta^{-1}g^{ij}(\hat{p}) = \kappa\theta^{-1}g^{ij}(q) + O(\kappa\theta^{-2}).$$

Therefore,

$$\begin{aligned} \kappa(\lambda^i - \omega^i) &= \beta_s\theta^{-1}\left(1 + \kappa\frac{1 - \alpha}{2}\right)g^{il}(\hat{p})s_l \\ &\quad - \kappa\hat{p}^i + \beta_s\left(\frac{1 + \alpha}{2}\right)\kappa\theta^{-1}v_i g^{il}(q) + O(\theta^{-3} + \kappa\theta^{-2}). \end{aligned}$$

Solving for the security design,

$$s_l\left[\beta_s + \frac{1 - \alpha}{2}\beta_s\kappa\right]\theta^{-1} = \kappa(\hat{p}^i - \lambda^i + \omega^i)g_{il}(\hat{p}) - \frac{1 + \alpha}{2}\beta_s\kappa\theta^{-1}v_l + O(\theta^{-3} + \kappa\theta^{-2}).$$

This expression can be rewritten (ignoring higher order terms) as

$$\beta_s s_l(1 + \kappa) + \frac{1 + \alpha}{2}\kappa\eta_l(s) = \kappa\theta(\hat{p}^i - \lambda^i + \omega^i)g_{il}(\hat{p}).$$

Taking expected values under the  $\hat{p}$  distribution,

$$\beta_s \hat{p}^l s_l(1 + \kappa) + \frac{1 + \alpha}{2}\kappa \hat{p}^l \eta_l(s) = \kappa\theta \hat{p}^l (\hat{p}^i - \lambda^i + \omega^i)g_{il}(\hat{p}).$$

Recalling that

$$g_{il}(\hat{p}) = \left(\frac{\delta_{il}}{\hat{p}^i} + \frac{1}{\hat{p}^0}\right),$$

$$\beta_s \hat{p}^l s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \hat{p}^l \eta_l(s) = \kappa \theta \sum_i [\hat{p}^i - \lambda^i + \omega^i] + \kappa \theta \sum_i [\hat{p}^i - \lambda^i + \omega^i] \frac{1 - \hat{p}^0}{\hat{p}^0}.$$

Therefore,

$$\beta_s \hat{p}^l s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \hat{p}^l \eta_l(s) = \frac{\kappa \theta}{\hat{p}^0} \sum_i [\hat{p}^i - \lambda^i + \omega^i].$$

Plugging this back in,

$$\beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \eta_l(s) = \beta_s \hat{p}^l s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \hat{p}^l \eta_l(s) + \kappa \theta \sum_{i>0} [\hat{p}^i - \lambda^i + \omega^i] \frac{\delta_{ij}}{\hat{p}^i}.$$

Define

$$\bar{v} = \hat{p}^l s_l + \frac{\frac{1+\alpha}{2} \kappa \hat{p}^l \eta_l(s) + \kappa \theta}{\beta_s (1 + \kappa)}.$$

Suppose that  $\omega^i > 0$ , and therefore  $s_l = 0$ ,  $\eta_l = \beta_s v_l$ ,  $\lambda_l = 0$ . Then

$$\frac{1 + \alpha}{2} \beta_s \kappa v_l = \beta_s (1 + \kappa) \bar{v} + \kappa \theta \frac{\omega^i}{\hat{p}^i}.$$

This can never occur if  $\alpha \leq -1$  and  $\bar{v} \geq 0$ , by the requirement that  $v_l > 0$ . If  $\alpha > -1$ , we know that  $\bar{v} > 0$ , and the condition occurs only if

$$\frac{1 + \alpha}{2} \beta_s \kappa v_l - \beta_s (1 + \kappa) \bar{v} > 0,$$

which requires large values of  $v_l$ . Next, consider the  $\lambda^i > 0$  case, when  $s_l = v_l$  and  $\eta_l = 0$  (and  $\omega_l = 0$ ). In this case,

$$\beta_s v_l (1 + \kappa) = \beta_s (1 + \kappa) \bar{v} - \kappa \theta \frac{\lambda^i}{\hat{p}^i}.$$

For any  $v_l < \bar{v}$ , this condition can hold. Finally, consider the case when  $\lambda^i = \omega^i = 0$ . In this case,

$$\beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \eta_l(s) = \beta_s (1 + \kappa) \bar{v}.$$

First, assume that  $\alpha \leq -1$ . In this case, it follows that if  $v_l < \bar{v}$ , this condition cannot hold, and otherwise it can. Therefore, there are two regions for  $\alpha \leq -1$ : a region where  $s_l = v_l$ , for  $v_l < \bar{v}$ , and a region where

$$s_l = \frac{(1 + \kappa)}{(1 + \kappa \frac{1-\alpha}{2})} \bar{v} - \frac{\frac{1+\alpha}{2} \kappa}{(1 + \kappa \frac{1-\alpha}{2})} v_l$$

for  $v_l > \bar{v}$ . This simplifies to

$$s_l = \bar{v} - \frac{\kappa(1 + \alpha)}{(2 + \kappa(1 - \alpha))} (v_l - \bar{v}).$$

It follows that

$$\eta_l = -\frac{\beta_s(1 + \kappa)}{(1 + \kappa \frac{1-\alpha}{2})} \bar{v} + \frac{\beta_s(1 + \kappa)}{(1 + \kappa \frac{1-\alpha}{2})} v_l$$

in this region. The equation defining  $\bar{v}$  becomes

$$\bar{v} \sum_{i: v_i < \bar{v}} \hat{p}^i = \sum_{i: v_i < \bar{v}} \hat{p}^i v_i + \frac{\kappa \theta}{\beta_s(1 + \kappa)}$$

which is guaranteed to be greater than zero. Note that the ‘‘option’’ value of the debt is the same as in the KL divergence case, but under the  $\hat{p}$  probability distribution. Next, consider  $\alpha \in (-1, 1 + \frac{2}{\kappa})$ . It follows that

$$\frac{1 + \alpha}{2} \kappa \in (0, 1 + \kappa)$$

and therefore  $\bar{v} > 0$ . The condition that

$$\frac{1 + \alpha}{2} \beta_s \kappa v_l - \beta_s (1 + \kappa) \bar{v} > 0$$

is satisfied only for some  $v_l > v_{max} > \bar{v}$ . Suppose that  $v_l > \bar{v}$ . The left hand side of the zero-multiplier condition below,

$$\beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \eta_l(s) - \beta_s (1 + \kappa) \bar{v} = 0,$$

achieves its minimum when  $\eta_l(s) = \beta_s v_l$ , and therefore this cannot hold if  $v_l > v_{max}$ . By a similar argument, it achieves its maximum when  $s_l = v_l$ , and therefore cannot hold if  $v_l < \bar{v}$ . It follows that there are three regions: a low  $v_l$  region, in which  $v_l < \bar{v}$ , an intermediate  $v_l \in (\bar{v}, v_{max})$  region in which

$$s_l = \bar{v} - \frac{\kappa(1 + \alpha)}{(2 + \kappa(1 - \alpha))} (v_l - \bar{v}),$$

and a high  $v_l$  region in which  $s_l = 0$ . Therefore,  $s_{debt-eq}$  is the second-order optimal security design.

The statement that

$$U(s^*(\theta^{-1}, \kappa); \theta^{-1}, \kappa) - U(s_{debt-eq}(\theta^{-1}, \kappa); \theta^{-1}, \kappa) = O(\theta^{-3} + \kappa\theta^{-2})$$

follows, from the fact that  $s_{debt-eq}(\theta^{-1}, \kappa)$  maximizes the non- $O(\theta^{-3} + \kappa\theta^{-2})$  terms of  $U(s; \theta^{-1})$  (see the proof of corollary 1).

**E.8. Proof of proposition 5.** This proof is similar in structure to proposition 3. I will continue to use the summation notation. We define the utility generated by a particular contract as

$$U(s; M_\xi, \theta^{-1}, \kappa) = \beta_s(1 + \kappa)p^i(\eta(s); M_\xi, \theta^{-1})s_i + \phi(\eta(s); M_\xi, \theta^{-1}).$$

It will also be useful to define  $p^i(\xi)$ , which is the probability distribution induced by the parameters  $\xi$ . Define

$$B_a^i(\bar{p}) = \frac{\partial p^i(\xi)}{\partial \xi^a} \Big|_{p=\bar{p}},$$

$$B_{ab}^i(\bar{p}) = \frac{\partial^2 p^i(\xi)}{\partial \xi^a \partial \xi^b} \Big|_{p=\bar{p}},$$

The first-order condition for the moral hazard problem is

$$B_a^i(p(\xi))(\eta_i - \partial_i \psi(p(\xi))) = 0.$$

As mentioned in the text, there may be multiple  $\xi$  for which this FOC is satisfied. By assumption, however, the FOC holds at all  $\xi$  that maximize the seller's problem. In the neighborhood of  $\theta \rightarrow \infty$ ,

$$\lim_{\theta \rightarrow \infty} p(\eta; M_\xi, \theta^{-1}) = q,$$

because otherwise the seller would receive unbounded negative utility. By assumption,  $q \in M_\xi$ , and therefore there are coordinates  $\xi_q$  such that  $p(\xi_q) = q$ . Expanding the functions  $B_a^i(p(\xi))$  and  $B_a^i(p(\xi))\partial_i \psi(p(\xi))$  around  $\xi = \xi_q$ , using the fact that  $\partial_i \psi(q) = 0$ ,

$$B_a^i(p(\xi)) = B_a^i(q) + B_{ab}^i(q)(\xi^b - \xi_q^b) + B_{abc}^i(\hat{p})(\xi^b - \xi_q^b)(\xi^c - \xi_q^c),$$

$$\begin{aligned}
B_a^i(p(\xi))\partial_i\psi(p(\xi)) &= B_a^i(q)B_b^j(q)\partial_i\partial_j\psi(q)(\xi^b - \xi_q^b) + \\
&B_a^i(p^*)B_{bc}^j(p^*)\partial_i\partial_j\psi(p^*)(\xi^b - \xi_q^b)(\xi^c - \xi_q^c) + \\
&\frac{1}{2}B_a^i(p^*)B_b^j(p^*)B_c^k(p^*)\partial_i\partial_j\partial_k\psi(p^*)(\xi^b - \xi_q^b)(\xi^c - \xi_q^c) + \\
&\frac{1}{2}B_{ab}^i(p^*)B_c^j(p^*)\partial_i\partial_j\psi(p^*)(\xi^b - \xi_q^b)(\xi^c - \xi_q^c),
\end{aligned}$$

where  $p^* = q + c^*(p - q)$  for some  $c^* \in (0, 1)$  and  $\hat{p}$  similarly defined. Define  $g_{ab}(q) = B_a^i(q)B_b^j(q)g_{ij}(q)$  and  $g^{ab}(q)$  as its inverse. Define

$$m_{ab}(q, \eta) = \theta g_{ab}(q) - B_{ab}^i(q)\eta_i$$

and

$$\begin{aligned}
m_{abc}(\hat{p}, p^*, \eta_i) &= B_a^i(p^*)B_{bc}^j(p^*)\partial_i\partial_j\psi(p^*) + \frac{1}{2}B_a^i(p^*)B_b^j(p^*)B_c^k(p^*)\partial_i\partial_j\partial_k\psi(p^*) + \\
&\frac{1}{2}B_{ab}^i(p^*)B_c^j(p^*)\partial_i\partial_j\psi(p^*) - B_{abc}^i(\hat{p})\eta_i.
\end{aligned}$$

The FOC can be written, using the invariance property of  $f$ -divergences, as

$$B_a^i(q)\eta_i = m_{ab}(q, \eta)(\xi^b - \xi_q^b) + m_{abc}(\hat{p}, p^*, \eta_i)(\xi^b - \xi_q^b)(\xi^c - \xi_q^c).$$

Multiplying by  $g^{ab}(q)\theta^{-1}$ ,

$$\begin{aligned}
\xi^b - \xi_q^b &= \theta^{-1}g^{ab}(q)B_a^i(q)\eta_i + \theta^{-1}(\xi^c - \xi_q^c)g^{ab}(q)B_{ac}^i(q)\eta_i + \\
&\theta^{-1}g^{ab}(q)m_{acd}(\hat{p}, p^*, \eta)(\xi^d - \xi_q^d)(\xi^c - \xi_q^c).
\end{aligned}$$

Note that  $\theta^{-1}m_{acd}(\hat{p}, p^*, \eta) = O(1)$  in  $\theta$ . By the argument that  $\lim_{\theta \rightarrow \infty} p(\eta; M_\xi, \theta^{-1}) = q$ , we know that  $\xi^b - \xi_q^b = o(1)$ . It follows that

$$\xi^b - \xi_q^b = \theta^{-1}g^{ab}(q)B_a^i(q)\eta_i + O(\theta^{-2}).$$

This demonstrates that  $\xi^b(\eta)$ , and therefore  $p(\eta; M_\xi, \theta^{-1})$ , is locally unique when  $\theta$  is large. It follows that  $p(\eta; M_\xi, \theta^{-1})$  is differentiable in the neighborhood of the expansion, regardless of the arbitrary rule used to “break ties” when multiple  $p$  maximize the moral hazard sub-problem.

We now proceed as before, substituting in for the definition of  $\beta_b$  and  $\phi(\eta)$ ,

$$U(s; M_\xi, \theta^{-1}, \kappa) = \beta_s(1 + \kappa)p^i(\eta(s); M_\xi, \theta^{-1})v_i - \kappa p^i(\eta(s); M_\xi, \theta^{-1})\eta_i(s) - \psi(p(\eta(s); M_\xi, \theta^{-1})).$$

Expanding the function  $p(\xi)$  around  $\xi = \xi_q$ , and using the result above,

$$p^i(\eta; M_\xi, \theta^{-1}) - q^i = \theta^{-1} B_a^i(q) g^{ab}(q) B_b^j(q) \eta_j + O(\theta^{-2})$$

Expanding  $\psi(p(\xi))$ , again using the result from above,

$$\psi(p) = \frac{\theta^{-1}}{2} \eta_i \eta_j B_a^i(q) B_b^j(q) g_{ij}(q) + O(\theta^{-2}).$$

Plugging these two into the utility function,

$$\begin{aligned} U(s; M_\xi, \theta^{-1}, \kappa) &= \beta_s (1 + \kappa) q^i v_i + \beta_s \theta^{-1} g^{ab}(q) B_a^i(q) B_b^j(q) \eta_i(s) v_j - \\ &\quad \kappa q^i \eta_i(s) - \frac{1}{2} \theta^{-1} B_a^i(q) B_b^j(q) g^{ab}(q) \eta_i(s) \eta_j(s) + O(\theta^{-2} + \theta^{-1} \kappa). \end{aligned}$$

The no-trade utility is

$$U(0; M_\xi, \theta^{-1}, \kappa) = \beta_s q^i v_i + \frac{1}{2} \beta_s^2 \theta^{-1} B_a^i(q) B_b^j(q) g^{ab}(q) v_i v_j + O(\theta^{-2} + \theta^{-1} \kappa).$$

Taking the difference,

$$\begin{aligned} U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) &= \beta_s \kappa q^i s_i + \beta_s \theta^{-1} B_a^i(q) B_b^j(q) g^{ab}(q) \eta_i(s) v_j - \\ &\quad - \frac{1}{2} \theta^{-1} B_a^i(q) B_b^j(q) g^{ab}(q) \eta_i(s) \eta_j(s) - \\ &\quad \frac{1}{2} \beta_s^2 \theta^{-1} B_a^i(q) B_b^j(q) g^{ab}(q) v_i v_j + O(\theta^{-2} + \theta^{-1} \kappa) \end{aligned}$$

which simplifies to

$$U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) = \beta_s \kappa q^i s_i - \frac{1}{2} \beta_s^2 \theta^{-1} B_a^i(q) B_b^j(q) g^{ab}(q) s_i s_j + O(\theta^{-2} + \theta^{-1} \kappa).$$

The lower bound,

$$U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) \geq \kappa E^q[\beta_s s_i] - \frac{\theta^{-1}}{2} \text{Var}^q[\beta_s s_i] + O(\theta^{-2} + \theta^{-1} \kappa),$$

follows from theorems 2.7 and 2.8 of Amari and Nagaoka [2007]. Note that

$$\frac{\partial}{\partial \xi^a} E^p[s_i] \Big|_{p=q} = s_i B_a^i(q).$$

From theorems 2.7 and 2.8, we have

$$Var^q[s_i] = s_i s_j g_{ij}(q) \geq s_i s_j B_a^i(q) B_b^j(q) g^{ab}(q).$$

The tightness of the lower bound applies when  $M_\xi$  is an exponential family, and  $s_i$  is a linear combination of its sufficient statistics, by the definition of a sufficient statistic.

**E.9. Proof of proposition 8.** This proof follows the proof of proposition 5. I will use the summation convention. We can write the utility from a particular security as

$$U_{RI}(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) p^i(\eta(s); \theta^{-1}) s_i + \phi_{RI}(\eta(s); \theta^{-1}),$$

$$\phi_{RI}(\eta; \theta^{-1}) = \max_{p \in M_{RI}} \{p^i \eta_i - \theta D_{KL}(p||q(p))\}.$$

Define a (for now) arbitrary distribution  $\bar{p}$ , with the property that actions are independent of states, and the marginal distribution of states is  $g(x)$ . We can write

$$\psi(p) = \theta D_{KL}(p||q(p)) = \theta D_{KL}(p||\bar{p}) - \theta D_{KL}(q(p)||\bar{p}),$$

where  $q_a(p)$  refers to the marginal distribution over actions of  $q$ , and  $\bar{p}_a$  is the marginal distribution of actions over  $\bar{p}$ . The key to this idea is that  $q(p)$  has actions independent of states, and  $q_a(p) = p_a$ , and therefore this equation holds for any full support  $\bar{p}$ .

As noted earlier, the set of feasible probability distributions  $M_{RI}$  is an  $|X| \cdot (|A| - 1)$  dimensional space, not an  $N$  dimensional space. It is an exponential family embedded in the space of all probability distributions. Denote a flat coordinate system  $\xi$ , such that

$$\frac{\partial p^i(\xi)}{\partial \xi^b} = B_b^i.$$

Note that unlike the general parametric model,  $B_b^i$  is a constant matrix. Similarly, define

$$\frac{\partial q^i(p)}{\partial p^j} = C_j^i.$$

The first-order condition for the moral hazard problem is

$$B_b^i (\eta_i - \partial_i \psi(p(\xi))) = 0.$$

As in the parametric problem, there may be multiple  $\xi$  for which this FOC is satisfied. By assumption, however, the FOC holds at all  $\xi$  that maximize the seller's problem. In the neighborhood of  $\theta \rightarrow \infty$ ,

$$\lim_{\theta \rightarrow \infty} p(\eta; \theta^{-1}) = \lim_{\theta \rightarrow \infty} q(p(\eta; \theta^{-1})),$$

because otherwise the seller would receive unbounded negative utility. This is different from the parametric problem, because even in the limit as  $\theta$  becomes large, the unconditional distribution of actions might depend on the retained tranche. The only requirement in this limit is that actions are independent of states. However, by the assumption of uniqueness, for each  $\eta$  there exists a unique  $\bar{p}(\eta)$  such that

$$\lim_{\theta \rightarrow \infty} p(\eta; \theta^{-1}) = \bar{p}(\eta).$$

By assumption,  $\bar{p} \in M_\xi$ , and therefore there are coordinates  $\xi_{\bar{p}}$  such that  $p(\xi_{\bar{p}}) = \bar{p}$ . Expanding the function  $\partial_i \psi(p(\xi))$  around  $\xi = \xi_{\bar{p}}$ , using the fact that  $\partial_i \psi(\bar{p}) = 0$ ,

$$\begin{aligned} B_b^i \partial_i \psi(p(\xi)) &= B_b^i B_c^j \partial_i \partial_j \psi(\bar{p}(\eta)) (\xi^c - \xi_{\bar{p}}^c) + \\ &\quad \frac{1}{2} B_d^i B_b^j B_c^k \partial_i \partial_j \partial_k \psi(p^*) (\xi^d - \xi_{\bar{p}}^d) (\xi^c - \xi_{\bar{p}}^c), \end{aligned}$$

where  $p^* = q + c^*(p - q)$  for some  $c^* \in (0, 1)$ . Using the properties of the KL divergence,

$$B_b^i B_c^j \partial_i \partial_j \psi(\bar{p}(\eta)) = \theta B_b^i B_c^j [g_{ij}(\bar{p}(\eta)) - C_i^k C_j^l g_{kl}(\bar{p}(\eta))] = \theta m_{bc}(\bar{p}(\eta)),$$

$$B_b^i B_c^j B_d^k \partial_i \partial_j \partial_k \psi(p) = \theta B_b^i B_c^j B_d^k [h_{ijk}(p) - C_i^l C_j^m C_k^n g_{lmn}(p)] = \theta m_{bcd}(p).$$

The FOC can be rearranged to

$$m_{bc}(\bar{p}(\eta)) (\xi^c - \xi_{\bar{p}}^c) = \theta^{-1} B_b^i \eta_i - \frac{1}{2} (\xi^d - \xi_{\bar{p}}^d) (\xi^c - \xi_{\bar{p}}^c) m_{bcd}(p^*).$$

Note that unlike the previous proof,  $m_{bc}(\bar{p})$  is positive-semidefinite but singular, and therefore not invertible. Nevertheless, this is sufficient to show that

$$m_{bc}(\bar{p}(\eta)) (\xi^c - \xi_{\bar{p}}^c) = \theta^{-1} B_b^i \eta_i + O(\theta^{-2}).$$

We now proceed as before, substituting in for the definition of  $\beta_b$  and  $\phi(\eta)$ ,

$$U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) p^i(\eta(s); \theta^{-1}) v_i - \kappa p^i(\eta(s); \theta^{-1}) \eta_i(s) - \psi(p(\eta(s); \theta^{-1})).$$

Expanding the function  $p(\xi)$  around  $\xi = \xi_{\bar{p}}$ , and using the result above,

$$p^i(\eta; \theta^{-1}) - \bar{p}^i(\eta) = \theta^{-1} B_b^i (\xi^b - \xi_{\bar{p}}^b) + O(\theta^{-2})$$

Expanding  $\psi(p(\xi))$ , again using the result from above,

$$\psi(p) = \frac{\theta}{2} m_{bc}(\bar{p})(\xi^b - \xi_{\bar{p}}^b)(\xi^c - \xi_{\bar{p}}^c) + O(\theta^{-2}).$$

Therefore,

$$\begin{aligned} U(s; \theta^{-1}, \kappa) &= \beta_s(1 + \kappa)\bar{p}^i(\eta(s))v_i + \beta_s B_b^i(\xi^b - \xi_{\bar{p}}^b)v_i - \\ &\quad \kappa\bar{p}^i(\eta(s))\eta_i(s) - \\ &\quad \frac{\theta}{2} m_{bc}(\bar{p}(\eta(s)))(\xi^b - \xi_{\bar{p}}^b)(\xi^c - \xi_{\bar{p}}^c) + O(\theta^{-2} + \theta^{-1}\kappa). \end{aligned}$$

When security is trading everything,

$$U(v; \theta^{-1}, \kappa) = \beta_s(1 + \kappa)\bar{p}^i(0)v_i,$$

where  $\bar{p}^i(0)$  is the arbitrary decision the seller will make about unconditional actions, when they have no incentive whatsoever. When the security is trading nothing,

$$\begin{aligned} U(0; \theta^{-1}, \kappa) &= \beta_s\bar{p}^i(\beta_s v)v_i + \beta_s B_b^i(\xi_0^b - \xi_{\bar{p}(0)}^b)v_i - \\ &\quad - \frac{\theta}{2} m_{bc}(\bar{p}(\beta_s v))(\xi_0^b - \xi_{\bar{p}(0)}^b)(\xi_0^c - \xi_{\bar{p}(0)}^c) + O(\theta^{-2} + \theta^{-1}\kappa), \end{aligned}$$

where  $\xi_0^b$  is the endogenous distribution for the zero security, and  $\xi_{\bar{p}(0)}^b$  is the associated limiting distribution. Because the expected payoff is the same for all actions, when those actions are independent of state,

$$\bar{p}^i(\beta_s v)v_i = \bar{p}^i(\eta)v_i = \bar{p}^i(0)v_i,$$

and

$$C_j^i v_i = 0$$

for all  $j$ . We know that for any  $\xi_{ind}^b$  that generates independence between actions and states,

$$(\xi_{ind}^b - \xi_{\bar{p}}^b)m_{ab}(\bar{p})(\xi_{ind}^c - \xi_{\bar{p}}^c) = 0,$$

and therefore that  $\xi_{ind}^b - \xi_{\bar{p}}^b$  is in the null space of  $m_{bc}(\bar{p})$ . Therefore, the nullity of  $m_{bc}(\bar{p})$  is at least  $|A| - 1$ . The rank of  $B_b^i B_c^j g_{ij}(\bar{p})$  is  $(|A| - 1) \cdot |X|$  (full rank), and the rank of  $B_b^i B_c^j C_i^k C_j^l g_{kl}(\bar{p})$  is  $|A| - 1$ , so the rank of  $m_{ab}(\bar{p})$  satisfies

$$(|A| - 1) \cdot |X| \leq \text{rank}(m_{bc}(\bar{p})) + |A| - 1.$$

It follows that the nullity of  $m_{bc}(\bar{p})$  is exactly equal to  $|A| - 1$ , the dimension of marginal distributions of actions. This fact is useful later in the proof. Moreover, because  $B_b^j C_j^i v_i = 0$ ,  $B_b^i v_i$  lies entirely in the column-space of  $m_{ab}(\bar{p})$ . Therefore, there exists a vector  $v^c(\bar{p})$  such that

$$B_b^j v_j = m_{bc}(\bar{p})v^c(\bar{p}).$$

We can therefore rewrite

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(v; \theta^{-1}, \kappa) &= \beta_s(\xi^b - \xi_{\bar{p}}^b)m_{bc}(\bar{p}(\eta(s)))v^c(\bar{p}(\eta(s))) - \kappa\bar{p}^i(\eta(s))\eta_i(s) \\ &\quad - \frac{\theta}{2}m_{bc}(\bar{p}(\eta(s)))(\xi^b - \xi_{\bar{p}}^b)(\xi^c - \xi_{\bar{p}}^c) + O(\theta^{-2} + \theta^{-1}\kappa). \end{aligned}$$

Consider a generalized inverse of  $m_{bc}(\bar{p})$ ,  $m_+^{bc}(\bar{p})$ , which has the property that

$$m_{bc}(\bar{p})m_+^{cd}(\bar{p})m_{de}(\bar{p}) = m_{be}(\bar{p}).$$

Rewriting the utility condition,

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(v; \theta^{-1}, \kappa) &= \beta_s(\xi^b - \xi_{\bar{p}}^b)m_{bc}(\bar{p}(\eta(s)))v^c(\bar{p}(\eta(s))) - \kappa\bar{p}^i(\eta(s))\eta_i(s) \\ &\quad - \frac{\theta}{2}m_{bc}(\bar{p}(\eta(s)))m_+^{cd}(\bar{p}(\eta(s)))m_{be}(\bar{p}(\eta(s)))(\xi^b - \xi_{\bar{p}}^b)(\xi^e - \xi_{\bar{p}}^e) \\ &\quad + O(\theta^{-2} + \theta^{-1}\kappa). \end{aligned}$$

Applying the first-order condition,

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(v; \theta^{-1}, \kappa) &= \beta_s\theta^{-1}B_b^j B_c^i m_+^{bc}(\bar{p}(\eta(s)))\eta_i v_j - \kappa\bar{p}^i(\eta(s))\eta_i(s) \\ &\quad - \frac{\theta^{-1}}{2}m_+^{bc}(\bar{p}(\eta(s)))B_b^i B_c^j \eta_i \eta_j + O(\theta^{-2} + \theta^{-1}\kappa). \end{aligned}$$

For the sell-nothing security,

$$U(0; \theta^{-1}) - U(v; \theta^{-1}) = \frac{1}{2}\beta_s^2\theta^{-1}B_b^j B_c^i m_+^{bc}(\bar{p}(\beta_s v))v_i v_j - \kappa\theta^{-1}\bar{p}^i(\beta_s v)v_i + O(\theta^{-2}).$$

Because

$$g_{ab}(\bar{p}) = m_{ab}(\bar{p}) + B_b^i B_c^j C_i^k C_j^l g_{kl}(\bar{p})$$

is non-singular, it follows from Hearon [1967] that a generalized inverse can be constructed as

$$m_+^{bc}(\bar{p}) = g^{bd}(\bar{p})m_{de}(\bar{p})g^{ce}(\bar{p}) = g^{bc}(\bar{p}) - g^{bd}(\bar{p})B_d^i B_e^j C_i^k C_j^l g_{kl}(\bar{p})g^{ce}(\bar{p}).$$

Because  $B_b^i v_i$  is entirely in the column space of  $m_{bc}(\bar{p})$ , it is entirely in the row-space of  $m_+^{bc}(\bar{p})$ , and therefore

$$m_+^{bc}(\bar{p})B_b^i B_c^j v_i v_j = g^{bc}(\bar{p})B_b^i B_c^j v_i v_j.$$

Because the unconditional variance of asset values is identical across assets,

$$g^{bc}(\bar{p}(\beta_s v))B_b^i B_c^j v_i v_j = g^{bc}(\bar{p}(\eta))B_b^i B_c^j v_i v_j$$

for all  $\eta$ . It therefore follows that

$$\begin{aligned} U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) &= \beta_s \kappa \bar{p}^i(\eta(s)) s_i \\ &\quad - \frac{\theta^{-1}}{2} \beta_s^2 m_+^{bc}(\bar{p}(\eta(s))) B_b^i B_c^j s_i s_j + O(\theta^{-2} + \theta^{-1} \kappa). \end{aligned}$$

By the positive-definiteness of  $g^{bd}(\bar{p})B_d^i B_e^j C_i^k C_j^l g_{kl}(\bar{p})g^{ce}(\bar{p})$ , and the monotonicity of the Fisher metric,

$$U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) \geq \beta_s \kappa \bar{p}^i(\eta(s)) s_i - \frac{\theta^{-1}}{2} \beta_s^2 g^{ij}(\bar{p}^i(\eta(s))) s_i s_j + O(\theta^{-2} + \theta^{-1} \kappa),$$

which is the desired result.

**E.10. Proof of proposition 6.** The relaxed moral hazard problem can be written as

$$\phi_{CT}(\eta) = \sup_{\frac{dP}{dQ}} \{E^P[\eta(X)] - \theta D_{KL}(P||Q)\}.$$

A suitably modified version of the proof of proposition 1 could be applied to this problem, but the extension of the some of the shortcuts used in that proof is not straightforward. Instead, I will use a calculus-of-variations approach, similar to the earlier drafts of this paper. The result is a specialized version of Cvitanić et al. [2009], and I will rely on that paper for the proof of the existence of an optimal security design. I will also assume that the space of allowed security designs is restricted to depend on the asset value at an arbitrary set of times, which includes the final value. This assumption is necessary to use a theorem from the calculus of variations, but is not required for the proof of Cvitanić et al. [2009].

Let  $\hat{P}$  be an alternative measure on  $\Omega$  that is absolutely continuous with respect to  $Q$ . Define

$$\frac{dP(\eta, \alpha)}{dQ} = (1 - \alpha) \frac{dP^*(\eta)}{dQ} + \alpha \frac{d\hat{P}}{dQ},$$

where  $P^*(\eta)$  is the measure that maximizes the moral hazard problem. The retained tranche  $\eta : \Omega \rightarrow \mathbb{R}$  is a  $\mathcal{F}_1^B$ -measurable function on the sample space  $\Omega$ . To keep notation compact, I use  $\eta$  instead of  $\eta(X)$  and  $\frac{dP}{dQ}$  instead of  $\frac{dP}{dQ}(B)$  when the meaning is not ambiguous.

$$\phi_{CT}(\eta, \alpha) = \int_{\Omega} \frac{dP(\eta, \alpha)}{dQ} dQ - \theta \int_{\Omega} \frac{dP(\eta, \alpha)}{dQ} \ln\left(\frac{dP(\eta, \alpha)}{dQ}\right) dQ + \theta \int_{\Omega} \left[\frac{dP(\eta, \alpha)}{dQ} - 1\right] dQ,$$

we must have

$$\left. \frac{\partial \phi_{CT}(\eta, \alpha)}{\partial \alpha} \right|_{\alpha=0} \leq 0$$

for all  $\hat{P}$ . It follows that, for all  $\hat{P}$ ,

$$\int_{\Omega} \left[ \frac{d\hat{P}}{dQ} - \frac{dP^*(\eta)}{dQ} \right] \left[ \eta - \theta \ln\left(\frac{dP^*(\eta)}{dQ}\right) \right] dQ \leq 0.$$

This will be satisfied only for

$$\frac{dP^*(\eta)}{dQ} = \exp(\theta^{-1}\eta - \lambda),$$

for some constant  $\lambda$ . Of course,  $\frac{dP^*(\eta)}{dQ}$  must be a valid Radon-Nikodym derivative, and therefore

$$\lambda = \ln(E^Q[\exp(\theta^{-1}\eta)]) > 0.$$

The integrability assumption is  $E^Q[\exp(\theta^{-1}X_1)] < \infty$ , and therefore  $E^Q[\exp(\theta^{-1}\eta)]$  is finite for all limited liability  $\eta$ .

The security design problem can be written as

$$\begin{aligned} U_{CT}(\eta) &= \beta_b E^P[X_1 - \beta_s^{-1}\eta] + \phi_{CT}(\eta) \\ &= \beta_b E^P[X_1 - \beta_s^{-1}\eta] + \theta \ln(E^Q[\exp(\theta^{-1}\eta)]). \end{aligned}$$

This equation is identical to the one in Yang [2012] (proposition 3), and the proof (from this point) is essentially the same. I define

$$\eta(X, \epsilon) = \eta^*(X) + \epsilon\tau(X),$$

where  $\tau : \Omega \rightarrow \mathbb{R}$  is another measurable function, restricted to the same set of payoff-relevant times. Using the calculus-of-variations approach again,

$$\frac{\partial U_{CT}(\eta(X, \epsilon))}{\partial \epsilon} \Big|_{\epsilon=0} \leq 0$$

for all  $\tau$  for which there exists some  $\epsilon > 0$  such that  $\eta(X, \epsilon)$  is a limited liability security. We have

$$\frac{\partial}{\partial \epsilon} \frac{dP^*(\eta)}{dQ}(B) \Big|_{\epsilon=0} = \theta^{-1} \frac{dP^*(\eta^*)}{dQ}(B) (\tau(X(B)) - E^{P^*(\eta^*)}[\tau]),$$

and

$$\frac{\partial}{\partial \epsilon} \phi(\eta) \Big|_{\epsilon=0} = E^{P^*(\eta^*)}[\tau].$$

It follows that, for all  $\tau$ , if an  $\eta^*$  that maximizes  $U_{CT}$  exists, then

$$(1 - \frac{\beta_b}{\beta_s}) \int_{\Omega} \frac{dP^*(\eta^*)}{dQ} \tau dQ + \theta^{-1} \beta_b \int_{\Omega} \frac{dP^*(\eta^*)}{dQ} (\tau - E^{P^*(\eta^*)}[\tau]) s dQ \leq 0.$$

This can be rearranged to

$$(1 - \frac{\beta_b}{\beta_s}) \int_{\Omega} \frac{dP^*(\eta^*)}{dQ} \tau dQ + \theta^{-1} \beta_b \int_{\Omega} \frac{dP^*(\eta^*)}{dQ} \tau (s - E^{P^*(\eta^*)}[s]) dQ \leq 0.$$

Because the set of payoff-relevant times is fixed and finite, these integrals can be rewritten as integrals over  $\mathbb{R}^M$ , where  $M$  is the number of payoff-relevant times. By the du Bois-Reymond lemma, for any interior  $s(X)$ ,

$$\theta \frac{\beta_b - \beta_s}{\beta_b \beta_s} = s(X) - E^{P^*(\eta^*)}[s(X)].$$

Note that if  $s(X)$  is zero, decreasing  $\eta(X)$  (increasing  $s(X)$ ) will increase utility, and therefore  $s(X) > 0$  for all paths with  $X_1 > 0$ . By an argument similar to proposition 1, the optimal contract is a debt security. The security depends only on the time-one value of the asset.

**E.11. Proof of lemma 2.** The first part of the lemma is a restatement of lemma 7.1 in Cvitanic et al. [2009]. The second part essentially restates a result found Bierkens and Kappen [2012], proposition 3.2. The uniqueness, up to an evanescence, of  $u(X, t)$  is shown below.

For all  $P \in M$ ,

$$E^Q[(\frac{dP}{dQ})^2] = E^P[\frac{dP}{dQ}] < \infty.$$

By the inequality that  $\ln x < x - 1$  for all  $x > 0$ , and the absolute continuity of  $\frac{dP}{dQ}$ ,

$$E^P[\ln(\frac{dP}{dQ})] + 1 < E^P[\frac{dP}{dQ}] < \infty.$$

Therefore,  $D_{KL}(P||Q)$  is finite. It follows that the process  $u(X, t)$  given by proposition 3.2 of Bierkens and Kappen [2012] is square integrable, and therefore in the set  $\mathcal{U}$ .

The semimartingale  $M_t$  that solves

$$Z_t = E[\frac{dP}{dQ} | \mathcal{F}_t^B] = \exp(M_t - [M, M]_t)$$

is unique, up to an evanescence (see theorem 8.3 in Jacod and Shiryaev [2003]). Therefore, it has a version such that

$$M_t = \int_0^t u(X, s) dB_s,$$

and is a square integrable martingale. By the Ito representation theorem,  $u(X, t)$  is the only square-integrable process for which this equation is satisfied.

**E.12. Proof of proposition 7.** I will proceed in four steps. In the first two steps, I will establish convergence results. In the third and fourth step, I will Taylor-expand the indirect utility function and buyer's security valuation (the two components of the security design utility function). The extra steps in this proof, relative to the static models, arise because of the need to ensure integrability, and to prove that certain limits and integrals can be interchanged.

Define the retained tranche as a function of the  $Q$ -Brownian motion,  $\hat{\eta}(B) = \eta(X(B))$ . By limited liability,  $\hat{\eta}(B) \in [0, \beta_s^{-1} X(B)]$ , and therefore  $E^P[\hat{\eta}(B)^2] < \infty$ . It follows that  $\hat{\eta}$  is Hida-Malliavin differentiable (Di Nunno et al. [2008]). Define  $h_t = \int_0^t u_s ds$ . Following Monoyios [2013], the first-order condition for  $u^*$  to be optimal (assuming the bounds do not bind) is

$$\psi'(u_t^*) = \theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B],$$

where  $h^* = \int_0^1 u_s^* ds$ . If the bounds do bind, so that  $|u^*| = \bar{u}$  at some time and state, then

$$|\psi'(u_t^*)| \leq |\theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B]|$$

By the mean value theorem,

$$\psi''(\hat{u}_t) u_t^* = \theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B],$$

for some  $|\hat{u}_t| \leq |u_t^*|$ , if the bounds do not bind, and

$$\psi''(\hat{u}_t) |u_t^*| \leq |\theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B]|,$$

if they do. Define  $e_t$  as

$$e_t = E^Q[D_t \hat{\eta}(B) | \mathcal{F}_t^B].$$

Additionally, define

$$f_t = E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B].$$

Note that  $e_t$  does not depend on  $\theta$ , whereas  $f_t$  depends on  $\theta$  through its dependence on  $h^*$ .

The proof proceeds in several steps. First, I will show that  $u_t^*$  converges to zero, in the  $L^2(Q \times [0, 1])$  sense, meaning that

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q \left[ \int_0^1 (u_t^*)^2 dt \right] = 0.$$

For all  $|u| \leq \bar{u}$ ,  $\psi''(u) \geq K_1 > 0$ , and therefore

$$(u_t^*)^2 \leq \theta^{-2} K_1^{-2} f_t^2,$$

regardless of whether the bounds on  $u_t^*$  bind. By the assumption that  $\hat{\eta}(B + h^*)$  is in  $L^2(Q)$ , for all  $h$ , and the Clark-Ocone theorem for  $L^2(Q)$  (theorem 6.35 in Di Nunno et al. [2008]),  $f_t \in L^2(Q \times [0, 1])$ .

By the Ito isometry,

$$E^Q \left[ \int_0^1 f_t^2 dt \right] = E^Q \left[ \left( \int_0^1 f_t dB_t \right)^2 \right].$$

By the Clark-Ocone theorem,

$$\int_0^1 f_t dB_t = \eta(B + h^*) - E^Q[\eta(B + h^*)].$$

Putting these two together,

$$(E.2) \quad E^Q \left[ \int_0^1 f_t^2 dt \right] = E^Q \left[ (\eta(B + h^*) - E^Q[\eta(B + h^*)])^2 \right].$$

Because  $E^Q[(\eta(B + h^*))^2] < \infty$  for all feasible  $u$ , and the set of feasible  $u$  is compact, it follows that

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q \left[ \int_0^1 f_t^2 dt \right] < \infty.$$

Using this result,  $\lim_{\theta^{-1} \rightarrow 0^+} \theta^{-2} K_1^{-2} E^Q \left[ \int_0^1 f_t^2 dt \right] = 0$ . By the squeeze theorem,

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q \left[ \int_0^1 (u_t^*)^2 dt \right] = 0.$$

A similar application of the Ito isometry and Clark-Ocone theorem shows that

$$(E.3) \quad E^Q \left[ \int_0^1 e_t^2 dt \right] = E^Q [(\eta(B) - E^Q[\eta(B)])^2],$$

which is useful later in the proof.

Next, I will show that  $\lim_{\theta^{-1} \rightarrow 0^+} \theta u_t^* = \lim_{\theta^{-1} \rightarrow 0^+} f_t = e_t$ , with convergence in  $L^2(Q \times [0, 1])$ . First, note that

$$\lim_{\theta^{-1} \rightarrow 0^+} \theta u_t^* = \lim_{\theta^{-1} \rightarrow 0^+} \frac{1}{\psi''(\hat{u}_t)} f_t.$$

Because  $|\hat{u}_t| \leq |u_t^*|$ , it converges to zero in measure ( $L^2$  convergence implies convergence in measure). Therefore,  $\lim_{\theta^{-1} \rightarrow 0^+} \frac{1}{\psi''(\hat{u}_t)} = 1$ , with convergence in measure. Because the measure  $Q \times \mu([0, 1])$  is finite ( $\mu([0, 1])$  is the Lebesgue measure), the product of two sequences that converge in measure converges in measure to the product of the limits. Therefore, to show that  $\lim_{\theta^{-1} \rightarrow 0^+} \theta u_t^* = e_t$ , with convergence in measure, it is sufficient to show that  $\lim_{\theta^{-1} \rightarrow 0^+} f_t = e_t$ , with convergence in measure.

Therefore, if

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[\eta(B + h^*)^2] = E^Q[\eta(B)^2],$$

it would follow that  $f_t$  converges to  $e_t$  in the  $L^2(Q \times [0, 1])$  sense.

Define

$$Z_1 = \exp\left(\int_0^1 u_t^* dB_t - \frac{1}{2} \int_0^1 (u_t^*)^2 dt\right).$$

By Girsanov's theorem,

$$\begin{aligned} E^Q[\eta(B + h^*)^2] &= E^Q[Z_1 \eta(B)^2] \\ &= E^Q[\eta(B)^2] + E^Q[(Z_1 - 1)\eta(B)^2]. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$E^Q[\eta(B + h^*)^2] = E^Q[\eta(B)^2] + E^Q[(Z_1 - 1)^2]^{0.5} E^Q[\eta(B)^4]^{0.5}.$$

By limited liability,  $E^Q[\eta(B)^4] \leq E^Q[X_1(B)^4]$ , and by assumption is finite. By construction,  $E^Q[Z_1] = 1$ .

$$E^Q[Z_1^2] = E^Q[\exp(2 \int_0^1 u_t^* dB_t - \int_0^1 (u_t^*)^2 dt)].$$

Therefore,

$$\begin{aligned} E^Q[Z_1^2] &\leq E^Q[\exp(2 \int_0^1 u_t^* dB_t)] \\ &\leq E^Q[\exp(2 \int_0^1 (u_t^*)^2 dt)]. \end{aligned}$$

Using the inequality that  $(u_t^*)^2 \leq K_1^2 \theta^{-2} f_t^2$ ,

$$\begin{aligned} E^Q[(Z_1 - 1)^2] &\leq E^Q[\exp(2 \int_0^1 K_1^{-2} \theta^{-2} f_t^2 dt)] - 1 \\ &\leq E^Q[\exp(2 \int_0^1 K_1^{-1} \theta^{-1} f_t dB_t)] - 1 \\ &\leq E^Q[\exp(2K_1^{-1} \theta^{-1} (\eta(B + h^*) - E^Q[\eta(B + h^*)]))] - 1 \\ &\leq E^Q[\exp(2K_1^{-1} \theta^{-1} X_1(B + h^*))] - 1 \\ &\leq E^Q[\exp(2K_1^{-1} \theta^{-1} X_1(B + \bar{u}))] - 1. \end{aligned}$$

The first inequality follows from the inequality that  $(u_t^*)^2 \leq K_1^2 \theta^{-2} f_t^2$ . The second follows from the expectation of the stochastic exponential. The third applies the Clark-Ocone theorem. The fourth follows from limited liability. The fifth follows from the monotonicity of the asset value in effort.

By the assumption that, for all effort strategies,  $X_1$  is square-integrable, it follows that the moment-generating function exists in some neighborhood around zero. Therefore,

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[\exp(\sqrt{2} K_1^{-1} \theta^{-1} X_1(B + \bar{u}))] = 1.$$

It follows that

$$(E.4) \quad \lim_{\theta^{-1} \rightarrow 0^+} E^Q[(Z_1 - 1)^2] = 0.$$

I have shown that  $f_t$  converges to  $e_t$ , in the sense of  $L^2(Q \times [0, 1])$  convergence. It follows that  $f_t$  and  $\theta u_t^*$  converge in measure to  $e_t$ . Moreover,  $\theta^2 (u_t^*)^2$  and  $f_t^2$  converge to  $e_t^2$  in measure, which will be useful below. Additionally,  $Z_1$  converges in measure to 1.

The third step is to Taylor-expand the indirect utility function from the moral hazard problem, in terms of  $\theta^{-1}$ . To start, consider the first-order term:

$$\frac{\partial}{\partial \theta^{-1}} \phi_{CT}(\eta; \theta) = \theta^2 E^P \left[ \int_0^1 \psi(u_t^*) dt \right].$$

By assumption,  $\forall |u| < \bar{u}$ ,  $\psi''(u) \in [K_1, K_2]$  for some positive constants  $K_1$  and  $K_2$ . Therefore, for some  $|\tilde{u}_t| < |u_t^*|$ ,

$$\begin{aligned} \theta^2 \psi(u_t^*) &= \frac{1}{2} \psi''(\tilde{u}_t) \theta^2 (u_t^*)^2 \\ &= \frac{1}{2} \frac{\psi''(\tilde{u}_t)}{(\psi''(\hat{u}_t))^2} f_t^2 \\ &\leq K f_t^2, \end{aligned}$$

for some finite positive constant  $K = \frac{1}{2} \frac{K_2}{(K_1)^2}$ . Therefore,

$$E^P \left[ \int_0^1 \theta^2 \psi(u_t^*) dt \right] \leq E^Q \left[ \int_0^1 K f_t^2 dt \right] + E^Q \left[ (Z_1 - 1) \int_0^1 K f_t^2 dt \right].$$

To demonstrate that the second term on the right-hand side of this equation converges to zero, I prove the following lemma. The purpose of this lemma is only to establish that the second term converges to zero.

**Lemma 4.** *With  $Z_1 - 1$  and  $f_t$  defined as above,*

$$E^Q \left[ (Z_1 - 1) \int_0^1 f_t^2 dt \right] = \frac{1}{3} E^Q \left[ (Z_1 - 1) (\eta(B + h^*) - E^Q[\eta(B + h^*)])^2 \right].$$

*Proof.* See appendix, section E.13. □

Using this lemma and the Cauchy-Schwarz inequality,

$$E^Q \left[ (Z_1 - 1) (\eta(B + h^*) - E^Q[\eta(B + h^*)])^2 \right] \leq E^Q \left[ (Z_1 - 1)^2 \right]^{0.5} E^Q \left[ (\eta(B + h^*) - E^Q[\eta(B + h^*)])^4 \right]^{0.5}.$$

Therefore, by the assumption that  $E^Q[\eta(B + h^*)^4] < \infty$ , and equation (E.4),

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q \left[ (Z_1 - 1) \int_0^1 f_t^2 dt \right] = 0.$$

Using equation (E.2) and Girsanov's theorem,

$$\begin{aligned}
E^Q\left[\int_0^1 f_t^2 dt\right] &= E^Q[(\hat{\eta}(B + h^*) - E^Q[\hat{\eta}(B + h^*)])^2] \\
&= E^Q[Z_1(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2] \\
&= E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2] + \\
&\quad E^Q[(Z_1 - 1)(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2].
\end{aligned}$$

By the Cauchy-Schwarz inequality and  $\lim_{\theta^{-1} \rightarrow 0^+} E^Q[(Z_1 - 1)^2] = 0$ ,

$$\begin{aligned}
\lim_{\theta^{-1} \rightarrow 0^+} E^Q[Z_1 \int_0^1 f_t^2 dt] &= \lim_{\theta^{-1} \rightarrow 0^+} E^Q\left[\int_0^1 f_t^2 dt\right] \\
&= E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2] \\
&= E^Q\left[\int_0^1 e_t^2 dt\right].
\end{aligned}$$

The second step follows from equation (E.3).

Consider the sample space  $\Omega \times [0, 1]$ , with the standard tensor-product sigma algebra and product measure  $Q \times \mu([0, 1])$ . The above result establishes convergence in  $L^1$  of  $Z_1 f_t^2$  to  $e_t^2$ , and therefore convergence in measure. By theorem 5 in chapter 11.6 of Shiryaev [1996],  $Z_1 f_t^2$  is uniformly integrable. I will next argue that  $Z_1 \theta^2 \psi(u_t^*)$  is uniformly integrable.

By lemma 2 in chapter 11.6 of Shiryaev [1996], a necessary and sufficient condition for uniform integrability (which some authors use as the definition of uniform integrability) is that, for some random variable  $x_t$  that depends on  $\theta$ , and for all  $\theta$ ,

$$E^Q\left[\int_0^1 |x_t| dt\right] < \infty,$$

and, for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that, for any set  $A \subseteq \Omega \times [0, 1]$  with  $E^Q\left[\int_0^1 \mathbf{1}(A) dt\right] \leq \delta$ , and all  $\theta$ ,

$$E^Q\left[\int_0^1 |x_t| \mathbf{1}(A) dt\right] \leq \epsilon.$$

By the inequality that  $\psi(u) \in [0, K f_t^2]$ , it follows that

$$\frac{1}{K} E^Q\left[\int_0^1 |Z_1 \theta^2 \psi(u_t^*)| dt\right] \leq E^Q\left[\int_0^1 |Z_1 f_t^2| dt\right] < \infty,$$

and

$$\frac{1}{K} E^Q \left[ \int_0^1 |Z_1 \theta^2 \psi(u_t^*)| \mathbf{1}(A) dt \right] \leq E^Q \left[ \int_0^1 |Z_1 f_t^2| \mathbf{1}(A) dt \right] \leq \epsilon.$$

Therefore  $\frac{1}{K} Z_1 \theta^2 \psi(u_t^*)$  is also uniformly integrable. It follows that  $Z_1 \theta^2 \psi(u_t^*)$  is uniformly integrable. Taylor-expanding around  $u_t^* = 0$ , to second order,

$$Z_1 \theta^2 \psi(u_t^*) = \frac{1}{2} Z_1 \psi''(\tilde{u}_t) (\theta u_t^*)^2,$$

for some  $|\tilde{u}_t| \leq u_t^*$ . By the convergence in measure of  $Z_1$  to 1,  $u_t^*$  (and therefore  $\tilde{u}_t$ ) to zero, and  $\theta u_t^*$  to  $e_t$ ,  $Z_1 \theta^2 \psi(u_t^*)$  converges in measure to  $\frac{1}{2} e_t^2$ . Because  $Z_1 \theta^2 \psi(u_t^*)$  converges in measure and is uniformly integrable,

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q \left[ Z_1 \int_0^1 \theta^2 \psi(u_t^*) dt \right] = E^Q \left[ \int_0^1 e_t^2 dt \right] = \frac{1}{2} V^Q[\hat{\eta}(B)].$$

I have shown that the first-order term in the Taylor expansion converges to the variance. The zero-order term converges to the expected value of the retained tranche. Therefore, the Taylor expansion around the limit  $\theta^{-1} \rightarrow 0^+$  is

$$\phi_{CT}(\eta; \theta) = E^Q[\hat{\eta}] + \theta^{-1} \frac{1}{2} V^Q[\hat{\eta}(B)] + O(\theta^{-1}).$$

The fourth step is to consider a Taylor expansion of the buyer's security valuation, again around  $\theta^{-1} \rightarrow 0^+$ . Define  $\hat{s}(B) = s(X(B))$ . The buyer's expected value is

$$E^Q[Z_1 \hat{s}(B)] = E^Q[\hat{s}(B)] + E^Q[(Z_1 - 1) \hat{s}(B)].$$

I will show that

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[\theta(Z_1 - 1) \hat{s}(B)] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)]) \hat{s}(B)].$$

Using the stochastic logarithm,

$$Z_1 - 1 = \int_0^1 Z_t u_t^* dB_t,$$

where  $Z_t u_t^*$  is in  $L^2(\Omega \times [0, 1])$ . By definition,  $Z_t = E^Q[Z_1 | \mathcal{F}_t]$ . By equation (E.4), it follows that  $Z_t$  converges in measure to 1, and therefore that  $\theta Z_t u_t^*$  converges in measure to  $e_t$ .

Because  $\hat{s}(B) \leq X_1(B)$  is in  $L^2(\Omega)$ , we can define

$$r_t = E^Q[D_t \hat{s}(B) | \mathcal{F}_t^B]$$

and see that it is in  $L^2(\Omega \times [0, 1])$ . By the Clark-Ocone theorem,

$$E^Q[\theta(Z_1 - 1)(\hat{s}(B) - E^Q[\hat{s}(B)])] = E^Q[(\int_0^1 \theta Z_t u_t^* dB_t)(\int_0^1 r_t dB_t)].$$

Using the Ito isometry,

$$E^Q[(\int_0^1 \theta Z_t u_t^* dB_t)(\int_0^1 r_t dB_t)] = E^Q[\theta \int_0^1 Z_t u_t^* r_t dt].$$

By construction,  $\theta Z_t u_t^*$  is square-integrable for all  $\theta$ . Because  $\theta Z_t u_t^*$  converges in measure, it converges weakly on the space  $L^2(\Omega \times [0, 1])$ , which is a Hilbert space, and therefore

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[\theta \int_0^1 Z_t u_t^* r_t dt] = E^Q[\int_0^1 e_t r_t dt].$$

Reversing the use of the Ito isometry and Clark-Ocone theorem,

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[\theta(Z_1 - 1)(\hat{s}(B) - E^Q[\hat{s}(B)])] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])(\hat{s}(B) - E^Q[\hat{s}(B)])].$$

The term

$$\begin{aligned} \lim_{\theta^{-1} \rightarrow 0^+} E^Q[\theta(Z_1 - 1)]E^Q[\hat{s}(B)] &= \lim_{\theta^{-1} \rightarrow 0^+} E^Q[\int \theta Z_t u_t^* dB_t]E^Q[\hat{s}(B)] \\ &= 0. \end{aligned}$$

Therefore,

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[\theta(Z_1 - 1)\hat{s}(B)] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])\hat{s}(B)].$$

The zero-order term in the buyer's security valuation converges to

$$\lim_{\theta^{-1} \rightarrow 0^+} E^Q[Z_1 \hat{s}(B)] = E^Q[\hat{s}(B)].$$

The first-order term converges to

$$\lim_{\theta^{-1} \rightarrow 0^+} \frac{\partial}{\partial(\theta^{-1})} E^Q[\frac{\partial Z_1}{\partial \theta} \hat{s}(B)] = \lim_{\theta^{-1} \rightarrow 0^+} E^Q[\theta(Z_1 - 1)\hat{s}(B)].$$

Combining the results in the third and fourth steps, and also computing the first-order expansion of the buyer's security valuation in terms of  $\kappa$ , it follows that

$$\begin{aligned} U(s; \theta^{-1}, \kappa) &= (\beta_s + \kappa\beta_s)E^Q[\hat{s}(B)] + \theta^{-1}\beta_s E^Q[\hat{s}(B)\hat{\eta}(B)] - \\ &\quad \theta^{-1}\beta_s E^Q[\hat{s}(B)]E^Q[\hat{\eta}(B)] + E^Q[\hat{\eta}(B)] + \\ &\quad \frac{1}{2}\theta^{-1}V^Q[\hat{\eta}(B)] + O(\theta^{-2} + \theta^{-1}\kappa). \end{aligned}$$

This equation is equivalent to the first-order terms of equation (E.1) in appendix section E.4. The remainder of the proof is identical to the algebra in that section.

**E.13. Proof of lemma 4.** The purpose of this lemma is to derive an alternate form of a particular expression, which is shown to go to zero in the above proof. To accomplish this, I use a result regarding the relationship between Malliavin derivatives, Skorohod integrals, and cumulants, from Privault [2013]. I specialize that paper's results to a simple case, involving adapted, square integrable processes that are themselves Malliavin derivatives. The proof relies heavily on the theorems of Malliavin calculus described in Di Nunno et al. [2008].

Using the martingale representation theorem, we can define

$$Z_1 - 1 = \int_0^1 z_t dB_t,$$

where  $z_t$  is a square-integrable,  $\mathcal{F}_t^B$ -adapted process. Note that  $f_t$  also has these properties. It follows that both  $z_t$  and  $f_t$  are themselves Hida-Malliavin differentiable.

Using lemma 4.2 of Privault [2013],

$$\begin{aligned} E^Q[(Z_1 - 1)(\int_0^1 f_t dB_t)^2] &= E^Q[(Z_1 - 1) \int_0^1 f_t^2 dt] + E^Q[\int_0^1 \int_0^1 (D_t(Z_1 - 1))(D_s f_t) f_s ds dt] + \\ &\quad E^Q[(\int_0^1 f_t dB_t) \int_0^1 f_t (D_t(Z_1 - 1)) dt]. \end{aligned}$$

Using theorem 3.18 of Di Nunno et al. [2008], the fundamental theorem of Malliavin calculus,

$$\begin{aligned} D_t(Z_1 - 1) &= D_t(\int_0^1 z_r dB_r) \\ &= \int_0^1 (D_t z_r) dB_r + z_t. \end{aligned}$$

Therefore,

$$E^Q\left[\int_0^1 \int_0^1 (D_t(Z_1 - 1))(D_s f_t) f_s ds dt\right] = E^Q\left[\int_0^1 \int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) (D_s f_t) f_s ds dt\right] + E^Q\left[\int_0^1 \int_0^1 z_t (D_s f_t) f_s ds dt\right].$$

Analyzing the first term of this expression, using the integration by parts formula (theorem 6.15 of Di Nunno et al. [2008]), along with the Clark-Ocone theorem,

$$E^Q\left[\int_0^1 \int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) (D_s f_t) f_s ds dt\right] = E^Q\left[\int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) f_t (\hat{\eta}(B + h^*) - \bar{\eta}) dt\right] - E^Q\left[\int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) \int_0^1 f_t f_s dB_s dt\right].$$

To minimize notation, I have used  $\bar{\eta} = E^Q[\hat{\eta}(B + h^*)]$  in the previous expression. Analyzing the first term of this expression,

$$\begin{aligned} E^Q\left[\int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) f_t (\hat{\eta}(B + h^*) - \bar{\eta}) dt\right] &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) D_r (f_t (\hat{\eta}(B + h^*) - \bar{\eta})) dr dt\right] \\ &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t D_r (\hat{\eta}(B + h^*) - \bar{\eta}) dr dt\right] \\ &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t f_r dt dr\right] - \\ &\quad E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t \left(\int D_r f_s dB_s\right) dr dt\right] \\ &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t f_r dt dr\right] - \\ &\quad E^Q\left[\int_0^1 \int_0^1 \int_0^1 (D_r f_s) (D_t z_r) (D_s f_t) ds dr dt\right] \\ &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t f_r dt dr\right]. \end{aligned}$$

The first step follows from integration by parts, the second by the fact (for adapted processes  $z_r$  and  $f_t$ ) that  $(D_t z_r)(D_r f_t)$  integrates to zero (see remark 6.18 of Di Nunno et al. [2008]). The third step follows from the fundamental theorem, and the fourth from integration by parts. The fifth step applies the same step about adapted processes.

By the Ito isometry,

$$E^Q\left[\int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) \int_0^1 f_t f_s dB_s dt\right] = E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t f_r dt dr\right].$$

Putting these last two results together,

$$E^Q\left[\int_0^1 \int_0^1 \left(\int_0^1 (D_t z_r) dB_r\right) (D_s f_t) f_s ds dt\right] = 0.$$

Next, consider the term

$$\begin{aligned} E^Q\left[\left(\int_0^1 f_t dB_t\right) \int_0^1 f_t (D_t (Z_1 - 1)) dt\right] &= E^Q\left[\left(\int_0^1 f_t dB_t\right) \int_0^1 f_t \left(\int_0^1 (D_t z_s) dB_s\right) dt\right] + \\ &E^Q\left[\left(\int_0^1 f_t dB_t\right) \int_0^1 f_t z_t dt\right], \end{aligned}$$

where I have used integration by parts. Considering the first term in this expression,

$$\begin{aligned} E^Q\left[\left(\int_0^1 f_t dB_t\right) \int_0^1 f_t \left(\int_0^1 (D_t z_s) dB_s\right) dt\right] &= \int_0^1 E^Q\left[\left(\int_0^1 (D_t z_s) dB_s\right) (f_t (\eta(B + h^*) - \bar{\eta}))\right] dt \\ &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) D_r (f_t (\hat{\eta}(B + h^*) - \bar{\eta})) dr dt\right] \\ &= E^Q\left[\int_0^1 \int_0^1 (D_t z_r) f_t f_r dt\right]. \end{aligned}$$

The first step is Fubini's theorem and the Clark-Ocone theorem, the second is integration by parts, and the third step reuses the algebra from above.

Putting it together, and using the Clark-Ocone theorem on the last term,

$$\begin{aligned} E^Q\left[(Z_1 - 1) \left(\int_0^1 f_t dB_t\right)^2\right] &= E^Q\left[(Z_1 - 1) \int_0^1 f_t^2 dt\right] + E^Q\left[\int_0^1 \int_0^1 z_t (D_r f_t) f_r dr dt\right] + \\ &E^Q\left[\int_0^1 \int_0^1 (D_r z_t) f_t f_r dr dt\right] + E^Q\left[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 f_t z_t dt\right]. \end{aligned}$$

Using the chain rule of Malliavin calculus,

$$\begin{aligned} E^Q\left[(Z_1 - 1) \left(\int_0^1 f_t dB_t\right)^2\right] &= E^Q\left[(Z_1 - 1) \int_0^1 f_t^2 dt\right] + E^Q\left[\int_0^1 \int_0^1 (D_r (z_t f_t)) f_r dr dt\right] + \\ &E^Q\left[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 f_t z_t dt\right]. \end{aligned}$$

Integrating by parts,

$$E^Q[(Z_1 - 1)(\int_0^1 f_t dB_t)^2] = E[(Z_1 - 1) \int_0^1 f_t^2 dt] + 2E^Q[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 f_t z_t dt].$$

Finally, I figure out what the second term in this expression is.

$$\begin{aligned} \frac{1}{2}E^Q[(\hat{\eta}(B + h^*) - \bar{\eta})^2(Z_1 - 1)] &= E^Q[\int_0^1 \frac{1}{2}z_t D_t[(\int_0^1 f_s dB_s)^2]] \\ &= E^Q[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 f_t z_t dt] + \\ &E^Q[(\eta(B + h^*) - \bar{\eta}) \int_0^1 z_t (\int_0^1 (D_t f_s) dB_s) dt]. \end{aligned}$$

The first step uses integration by parts and the Clark-Ocone theorem. The second uses the chain rule, Clark-Ocone theorem, and then fundamental theorem. Considering the second term,

$$\begin{aligned} E^Q[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 z_t (\int_0^1 (D_t f_s) dB_s) dt] &= E^Q[\int_0^1 \int_0^1 (D_t f_s) D_s(z_t(\hat{\eta}(B + h^*) - \bar{\eta})) ds dt] \\ &= E^Q[\int_0^1 \int_0^1 (D_t f_s) z_t D_s((\hat{\eta}(B + h^*) - \bar{\eta})) ds dt] \\ &= E^Q[\int_0^1 \int_0^1 (D_t f_s) z_t f_s dt ds] - \\ &E^Q[\int_0^1 \int_0^1 (D_t f_s) z_t (\int_0^1 D_s f_r dB_r) ds dt] \\ &= \frac{1}{2}E^Q[\int_0^1 \int_0^1 (D_t f_s^2) z_t dt ds] - \\ &E^Q[\int_0^1 \int_0^1 \int_0^1 (D_t f_s)(D_r z_t)(D_s f_r) ds dr dt] \\ &= \frac{1}{2}E^Q[(Z_1 - 1) \int_0^1 f_s^2 ds]. \end{aligned}$$

The first step is integration by parts, and the second applies the previously mentioned fact about adapted processes. The third uses the fundamental theorem, and the fourth uses both the chain rule (in the first term) and integration by parts (in the second term). The last step uses the fact about adapted processes and integration by parts, along with the Clark-Ocone theorem. It follows that

$$E^Q[(\hat{\eta}(B + h^*) - \bar{\eta})^2(Z_1 - 1)] = 3E^Q[(\hat{\eta}(B) - \bar{\eta}) \int_0^1 f_t z_t dt].$$

Plugging in this into the earlier equation,

$$E^Q[(Z_1 - 1)(\int_0^1 f_t dB_t)^2] = E[(Z_1 - 1) \int_0^1 f_t^2 dt] + \frac{2}{3}E^Q[(\hat{\eta}(B + h^*) - \bar{\eta})^2(Z_1 - 1)].$$

Using the Clark-Ocone theorem,

$$E^Q[(\hat{\eta}(B + h^*) - \bar{\eta})^2(Z_1 - 1)] = 3E^Q[(Z_1 - 1) \int_0^1 f_t^2 dt],$$

which proves the lemma.