Continuous time noisy signalling

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Abstract

Most real-life signalling is noisy and in many cases takes time. Information may be revealed gradually (many online customer reviews of a gadget) or by discrete events at random times (scientific breakthrough, oil spill, exposure of a corrupt politician). Both settings lead to behaviour different from noiseless or one-shot signalling situations. Signalling may occur in multiple disjoint intervals of beliefs. There may be no ‘most informative’ equilibrium with a signalling region that contains the signalling regions of all other equilibria. A higher prior may raise the payoff to the high type from separation relative to pooling. In noisy one-shot or repeated signalling, no signalling occurs at extreme priors, unlike in the noiseless case.

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1 Introduction

Most signalling situations are noisy. An employer looking at the diploma certifying the education of a job applicant cannot be sure the diploma is not...
forged, or that an educated applicant has not lost the diploma. A politician refusing bribes to signal honesty cannot guarantee the absence of accusations of corruption from political rivals, just like a corrupt politician is not certain to be caught. A security guard sleeping on the job does not guarantee a theft. A safety inspector being lax is not certain to miss a fault that causes an accident.

Many signalling situations also feature effort and signals over time. A researcher may work decades before a breakthrough occurs. A smartphone producer may hire people to post positive reviews of its latest gadget, but the reviews had better be spread out over time to prevent discovery of the practice. An IT consultant can spend a lifetime accumulating certificates from courses and workshops on the latest technology.

If information is revealed over time, it may take the form of frequent not very informative signals or rare informative ones. Frequent signals are for example online product reviews by customers, of which there may be hundreds on a popular website. Rare and significant signals are scientific breakthroughs, environmental catastrophes and political scandals. In this paper, signalling with frequent small signals is modelled as the sender controlling the drift of a Brownian motion. Rare signals are modelled as a Poisson process for which the sender controls the intensity.

With both the Poisson and the Brownian signal process, there is a long-lived strategic sender and a competitive market of receivers. The sender has two types. The type is known to the sender, but not to the receivers. The receivers offer higher compensation to the sender when they put higher probability on the high type. Other things equal, both types would like to pretend to be the high type. The sender has a continuum of signalling actions available, with higher actions more costly. The cost is linear and the marginal cost is higher for the low type. The sender’s action is not observable to the receivers, but it generates a public signal. A higher action by the sender leads to a higher signal distribution, with signal distributions ordered by first order stochastic dominance. Observing the signal, the receivers use Bayes’ rule to update their belief about the sender’s type. All signal histories are on the equilibrium path with a Brownian signal process, so refinements are not needed in that case. With a Poisson signal process, after zero-probability histories belief does not respond to the signal.

The Poisson signal can be good news (a breakthrough) or bad news (a breakdown). In the good news case, the intensity of the signal process is zero at zero signalling effort and rises linearly in effort. In the research
breakthrough example, the harder a researcher works, the greater the probability of making a discovery. In the bad news case, the intensity is maximal at zero effort and falls linearly in effort. In the corruption scandal example, a politician refusing bribes is in effect paying a cost to lower the probability of being exposed.

If the types of the sender are expected to take equal efforts, beliefs do not move. If the efforts are expected to be different, then the cost structure leads the high type to take a higher effort than the low type. Beliefs in the good news case fall in the absence of a signal and jump up when a breakthrough occurs. Symmetrically, under bad news beliefs rise in the absence of a signal and jump down when a breakdown occurs. If the low type is expected to take zero effort at the time when a signal occurs and the high type expected to take positive effort, the jump in beliefs is to certainty of the high type in the good news case. Similarly, under bad news if the high type is expected to take maximal effort and the low type less than maximal, then the jump in beliefs is to certainty of the low type.

The Brownian signal never resolves all uncertainty. Beliefs move as a diffusion process, the variance of which increases in the difference of the efforts expected from the types of the sender. If the expected efforts are the same, the variance is zero and beliefs stay constant. If the efforts are expected to be different, then the cost structure leads the high type to take a higher effort than the low type. In that case when the sender takes effort greater than the beliefs-weighted average effort of the types, beliefs can be expected to drift up. Otherwise the beliefs drift down on average.

In both the Poisson and the Brownian case, the sender is restricted to stationary Markov strategies with the common posterior as the state variable. The Markov restriction eliminates history-dependent strategies that are well known from repeated games, but technically difficult to define in continuous time.

As benchmarks for the continuous time noisy signalling games, a one-shot noisy signalling game is solved and an overview of a noiseless one-shot and a noiseless repeated signalling game is given. The repeated noiseless game gives results rather similar to Spence (1973): the existence and payoffs of a separating equilibrium do not depend on the prior and pooling equilibria with positive effort exist for all parameter values. Separating equilibria not depending on the prior is considered unrealistic (Kreps and Sobel, 1994) and this issue is addressed in the current paper.

The one-shot noisy game in this paper differs from the noiseless games
in several respects. The existence and payoffs of the maximally informative equilibrium do depend on the prior and pooling on a positive effort level is impossible in equilibrium. There are only four pure-strategy equilibria in the one-shot game with linear cost, and two equilibria (in pure or mixed strategies) with quadratic cost. Adding a small type-dependent drift to the signal in the game with quadratic cost eliminates the pooling equilibrium, leading to a unique equilibrium with partial information revelation. In one of the equilibria of the one-shot noisy signalling game with linear cost, learning increases in the amount of noise. This is because with more noise, the equilibrium actions of the types become more distinct as the low type has less incentive to imitate the high type (the benefit to effort falls in the amount of noise).

Similarly to the one-shot noisy games, payoff depends on the prior in informative equilibria also in the continuous time games with Poisson or Brownian signals. The existence of nonpooling equilibria also depends on the prior. Pooling on zero effort is an equilibrium for all parameter values, but pooling on positive effort is never an equilibrium. If a pure strategy nonpooling equilibrium exists, then there is a continuum of such equilibria, unlike in the one-shot noisy signalling games.

In the one-shot noisy game and in the bad news Poisson game, both types prefer the maximally informative equilibrium to the pooling equilibrium for some parameter values. In the good news Poisson game, the low type always prefers the pooling equilibrium to any informative equilibrium.

In both the good and the bad news Poisson games, if a type of the sender prefers an informative equilibrium to the pooling equilibrium, then that type gets the highest payoff among equilibria from the most informative equilibrium for all priors. Similarly, if the sender prefers pooling to an informative equilibrium at one prior, then the sender prefers pooling to all informative equilibria at all priors. This feature differs from the game with a Brownian signal process, where for some parameters the high type prefers a given informative equilibrium to pooling at one prior, but not at another prior.

The game with a Brownian signal process is distinguished from the Poisson game also by the feature that for some parameter values there is no ‘most informative’ equilibrium. To make this statement precise, the concept of a signalling region must be introduced. The set of beliefs at which both types of the sender take zero effort is called the pooling region and its complement the signalling region. In a Markov equilibrium, once beliefs reach the pooling region, both types take zero effort forever and beliefs stay constant. There
are parameter values for which there is no equilibrium with a signalling region containing the signalling regions of all other equilibria. In other words, the union of equilibrium signalling regions need not be an equilibrium signalling region.

1.1 Literature

Repeated noiseless signalling and one-shot noisy signalling have been studied by many authors. Nöldeke and Van Damme (1990) use a job market signalling model where time is split into periods of length $\Delta$ and then take $\Delta$ to zero. The signal is delay in accepting a wage offer. For fixed $\Delta$, many sequential equilibria are possible. In the limit, the unique equilibrium corresponds to the one in the static game that satisfies independence of never weak best response. Swinkels (1999) studies a variation of the same model where wage offers to the signalling worker are private. He finds that only the pooling equilibrium is possible.

Infinitely repeated noiseless signalling in discrete time is considered in Kaya (2009); Roddie (2011). A large set of equilibria is found in Kaya (2009). One of the sources of the multiplicity of separating equilibria is that the signalling cost can be distributed over time in many ways. Roddie (2011) focuses on the least-cost separating equilibrium and finds conditions under which the signalling level is higher than in the one-shot game.

An early work on one-shot noisy signalling is Matthews and Mirman (1983), where a monopolist tries to deter entry by setting a low price to convince the entrant that the market is unprofitable. The entrant observes the price subject to noise generated by a demand shock. Daley and Green (2012a) consider one-shot signalling where the receivers observe both the signaller’s action and a noisy grade with a distribution that depends on the signaller’s type and action. Carlsson and Dasgupta (1997) study one-shot signalling games where the receiver has only two actions. They prove the existence of noise-proof equilibrium, which is an equilibrium in the noiseless game that is the limit of equilibria in games perturbed by noise.

Experimental papers on signalling have used noisy one-shot signalling models, which can reasonably be expected to fit the data better. Examples are de Haan, Offerman, and Sloof (2011) and Jeitschko and Normann (2012).

There have also been works on repeated noisy signalling. Kremer and Skrzypacz (2007) analyze both noisy and noiseless cases when the informed party signals by delaying trade and after a finite amount of time the infor-
mation is exogenously revealed. Signalling by delaying trade also occurs in Hörner and Vieille (2009).

Continuous time signalling with Brownian noise is considered in Daley and Green (2012b); Gryglewicz (2009); Dilme (2012). In Daley and Green (2012b) the uninformed players receive information (observations of a diffusion process) exogenously over time and the informed player decides when to stop the game (execute the trade) and receive a final payoff. Their equilibrium has three regions: if the probability on the good type is high, there is immediate trade, which is efficient. For low probability on the good type, the good type rejects and with positive probability the bad type accepts. For intermediate beliefs, no trade occurs.

The case where the good type of the informed player is a commitment type is examined in Gryglewicz (2009). Both players can stop the game and the payoffs received upon stopping depend on the type of the informed player. The uninformed player optimally stops at a high belief threshold and the low type informed player at a low belief threshold.

The model closest to the one in the present paper featuring Brownian noise is Dilme (2012), which has no discounting and has the signaller receiving a payoff only when he decides to stop the game. Both these characteristics are different from the current paper. Dilme (2012) finds that the more similar the costs of different types, the worse off they are (competition effect) and that the payoff of the signaller does not depend on the volatility of the noise process. These effects are absent in the current work.

To the author’s knowledge, signalling with Poisson signals has not been studied previously.

The next section studies the benchmark cases of one-shot noiseless and noisy signalling and repeated noiseless signalling, in order to provide comparisons to the main models of noisy signalling over time studied in later sections.

2 Benchmarks

2.1 One-shot noiseless signalling

The game of Spence (1973) is presented in this subsection to introduce notation and remind the reader. Modifications to Spence’s model will be pointed out as they are made.
The players are a sender and a competitive market of receivers. The sender is one of two types, $H$ or $L$, with initial log likelihood ratio $l_0 = \ln \frac{\Pr(H)}{\Pr(L)} \in \mathbb{R}$. The log likelihood ratio corresponding to $\Pr(H) = 1$ is $l = \infty$ and corresponding to $\Pr(H) = 0$ is $l = -\infty$. The sender knows his type, but the receivers only know $l_0$.

Both types of sender have action set $[0, \infty)$, with generic action $e$ (effort or education). A mixed action of type $\theta$ is a cdf $F_\theta$ on $[0, \infty)$. Pure actions are written as $e$ for simplicity. Effort $e$ costs $A_\theta e$ to type $\theta \in \{H, L\}$, with $A_L > A_H > 0$.

Upon observing an action $e$, the receivers update the log likelihood ratio from $l_0$ to $l$, using Bayes’ rule where possible. The sender is assumed to derive benefit $R(l)$ from the receivers’ updated log likelihood ratio directly, unlike in Spence’s game. This can be microfounded by assuming the receivers have a unique best response $a^*(l)$ to every log likelihood ratio $l \in \mathbb{R}$ and the sender derives benefit $\hat{R}(a^*(l))$ from the receivers’ action $a^*(l)$.

Type $\theta$ sender’s utility from effort $e$ and the receivers’ log likelihood ratio is $u_\theta = R(l) - A_\theta e$. Assume $R$ is continuous, strictly increasing and bounded. In Spence (1973), the sender’s benefit from the receivers’ belief is one plus the belief, i.e. $R(l) = 1 + \frac{\exp(l)}{1+\exp(l)}$.

Denote $\lim_{l \to -\infty} R(l)$ by $R_{\min}$ and $\lim_{l \to \infty} R(l)$ by $R_{\max}$. Finite limits $R_{\min}, R_{\max}$ exist due to the assumptions on $R$.

**Definition 1.** A perfect Bayesian equilibrium consists of (possibly mixed) actions $F^*_L, F^*_H$ for types $L, H$ of the sender and, for each action $e$ of the sender, a log likelihood ratio $l(e)$ s.t.

1. if $F^*_\theta$ puts positive probability on $e_\theta$, then $e_\theta \in \arg \max_e R(l(e)) - A_\theta e$,

2. $l(e)$ is derived from Bayes’ rule after efforts $e$ that have positive probability under $F^*_L$ or $F^*_H$ and is arbitrary elsewhere.

There is a continuum of separating equilibria with $e_L^* = 0$ and $e_H^* = e^* > 0$ s.t. $R_{\min} \geq R_{\max} - A_L e^*, R_{\min} \leq R_{\max} - A_H e^*$, for $e < e^*$, $l(e) = -\infty$ and for $e \geq e^*$, $l(e) = \infty$. The effort of $H$ must satisfy $e^* \in \left[\frac{R_{\max} - R_{\min}}{A_L}, \frac{R_{\max} - R_{\min}}{A_H}\right]$. The expected utility of $L$ is the same from all separating equilibria, while

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1 Throughout this paper, log likelihood ratio $l$ is used instead of belief $\Pr(H) = \frac{\exp(l)}{1+\exp(l)}$, because in the continuous time models this simplifies formulas significantly. There is a one-to-one correspondence between log likelihood ratio and belief, so all results can be given in terms of belief.
the utility of $H$ decreases in $e^*$. Neither the utilities of the types nor the minimal separating effort $\frac{R_{\text{max}} - R_{\text{min}}}{A_L}$ depends on $l_0$, which Kreps and Sobel (1994) consider unintuitive.

There is also a continuum of pooling equilibria with $e^*_L = e^*_H = \hat{e} > 0$ s.t. $R_{\text{min}} \leq R(l_0) - A_L \hat{e}$. For $e < \hat{e} \leq \frac{R(l_0)}{A_L}$, $l(e) = -\infty$ and for $e \geq \hat{e}$, $l(e) = l_0$. These equilibria exist even if $A_L = A_H$.

A pooling equilibrium where $e^*_L = e^*_H = 0$ exists for all parameter values, supported by the updating rule $l(e) = l_0 \forall e$. The $L$ type has a higher expected utility in the equilibrium with pooling on $e = 0$ than in all the equilibria mentioned previously.

The utility of $H$ is higher in a separating equilibrium prescribing $e^*$ than under pooling on $e = 0$ iff $R_{\text{max}} - A_H e^* > R(l_0)$. A separating equilibrium in which $H$ gets higher utility than in pooling on $e = 0$ exists iff $R_{\text{max}} - A_H \frac{R_{\text{max}} - R_{\text{min}}}{A_L} > R(l_0)$.

The receivers get zero profit from any equilibrium due to competition, so welfare is \(\frac{\exp(l_0)}{1+\exp(l_0)} u_H + \frac{1}{1+\exp(l_0)} u_L\). Pooling on zero effort yields lower welfare than the least costly separating equilibrium iff $R(l_0) \leq \frac{\exp(l_0)}{1+\exp(l_0)} [R_{\text{max}} - A_H \frac{R_{\text{max}} - R_{\text{min}}}{A_L}] + \frac{1}{1+\exp(l_0)} R_{\text{min}}$. This fails if $R(l) = \frac{\exp(l)}{1+\exp(l)}$, so pooling on zero gives the highest welfare in that case.

If in the game described above the sender’s action is restricted to $[0, 1]$, then a separating equilibrium exists iff $A_L \geq R_{\text{max}} - R_{\text{min}}$. The other results are unchanged.

### 2.2 One-shot noisy signalling

Consider the game described in the previous subsection, with the action set of both types being $[0, 1]$. Assume the receivers do not observe the sender’s effort, but only see one of two signals, $g$ or $b$, interpreted as good and bad respectively. Effort increases the probability of $g$, specifically $\Pr(g|e) = \lambda e + \frac{1-\lambda}{2}$, with $\lambda \in (0, 1)$ being the precision of the signal.

Both signals have positive probability after any $e$. The receivers Bayesian update in each of their two information sets (after $g$ and after $b$). Denote the log likelihood ratio after signal $x \in \{g, b\}$ by $l(x)$. The strategy the receivers expect from type $\theta$ is the cdf $F^*_\theta$. The updating rule is

\[
l(x) = l_0 + \ln \frac{\int_0^1 \Pr(x|e)dF^*_H(e)}{\int_0^1 \Pr(x|e)dF^*_L(e)}.
\]
Type $\theta$ sender’s expected utility from pure action $e$ is

$$-A\theta e + \left[ \lambda e + \frac{1 - \lambda}{2} \right] R(l(g)) + \left[ -\lambda e + \frac{1 + \lambda}{2} \right] R(l(b)).$$  \hfill (2)

**Definition 2.** A perfect Bayesian equilibrium of the one-shot noisy signalling game consists of (possibly mixed) actions $F_L^*$, $F_H^*$ for types $L, H$ of the sender and log likelihood ratios $l(g), l(b)$ after signals $g, b$ s.t.

1. if $F_\theta^*$ puts positive probability on $e$, then $e$ maximizes (2),

2. given $F_H^*, F_L^*$, the log likelihood ratios $l(g), l(b)$ are derived from (1).

If the two types are expected to take (mixed) actions leading to the same signal distribution, then $l(g) = l(b) = l_0$. In that case both types would choose the pure action 0 to avoid the signalling cost. Unlike in Spence (1973), there can be no equilibrium with both types taking the same positive effort.

If the receivers expect pure actions $e_L^* > e_H^*$, then $l(g) < l(b)$ and therefore $R(l(g)) < R(l(b))$. Then both types would switch to $e = 0$ to avoid the signalling cost and reduce the probability of the $g$ signal. So in a pure strategy equilibrium either $e_L^* = e_H^* = 0$ or $e_H^* > e_L^*$.

For any mixed strategy of the sender there is a pure strategy yielding the same signal distribution and the same cost. The set of payoffs achievable with pure strategies is the same as with mixed strategies. Subsequently the focus is on pure strategy equilibria.

It cannot be that both types are taking $e \in (0, 1)$ in equilibrium. The benefit from a good signal is the same for both types and the cost of effort is lower for $H$, so when $H$ is indifferent between two different efforts, $L$ strictly prefers $e = 0$.

The set of pure strategy equilibria is characterized in Proposition 1. Define $l(g) = l_0 + \ln \frac{Pr(g|1)}{Pr(g|0)}$ and $l(b) = l_0 + \ln \frac{Pr(b|1)}{Pr(b|0)}$. These are the log likelihood ratios after signals $g, b$ when type $L$ is expected to take the minimal effort $e_L^* = 0$ and type $H$ the maximal effort $e_H^* = 1$. The marginal cost of effort to type $\theta$ is $A_\theta$ at any $e$. The marginal benefit to both types under the expectation $e_L^* = 0, e_H^* = 1$ is $\lambda[R(l(g)) - R(l(b))]$, as can be seen from (2). This is the maximal feasible marginal benefit of effort.

**Proposition 1.** 1. If $A_H > \lambda[R(l(g)) - R(l(b))]$, then the unique equilibrium is $e_L^* = e_H^* = 0$, called pooling.
2. If \( A_H = \lambda \left[ R(l(g)) - R(l(b)) \right] \), then the equilibria are pooling and \( e^*_L = 0 \), \( e^*_H = 1 \), called the max info equilibrium.

3. If \( A_H < \lambda \left[ R(l(g)) - R(l(b)) \right] \leq A_L \), then the equilibria are pooling, max info and \( e^*_L = 0, 0 < e^*_H < 1 \), called interior \( e^*_H \).

4. If \( A_L < \lambda \left[ R(l(g)) - R(l(b)) \right] \), then the equilibria are pooling, interior \( e^*_H \) and \( 0 < e^*_L < 1, e^*_H = 1 \), called interior \( e^*_L \).

All proofs omitted from the text are in Appendix A.

The parameter regions for the equilibria are illustrated in Fig. 1 for \( R(l) = \frac{\exp(l)}{1+\exp(l)} \) (the sender’s benefit from the receivers’ belief equals the belief). The prior \( \Pr(H) = \frac{\exp(l_0)}{1+\exp(l_0)} \) is on the horizontal axis and the signal precision \( \lambda \) on the vertical axis. The equilibria in the top (unshaded) region are pooling, interior \( e^*_H \) and interior \( e^*_L \). The equilibria in the middle (blue) region are pooling, max info and interior \( e^*_H \). Pooling is the unique equilibrium in the bottom (purple) region. As the comparison of the panels indicates, decreasing \( A_H \) expands the region where the max info equilibrium exists at the expense of the region where only pooling exists. Decreasing \( A_L \) expands the region where the interior \( e^*_L \) equilibrium exists and correspondingly shrinks the region where max info exists.

Fig. 1 also shows that when \( l_0 \to -\infty \) or \( l_0 \to \infty \), pooling is eventually the unique equilibrium. This is in contrast to Spence (1973) where separating equilibria exist for all parameter values. Restricting the sender to action set \([0,1]\) in the Spence game, the existence of separating equilibria depends on \( A_L, R_{\max} \) and \( R_{\min} \), but not on \( l_0 \).

An interesting feature of the interior \( e^*_L \) equilibrium is that the expected posterior variance decreases in signal precision. This is because \( L \) must be indifferent: \( A_L = \lambda \left[ R(l(g)) - R(l(b)) \right] \). If \( \lambda \) rises, \( R(l(g)) - R(l(b)) \) must fall to keep equality, therefore \( l(g) - l(b) \) must fall. It turns out that the expected posterior variance increases in \( l(g) - l(b) \). The fall in \( l(g) - l(b) \) comes from \( e^*_L \) rising to become closer to 1 = \( e^*_H \). Expected posterior variance is a measure of the informativeness of an equilibrium, so in the interior \( e^*_L \) equilibrium, learning increases in the amount of noise.

With \( R \) concave over the range \([l(b), l(g)]\), the \( L \) type expected utility is higher in pooling than in the other three equilibria. This is because \( L \) pays no signalling cost in the pooling equilibrium and expects the receivers’ belief to go down if there is positive signalling. The concavity of \( R \) implies
that randomness in the posterior also lowers $L$’s expected utility. This utility comparison for $L$ across equilibria fails with $R$ sufficiently convex over the range $[l(b), l(g)]$. If $l_0 = 0$, $\lambda = \frac{1}{2}$ and $R(l) = \left(\frac{\exp(l)}{1+\exp(l)}\right)^n$, then for $n \geq 4$, the utility of $L$ from pooling is lower than from separating. With these parameters, $l(g) = \ln 3$ and $l(b) = \ln \frac{1}{3}$, so the pooling utility is $2^{-n}$ and the separating utility is $\frac{3}{4^n} + \frac{1}{4^n}$.

The $H$ type utility comparison of equilibria depends on the parameters, because if there is any signalling, then $H$ expects the receivers’ log likelihood ratio to go up, but it has greater variance and $H$ must pay a signalling cost. If $R(l) = \frac{\exp(l)}{1+\exp(l)}$, then in the max info equilibrium, $u_H^{\text{max info}} = -A_H + \frac{1+\lambda}{2} \frac{(1+\lambda)\exp(l_0)}{1-\lambda+(1+\lambda)\exp(l_0)} + \frac{1-\lambda}{2} \frac{(1-\lambda)\exp(l_0)}{1+\lambda+(1-\lambda)\exp(l_0)}$. Unlike in Spence (1973), $u_H^{\text{max info}}$ depends on the prior.

With $R(l) = \frac{\exp(l)}{1+\exp(l)}$, the pooling utility $R(l_0)$ is less than $u_H^{\text{max info}}$ iff $A_H \leq \frac{4\lambda^2 \exp(l_0)}{(1+\exp(l_0))^3 - \lambda^2(1-\exp(l_0))^2(1+\exp(l_0))}$, which depends on $l_0$, unlike the con-

Figure 1: Equilibrium regions for $R(l) = \frac{\exp(l)}{1+\exp(l)}$
Figure 2: Region in which $H$ gets a higher payoff from the max info equilibrium than from pooling.

The parameter region where $H$ gets a higher payoff from max info than from pooling is illustrated as the shaded area in Fig. 2. The prior $\Pr(H) = \frac{\exp(l_0)}{1+\exp(l_0)}$ is on the horizontal axis and the signal precision $\lambda$ on the vertical axis.

Since the receivers are a competitive market, they get zero profit from any equilibrium and do not affect welfare calculations. Welfare from max info $\frac{\exp(l_0)}{1+\exp(l_0)}u^\text{max info}_H + \frac{1}{1+\exp(l_0)}u^\text{max info}_L$ decreases in $A_H$, is independent of $A_L$, increases in $l_0$. For $R$ concave over the range $[l(b), l(g)]$, welfare from max info is always lower than from pooling, due to the signalling cost and the variance of the posterior. If $R(l) = \frac{\exp(l)}{1+\exp(l)}$, welfare increases in the signal precision $\lambda$ iff $\frac{\exp(l_0)}{1+\exp(l_0)} \geq \frac{(1+\lambda)^2}{2(1+\lambda)^2-2\lambda}^{-1}$.

One may well wonder whether the results in this subsection are driven by the linear cost. With a strictly convex cost, the interior $e^*_H$ and interior $e^*_L$ equilibria disappear, because a signal distribution from a mixed action can be replicated by a pure action at strictly lower cost. Appendix B solves a game with type $\theta$ sender’s utility $\frac{\exp(l)}{1+\exp(l)} - \frac{A_0}{2}e^2$, with other elements the same as in this subsection. There are at most two equilibria: pooling and for some parameter values an informative equilibrium with $e^*_H > e^*_L$. The efforts of both types in the informative equilibrium increase in $\lambda$, the precision of the signal. In addition, $e^*_H$ increases in $A_L$ and decreases in $A_H$. For $l_0 \geq 0$, $e^*_H$ and $e^*_L$ decrease in $l_0$, so unlike in Spence (1973), the informative equilibrium efforts, as well as its existence, depend on the prior. If the game is modified so that at equal efforts, $H$ has an $\epsilon > 0$ higher probability of the $g$ signal, then the pooling equilibrium does not exist and for all parameter values there
is a unique separating equilibrium.

2.3 Repeated noiseless signalling

An overview of the game of Kaya (2009) is given in this subsection. The notation is changed to match the present paper.

Time is discrete, the horizon infinite. The discount factor is $\delta \in (0, 1)$. The players are a sender and in each period, a myopic receiver. The sender has types $H$ and $L$, with initial log likelihood ratio $l_0$. The sender’s action set is a closed real interval. The receiver at time $t$ observes the sender’s actions up to $t$ and updates the log likelihood ratio to $l_t$.

Type $\theta$ sender’s per-period utility is $u_\theta(l, e)$, which is assumed to satisfy the following.

1. $u_\theta(l, e)$ is continuously differentiable in $e$ and increasing in $l$,

2. for each $\theta$ and $l$, $u_\theta(l, \cdot)$ is strictly quasiconcave and has a unique maximum at $e_\theta(l)$, with $e_H(l) \geq e_L(l)$,

3. for all $e \leq e_L(\infty)$ there exists $e' \geq e_H(\infty)$ s.t. $u_L(\infty, e) \geq u_L(\infty, e')$,

4. $\exists e$ s.t. $u_H(\infty, e) > u_H(-\infty, e_H(-\infty))$ and $u_L(\infty, e) < u_L(\infty, e_L(\infty))$,

5. $\exists e$ s.t. $u_L(\infty, e) + \delta \frac{u_L(\infty, e_L(\infty))}{1-\delta} \leq \frac{u_L(-\infty, e_L(-\infty))}{1-\delta}$.

The time $t$ history is defined as the sender’s actions up to $t$. A pure strategy of the sender is a map from types and histories to the action set. A log likelihood ratio system is a map from histories to $[-\infty, \infty]$. A perfect Bayesian equilibrium consists of a strategy and a log likelihood ratio system s.t. given the log likelihood ratio system, the strategy maximizes the discounted future payoff of each type after every history and the log likelihood ratios are derived from the strategy using Bayes’ rule when possible. A separating equilibrium is an equilibrium in which the first period actions of the types differ. In a separating equilibrium, the log likelihood ratio is $\infty$ or $-\infty$ after every on-path history. The focus of Kaya (2009) is on separating pure strategy perfect Bayesian equilibria.

There is a continuum of separating equilibria. In these, the signalling cost can be distributed arbitrarily over time, provided the total cost up to each time $t$ is high enough to deter $L$ from imitating $H$ up to $t$. Formally, the
equilibrium effort sequence \((e_\tau)_{\tau=1}^{\infty}\) of \(H\) must satisfy \(\sum_{\tau=1}^{t} \delta^{\tau-1}[u_L(\infty, e_\tau) - u_L(-\infty, e_L(-\infty))] \leq 0\) for all \(t\).

Kaya (2009) does not discuss this, but there is also a continuum of pooling equilibria with positive signalling cost in all or some periods, provided the total cost up to each time \(t\) is low enough to deter \(L\) from switching to minimal effort. Formally, the effort sequence of both types \((e'_\tau)_{\tau=1}^{\infty}\) must satisfy \(\sum_{\tau=1}^{t} \delta^{\tau-1}[u_L(l_0, e'_\tau) - u_L(-\infty, e_L(-\infty))] \geq 0\) for all \(t\). The log likelihood ratio system assigns \(l_0\) to on-path histories and \(-\infty\) to off-path histories.

As in Spence (1973), the existence and payoffs of a separating equilibrium do not depend on the initial log likelihood ratio and pooling equilibria with positive effort exist for all parameter values.

### 3 Poisson signalling

The preceding one-shot and noiseless models cannot describe what happens when noisy signalling occurs over a period of time. This section turns to the main model where effort increases the intensity of a Poisson signal process.

Time is continuous and the horizon is infinite. The discount rate is \(r > 0\). There is a strategic sender and a competitive market of receivers. The sender has two types, \(H\) and \(L\), with initial log likelihood ratio \(l_0\).

The sender has action set \([0, 1]\) (endowed with the natural Borel \(\sigma\)-algebra) at each instant of time. The action 0 is interpreted as no effort of signalling and the action 1 as maximal effort. Effort \(e\) costs type \(\theta\) sender \(A_\theta e\), with \(A_L > A_H > 0\). Effort benefits the sender via its influence on the signal process, which influences the market’s log likelihood ratio \(l\), which determines the flow payoff. Before discussing the benefit in more detail, the signal process, strategies and market expectations must be defined.

Two Poisson signal processes are considered below. In the good news (breakthrough) case, the intensity of the Poisson signal process at time \(t\) is \(e_t \lambda\), with \(\lambda \in (0, \infty)\) the informativeness of effort and \(e_t\) the effort at \(t\). The intensity increases in the sender’s effort, so the occurrence of the signal is good news about the sender. In the bad news (breakdown) case, the intensity is \((1 - e_t) \lambda\), which decreases in effort. The receivers observe the occurrence or absence of the signal, but not the sender’s type or effort.

A signal sequence is a sequence \((\tau_k)_{k=1}^{\infty}\) of signal times satisfying \(0 = \tau_0 < \tau_1 < \tau_2 < \cdots\) and having no accumulation points. The set of signal sequences \(H_\infty\) is endowed with the \(\sigma\)-algebra generated by cylinders. An \(n\)-signal public
history is a finite sequence \((\tau_1, \ldots, \tau_n, t)\) satisfying \(\tau_1 < \cdots < \tau_n < t\), with \(t \in (0, \infty)\). The set of \(n\)-signal histories is \(H_n\). It inherits a \(\sigma\)-algebra from \(H_\infty\). The set of public histories is \(H = \bigcup_{n \in \mathbb{N}} H_n\). \(H\) inherits a \(\sigma\)-algebra from \(\{H_n\}\) and is a Borel space (Yushkevich, 1980). The truncation of a history \(h = (\tau_1, \ldots, \tau_n, t)\) to time \(s \leq t\) is \(h_s = (\tau_1, \ldots, \tau_m, s)\), with \(\tau_m < s\).

A pure public strategy is a pair of measurable maps \((e_H, e_L)\) from \(H\) into the action set \([0, 1]\). Throughout this section, only pure public strategies are considered.

If the market expects the strategy \((e^*_H, e^*_L)\) and observes a history \(h = (\tau_1, \ldots, \tau_n, t)\) up to \(t\), then under good news signals, the log likelihood ratio at \(t\) is given by

\[
l_t = l_0 - \lambda \int_0^t e^*_H(h_s) - e^*_L(h_s) ds + \sum_{k=1}^n \ln \left( \frac{e^*_H(h_{\tau_k})}{e^*_L(h_{\tau_k})} \right).
\]

The convention \(\frac{0}{0} = 1\) is used when \(e^*_H(h_{\tau_k}) = e^*_L(h_{\tau_k}) = 0\). The solution to (3) is the log likelihood ratio process \((l_t)_{t \geq 0}\). To make \((l_t)_{t \geq 0}\) well-defined, assume that if for some signal time \(\tau_k\), \(e^*_H(h_{\tau_k}) > e^*_L(h_{\tau_k}) = 0\) or \(e^*_H(h_{\tau_k}) = 0 < e^*_L(h_{\tau_k})\), then for \(t > \tau_k\), \(e^*_H(h_t) = e^*_L(h_t) = 0\). This makes the sum in (3) uniquely defined, as it cannot contain both \(\infty\) and \(-\infty\). The interpretation is that after the type is revealed, the market expects the sender not to signal.

The integral in (3) is uniquely defined, because \(e_L, e_H\) are bounded and measurable in the \(\sigma\)-algebra of histories, which contains singletons. Fixing a signal sequence, a history is determined by its length \(t\), so \(e_L, e_H\) are measurable functions from time to actions.

The derivation of (3) is from Bayes’ rule as follows (see Keller and Rady (2010) for more detail). The probability that from time \(s\) to \(t\) there are no signals when the effort process is \(e_\theta\) is exp \(\left(-\lambda \int_s^t e_\theta(h_z) dz\right)\), so the probability \(\mu_t\) on type \(H\) at time \(t\) when no signals were observed from \(s\) to \(t\) is

\[
\mu_t = \frac{\mu_s \exp \left(-\lambda \int_s^t e_H(h_z) dz\right)}{\mu_s \exp \left(-\lambda \int_s^t e_H(h_z) dz\right) + (1 - \mu_s) \mu_s \exp \left(-\lambda \int_s^t e_L(h_z) dz\right)}.
\]

By the transformation \(\ln \frac{\mu_t}{1 - \mu_t}\), the integral term in (3) is obtained. For the sum in (3), the instantaneous probability of a jump at \(t\) is \(\lambda e_\theta(h_t)\), so the probability of \(H\) after a jump is \(\frac{\mu_t \lambda e_H(h_t)}{\mu_t - \lambda e_H(h_t) + (1 - \mu_t) \lambda e_L(h_t)}\), where \(\mu_t\) is the
left limit of $\mu$ at $t$. Cancelling $\lambda$ and transforming by $\ln \frac{\mu_t}{1-\mu_t}$, the jump $j(l_{t-}) = l_{t-} + \ln \frac{e_H^*(h_t)}{e_L^*(h_t)}$ in log likelihood ratio is obtained.

Under bad news signals, the log likelihood ratio at $t$ is given by

$$l_t = l_0 + \lambda \int_0^t e_H^*(h_s) - e_L^*(h_s) ds + \sum_{k=1}^n \ln \frac{1 - e_H^*(h_{\tau_k})}{1 - e_L^*(h_{\tau_k})}. \quad (4)$$

A Markov stationary strategy depends on the history only through the log likelihood ratio induced by the history and can thus be written as a pair of functions $(e_L, e_H) : [-\infty, \infty] \to [0, 1]^2$. The state variable is the left limit of the log likelihood ratio, so jumps are not anticipated by a strategy. Subsequently only pure Markov stationary strategies are considered.

Now that strategies and the log likelihood ratio process have been described, the sender’s payoff can be defined. The sender is assumed to derive flow benefit $R(l)$ directly from the market’s log likelihood ratio $l$. This can be microfounded by assuming that each receiver has a unique one-shot best response $a^*(l)$ to each log likelihood ratio $l \in \mathbb{R}$. Since each receiver is infinitesimal, their current action does not influence the future, so in any equilibrium each receiver must play the one-shot best response. The sender is then assumed to derive flow benefit $\hat{R}(a^*(l))$ from the receivers’ action $a^*(l)$.

The sender’s flow utility from effort $e$ and the market’s log likelihood ratio $l$ is $R(l) - A_\theta e$, where $R$ is assumed strictly increasing, bounded and continuously differentiable. Denote the flow benefit from $l = \infty$ (corresponding to $\Pr(H) = 1$) by $R_{\max}$ and from $l = -\infty$ by $R_{\min}$.

The expected discounted payoff of type $\theta$ from effort function $e_\theta(\cdot)$ and log likelihood ratio process $(l_t)_{t \geq 0}$ is

$$\mathbb{E} \left[ \int_0^\infty \exp(-rt) [R(l_t) - A_\theta e_\theta(l_t)] dt \right], \quad (5)$$

where the expectation is over the stochastic process $(l_t)_{t \geq 0}$. Except for jumps, $l$ evolves deterministically given the market expectations $(e_L^*, e_H^*)$. The jumps occur at Poisson times. Given the time of a jump, its size is deterministic. The expectation in (5) is thus over the jump times of the Poisson signal process induced by $e_\theta(\cdot)$.

If the $(l_t)_{t \geq 0}$ process starts at $l_0$, the supremum of (5) for type $\theta$ is the value function $V_\theta(l_0)$, discussed in more detail in Lemma 9 in Appendix A.
**Definition 3.** A Markov stationary equilibrium consists of a Markov stationary strategy \((e^*_H, e^*_L)\) of the sender and a log likelihood ratio process \((l_t)_{t \geq 0}\) s.t.

1. given \((l_t)_{t \geq 0}\), \(e^*_\theta\) maximizes (5) for \(\theta = H, L\) over strategies,
2. given \((e^*_H, e^*_L), (l_t)_{t \geq 0}\) is derived from (3) under good news and (4) under bad news.

### 3.1 The good news case

The results about the good news model are presented next. Some preliminary observations about the equilibrium efforts are collected in the following lemma.

**Lemma 2.** For all \(l\), the equilibrium efforts must satisfy \(e^*_L(l) = e^*_H(l) = 0\) or \(e^*_L(l) < e^*_H(l)\). Moreover, \(e^*_L(l) = 0 < e^*_H(l) < 1\) cannot occur over an interval of positive length.

Lemma 2 implies that, restricting attention to simple strategies in which the range of \(e_H\) and \(e_L\) is \(\{0, 1\}\), in equilibrium the only possible effort combinations are \(e^*_H = e^*_L = 0\) and \(e^*_H = 1, e^*_L = 0\). An equilibrium in simple strategies must consist of intervals of log likelihood ratios \((l, \hat{l})\) on which \(e^*_H = 1, e^*_L = 0\), separated by regions where \(e^*_H = e^*_L = 0\). Call the union of the intervals on which \(e^*_H = 1, e^*_L = 0\) the signalling region and its complement the pooling region. Lemma 3 shows the signalling region cannot extend to log likelihood ratios at which the receivers are almost certain of the sender’s type being \(H\).

**Lemma 3.** If the range of \(e_H\) and \(e_L\) is restricted to \(\{0, 1\}\), then \(\exists \hat{l} \in (0, \infty)\) s.t. in equilibrium, \(e^*_H(l) = e^*_L(l) = 0\) for all \(l \geq \hat{l}\).

If \(e^*_H(l_0) = e^*_L(l_0) = 0\), then \(l\) stays constant at \(l_0\). If \(l\) is in the signalling region, then after a signal, \(l = \infty\) and in the absence of a signal, \(l\) drifts down until reaching the pooling region. In either case, \(l\) cannot move from one signalling interval to another, so in the class of simple strategies it is w.l.o.g. to consider only equilibria in which there is only one (possibly empty) interval \((\underline{l}, \hat{l})\) on which \(e^*_H = 1, e^*_L = 0\). Such equilibria are called interval equilibria and will be the focus of this subsection. An equilibrium is called pooling if \(e^*_H = e^*_L \equiv 0\). The pooling equilibrium exists for all parameter
values, because if \( e^*_H = e^*_L \equiv 0 \), then \( l \) does not respond to signals, so both types will avoid the signalling cost by choosing \( e = 0 \).

Outside \((l, l)\), the value functions of both types are \( V_\theta(l) = \frac{R(l)}{r} \). In \((l, l)\), the value functions are solved for using the Bellman equation and a verification theorem (Theorem 4.6 in Presman, Sonin, Medova-Dempster, and Dempster (1990) as modified for the discounted case in Yushkevich (1988)). The Bellman equation for type \( \theta \) is

\[
r V_\theta(l) = R(l) - \lambda V'_\theta(l) + \max_e \left\{ \lambda \left[ \frac{R_{\max}}{r} - V_\theta(l) \right] - A_\theta \right\}.
\]

The \( -\lambda V'_\theta(l) \) term is due to the log likelihood ratio falling at rate \( \lambda (e^*_H - e^*_L) \) in the absence of a signal. The expected equilibrium actions are \( e^*_H = 1, \ e^*_L = 0 \) in the signalling region. The \( e\lambda \left[ \frac{R_{\max}}{r} - V_\theta(l) \right] \) term describes the benefit from the flow probability \( e\lambda \) of the log likelihood ratio jumping to \( \infty \), which yields benefit equal to the difference of receiving \( R_{\max} \) forever and receiving the current value. The log likelihood ratio jumps to \( \infty \) after a signal because the expected equilibrium actions are \( e^*_H = 1, \ e^*_L = 0 \) and \( j(l) = l + \ln \frac{e^*_H}{e^*_L} = \infty \). The \( -A e \) term is the flow cost of signalling with intensity \( e \).

In the pooling region, incentives are trivial. At every \( l \) in the signalling region, the incentive constraints

\[
\lambda \left[ \frac{R_{\max}}{r} - V_H(l) \right] - A_H \geq 0 \quad \text{and} \quad \lambda \left[ \frac{R_{\max}}{r} - V_L(l) \right] - A_L \leq 0
\]

must be satisfied in order for \( H \) to choose \( e_H(l) = 1 \) and \( L \) to choose \( e_L(l) = 0 \). These incentive constraints restrict the set of possible signalling regions and must be checked after solving for the value functions. The constraints have a simple interpretation: the marginal benefit of an increase in effort is the increased probability of jumping to \( l = \infty \) and getting \( R_{\max} \) forever instead of the current value \( V_\theta(l) \). The probability increases with effort at rate \( \lambda \). The marginal cost of effort is \( A_\theta \). If marginal cost minus marginal benefit is positive, type \( \theta \) chooses \( e = 1 \), otherwise \( e = 0 \).

To solve for the value functions, substitute the equilibrium strategies \( e^*_H = 1 \) and \( e^*_L = 0 \) into the Bellman equations of \( H \) and \( L \). The Bellman equations become the ODEs \( \lambda V'_H(l) + (\lambda + r)V_H(l) = R(l) + \frac{\lambda R_{\max}}{r} - A_H \) and \( \lambda V'_L(l) + r V_L(l) = R(l) \). In the absence of a signal, starting in \((l, l)\), the log likelihood ratio falls continuously to \( l \), so \( \lim_{l \to l} V_\theta(l) = V_\theta(l) = \frac{R(l)}{r} \). The
boundary condition for the ODEs is $V_0(l) = \frac{R(l)}{r}$. The solutions of the ODEs are

$$V_H(l) = \exp \left( - (r + \lambda) \frac{l - l}{\lambda} \right) \frac{R(l)}{r} + \int_l^l \left[ \frac{R(z) - A_H}{\lambda} + \frac{R_{\max}}{r} \right] \exp \left( - (r + \lambda) \frac{l - z}{\lambda} \right) dz,$$

$$V_L(l) = \exp \left( - r \frac{l - l}{\lambda} \right) \frac{R(l)}{r} + \int_l^l \frac{R(z)}{\lambda} \exp \left( - r \frac{l - z}{\lambda} \right) dz.$$

The solutions to the Bellman equations of the types are continuously differentiable, so by the verification theorem in Yushkevich (1988), the solutions are the value functions and the Markov controls maximizing the Bellman equations are optimal.

Both value functions increase in $l$, which is intuitive: the value after the log likelihood ratio jumps is $\frac{R_{\max}}{r}$ regardless of which $l$ the jump occurs at, the cost is constant in the signalling region and the benefit $R(l)$ increases in $l$. In contrast to Spence (1973), the payoffs of the types in an informative equilibrium depend on the initial log likelihood ratio.

The value functions have a nice interpretation. In $V_H$, the exponential term $\exp \left( - (r + \lambda) \frac{l - l}{\lambda} \right)$ describes discounting at rate $r + \lambda$ for a time period $\frac{l - l}{\lambda}$. The distance in log likelihood ratio space from the current $l$ until pooling is reached at $l$ is $l - l$. In the absence of a signal, $l$ changes at rate $\lambda$, so the time until pooling is $\frac{l - l}{\lambda}$, distance divided by speed. Discounting at rate $r + \lambda$ instead of $r$ captures the flow probability of not reaching $l$ due to a jump up to $l = \infty$. The benefit upon reaching $l$ is $R(l)$ forever.

The integral in the value functions describes the discounted sum of the flow benefit until $l$ either reaches $l$ or jumps to $\infty$. The time until reaching log likelihood ratio $z$ is $\frac{z - l}{\lambda}$ and the discount rate for $H$ is $r + \lambda$. The benefit from log likelihood ratio $z$ is $R(z)$ and the signalling cost to $H$ is $A_H$, both per unit of time. The time it takes to move a unit of distance in log likelihood ratio space is $\frac{1}{\lambda}$. The benefit of $l$ jumping to $\infty$ is getting $R_{\max}$ forever. The jump occurs at rate $\lambda$ per unit of time and the time it takes to move a unit of distance is $\frac{1}{\lambda}$, which cancels with $\lambda$ in the $R_{\max}$ term.

Since $L$ has no chance of jumping to $l = \infty$, the discount rate is simply $r$ in $V_L$ and the is no $R_{\max}$ term. The flow cost of signalling is similarly absent.

The signalling regions $(l, \hat{l})$ for which the incentive constraints are satisfied can now be characterized. The minimal lower boundary $\min \hat{l}$ that an
equilibrium signalling region can have must satisfy the incentive constraint \( \lim_{l \to \min l^+} V_L(l) = \frac{R_{\max}}{r} - \frac{A_L}{\lambda} \). Due to \( V_L' > 0 \), if \( V_L(l) \geq \frac{R_{\max}}{r} - \frac{A_L}{\lambda} \) holds at \( l \), then it holds for all \( \hat{l} > l \). If \( L \) is deterred from imitating \( H \) at \( l \), then \( L \) is also deterred at all \( \hat{l} > l \), because the benefit to imitation is the difference between the payoff of being believed to be the \( H \) type and the current value. The higher \( l \), the higher the current value, so the lower the incentive to exert effort. Since \( l \) drifts down in the absence of a signal, the \( L \) type incentive constraint determines the log likelihood ratio below which signalling must stop. If the expectation was for signalling to continue at \( \min l \), then \( L \) would deviate to \( e = 1 \).

The boundary condition for the ODE determining \( V_L \) is \( V_L(l) = \frac{R(l)}{r} \), so \( l \geq R^{-1} \left( \frac{R_{\max}}{r} - A_L \lambda \right) \), with equality for \( \min l \). A more patient sender means the set of equilibrium signalling regions is smaller for a given \( l_0 \), specifically \( \frac{\partial \min l}{\partial r} < 0 \) and \( \lim_{r \to 0} \min l = \infty \). The lower \( l \) is, the more informative the equilibrium. Since \( \frac{\partial \min l}{\partial r} < 0 \), a less patient sender means more information can be revealed.

The maximal upper boundary that an equilibrium signalling region can have is determined by \( V_H \), which depends on \( \bar{l} \). Denote the maximal upper boundary given \( \bar{l} \) by \( \max \bar{l} \). The incentive constraint \( \lim_{l \to \max \bar{l}(\bar{l})} V_H(l) \leq \frac{R_{\max}}{r} - \frac{A_H}{\lambda} \) must hold with equality at \( \max \bar{l}(\bar{l}) \). For all parameter values, \( \max \bar{l}(\bar{l}) > l \), because \( V_H \) is strictly increasing and

\[
V_H(\min l) = V_L(\min l) = \frac{R_{\max}}{r} - \frac{A_L}{\lambda} < \frac{R_{\max}}{r} - \frac{A_H}{\lambda} = \lim_{l \to \max \bar{l}(\bar{l})} V_H(l).
\]

The result \( \max \bar{l}(\bar{l}) > l \) implies that a nontrivial interval equilibrium exists whenever \( l_0 = \min l + \epsilon \), with \( \epsilon > 0 \) sufficiently small. In fact, there is a continuum of such equilibria, each corresponding to a particular \( l \in [\min l, l_0) \). The existence of an informative equilibrium depends on the initial log likelihood ratio, which differs from Spence (1973).

Due to \( V_H' > 0 \), if the incentive constraint \( V_H(l) \leq \frac{R_{\max}}{r} - \frac{A_H}{\lambda} \) holds at \( l \), then for all \( l < \hat{l} \) we have \( V_H(\hat{l}) < \frac{R_{\max}}{r} - \frac{A_H}{\lambda} \). If \( H \) is incentivized to signal at \( l \), then \( H \) is also incentivized at all \( \hat{l} < l \), because the benefit to signalling is the difference between the payoff of being believed to be the \( H \) type and the current value. Since \( l \) drifts down in the absence of a signal, the \( H \) type incentive constraint determines the log likelihood ratio above which signalling cannot start. If the expectation was for signalling to start above \( \max \bar{l}(\bar{l}) \), then \( H \) would deviate to \( e = 0 \).
Comparing interval equilibria, the payoff of $L$ strictly increases in $l$ and pooling gives $L$ the highest payoff. This result differs in the bad news model, where for some benefit functions $R$, the payoff of $L$ may be highest in the most informative equilibrium.

The payoff of $H$ increases in $l$ iff $R'(l) - R_{\text{max}} + R(l) + \frac{A_H r}{\lambda} \geq 0$. If the condition holds, then pooling gives $H$ the highest payoff, otherwise the most informative interval equilibrium maximizes the payoff of $H$. The condition has a straightforward interpretation. Increasing $l$ increases the payoff upon reaching $l$ at a rate $R'(l)$, lowers the chance of jumping to $l = \infty$ (the $-R_{\text{max}}$ term), increases the chance of reaching $l$ (the $R(l)$ term) and reduces the time during which the signalling cost is paid. The balance of these effects determines whether $V_H$ increases or decreases in $l$.

If $R(l) = \frac{\exp(l)}{1+\exp(l)}$, then $R'(l) + R(l) - R_{\text{max}} = -\frac{1}{(1+\exp(l))^2}$. Then $\frac{\partial V_H(l)}{\partial l} \geq 0$ iff $A_H \geq \frac{\lambda}{r(1+\exp(l))^2}$. As in the benchmark models in Section 2, $H$ may prefer pooling or the maximally informative equilibrium, depending on the parameters.

Welfare $\frac{\exp(l)}{1+\exp(l)}V_H(l) + \frac{1}{1+\exp(l)}V_L(l)$ increases in $l$ iff

$$\exp(l) \left[ R'(l) - R_{\text{max}} + R(l) + \frac{A_H r}{\lambda} \right] + R'(l) \geq 0.$$ 

This condition holds when $R(l) = \frac{\exp(l)}{1+\exp(l)}$, so pooling gives the highest welfare when the sender’s benefit from the receivers’ belief is linear.

### 3.2 The bad news case

In some environments, the good news and bad news Poisson signals may lead to radically different results (Board and Meyer-ter-Vehn, 2013). In the current paper, if the signal is bad news (its probability decreases in effort), the results are similar to the good news case, but not exactly the same. In the good news case, $V_L$ is always highest in pooling, but in the bad news case, there are reward functions $R$ for which the $L$ type prefers the maximally informative equilibrium to pooling. This aspect of the bad news model is similar to the one-shot noisy signalling game.

Some effort patterns can be ruled out in equilibrium, similarly to Lemma 2 in the good news case.
Lemma 4. In equilibrium, it cannot be that \( e^*_L(l) = e^*_H(l) > 0 \) for some \( l \) or that for an \( l \) s.t. \( j(l) = l + \ln \frac{1 - e^*_H(l)}{1 - e^*_L(l)} \) is in the pooling region, \( e^*_L(l) > e^*_H(l) \). In particular, there is no \( l \) for which \( e^*_L(l) = 1 > e^*_H(l) \). An interval \((l, 1)\) on which \( e^*_L > e^*_H \) holds must be bounded and must contain \( l \) s.t. \( j(l) \notin (l, 1) \). Among the intervals on which \( e^*_L > e^*_H \), at least one must contain \( l'' \) s.t. \( j(l'') \) lies outside the \( e^*_L > e^*_H \) region.

The focus in this section is on interval equilibria in which \( e^*_L \equiv 0 \) and \( e^*_H(l) = 1 \) if \( l \in (l, 1) \), with \( e^*_H(l) = 0 \) otherwise.

Outside \((l, 1)\), the value functions of both types are \( V_\theta(l) = \frac{R(l)}{r} \). In \((l, 1)\), the value functions are solved for using Bellman equations and a verification theorem is used to check that the solutions coincide with the value functions. The Bellman equation of type \( \theta \) is

\[
rv_\theta(l) = R(l) + \lambda V_\theta(l) + \max_e \left\{ \lambda(1 - e) \left[ \frac{R_{\min}}{r} - V_\theta(l) \right] - A_\theta e \right\}.
\]

The \( \lambda V_\theta(l) \) term is due to the log likelihood ratio rising at rate \( \lambda(e^*_H - e^*_L) \) in the absence of a signal. The expected equilibrium actions are \( e^*_L = 1 \), \( e^*_L = 0 \) in the signalling region. The \((1 - e)\lambda \left[ \frac{R_{\min}}{r} - V_\theta(l) \right] \) term describes the (negative) benefit from the flow probability \((1 - e)\lambda \) of the log likelihood ratio jumping to \(-\infty \). A jump yields benefit equal to the difference of receiving \( R_{\min} \) forever and receiving the current value. The log likelihood ratio jumps to \(-\infty \) after a signal because the expected equilibrium actions are \( e^*_L = 1 \), \( e^*_L = 0 \) and \( l_0 + \ln \frac{1 - e^*_H}{1 - e^*_L} = -\infty \) in this case. The \(-A_\theta e \) term is the flow cost of signalling with intensity \( e \).

In the pooling region, incentives are trivial. In the signalling region, type \( \theta \) chooses \( e_\theta = 1 \) if \(-\lambda[\frac{R_{\min}}{r} - V_\theta(l)] - A_\theta \geq 0 \). Rearranging this and assuming \( V_\theta > 0 \), one obtains the incentive constraints \( \frac{A_H}{\lambda} + \frac{R_{\min}}{r} \leq \lim_{l \to l} V_H(l) \) and \( \frac{A_L}{\lambda} + \frac{R_{\min}}{r} \geq \lim_{l \to 1} V_L(l) \), which restrict the set of possible signalling regions and must be checked after solving for the value functions.

To solve for the value functions, substitute the equilibrium strategies \( e^*_H = 1 \) and \( e^*_L = 0 \) into the Bellman equations of \( H \) and \( L \). The Bellman equations become the ODEs \( rV_H(l) = R(l) - A_H + \lambda V'_H(l) \) and \( rV_L(l) = R(l) + \lambda V'_L(l) + \frac{R_{\min}}{r} - \lambda V_L(l) \). In the absence of a signal, starting in \((l, 1)\), the log likelihood ratio rises continuously to \( l \), so \( \lim_{l \to \frac{R_{\min}}{r}} V_\theta(l) = V_\theta(l) = R_{\min} \), which provides the boundary condition for the ODEs. The solutions of the
ODEs are

\[ V_H(l) = \exp \left( -rl - l \right) \frac{R(l)}{r} + \int_l^T \frac{R(z) - A_H}{\lambda} \exp \left( -rz - l \right) \, dz, \]

\[ V_L(l) = \exp \left( -(r + \lambda)l - l \right) \frac{R(l)}{r} \]

\[ + \int_l^T \left[ \frac{R(z) + R_{\min}}{\lambda} \right] \exp \left( -(r + \lambda)z - l \right) \, dz. \]

These are continuously differentiable, so by the verification theorem in Yushkevich (1988), they coincide with the value functions and the Markov control maximizing the Bellman equation is optimal.

Both value functions increase in \( l \), which is intuitive: the value after the belief jumps is \( \frac{R_{\min}}{r} \) regardless of which \( l \) the jump occurs at, the cost is constant in the signalling region and the benefit \( R(l) \) increases in \( l \). The interpretation of the terms in the value functions is similar to the good news case.

The signalling regions \((l, \bar{l})\) for which the incentive constraints are satisfied can now be characterized. The maximal upper boundary \( \max \bar{l} \) that an equilibrium signalling region can have must satisfy the incentive constraint \( \lim_{l \to \max} V_L(l) = \frac{R_{\min}}{r} + \frac{A_L}{\lambda} \). The benefit to signalling is avoiding the bad signal, so the larger the difference between \( \frac{R_{\min}}{r} \) and \( V_L(l) \), the greater the incentive of \( L \) to imitate \( H \). The \( L \) type incentive constraint determines the log likelihood ratio above which signalling cannot continue.

The boundary condition for the ODE determining \( V_L \) is \( V_L(\bar{l}) = \frac{R(\bar{l})}{r} \), so \( \bar{l} \leq R^{-1} \left( \frac{R_{\min}}{r} + \frac{A_L}{\lambda} \right) \), with equality for max \( \bar{l} \). A more patient sender means the set of equilibrium signalling regions is smaller for a given \( l_0 \), specifically \( \frac{\partial \max \bar{l}}{\partial r} > 0 \) and \( \lim_{r \to 0} \max \bar{l} = -\infty \). The higher \( l \) is, the more informative the equilibrium. Since \( \frac{\partial \max \bar{l}}{\partial r} > 0 \), a less patient sender means more information can be revealed, just as in the good news case.

The minimal lower boundary that an equilibrium signalling region can have is determined by \( V_H \), which depends on \( \bar{l} \). Denote the minimal lower boundary given \( l \) by \( \min l(\bar{l}) \). The incentive constraint \( \lim_{l \to \min} V_H(l) \geq \frac{R_{\min}}{r} + \frac{A_H}{\lambda} \) must hold with equality at \( \min l(\bar{l}) \). For all parameter values, \( \min l(\bar{l}) > \bar{l} \), because \( V_H \) is strictly increasing and

\[ V_H(\max \bar{l}) = V_L(\max \bar{l}) = \frac{A_L}{\lambda} + \frac{R_{\min}}{r} > \frac{A_H}{\lambda} + \frac{R_{\min}}{r} = \lim_{l \to \min l(\bar{l})} V_H(l). \]
The result $\min \mu(\bar{l}) > \bar{l}$ implies that a nontrivial interval equilibrium exists whenever $l_0 = \max \bar{l} - \epsilon$, with $\epsilon > 0$ sufficiently small. In fact, there is a continuum of such equilibria, each corresponding to a particular $\bar{l} \in (l_0, \max \bar{l}]$. The existence of an informative equilibrium again depends on the initial log likelihood ratio.

Comparing interval equilibria, the payoff of $L$ increases in $\bar{l}$ iff $R'(\bar{l}) - R(\bar{l}) + R_{\min} > 0$. This condition holds e.g. when $R(l) = \left(\frac{\exp(l)}{1+\exp(l)}\right)^n$, $n \in \mathbb{N}$, $n \geq 2$, $\bar{l} < \ln(n)$. The effects of raising $\bar{l}$ on $V_L$ are a higher payoff upon reaching $\bar{l}$ (the $R'(\bar{l})$ term), but a lower chance of reaching it (the $-R(\bar{l})$ term) and a higher chance of jumping to $l = -\infty$. If $R'(\bar{l}) - R(\bar{l}) + R_{\min} > 0$, then the interval equilibrium maximizing $V_L$ is the maximally informative one, otherwise it is the pooling equilibrium. Unlike in the good news model, but similarly to the one-shot noisy signalling game with a convex benefit, it may be the maximally informative equilibrium that maximizes the payoff of $L$. If $R(l) = \frac{\exp(l)}{1+\exp(l)}$, then pooling gives $L$ the highest payoff.

The payoff of $H$ increases in $\bar{l}$ iff $\frac{R'(\bar{l})}{r} \geq \frac{Ah}{\lambda}$. If the condition holds, then the maximally informative equilibrium gives $H$ the highest payoff, otherwise pooling maximizes the payoff of $H$. The interpretation of the condition is that increasing $\bar{l}$ increases the payoff upon reaching $\bar{l}$ at a rate $R'(\bar{l})$ and increases the time during which the signalling cost is paid. Whether $V_H$ increases or decreases in $\bar{l}$ depends on which effect dominates.

Welfare increases in $\bar{l}$ iff $\exp(\bar{l}) \left[ R'(\bar{l}) - \frac{Ah}{\lambda} \right] + R'(\bar{l}) - R(\bar{l}) + R_{\min} \geq 0$. This condition fails when $R(l) = \frac{\exp(l)}{1+\exp(l)}$. In that case pooling gives the highest welfare.

Including both good and bad news in the game, with good signals occurring at rate $\lambda_g e$ and bad signals at rate $\lambda_b (1 - e)$, the behaviour of the model is similar to the good news case when $\lambda_g > \lambda_b$ and similar to the bad news case when $\lambda_g < \lambda_b$. If $\lambda_g = \lambda_b$, then the log likelihood ratio stays constant in the absence of signals and jumps when a signal occurs. The game with both good and bad news is discussed in Appendix C.

## 4 Signalling with Brownian noise

The Poisson signalling game corresponds to an environment where information is revealed by rare and significant events. There are also situations in which a gradual and continuous information revelation is realistic. This sec-
tion turns to the gradual information revelation case and models the signal process as a Brownian motion.

Time is continuous and the horizon is infinite. The discount rate is \( r > 0 \). The players are an infinitely lived strategic sender and a market of competitive receivers. The sender can be one of two types, \( H \) or \( L \), with initial log likelihood ratio \( l_0 \). Both types have action set \([0, 1]\), with generic action \( e \) at each instant of time \( t \in [0, \infty) \). The cost of action \( e \) is \( A_\theta e \) for type \( \theta \), with \( A_L > A_H > 0 \).

The process of the sender’s actions \((e_t)_{t \geq 0}\) controls the drift of a signal process \((X_t)_{t \geq 0}\) given by 
\[
dX_t = e_t dt + \sigma dB_t,
\]
where \( B_t \) is standard Brownian motion and \( \sigma > 0 \). Denote the filtration generated by \((X_t)_{t \geq 0}\) by \((\mathcal{F}_t)_{t \geq 0}\). The receivers at time \( t \) observe \((X_\tau)_{\tau \in [0,t]}\), but not the sender’s type or present or past actions. Based on the signal, the receivers update their log likelihood ratio. The log likelihood ratio process \((l_t)_{t \geq 0}\) is adapted to \((\mathcal{F}_t)_{t \geq 0}\).

The flow utility of a sender of type \( \theta \) from action \( e \) and the receivers’ log likelihood ratio \( l \) is \( R(l) - A_\theta e \). Assume \( R \) is bounded, Lipschitz, strictly increasing and twice continuously differentiable on \([-\infty, \infty]\).

A Markov stationary strategy of the sender is a pair of measurable functions \( e^*_H, e^*_L : [-\infty, \infty] \to [0, 1] \). The state variable is the receivers’ log likelihood ratio \( l \). Given expected strategies \( e^*_L, e^*_H \) of the types of the sender, the receivers update the log likelihood ratio using Bayes’ rule
\[
dl_t = \sigma^{-2}(e^*_H(l_t) - e^*_L(l_t))[dX_t - \frac{1}{2}e^*_H(l_t)dt - \frac{1}{2}e^*_L(l_t)dt]. \tag{6}
\]

**Definition 4.** A Markov stationary equilibrium consists of a strategy \((e^*_H, e^*_L)\) and a log likelihood ratio process \((l_t)_{t \geq 0}\) s.t.

1. given \((l_t)_{t \geq 0}\), \( e^*_\theta \) solves
\[
\sup_{e_\theta(\cdot)} \mathbb{E} \int_t^\infty \exp(-rs) \left[ R(l_s) - A_\theta e_\theta(l_s) \right] ds,
\]
where the expectation is over the process \((l_t)_{t \geq 0}\).

2. given \((e^*_H, e^*_L)\), \((l_t)_{t \geq 0}\) is derived from Bayes’ rule (6).

The existence of an equilibrium is established in the following lemma, which also gives a necessary condition for optimality of a strategy.

\(^2\)The updating rule for the log likelihood ratio is derived from the continuous time Bayes’ rule for probability (Liptser and Shiryaev (1977) Theorem 9.1) using Itô’s formula.
Lemma 5. Pooling \((e_L^* = e_H^* = 0)\) is an equilibrium for all parameter values. In equilibrium \(\exists l_1, l_2\) s.t. \(l_1 < l_2\) and \(\forall l \in (l_1, l_2),\ e_L^*(l) = e_H^*(l) > 0\) or \(e_L^*(l) > e_H^*(l)\).

For an equilibrium strategy \((e_H^*, e_L^*)\), call \(\{ l : e_H^*(l) = e_L^*(l) = 0 \}\) the pooling region and its complement the signalling region. It is clear from (6) that once the log likelihood ratio process reaches the pooling region, \(l\) stays constant forever. Starting from the signalling region, if the \(l\) process reaches the boundary of the signalling region, then in the next instant it enters the pooling region due to the rapidly varying Brownian motion driving \(l\). For this reason, it is w.l.o.g. to consider only open signalling regions. An open subset of \(\mathbb{R}\) is a countable union of disjoint open intervals. Starting in one interval, the \(l\) process cannot move to another, because the sample paths of \(l\) are continuous and \(l\) stops moving when it enters the pooling region. In light of this and Lemma 5, it is w.l.o.g. to consider only equilibria in which outside an interval of log likelihood ratios \((l, \bar{l})\), both types choose action 0 and inside that interval, \(e_L^* < e_H^*\).

The subsequent focus of this section is on interval equilibria in which outside an interval of log likelihood ratios \((l, \bar{l})\), both types choose action 0 and inside that interval, \(e_L^* = 0, e_H^* = 1\). In \((l, \bar{l})\), the \(l\) process is a simple Brownian motion with drift either \(\frac{1}{2}\) or \(-\frac{1}{2}\), depending on whether the sender’s chosen action is \(e = 1\) or 0.

Define \(\hat{T}_{t, l}\) as the first exit time after \(t\) of the \(l\) process from \((l, \bar{l})\), i.e. \(\hat{T}_{t, l} = \inf \{ \tau > t : l_\tau / \ell \notin (l, \bar{l}) \} \leq \infty\). In the signalling region, the value function of type \(\theta\) is

\[
V_\theta(l_t) = \sup_{e_\theta(\cdot)} \mathbb{E} \int_t^{\hat{T}_{t, l}} \exp(-rs) \left[ R(l_s) - A_\theta e_\theta(l_s) \right] ds
+ \exp(-r\hat{T}_{t, l}) \frac{R(l_{\hat{T}_{t, l}})}{r} \mathbf{1} \{ \hat{T}_{t, l} < \infty \},
\]

where \(\mathbf{1}\{A\}\) denotes the indicator function for the set \(A\).

The interpretation of the value function expression is straightforward: the agent gets flow benefit \(R(l)\) depending on \(l\) and chooses the optimal signalling effort \(e_\theta\) at each \(l\), which determines the flow cost. When the log likelihood ratio exits the signalling region (if ever), the sender gets \(R(l_{\hat{T}_{t, l}})\) forever, where \(l_{\hat{T}_{t, l}}\) is the value of \(l\) upon exit, which equals either \(l\) or \(\bar{l}\) due to the continuity of the sample paths of the \(l\) process. Some observations about the value functions are formalized in the following lemma.

26
Lemma 6. $V_\theta$ is finite for $\theta = H, L$. $V_H \geq V_L$, with strict inequality in the signalling region. $V_\theta$ is strictly increasing.

The result that $V_\theta$ is strictly increasing in $l_0$ stands in contrast to Spence (1973), where the payoffs of the types from a separating equilibrium do not depend on the prior.

The next proposition shows that with $R$ concave in the (nonempty) signalling region of an interval equilibrium, the payoff of $L$ is higher under pooling than in that interval equilibrium. Comparing equilibria with signalling, $L$’s payoff is higher in an equilibrium with a smaller signalling region.

Proposition 7. Assume $R$ is concave on $(l_2, \bar{l}_2)$. Then the value $V_L$ of $L$ in the equilibrium with signalling region $(l_2, \bar{l}_2)$ satisfies $V_L(l) < \frac{R(l)}{r}$ for all $l \in (l_2, \bar{l}_2)$. The value $V_L$ of $L$ in another equilibrium with signalling region $(l_1, \bar{l}_1) \subsetneq (l_2, \bar{l}_2)$ satisfies $V_{L1}(l) > V_L(l)$ for all $l \in (l_2, \bar{l}_2)$.

To solve the control problems of the types of the sender, the HJB equations are solved and a verification theorem is used to check that the solutions of the HJB equations coincide with the value functions. The HJB equation of type $\theta$ is

$$rv_\theta(l) = R(l) + \max \left\{ -A_\theta + \frac{1}{2}v_\theta'(l)\sigma^{-2}, -\frac{1}{2}v_\theta'(l)\sigma^{-2} \right\} + \frac{1}{2}v_\theta''(l)\sigma^{-2}. $$

Given the signalling region $(l, \bar{l})$ the receivers expect, the optimal strategy of type $\theta$ is to choose

$$e_\theta(l) = \begin{cases} 
1 & \quad -A_\theta + \frac{1}{2}v_\theta'(l)\sigma^{-2} \geq -\frac{1}{2}v_\theta'(l)\sigma^{-2} \\
0 & \quad \text{if } l \in (l, \bar{l}), \\
0 & \quad \text{if } l \notin (l, \bar{l}).
\end{cases}$$

The conditions for $H$ to choose $e_H(l) = 1$ and $L$ to choose $e_L(l) = 0$ in the signalling region are

$$v_H'(l) \geq A_H\sigma^2, \quad v_L'(l) \leq A_L\sigma^2. \quad (7)$$

Call these constraints IC$_H$ and IC$_L$. After finding the candidate equilibrium strategies, it must be verified that the IC constraints hold at every point in the signalling region.
Set the chosen actions equal to the equilibrium actions. The HJB equations become the pair of linear second-order ODEs

\[ \begin{align*}
rv_H(l) &= R(l) - A_H + \frac{1}{2}v_H'(l)\sigma^{-2} + \frac{1}{2}v_H''(l)\sigma^{-2}, \\
rv_L(l) &= R(l) - \frac{1}{2}v_L'(l)\sigma^{-2} + \frac{1}{2}v_L''(l)\sigma^{-2}.
\end{align*} \]

This is where using the log likelihood ratio instead of the belief is helpful—in the case of belief, the ODEs would not have constant coefficients. After solving the ODEs for \(v_L, v_H\), the ICs as well as the smoothness conditions for the verification theorem must be checked at every point in the signalling region.

The solutions \(v_\theta\) to the ODEs are the sum of the general solution \(C_{\theta_1}y_{\theta_1} + C_{\theta_2}y_{\theta_2}\) of the homogeneous equation and a particular solution \(y_{\theta p}\) of the inhomogeneous equation. The general solutions for \(H\) and \(L\) respectively are

\[ \begin{align*}
C_{H1}\exp\left(\frac{-1 - \sqrt{1 + 8r\sigma^2}}{2}\right) + C_{H2}\exp\left(\frac{-1 + \sqrt{1 + 8r\sigma^2}}{2}\right), \\
C_{L1}\exp\left(\frac{1 - \sqrt{1 + 8r\sigma^2}}{2}\right) + C_{L2}\exp\left(\frac{1 + \sqrt{1 + 8r\sigma^2}}{2}\right).
\end{align*} \]

Using d’Alembert’s method, the particular solutions are

\[ \begin{align*}
y_{Hp} &= -\frac{A_H}{r} \\
&\quad + \frac{2\sigma^2}{\sqrt{1 + 8r\sigma^2}}\exp\left(\frac{-1 - \sqrt{1 + 8r\sigma^2}}{2}\right) \int R(l)\exp\left(\frac{-1 + \sqrt{1 + 8r\sigma^2}}{2}\right) dl \\
&\quad - \frac{2\sigma^2}{\sqrt{1 + 8r\sigma^2}}\exp\left(\frac{1 - \sqrt{1 + 8r\sigma^2}}{2}\right) \int R(l)\exp\left(\frac{1 + \sqrt{1 + 8r\sigma^2}}{2}\right) dl,
\end{align*} \]

\[ \begin{align*}
y_{Lp} &= \\
&\quad + \frac{2\sigma^2}{\sqrt{1 + 8r\sigma^2}}\exp\left(\frac{1 - \sqrt{1 + 8r\sigma^2}}{2}\right) \int R(l)\exp\left(\frac{-1 + \sqrt{1 + 8r\sigma^2}}{2}\right) dl \\
&\quad - \frac{2\sigma^2}{\sqrt{1 + 8r\sigma^2}}\exp\left(\frac{1 + \sqrt{1 + 8r\sigma^2}}{2}\right) \int R(l)\exp\left(\frac{-1 - \sqrt{1 + 8r\sigma^2}}{2}\right) dl.
\end{align*} \]
where the integrals are nonelementary even for simple functional forms of $R$, e.g. for $R(l) = \frac{\exp(l)}{1+\exp(l)}$, which is equivalent to $\dot{R}(\mu) = \mu$.

Imposing the boundary conditions $v_{\theta}(l) = \frac{R(l)}{r}$ and $v_{\theta}(\bar{l}) = \frac{R(\bar{l})}{r}$, the constants in the general solution for $H$ are

$$C_{H1} = \frac{y_{H2}(\bar{l})[\frac{R(l)}{r} - y_{Hp}(l)] - y_{H2}(l)[\frac{R(\bar{l})}{r} - y_{Hp}(\bar{l})]}{y_{H1}(l)y_{H2}(\bar{l}) - y_{H2}(l)y_{H1}(\bar{l})},$$

$$C_{H2} = \frac{-y_{H1}(\bar{l})[\frac{R(l)}{r} - y_{Hp}(l)] + y_{H1}(l)[\frac{R(\bar{l})}{r} - y_{Hp}(\bar{l})]}{y_{H1}(l)y_{H2}(\bar{l}) - y_{H2}(l)y_{H1}(\bar{l})}.$$

The constants for $L$ are determined by a similar expression, replacing the $H$ subscripts with $L$.

Now that all components of the solutions of the HJB equations have been found, it can be verified that they coincide with the value functions.

**Lemma 8.** The solutions $v_H, v_L$ of the HJB equations equal the value functions $V_H, V_L$ in the signalling region. The Markov controls for the HJB equations maximize the value functions.

Closed form comparative statics results are not available for parameters other than $A_L$ due to the complexity of the $V_{\theta}$ expressions. Numerical simulations will be used instead. Examining IC$_L$ in (7), the LHS does not contain $A_L$, so there exists $\hat{A}_L$ s.t. for $A_L < \hat{A}_L$, IC$_L$ fails and for $A_L \geq \hat{A}_L$, IC$_L$ holds.

Numerical results on how the set of signalling regions depends on the parameters are presented next. As in the Poisson signalling game, for some initial log likelihood ratios there is a continuum of informative equilibria.

Until the end of this section, it is assumed that $R(l) = \frac{\exp(l)}{1+\exp(l)}$, so the sender’s benefit from the receivers’ belief equals the belief. The figures to follow depict signalling intervals $(l, \bar{l})$ as points on a plane, with the $x$-coordinate of the point equalling $l$ and the $y$-coordinate equalling $\bar{l}$.

For $A_H = 0.1$, $A_L = 0.24$ and $r = \sigma^2 = 1$, the region where the ICs hold is depicted in the right panel of Figure 3 as the shaded area. The left panel shows the area where IC$_H$ holds and the middle panel the area where IC$_L$ holds. The shaded area on the right panel is the intersection of the left and middle panels.

If $A_H = 0.15$, $A_L = 0.2$ and $r = \sigma^2 = 1$, then the region where both ICs hold is limited to the diagonal, so the only interval equilibrium is pooling.
The region where IC\textsubscript{H} holds for these parameters is depicted in the left panel of Figure 4. The region where IC\textsubscript{L} holds is in the right panel. The intersection of the two regions is the diagonal, depicting the empty signalling intervals.

The effect of increased patience or reduced noise on the ICs is shown in Figure 5, where \( A_H = 0.1, A_L = 0.24, r = 1 \) and \( \sigma^2 = 0.5 \). Note the different scale of the axes compared to Figure 3. Since \( r \) and \( \sigma^2 \) affect the ICs only through their product, reducing \( \sigma^2 \) by half has the same effect as reducing \( r \) by half.

There need not exist a signalling region containing all others, as in the previous figures. Such a signalling region is the point at the upper left corner of the shaded area of the right panel, i.e. a point that is simultaneously at
maximal horizontal and vertical distance from the diagonal. Figure 6 shows that for \( A_H = 0.15, A_L = 0.28 \) and \( r = \sigma^2 = 1 \), a higher \( \ell \) permits a higher \( \ell \) for a signalling region. Starting from \( l_0 \in (l, l'] \), there is thus no ‘maximally informative’ equilibrium. In other words, the union of two equilibrium signalling regions need not be an equilibrium signalling region. This distinguishes the game with Brownian noise from the Poisson signalling game, the repeated noiseless game and the one-shot noisy and noiseless games.

In a given informative equilibrium the \( H \) type payoff can be higher or lower than the pooling payoff \( \frac{\exp(\ell)}{r(1+\exp(\ell))} \) for different log likelihood ratios. This contrasts with the Poisson signalling game, where either pooling or the max info equilibrium gives \( H \) the highest payoff for all log likelihood ratios. The Brownian case is illustrated in Figure 7, where \( V_H \) is strictly higher than \( \frac{\exp(\ell)}{r(1+\exp(\ell))} \) for \( l \in (-1.5, -0.2) \) and strictly lower for \( l \in (-0.2, 3) \). In this equilibrium, the comparison of informative equilibrium and pooling payoffs of \( H \) accords well with Spence (1973), where for a higher fraction of \( H \) in the population, the payoff difference (separating minus pooling) for \( H \) is lower. In Spence’s model, the reason is that for a higher \( l_0 \) there is less scope for the log likelihood ratio to rise (\( \ell = \infty \) after the high action). In the present model this mechanism does not work, because the rise in belief after a good signal is highest for intermediate \( l_0 \). Correspondingly the Spence intuition does not always hold in the continuous time model. Close to the upper bound of the signalling region, the payoff from the informative equilibrium rises to the pooling one as \( l_0 \) increases, so in that region, the informative equilibrium payoff minus the pooling payoff rises in \( \ell \).

For the signalling region \((0, 3)\), with \( A_H = 0.1 \) and \( r = \sigma^2 = 1 \), the informative equilibrium payoff of \( H \) is below pooling in the whole signalling region. Close to the upper bound, the informative equilibrium payoff minus the pooling payoff rises in \( \ell \), while close to the lower bound, it falls in \( \ell \). This pattern is reversed in the informative equilibrium with signalling region \((-3, 0)\), \( A_H = 0.1 \) and \( r = \sigma^2 = 1 \). In that case, the informative equilibrium payoff is above pooling in the whole signalling region. Within the signalling region of a given informative equilibrium there is always a region where the payoff difference with pooling moves in the opposite direction to the prediction of Spence (1973). Comparing equilibria with \( \ell \leq 0 \) to those with \( \ell \geq 0 \), the Spence pattern holds—shifting the signalling region up raises the informative equilibrium payoff minus the pooling payoff.

In numerical simulations, as \( r \) or \( \sigma^2 \) increases, the payoff of \( H \) from an informative equilibrium minus the pooling payoff falls. Intuitively, patience
favours signalling and noise favours pooling. Across values of \( r \) and \( \sigma^2 \),
the payoff difference between an informative equilibrium and pooling can be positive or negative.

For the \( L \) type, as \( r \) increases, the payoff from an informative equilibrium
minus the pooling payoff rises. Since in the signalling region \( L \) expects the
receivers’ log likelihood ratio (and \( L \)’s own future payoff) to fall, the more
the future payoff matters, the worse off the occurrence of signalling makes \( L \).
As \( \sigma^2 \) increases, \( L \)’s payoff from an informative equilibrium increases—noise
is good for \( L \), since the receivers learn about the types more slowly.

So far, only equilibria with efforts of the types zero or one have been
considered. Other kinds of equilibria also exist: one class is where \( e^*_H \in (0, 1) \)
and \( e^*_L = 0 \), another class has \( e^*_H = 1 \) and \( e^*_L \in (0, 1) \). These interior effort
equilibria are more difficult to work with than the equilibria where \( e^*_\theta \in \{0, 1\} \)
and fewer results are available. They are discussed in Appendix D. A
continuum of interior effort equilibria in each class exists for some parameter
values.

As in the one-shot noisy signalling game, a natural question arising in
the above Brownian model is whether the results are driven by the linear
cost. Appendix E solves a Brownian signalling game with quadratic cost.
There is a continuum of equilibrium signalling intervals and on each signalling
interval, a continuum of equilibrium effort profiles. The reason why many
effort profiles on a given interval constitute equilibria is a linear dependence
in the first order conditions. This is a feature of the quadratic cost and
unlikely to generalize to other convex cost functions.

(A) Proofs omitted from the text

Proof of Proposition 1. Given the expectations of the receivers, which de-
dtermine \( l(g), l(b) \) via (1), action 1 is a best response for type \( \theta \) if
\[-A_\theta + \frac{1+\lambda}{2} R(l(g)) + \frac{1-\lambda}{2} R(l(b)) \geq \frac{1+\lambda}{2} R(l(g)) + \frac{1-\lambda}{2} R(l(b)) \],
which is equivalent to
\[ A_\theta \leq \lambda[R(l(g)) - R(l(b))]. \]
Action 0 is a best response if \( A_\theta \geq \lambda[R(l(g)) - R(l(b))]. \)

If pooling is expected, then \( l(g) = l(b) \) and \( \lambda[R(l(g)) - R(l(b))] = 0. \nIn that case 0 is the unique best response for both types, so pooling is an
equilibrium. This observation is independent of the parameters.

Maximizing \( R(l(g)) - R(l(b)) \) by choosing \( e^*_H, e^*_L \), we get \( e^*_L = 0 \) and
\( e^*_H = 1 \) as the unique solution, because \( R \) is strictly increasing. If \( A_H > \)
\(\lambda[R(\bar{l}(g)) - R(\bar{l}(b))]\), then for any expectations of the receivers, 0 is the unique best response for both types.

If \(A_H = \lambda[R(\bar{l}(g)) - R(\bar{l}(b))]\) and the receivers expect \(e_L^* = 0\) and \(e_H^* = 1\), then \(H\) is indifferent between actions 0 and 1, while \(L\) strictly prefers 0. Max info is thus an equilibrium. There are no other equilibria, because if the receivers expect \(e_H^* < 1\) or \(e_L^* > 0\), then \(R(l(g)) - R(l(b)) < R(\bar{l}(g)) - R(\bar{l}(b))\), so 0 is the unique best response for \(H\).

If \(A_H < \lambda[R(\bar{l}(g)) - R(\bar{l}(b))] \leq A_L\) and the receivers expect \(e_L^* = 0\) and \(e_H^* = 1\), then \(H\) strictly prefers 1 and \(L\) weakly prefers 0, so max info is an equilibrium. Due to the continuity of \(R\) and the continuity of \(l(g), l(b)\) in \(e_H^*, e_L^*\), by the Mean Value Theorem there exists \(\hat{e}_H \in (0, 1)\) such that \(A_H = \lambda[R(l(g)) - R(l(b))]\) when the receivers expect \(e_L = 0,\ e_H = \hat{e}_H\). Therefore there exists an equilibrium where \(e_L^* = 0,\ e_H^* = \hat{e}_H\). \(\hat{e}_H\) is unique, because \(R\) is strictly increasing, \(l(g)\) is strictly increasing in \(e_H^*\) and strictly decreasing in \(e_L^*\) and \(l(b)\) is strictly decreasing in \(e_H^*\) and strictly increasing in \(e_L^*\).

If \(A_L < \lambda[R(\bar{l}(g)) - R(\bar{l}(b))]\), then under expectations \(e_H^* = 1,\ e_L^* = 0\), \(L\) strictly prefers \(e = 1\). Under expectations \(e_H^* = 1,\ e_L^* = 1\), \(L\) strictly prefers \(e = 0\). By the Mean Value Theorem, there exists \(\hat{e}_L \in (0, 1)\) such that \(A_L = \lambda[R(l(g)) - R(l(b))]\) when the receivers expect \(e_L^* = \hat{e}_L,\ e_H^* = 1\). Again, \(\hat{e}_L\) is unique, so there is a unique interior \(e_L^*\) equilibrium. The reasoning in the previous paragraph showing the existence of a unique interior \(e_H^*\) equilibrium still holds.

**Lemma 9.** In the Poisson game, \(V_{\theta}(l_0)\) is finite. \(V_H(l_0) \geq V_L(l_0)\), with strict inequality if under the optimal \((e_L, e_H)\) starting at \(l_0\), the set of histories at which \(e_L > 0\) has positive probability.

**Proof.** \(V_{\theta}(l_0)\) is bounded above by \(\int_0^{\infty} \exp(-rt)R(\infty)dt = \frac{R(\infty)}{r} \in \mathbb{R}\) and below by \(\frac{R(-\infty)}{r} \in \mathbb{R}\).

\(V_H(l_0)\) is greater than the payoff to \(H\) from imitating an optimal strategy of \(L\). An optimal strategy gives \(L\) payoff \(V_L(l_0)\). \(H\) can imitate an optimal strategy of \(L\) at a lower cost, getting the same benefit, so the imitation payoff to \(H\) is greater than \(V_L(l_0)\). If the set of histories where \(e_L > 0\) has positive probability under the optimal strategy, then \(H\) can imitate \(L\) at a strictly lower cost, getting the same benefit.

**Proof of Lemma 2.** If \(e_L^* = e_H^*\), then the log likelihood ratio stays constant regardless of the occurrence or absence of signals. Then both types optimally
choose \( e_\theta = 0 \) to avoid the effort cost. This rules out \( e_L^* = e_H^* > 0 \) occurring in equilibrium.

If \( e_L^* > e_H^* \), then the log likelihood ratio drifts up in the absence of signals and jumps down after a signal. Consider a point \( \hat{l} \) in the region where \( e_L^* > e_H^* \). If type \( \theta \) takes action \( e = 0 \) forever, there will be no jumps, so the log likelihood ratio cannot fall below \( \hat{l} \). The log likelihood ratio may exit the \( e_L^* > e_H^* \) region at the upper boundary, but cannot drift back into it and, if \( e = 0 \), cannot jump back into it. This ensures \( V_\theta(\hat{l}) \geq \frac{R(\hat{l})}{r} \). To incentivize \( \theta \) to take costly \( e > 0 \), the value after a jump down to \( \hat{l} + \ln \frac{e_H^*}{e_L^*} \) must be strictly greater than \( V_\theta(\hat{l}) \).

Since \( R \) is strictly increasing, it is necessary for \( V_\theta(\hat{l} + \ln \frac{e_H^*}{e_L^*}) > \frac{R(\hat{l})}{r} \) that from some \( l < \hat{l} \) that is reached with positive probability from \( \hat{l} + \ln \frac{e_H^*}{e_L^*} \), there is a jump up to some \( l' > \hat{l} \). A jump up can only occur with positive probability in an interval \( (\bar{l}, \hat{l}) \) on which \( e_L^* < e_H^* \). For \( V_\theta(\hat{l} + \ln \frac{e_H^*}{e_L^*}) > \frac{R(\hat{l})}{r} \), it is necessary that in some \( (\bar{l}, \hat{l}) \) with \( \bar{l} < \hat{l} \), some \( l'' \in (\bar{l}, \hat{l}) \) satisfies \( V_\theta(l'') > V_\theta(\hat{l}) \).

Note that \( e_L^* < e_H^* \) implies \( e_L^* < 1 \). In an interval where \( e_L^* < 1 \), keeping expectations constant, \( V_L \) does not change when \( L \) chooses \( e = 0 \) instead of \( e = e_L^* \) in the whole interval. Since \( V_L \) does not change in that interval after the \( e = 0 \) substitution, no continuation values change, so \( V_L \) does not change outside the interval. With \( e = 0 \), \( L \) has zero probability of \( l \) jumping up. In an interval \( (\bar{l}, \hat{l}) \) in which \( e_L^* < e_H^* \), the drift of \( l \) never takes \( l \) above \( \hat{l} \). This establishes that \( V_L(l'') < V_L(\hat{l}) \) for every \( l'' \in (\bar{l}, \hat{l}) \). Therefore \( L \) cannot be incentivized to take \( e > 0 \) when \( e_L^* > e_H^* \) is expected.

Consider an interval \( (\bar{l}_1, \hat{l}_1) \) in which \( e_L^* = 0 < e_H^* < 1 \). Type \( H \) must be indifferent, so switching type \( H \)’s choice from \( e = e_H^* \) to \( e = 0 \) in the whole \( (\bar{l}_1, \hat{l}_1) \) does not change \( V_H \). If \( e = 0 \), then \( l \) drifts down deterministically to \( \bar{l}_1 \) and stops there forever. Compare \( l', l'' \in (\bar{l}_1, \hat{l}_1) \), with \( l' > l'' \). Starting at \( l' \) or \( l'' \) yields flow cost zero. Starting at \( l' \) yields initially a strictly higher flow benefit than starting at \( l'' \), and later (when \( \bar{l}_1 \) is reached) a weakly higher flow benefit. So \( V_H \) is strictly increasing in \( (\bar{l}_1, \hat{l}_1) \).

The jumps from \( (\bar{l}_1, \hat{l}_1) \) go to \( l = \infty \), due to \( e_H^* > e_L^* = 0 \). So if \( H \) is indifferent between \( e > 0 \) and \( e = 0 \) at some \( l^* \in (\bar{l}_1, \hat{l}_1) \), he is not indifferent at any \( l \in (\bar{l}_1, \hat{l}_1) \) s.t. \( l \neq l^* \). This rules out \( e_L^* = 0 < e_H^* < 1 \) occurring over intervals of positive length in equilibrium.

\( \square \)

\textit{Proof of Lemma 3.} If \( l = \infty \) or \( l = -\infty \), then clearly neither type will
take positive effort. By \( r > 0 \) and the continuity and boundedness of \( R \), for all \( \epsilon > 0 \) \( \exists \hat{l} \in (0, \infty) \) s.t. \( \frac{|R(\hat{l})-R_{\text{max}}|}{r} < \epsilon \). Type \( H \)'s cost of choosing \( e = 1 \) over a time interval \( \Delta \) is \( A_H \Delta \) and the benefit is bounded above by \( \frac{|R(\hat{l})-R_{\text{max}}|}{r} \left(1 - \exp(-\lambda \Delta)\right) \), so for all parameter values \( \exists \epsilon > 0 \) s.t. the optimal choice of \( H \) is \( e = 0 \).

Proof of Lemma 4. If \( e^*_L = e^*_H \) is expected by the receivers, then their log likelihood ratio does not respond to the presence or absence of signals. In that case both types of the sender will choose \( e = 0 \) to avoid the signalling cost.

\[ e^*_L(l) > e^*_H(l) \] implies that \( L \) weakly prefers \( e = 1 \) and \( H \) weakly prefers \( e = 0 \). If \( j(l) \) is in the pooling region, then \( V_H(j(l)) = V_L(j(l)) = \frac{R(j(l))}{r} \).

Due to \( V_H \geq V_L \), the jump in value \( \frac{R(j(l)) - V_\theta(l)}{r} \) after a signal is larger for \( L \). The cost of avoiding jumps is strictly larger for \( L \), because \( A_L > A_H \). It cannot be that \( L \) weakly prefers to avoid jumps and \( H \) weakly prefers to allow them.

If \( e^*_L(l) = 1 > e^*_H(l) \), then \( j(l) = \infty \). For all \( l < \infty \), \( V_\theta(l) < \frac{R(\infty)}{r} \), so both types would deviate to \( e = 0 \).

Next it is shown that the \( e^*_L > e^*_H \) region, denoted \((\hat{l}_1, \hat{l}_1)\), is bounded below. Suppose \( \hat{l}_1 = -\infty \). By indifference of \( L \), \( V_\theta \) is unchanged when \( e^*_L \) is replaced by 1 throughout \((\hat{l}_1, \hat{l}_1)\). Then for \( l \in (\hat{l}_1, \hat{l}_1) \), \( V_L(l) \) is bounded above by \( \frac{R(l) - A_L}{r} \). After a jump that goes outside \((\hat{l}_1, \hat{l}_1)\), taking \( e = 1 \) forever ensures the absence of jumps. If the jump is to pooling or to a region where \( e^*_L < e^*_H \), the value is bounded below by \( \frac{R(j(l)) - A_L}{r} \). Then at \( l \), \( L \) would deviate to \( e = 0 \).

If the jump starts and ends in \((\hat{l}_1, \hat{l}_1)\), then \( V_L(j(l)) > V_L(l) \), because \( e^*_L \) can be replaced by 1 without changing \( V_L \) in that region. The cost paid is the same starting from \( j(l) \) and \( l \): \( A_L \) forever. The benefit is strictly higher starting from \( j(l) \). So in \((\hat{l}_1, \hat{l}_1)\), \( L \) would deviate to \( e = 0 \).

To show that \( \hat{l}_1 < \infty \), recall that \( j(l) < \infty \) must hold for all \( l \in (\hat{l}_1, \hat{l}_1) \). Avoiding jumps is costly, so \( \exists \epsilon > 0 \) s.t. for \( L \) to optimally choose \( e > 0 \), it is necessary that \( V_L(j(l)) - V_L(l) < -\epsilon \). If \( \hat{l}_1 = \infty \), then \((\hat{l}_1, \hat{l}_1)\) must contain the infinite sequence \( j(l), j(j(l)), j^3(l), \ldots \). But \( V_L \) is bounded above and below, so eventually \( V_L(j(l)) - V_L(l) < -\epsilon \) must be violated.

The preceding paragraph rules out a sequence \( j(l), j(j(l)), j^3(l), \ldots \) contained in the \( e^*_L > e^*_H \) region. This implies that any interval on which \( e^*_L > e^*_H \) must contain an \( \hat{l} \) s.t. \( j(\hat{l}) \) is not in that interval. \qed
Proof of Lemma 5. If \( e^*_H(l) = e^*_L(l) \), then the signal is statistically uninformative about the type, so the log likelihood ratio does not respond to the signal. Given this, both types will optimally choose \( e_\theta(l) = 0 \). Therefore the pooling equilibrium, in which \( e^*_H(l) = e^*_L(l) = 0 \) for all \( l \), exists for all parameter values and in equilibrium, it cannot be that \( e^*_L(l) = e^*_H(l) > 0 \) for any \( l \).

A higher expected signal is more costly to both types. If \( e^*_L > e^*_H \), then based on (6), \( l \) falls in response to a higher signal. The benefit of signalling only depends on the log likelihood ratio, with a higher \( l \) giving a higher benefit. So if \( e^*_L(l) > e^*_H(l) \) is expected, then both types optimally choose \( e_\theta(l) = 0 \). \( \square \)

Proof of Lemma 6. Due to the boundedness of \( R(l) \) and \( e_\theta \), discounting ensures that \( V_\theta \) is finite—even without the expectation, the integral in the definition of \( V_\theta \) is finite for any path of \( l \) and any control \( e_\theta \).

It is clear that \( V_H > V_L \) in the signalling region, because \( H \) can follow \( L \)'s strategy at a strictly lower cost than \( L \). Outside the signalling region, \( V_H(l) = V_L(l) = R(l) \).

To prove \( V_\theta \) is strictly increasing, a standard coupling argument is used. Consider two diffusion processes: the \( l \) process with optimal effort starting from \( l_1 \) and the \( l \) process under zero effort starting from \( l_2 > l_1 \). Call the former process \( l^\ast \) and the latter \( l^0 \). Define the stopping time \( \tau^\ast = \inf \{ t > 0 : l^0_t - l^\ast_t = 0 \} \). The receivers expect the optimal strategy in both cases.

Starting at \( l_2 \), the strategy \( s = \text{"play 0 until } \tau^\ast \text{ and the optimal strategy thereafter"} \) yields a weakly lower payoff than \( V_\theta(l_2) \), the payoff to the optimal stationary strategy starting from \( l_2 \). This holds even though \( s \) is not stationary, because if the receivers expect a stationary strategy, then among the optimal strategies for the sender there is a stationary one. The argument is standard—the competitive receivers always play a static best response, which depends on their belief about the type, but not the sender’s strategy, so if at some \( l \), a sender action \( \hat{e} \) is optimal at one point in time, then \( \hat{e} \) is optimal at that \( l \) at another point in time.

Starting at \( l_2 \), the strategy \( s \) yields a strictly higher payoff than \( V_\theta(l_1) \), the payoff to the optimal strategy starting from \( l_1 \). This is because the revenue \( R(l^0) \) is strictly higher than \( R(l^\ast) \) before \( \tau^\ast \) and the same in expectation after \( \tau^\ast \). The cost of \( l^0 \) is zero while the cost of \( l^\ast \) is positive before \( \tau^\ast \). The costs of the two strategies are the same in expectation after \( \tau^\ast \). Overall,
$V_\theta(l_2) > V_\theta(l_1)$ for $l_1, l_2$ in the signalling region.

If both $l_1, l_2$ are outside the signalling region, then since $R$ was assumed strictly increasing, the payoffs are ordered $V_\theta(l_2) = \frac{R(l_2)}{r} > \frac{R(l_1)}{r} = V_\theta(l_1)$. If $l_2$ is above the signalling region while $l_1$ is in the signalling region, then the expected benefit is strictly higher from $l_2$ onwards and the expected cost is the lowest possible from $l_2$ onwards, so $V_\theta(l_2) > V_\theta(l_1)$. If $l_2$ is in the signalling region while $l_1$ is below the signalling region, then $V_\theta(l_2)$ is higher than the payoff to the strategy of taking zero effort forever starting from $l_2$. The cost of this strategy is the same as the cost of the optimal strategy from $l_1$ onwards, while the benefit is strictly greater, so again $V_\theta(l_2) > V_\theta(l_1)$. \hfill \Box

Proof of Proposition 7. $L$ takes no effort in either equilibrium, so the flow cost is the same in both cases. The flow benefit $R$ is increasing in the log likelihood ratio $l$. In the pooling equilibrium $l$ stays constant forever, while in an equilibrium with nonempty $(l_2, \bar{l}_2)$, $L$ expects $l$ to strictly decrease. With a concave $R$, there is no benefit from the noise in the $l$ process. This establishes $V_L(l) < \frac{R(l)}{r}$.

For $l \in (l_2, \bar{l}_2) \cup (l_1, \bar{l}_1)$, it follows from the above that $V_{L_1}(l) = \frac{R(l_1)}{r} > V_L(l)$. Since $(l_1, \bar{l}_1)$ is a proper subset of $(l_2, \bar{l}_2)$, at least one of $V_{L_1}(l_1) > V_L(l_1)$, $V_{L_1}(\bar{l}_1) > V_L(\bar{l}_1)$ holds (the other may be an equality).

From any point in $(l_1, \bar{l}_1)$, the log likelihood ratio process has positive probability of hitting $l_1$ and positive probability of hitting $\bar{l}_1$. The flow cost to $L$ is zero in all interval equilibria for all $l$. For the same $l$, the flow benefit to $L$ is the same in all interval equilibria. The distribution over paths of $l$ up to hitting $l_1$ or $\bar{l}_1$ starting from $l_0 \in (l_1, \bar{l}_1)$ is the same in the two equilibria with signalling regions $(l_1, \bar{l}_1)$ and $(l_2, \bar{l}_2)$, because in both equilibria in the region $(l_1, \bar{l}_1)$, $H$ takes action 1 and $L$ takes 0. Therefore the continuation value comparisons $V_{L_1}(l_1) \geq V_L(l_1)$ and $V_{L_1}(\bar{l}_1) \geq V_L(\bar{l}_1)$, at least one of which is strict, determine the payoff comparison $V_{L_1}(l) > V_L(l)$ for any $l \in (l_1, \bar{l}_1)$. \hfill \Box

Proof of Lemma 8. For any signalling region $(l, \bar{l})$, the solutions of the ODEs are differentiable at least as many times as $R$ on $(l, \bar{l})$ and continuous on $[l, \bar{l}]$. Since $R$ was assumed twice continuously differentiable, $v_L$ and $v_H$ are as well. Given the signalling region, $v_H, v_L$ are bounded for any path of $l$ and control $e_\theta$. Therefore $v_H(l)$, $v_L(l)$ are integrable in the probability law of the $l$ process that starts from $l_0$ and is controlled by $e_\theta$, uniformly over Markov controls $e_\theta$. So by Theorem 11.2.2 of Øksendal (2010), $v_L, v_H$ coincide with the value functions $V_L, V_H$. 37
Under the previous conditions, Theorem 11.2.3 of Øksendal (2010) shows that the optimal Markov control does as well as the optimal nonanticipating control, so if the receivers expect Markov strategies, then both types of the sender have a Markov best response among their best responses. This does not imply that the payoffs of all non-Markov equilibria can be attained with Markov equilibria, since in a non-Markov equilibrium the receivers expect non-Markov strategies.

\[ \square \]

### B One-shot noisy signalling with quadratic cost

Some of the features of the one-shot noisy signalling model are driven by the linear cost of effort, which for some applications is unrealistic. Signalling may feature increasing marginal cost for the usual reason: first the easiest ways to signal are used, then if these are exhausted, more costly methods must be employed. The most tractable convex cost is quadratic, which is assumed in this section.

The sender’s action $e \in [0, 1]$ generates a signal $g$ or $b$, with probability of $g$ being $\Pr(g|e) = \lambda e + \frac{1-\lambda}{2}$, with $\lambda \in (0, 1)$. Denote the pure action the receivers expect type $\theta$ to take by $e^*_\theta$. Mixed actions add nothing, since the signal distribution from a mixed action can be replicated at strictly lower cost by a pure action. The best response to any beliefs by the receivers is pure.

The receivers observe the signal, but not the effort and update their log likelihood ratio using Bayes’ rule. Regardless of the action taken, both signals have positive probability, so Bayes’ rule applies after both signals and there are no off-path information sets. Denote by $l(x)$ the updated log likelihood ratio after signal $x$. Then

\[
l(x) = l_0 + \ln \frac{\Pr(x|e_H^*)}{\Pr(x|e_L^*)}.
\]

(8)

Type $\theta$ sender’s utility from action $e$ and receivers’ log likelihood ratio $l$ is $\frac{\exp(l)}{1+\exp(l)} - \frac{A_\theta}{2} e^2$, with $A_L > A_H$. The sender’s benefit from the receivers’ belief is thus equal to the belief. Given a best response by the receivers to the realized signal and the sender’s equilibrium play, the sender’s expected
utility from action $e$ is

$$u_\theta = -\frac{A_\theta}{2} e^2 + \lambda e \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)$$

(9)

$$+ \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} + \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))}.$$  

(10)

**Definition 5.** A perfect Bayesian equilibrium is $(e^*_H, e^*_L, l(g), l(b))$ such that

(a) given $e^*_L, e^*_H$, the log likelihood ratios $l(g), l(b)$ are obtained from (8),

(b) given $l(g), l(b)$, for $\theta = H, L$, $e^*_\theta$ maximizes (9) over $e$.

If $e^*_H \leq e^*_L$, then both types of sender will choose $e_\theta = 0$, because there is no benefit to signalling, but there is a cost. Therefore pooling at $e^*_L = e^*_H = 0$ is an equilibrium for all parameter values.

If the expected actions of the types satisfy $e^*_H > e^*_L$, then $l(g) > l(b)$. The marginal cost of signalling is zero at $e = 0$, the marginal benefit is

$$\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)$$

everywhere. Due to this, both types will choose $e_\theta > 0$. Given the expected equilibrium actions, the chosen action of type $\theta$ must satisfy the FOC

$$A_\theta e_\theta = \lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right).$$

The FOC is necessary and sufficient for a global maximum due to the linear benefit and the quadratic cost. The FOC already allows the comparison of the signalling efforts and expected utilities of the types, formalized in the following proposition.

**Proposition 10.** The equilibrium signalling efforts of types $H$ and $L$ satisfy $e^*_H = \frac{A_L}{A_H} e^*_L$ and the expected utilities $u_H, u_L$ satisfy

$$u_H - \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} - \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))} = \frac{A_L}{A_H} \left[ u_L - \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} - \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))} \right].$$
Proof. From the FOCs,

\[
e^*_H = \frac{1}{A_H} \lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)
\]

\[
= \frac{A_L}{A_H A} \lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right) = \frac{A_L}{A_H} e^*_L.
\]

Substituting this relation into the expected utilities, we get

\[
u_H = -\frac{A_H}{2} \left( \frac{\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)}{A_H} \right)^2
\]

\[
+ \frac{\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right) \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)}{A_H} \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} + \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))}
\]

\[
= \frac{\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)}{2A_H} \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} + \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))},
\]

and

\[
u_L = \frac{\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)}{2A_H} \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} + \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))},
\]

which gives the desired relation between the utilities of the types. \(\square\)

Proposition 10 holds for both informative and pooling equilibria. Under pooling, \(e^*_H = e^*_L = 0\) and \(l(g) = l(b) = l_0\), so \(u_H = u_L = \frac{\exp(l_0)}{1 + \exp(l_0)}\). Under separation, it is clear that \(u_H \geq u_L\).

To find the equilibrium actions, equate the chosen and the expected action, \(e^*_H = e^*_L\) in the FOCs. One solution is \(e^*_H = e^*_L = 0\). There are two other solutions. The one for which there exist parameter values such that \(e^*_H, e^*_L \in (0, 1)\) is

\[
e^*_H = \frac{A_L A_H \lambda - \sqrt{A_L A_H} \sqrt{A_L A_H - 4(A_L - A_H) \frac{\exp(l_0)}{(1 + \exp(l_0))^2}} \lambda^2}{2A_H \left[ A_H \frac{1}{1 + \exp(l_0)} + A_L \frac{\exp(l_0)}{1 + \exp(l_0)} \right] \lambda}, \tag{11}
\]

\[
e^*_L = \frac{A_L A_H \lambda - \sqrt{A_L A_H} \sqrt{A_L A_H - 4(A_L - A_H) \frac{\exp(l_0)}{(1 + \exp(l_0))^2}} \lambda^2}{2A_L \left[ A_H \frac{1}{1 + \exp(l_0)} + A_L \frac{\exp(l_0)}{1 + \exp(l_0)} \right] \lambda}.
\]
Sufficient conditions for the existence of an informative equilibrium with interior $e^*_L, e^*_H$ are presented in the following proposition.

**Proposition 11.** A separating equilibrium with $e^*_H, e^*_L \in (0, 1)$ exists if

$$
\frac{4A_L \exp(l_0) \lambda^2}{A_L(1 + \exp(l_0))^2 + 4\lambda^2 \exp(l_0)} \leq A_H < \frac{A_L \lambda^2 \exp(l_0)}{A_L \frac{1 - \lambda^2}{4}(1 + \exp(l_0))^2 + \lambda^2 \exp(l_0)}.
$$

**Proof.** If $e^*_H, e^*_L$ satisfy (11) and $1 \geq e^*_H > e^*_L \geq 0$, then they are equilibrium strategies, because given the expected strategies, both types are optimizing and the chosen strategies equal the expected strategies.

In (11), the term under the square root is nonnegative iff

$$
A_H(A_L + 4 \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2) \geq 4A_L \frac{\exp(l_0)}{(1 + \exp(l_0))^2}\lambda^2,
$$

which holds iff $A_H \geq \frac{4A_L \frac{\exp(l_0)}{(1 + \exp(l_0))^2}\lambda^2}{A_L + 4 \frac{\exp(l_0)}{(1 + \exp(l_0))^2}}$. The fraction is always less than one and positive, because $\lambda \in (0, 1)$ and for any $l \in \mathbb{R}$, $\frac{\exp(l_0)}{(1 + \exp(l_0))^2} \in (0, \frac{1}{4})$.

Using Mathematica to simplify (11), it turns out that $e^*_H < 1$ always holds. Since $e^*_H = \frac{A_L}{A_H} e^*_L$, the inequality $e^*_L < e^*_H$ is implied by $e^*_L > 0$. Again using Mathematica to simplify (11), it is found that $e^*_L > 0$ iff

$$
\frac{4A_L \exp(l_0) \lambda^2}{A_L(1 + \exp(l_0))^2 + 4\lambda^2 \exp(l_0)} \leq A_H < \frac{A_L \lambda^2 \exp(l_0)}{A_L \frac{1 - \lambda^2}{4}(1 + \exp(l_0))^2 + \lambda^2 \exp(l_0)}.
$$

The comparative statics results are given in Proposition 12 below. The derivatives of $e^*_L$ with respect to $A_L$ or $A_H$ do not have clear signs.

**Proposition 12.** If the condition in Proposition 11 is satisfied, then the efforts of both types in an informative equilibrium increase in $\lambda$, the precision of the signal. $e^*_H$ increases in $A_L$ and decreases in $A_H$. For $l_0 \geq 0$, $e^*_H$ and $e^*_L$ decrease in $l_0$.

**Proof.** $rac{\partial e^*_H}{\partial \lambda} = \frac{A_H A^2_L}{2[A_H \frac{1}{1 + \exp(l_0)} + A_L \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2 \sqrt{A_H A_L[A_H A_L - 4(A_H - A_H) \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2]}} > 0$.

Based on Proposition 10, $\frac{\partial e^*_L}{\partial \lambda} = \frac{A_H}{A_L} \frac{\partial e^*_L}{\partial \lambda}$, which is also positive.

Rewriting the $e^*_H$ expression as

$$
e^*_H = \frac{A_H \lambda - \sqrt{A_H^2 - 4A_H(1 - \frac{A_H}{A_L}) \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2}}{2A_H[A_L(1 + \exp(l_0)) + \frac{\exp(l_0)}{1 + \exp(l_0)}]}. $$

41
it is clear that the denominator is decreasing in $A_L$. In the numerator, $(1 - \frac{A_H}{A_L})$ is increasing in $A_L$, so the square root is decreasing in $A_L$. The numerator is increasing in $A_L$ and the denominator decreasing, so $e^*_H$ is increasing in $A_L$.

Rewriting the $e^*_H$ expression as

$$e^*_H = \frac{A_L \lambda - \sqrt{A_L^2 - 4A_L(\frac{A_L}{A_H} - 1)\frac{\exp(l_0)}{1 + \exp(l_0)}\frac{\lambda^2}{\lambda^2}}}{2[AH^1\frac{1}{1 + \exp(l_0)} + A_L^1\frac{\exp(l_0)}{1 + \exp(l_0)}]}$$

the denominator is increasing in $A_H$. In the numerator, the square root is increasing in $A_H$, so the numerator and therefore $e^*_H$ are decreasing in $A_H$.

Increasing $l_0$ increases the denominator of the $e^*_H, e^*_L$ expressions, since $A_L > A_H$. Increasing $|l_0|$ decreases the numerator, since the numerator is increasing in $\frac{\exp(l_0)(1 + \exp(l_0))^2}{(1 + \exp(l_0))^2}$ and $\frac{\exp(l_0)(1 + \exp(l_0))^2}{(1 + \exp(l_0))^2}$ is globally strictly concave with the maximum at $l_0 = 0$. So for $l_0 \geq \frac{1}{2}$, it is clear that $e^*_H$ and $e^*_L$ are decreasing in $l_0$.

The expected utility of $L$ is higher in the pooling equilibrium than in the informative, because in an informative equilibrium $L$ pays an effort cost and expects $l$ to go down. The expected utility of $H$ may be higher in pooling or the informative equilibrium, but the expressions defining the region where $H$’s pooling payoff is higher are too long to present here.

Changing the game to incorporate a type-dependent drift $d$ in the signal, so that $H$ is more likely to generate signal $g$, eliminates the pooling equilibrium. If pooling is expected, then seeing $g$, the receivers update the probability of the $H$ type to $l(g) = l_0 + \ln \frac{1 - \lambda + 2d}{1 - \lambda} > l_0$ and seeing $b$, they update to $l(b) < l_0$. To ensure the probabilities are in $[0, 1]$ after any expected sender action, it must be that $d \leq \frac{1 - \lambda}{2}$. The marginal cost of signalling is zero at $e = 0$, the marginal benefit is $\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)$, which is positive for any $d > 0$. Due to this, both types will choose $e_\theta > 0$ even when pooling is expected.

An arbitrarily small positive type-dependent drift leading to a unique equilibrium is not just a feature of the quadratic cost. Any convex cost with zero derivative at zero effort exhibits the same effect when the signal structure is such that the marginal benefit to effort is linear and positive.
C  Both good and bad news in the Poisson signalling model

If the Poisson rate of good news signals occurring is \(\lambda_g e\), the rate of bad news is \(\lambda_b(1 - e)\) and the expected actions of the types are \(e^*_L\) and \(e^*_H\), then the Bellman equation for type \(\theta\) is

\[
rv_\theta(l) = R(l) + (\lambda_b - \lambda_g)(e^*_H(l) - e^*_L(l))V'_\theta(l) \\
+ \max_{e} \{\frac{\lambda_g e}{\lambda_g + \lambda_b} \left(V_\theta(l + \ln \frac{e^*_H(l)}{e^*_L(l)}) - V_\theta(l)\right) \\
+ \lambda_b(1 - e) \left(V_\theta(l + \ln \frac{1 - e^*_H(l)}{1 - e^*_L(l)}) - V_\theta(l)\right) - A_\theta e\}.
\]

Type \(\theta\) chooses \(e_\theta = 1\) if

\[
\lambda_g V_\theta(l + \ln \frac{e^*_H(l)}{e^*_L(l)}) - \lambda_g V_\theta(l) - \lambda_b V_\theta(l + \ln \frac{1 - e^*_H(l)}{1 - e^*_L(l)}) + \lambda_b V_\theta(l) - A_\theta > 0.
\]

As in the main text, the focus is on interval equilibria in which type \(L\) always chooses \(e = 0\) and type \(H\) chooses 1 if \(l \in (\underline{l}, \bar{l})\) and 0 elsewhere. In the signalling region, the jump after a good signal is to \(l = \infty\) and the jump after a bad signal to \(l = -\infty\). When \(e^*_H = 1\), \(e^*_L = 0\) is expected by the receivers, then the Bellman equation becomes

\[
rV_\theta(l) = R(l) + (\lambda_b - \lambda_g)\frac{R_l}{r} - \lambda_b V_\theta(l) \\
+ \max_{e} e \left\{\frac{\lambda_g R_{max}}{r} - \lambda_b \frac{R_{min}}{r} + (\lambda_b - \lambda_g) V_\theta(l) - A_\theta\right\}.
\]

Type \(\theta\) chooses \(e_\theta = 1\) if \(\lambda_g \frac{R_{max}}{r} - \lambda_b \frac{R_{min}}{r} + (\lambda_b - \lambda_g) V_\theta(l) - A_\theta > 0\).

If \(\lambda_b > \lambda_g\), then in the absence of a signal, \(l\) drifts up and eventually reaches \(\bar{l}\). This implies \(\lim_{l \rightarrow \bar{l}} V_\theta(l) = \frac{R(\bar{l})}{r}\), as in the bad news case. If \(V_L\) is increasing, then \(\max \bar{l}\) is determined by \(\lambda_g \frac{R_{max}}{r} - \lambda_b \frac{R_{min}}{r} + (\lambda_b - \lambda_g) \frac{R(\bar{l})}{r} - A_L \leq 0\), i.e. \(\bar{l} \leq R^{-1} (\frac{\lambda_b R_{min} - \lambda_g R_{max}}{\lambda_b - \lambda_g} + \frac{r}{A_L r})\).

If \(\lambda_b < \lambda_g\), then in the absence of a signal, \(l\) drifts down and \(\lim_{l \rightarrow \underline{l}} V_\theta(l) = \frac{R(\underline{l})}{r}\), as in the good news case. If \(V_L\) is increasing, then \(\min \underline{l}\) is determined by \(\lambda_g \frac{R_{max}}{r} - \lambda_b \frac{R_{min}}{r} + (\lambda_b - \lambda_g) \frac{R(\underline{l})}{r} - A_L \leq 0\), i.e. \(\underline{l} \geq R^{-1} (\frac{\lambda_g R_{max} - \lambda_b R_{min}}{\lambda_g - \lambda_b} - \frac{r}{A_L r})\).

43
Substituting $e_H = 1$ and $e_L = 0$ into the Bellman equations of the types, these become the ODEs

$$rV_H = R(l) + (\lambda_b - \lambda_g)V_H'(l) + \lambda_g \frac{R_{\text{max}}}{r} - \lambda_g V_H(l) - A_H,$$

$$rV_L = R(l) + (\lambda_b - \lambda_g)V_L'(l) + \lambda_b \frac{R_{\text{min}}}{r} - \lambda_b V_L(l).$$

The boundary condition depends on whether $\lambda_b > \lambda_g$ or vice versa. If $\lambda_b > \lambda_g$, then the boundary condition is $V_\theta(l) = \frac{R(l)}{r}$ and the solutions are

$$V_H(l) = \exp(-(r + \lambda_g) \frac{l - l}{\lambda_b - \lambda_g} \frac{R(l)}{r})$$

$$+ \int_l^T \left[ \frac{R(z) - A_H}{\lambda_b - \lambda_g} + \frac{\lambda_g R_{\text{max}}}{r(\lambda_b - \lambda_g)} \right] \exp(-(r + \lambda_g) \frac{z - l}{\lambda_b - \lambda_g}) dz,$$

$$V_L(l) = \exp(-(r + \lambda_b) \frac{l - l}{\lambda_b - \lambda_g} \frac{R(l)}{r})$$

$$+ \int_l^T \left[ \frac{R(z)}{\lambda_b - \lambda_g} + \frac{\lambda_b R_{\text{min}}}{r(\lambda_b - \lambda_g)} \right] \exp(-(r + \lambda_b) \frac{z - l}{\lambda_b - \lambda_g}) dz.$$

If $\lambda_b < \lambda_g$, the boundary condition is $V_\theta(l) = \frac{R(l)}{r}$ and the solutions are

$$V_H(l) = \exp(-(r + \lambda_g) \frac{l - l}{\lambda_g - \lambda_b} \frac{R(l)}{r})$$

$$+ \int_l^T \left[ \frac{R(z) - A_H}{\lambda_g - \lambda_b} + \frac{\lambda_g R_{\text{max}}}{r(\lambda_g - \lambda_b)} \right] \exp(-(r + \lambda_g) \frac{l - z}{\lambda_g - \lambda_b}) dz,$$

$$V_L(l) = \exp(-(r + \lambda_b) \frac{l - l}{\lambda_g - \lambda_b} \frac{R(l)}{r})$$

$$+ \int_l^T \left[ \frac{R(z)}{\lambda_g - \lambda_b} + \frac{\lambda_b R_{\text{min}}}{r(\lambda_g - \lambda_b)} \right] \exp(-(r + \lambda_b) \frac{l - z}{\lambda_g - \lambda_b}) dz.$$

In the knife-edge case of $\lambda_g = \lambda_b$, the log likelihood ratio stays constant at $l_0$ in the absence of signals, so in the signalling region, $V_H(l_0) = \frac{R(l_0) - A_H}{r + \lambda_g} + \frac{\lambda_g R_{\text{max}}}{r(r + \lambda_g)}$ and $V_L(l_0) = \frac{R(l_0)}{r + \lambda_b} + \frac{\lambda_b R_{\text{min}}}{r(r + \lambda_b)}$. 

44
D Interior effort equilibria in the Brownian signalling model

In the one-shot game interior effort equilibria exist for some parameter values, so a natural question is whether this is also the case in the Brownian signalling model.

Given expected strategies $e^*_L, e^*_H$ and the chosen strategy $e_\theta$, the log likelihood ratio process satisfies

$$dl_t = \sigma^{-2}(e^*_H - e^*_L)(e_\theta - \frac{1}{2}e^*_H - \frac{1}{2}e^*_L)dt + \frac{e^*_H - e^*_L}{\sigma}dB_t,$$

(12)
due to $dX_t = e_t dt + \sigma dB_t$ and (6).

If the expected strategies $e^*_L, e^*_H$ are Lipschitz in $l$, then by Theorem 3.1 of Touzi (2013), Eq. (12) has a unique strong solution. In that case the log likelihood ratio process is well defined. The Lipschitz condition must be verified after solving for the optimal strategies.

Since $H$ has a lower cost for any $e > 0$ than $L$, while the benefit from the future path of the receivers’ log likelihood ratio is the same, only one of the types can be taking interior effort at a given $l$. If $H$ takes $e \in (0, 1)$ and is therefore indifferent between $e = 0$ and $e = 1$, $L$ strictly prefers $e = 0$. If $L$ takes interior effort, then $H$ strictly prefers $e = 1$.

Following the same solution procedure as with pure strategies, the HJB equation of type $\theta$ is

$$rv_\theta(l) = R(l) + \frac{1}{2}v''_\theta(l)\sigma^{-2}(e^*_H(l) - e^*_L(l))^2$$

$$+ \max_{e \in [0, 1]} \left\{-A_\theta e + v'_\theta(l)\sigma^{-2}(e^*_H(l) - e^*_L(l))[e(l) - \frac{1}{2}e^*_H(l) - \frac{1}{2}e^*_L(l)]\right\}.$$

Outside the signalling region $(\bar{l}, \bar{l})$, both types always take action 0. If the equilibrium features $e^*_H \in (0, 1)$ and $e^*_L = 0$, then it must be that $e^*_H(l) - e^*_L(l) = \frac{\Delta_H \sigma^2}{v''_H(l)}$ (for $H$ to be indifferent between $e = 0$ and 1) and $v'_L(l)(e^*_H(l) - e^*_L(l)) \leq A_L \sigma^2$ (the IC$_L$ constraint). Using the indifference condition and $e^*_L = 0$, IC$_L$ reduces to $\frac{v'_H(l)}{v''_H(l)} \leq \frac{A_H}{A_H}$. IC$_L$, as well as the feasibility constraint $e^*_H(l) = \frac{\Delta_H \sigma^2}{v''_H(l)} \in [0, 1]$ must be checked after solving for candidate equilibrium strategies and value functions. The feasibility constraint is equivalent to $v'_H(l) \geq A_H \sigma^2$. 

45
Equating $e_\theta$ and $e^*_\theta$ and substituting $\frac{A_H \sigma^2}{v^*_H(l)}$ for $e^*_H$ and 0 for $e^*_L$, the following pair of second order ODEs obtains. The first ODE is nonlinear. The second is linear, but with variable coefficients.

$$rv_H(l) = R(l) - \frac{A^2_H \sigma^2}{2v'_H(l)} + \frac{A^2_H \sigma^2 v'_H(l)}{2(v_H(l))^2},$$

$$rv_L(l) = R(l) - \frac{A^2_H \sigma^2 v'_L(l)}{2(v_H(l))^2} + \frac{A^2_H \sigma^2 v''_L(l)}{2(v_H(l))^2}.$$

The boundary conditions are $v_\theta(l) = \frac{R(l)}{r}$ and $v_\theta(l) = \frac{R(l)}{r}$.

Since the $H$ type equation does not depend on the variables of $L$, it can be solved first. After that, $v'_H$ can be substituted into the $L$ type equation. It is clear from the similarity of the equations that the solution to the $L$ equation is $v_L = v_H$. If $R$ is concave on the signalling region, then both types are worse off in this interior effort equilibrium than in pooling. In both equilibria, $v_L = v_H$, but with interior effort, a signalling cost is paid. The concavity of $R$ implies that the variance in the posterior of the receivers does not benefit the sender.

Due to $v_L = v_H$, the IC$_L$ constraint $\frac{v'_L(l)}{v'_H(l)} \leq \frac{A_L}{A_H}$ is always satisfied. The set of possible signalling regions is limited only by the feasibility constraint $v'_H(l) \geq A_H \sigma^2$.

Unfortunately the ODEs resulting from mixed equilibria do not belong to a standard class for which simple solution methods exist. They must be solved numerically. Figure 8 shows $v_H$ and $e^*_H$ for $A_H = 0.1$, $r = \sigma^2 = 1$ and signalling region $l = -0.5, \tilde{l} = 0.5$.

If the equilibrium features $e^*_L \in (0, 1)$ and $H$ choosing 1, then the indifference condition of $L$ is $e^*_H - e^*_L = \frac{A_L \sigma^2}{v'_L(l)}$ and IC$_H$ is $v'_H(l)(e^*_H - e^*_L) \geq A_H \sigma^2$.

Using the indifference condition, the latter reduces to $\frac{v'_H(l)}{v'_L(l)} \geq \frac{A_H}{A_L}$, which is slack near $l$ due to $v_H \geq v_L$, but may bind elsewhere. IC$_H$ as well as the feasibility constraint $\frac{A_L \sigma^2}{v'_L(l)} \in [0, 1]$ must be checked after solving for candidate equilibrium strategies and value functions. The feasibility constraint is equivalent to $v'_L \geq A_L \sigma^2$.

Equating $e_\theta$ and $e^*_\theta$ and substituting $1 - \frac{A_L \sigma^2}{v'_L(l)}$ for $e^*_L$ and 1 for $e^*_H$, the
following pair of second order ODEs obtains.

\[
rv_H(l) = R(l) - A_H + \frac{A_H^2 \sigma^2 v_H'(l)}{2v_H'(l)} + \frac{A_H^2 \sigma^2 v_H''(l)}{2(v_H'(l))^2},
\]

\[
rv_L(l) = R(l) - A_L + \frac{A_L^2 \sigma^2 v_L'(l)}{2v_L'(l)} + \frac{A_L^2 \sigma^2 v_L''(l)}{2(v_L'(l))^2}.
\]

The boundary conditions are the usual \( v_\theta(l) = \frac{R(l)}{r} \) and \( v_\theta(\bar{l}) = \frac{R(\bar{l})}{r} \). An example of the solution of the ODEs and the optimal strategy for \( L \) is shown in Figure 9. The shapes of \( v_H, v_L \) and \( e^*_L \) for other equilibrium signalling regions are similar. Widening the signalling region will eventually violate the feasibility constraint \( v_L' \geq A_L \sigma^2 \) at \( l \). This means \( L \) does not want to signal close to the lower end of the signalling region, where reaching \( \bar{l} \) is unlikely due to the downward drift of \( l \) that \( L \) expects.

The payoff of \( H \) in the signalling region is strictly greater than that of \( L \), because \( L \) is indifferent between mixing and setting \( e^*_L = 1 \) to imitate \( H \), but the cost of \( e = 1 \) is lower for \( H \) than for \( L \).

The signalling regions that satisfy IC\(_H\) and the feasibility constraint for the equilibrium where \( L \) takes interior effort are depicted in Figure 10. The binding constraint is feasibility—the set of regions satisfying it is a proper subset of the set of regions satisfying IC\(_H\). Compared to equilibria in which \( e^*_\theta \in \{0, 1\} \) at the same parameter values, the signalling regions sustainable in equilibria where \( e^*_L \in (0, 1) \) are much narrower. This is because the difference between the expected efforts of the types is smaller, so the benefit to signalling is smaller, and the cost for \( L \) to signal is higher than that for \( H \).

E Quadratic cost in the Brownian signalling model

The setup in this section resembles the linear-cost Brownian model, but the results differ in an important way due to unusual mathematical behaviour of the model. Given a signalling region, a continuum of effort profiles of the types constitute equilibria. Perturbing the quadratic cost model by changing the cost function slightly or by making the signal depend on the type as well as the action removes the linear dependence of the FOCs that makes the continuum of equilibria arise for a fixed signalling region.
Time is continuous and the horizon is infinite. There is an infinitely lived strategic sender who is one of two types, $H$ or $L$, with initial log likelihood ratio $l_0$. Both types of sender have discount rate $r$ and a bounded action set $[0,1]$ at each $t$. The action (signalling effort) of type $\theta$ sender at time $t$ is denoted $e_{\theta t}$.

The sender’s effort $e_{\theta t}$ determines the drift a signal process $(X_t)_{t \geq 0}$, which is subject to Brownian noise $B_t$. The receivers at time $t$ observe $(X_\tau)_{\tau \leq t}$ and, given the expected strategies of the types of the sender, update the log likelihood ratio to $l_t$ using Bayes’ rule.

The flow utility of a sender of type $\theta$ is $R(l) - \frac{1}{2}e_{\theta}^2$, so the cost of effort is quadratic. Both types of sender have the same benefit $R(l)$ from the receivers’ log likelihood ratio, but the $H$ type has a lower cost of effort.

The benefit function $R$ is assumed bounded, with $R'$ and $R''$ bounded and continuous on $[0,1]$.

Overall, the game consists of two stochastic control problems, one for each type, related by an equilibrium condition. The solution procedure can be divided in two parts. First, given the equilibrium strategies the receivers expect from the two types of sender, a standard stochastic control problem is solved for each type to find the best response. Second, the chosen strategy is set equal to the expected strategy for each type and the strategies are solved for. The unusual mathematical features arise in the second part, where many strategies satisfy the equilibrium condition.

A Markov stationary strategy of the sender is a pair of measurable functions $e^*_H, e^*_L : [-\infty, \infty] \rightarrow [0,1]$. The state variable is the receivers’ log likelihood ratio $l$. Again, the log likelihood ratio process satisfies

$$dl_t = \sigma^{-2}(e^*_H - e^*_L)(e_\theta - \frac{1}{2}e^*_H - \frac{1}{2}e^*_L)dt + \frac{e^*_H - e^*_L}{\sigma}dB_t,$$

(13)
due to $dX_t = e_t dt + \sigma dB_t$ and (6). The updating rule is also applicable in the pooling region: if $e^*_H = e^*_L$, then $dl = 0$ and belief does not change. In the signalling region it must be that $e^*_H > e^*_L$, otherwise belief would fall or remain constant in the costly signal, which would make both types deviate to $e_\theta = 0$. As a sufficient condition for the belief process to be well-defined and unique, assume the receivers expect strategies $e^*_H, e^*_L$ that are Lipschitz in $l$. It will turn out that the sender has a best response that is Lipschitz.

**Lemma 13.** If the receivers expect strategies $e^*_H, e^*_L$ that are Lipschitz in $l$, then there is a unique belief process satisfying (13).
Proof. If for any control $e_\theta$, the drift and variance in (13) are bounded and Lipschitz in belief, then by Theorem 3.1 of Touzi (2013), Eq. (13) has a unique strong solution. Since $e^*_H, e^*_L$ are Lipschitz in $l$ by assumption and the action space $[0, 1]$ is bounded, the drift of the belief process $\sigma^{-2}(e^*_H(l) - e^*_L(l))(e_\theta(l) - \frac{1}{2}e^*_H(l) - \frac{1}{2}e^*_L(l))$ and the variance $\frac{(e^*_H(l) - e^*_L(l))^2}{\sigma^2}$ are both Lipschitz in $l$ and bounded.

Definition 6. A Markov stationary equilibrium consists of a strategy $(e^*_H, e^*_L)$ and a log likelihood ratio process $(l_t)_{t \geq 0}$ s.t.

1. given $(l_t)_{t \geq 0}$, $e^*_\theta$ solves
   \[
   \sup_{e_\theta(\cdot)} \mathbb{E} \int_t^\infty \exp(-rs) \left[ R(l_s) - \frac{A_\theta}{2} e^2_\theta(l_s) \right] ds,
   \]
   where the expectation is over the process $(l_t)_{t \geq 0}$.

2. given $(e^*_H, e^*_L), (l_t)_{t \geq 0}$ is derived from Bayes’ rule (13).

The focus is on pure-strategy Markov stationary equilibria (the state variable is the log likelihood ratio of the receivers), where outside an interval of log likelihood ratios $(\bar{l}, \tilde{l})$ both types choose $e_\theta = 0$ and inside that interval at least one type chooses $e_\theta > 0$. Such equilibria are called interval equilibria. The interval $(\bar{l}, \tilde{l})$ is called the signalling region and its complement the pooling region. By the same reasoning as in the linear-cost case, an interval equilibrium always exists.

Lemma 14. An interval equilibrium exists. In an interval equilibrium it cannot be the case that $e^*_L = e^*_H > 0$ or that $e^*_L > e^*_H$.

The proof is the same as for Lemma 5.

In the pooling region $l$ does not change and both types optimally choose $e_\theta = 0$, so if $l$ reaches some $\tilde{l}$ in the pooling region, both types get payoff $R(\tilde{l})$ forever. In an interval equilibrium the game essentially ends upon reaching $\tilde{l}$ in the pooling region, with a final payoff $\frac{R(\tilde{l})}{r}$ to both types.

Both types of the sender maximize their expected discounted payoff by choosing Markov stationary control processes $e_H, e_L$. The solutions to these control problems can be written as value functions. Define $\tilde{T}_{l,t}$ as the first exit.
time after $t$ of the $l$ process from $(l, \tilde{l})$, i.e. $\hat{T}_{t,l} = \inf \{ \tau > t : l_\tau \notin (l, \tilde{l}) \} \leq \infty$.

In the signalling region, the value function of type $\theta$ is

$$V_\theta(l_t) = \sup_{e_\theta(l)} \mathbb{E} \int_t^{\hat{T}_{t,l}} \exp(-rs) \left[ R(l_s) - \frac{A_\theta}{2} e_\theta^2(l_s) \right] ds$$

$$+ \exp(-r\hat{T}_{t,l}) \frac{R(l_{\hat{T}_{t,l}})}{r} 1 \{ \hat{T}_{t,l} < \infty \},$$

where $1 \{ A \}$ denotes the indicator function for the set $A$.

The same observations as in the linear cost case can be made about the value functions.

**Lemma 15.** $V_\theta$ is finite for $\theta = H, L$. $V_H \geq V_L$, with strict inequality in the signalling region. $V_\theta$ is strictly increasing.

The proof is the same as for Lemma 6.

With $R$ concave in the signalling region, the $L$ type strictly prefers pooling to an informative equilibrium, in the sense that for any log likelihood ratio of the receivers, the payoff of $L$ is higher in a pooling equilibrium than in an informative equilibrium.

**Proposition 16.** If $R$ is concave in the signalling region, then $V_L(l) < \frac{R(l)}{r}$ for all $l$ in the signalling region.

The proof is the same as for Proposition 7.

To solve the control problems of the types of the sender, the HJB equations are solved and a verification theorem is used to check that the solutions of the HJB equations coincide with the value functions. To use Theorem 11.2.2 of Øksendal (2010) to prove that the solutions $v_H, v_L$ of the HJB equations equal the value functions $V_H, V_L$, it is sufficient that $v_H, v_L$ are twice continuously differentiable on $(l, \tilde{l})$, continuous on $[l, \tilde{l}]$ and integrable in the probability law of $l$ given the starting state $l_0$, uniformly over Markov controls $e_H, e_L$. As will be seen, these conditions are satisfied by the solutions of the HJB equations.

Under these conditions, Theorem 11.2.3 of Øksendal (2010) shows that the optimal Markov control does as well as the optimal nonanticipating control, so if the receivers expect Markov strategies, then both types of the sender have a Markov best response. This does not imply that the payoffs of all non-Markov equilibria can be attained with Markov equilibria, since in a non-Markov equilibrium the receivers expect non-Markov strategies.
The HJB equation of type \( \theta \) is

\[
rv_\theta(l) = R(l) + \frac{(e_\theta^* - e_L^*)^2}{2\sigma^2} v''_\theta(l) + \max_e \left\{ -\frac{A_\theta}{2} e^2 + v'_\theta(l) \sigma^{-2}(e_\theta^* - e_L^*) \left[ e_\theta - \frac{1}{2} e_\theta^* - \frac{1}{2} e_L^* \right] \right\}.
\]

The FOC of the type \( \theta \) HJB equation is \(-A_\theta e_\theta + v'_\theta \sigma^{-2}(e_\theta^* - e_L^*) = 0\), so given the expected equilibrium actions, both types have a unique optimal action \( e_\theta = (e_\theta^* - e_L^*) v'_\theta \). The SOC is \(-A_\theta < 0\) for all \( e_\theta \), so the FOCs are necessary and sufficient for a strict global maximum. This is not surprising, because the cost is quadratic and the benefit is linear in the control variable \( e_\theta \).

Thus far, the control problems of the two types of the sender were solved for a given pair of equilibrium strategies expected by the receivers. For the second part of the solution of the signalling game, the equilibrium condition \( e_\theta = e_\theta^* \) is imposed. The \( L \) type FOC then becomes

\[
e_L^*(l) = \frac{v'_L(l)}{A_L v''_L(l)} e^*_H(l).
\]

Substituting for \( e_L^* \) in the \( H \) type FOC gives \( A_H e_H^* = v'_H \sigma^{-2} \left[ 1 - \frac{v'_L \sigma^{-2}}{A_L + v'_L \sigma^{-2}} \right] e_H^* \). Therefore for any \( l \), either \( e_H^*(l) = 0 = e_L^*(l) \) or \( A_H A_L + A_H v'_L(l) \sigma^{-2} = A_L v'_H(l) \sigma^{-2} \). The latter is equivalent to

\[
v'_H(l) = A_H \sigma^2 + \frac{A_H}{A_L} v'_L(l).
\]

The only corner solution is \( e_H^* = e_L^* = 0 \). Other corner solutions would involve \( e_H^*(l) > 0 \) and \( e_L^*(l) = 0 \) for some \( l \) in the signalling region. The \( H \) type control problem has an interior solution if \( e_H^*(l) > 0 \) and the \( H \) type FOC is then satisfied. This implies an equation similar to (14), except derived from the \( H \) type FOC, requiring both \( e_H^* \) and \( e_L^* \) to be positive or both zero.

The relationship between the efforts and payoffs of the two types given by Eqs. (14) and (15) is similar to the one-shot model with quadratic cost, where \( e_L^* = \frac{A_H}{A_L} e_H^* \) and the expected utilities of the types are linearly related. This is not surprising, as in both cases the conditions are derived from the FOCs of quadratic problems with a similar structure.

The following lemma gives conditions on solutions of the HJB equations that are sufficient for these solutions to form an interval equilibrium with

51
a nonempty signalling region (an informative equilibrium). Subsequently, Proposition 18 provides restrictions on parameters that are sufficient for the existence of a particular kind of separating equilibrium.

**Lemma 17.** $e^*_L, e^*_H, v_L$ and $v_H$ constitute an interval equilibrium with signalling region $(\underline{l}, \overline{l})$, where $\underline{l} < \overline{l}$, if all of the following hold

1. $e^*_L, e^*_H, v_L$ satisfy Eq. (14) for all $l \in (\underline{l}, \overline{l})$,
2. $v_L$ and $v_H$ satisfy Eq. (15) for all $l \in (\underline{l}, \overline{l})$,
3. $v_L, v_H$ are twice continuously differentiable on $(\underline{l}, \overline{l})$,
4. $v_L, v_H$ are continuous on $[\underline{l}, \overline{l}]$,
5. $v_L, v_H$ are integrable in the probability law of $l$ given the starting state $l_0$, uniformly over Markov controls $e_H, e_L$,
6. $e^*_L, e^*_H$ are Lipschitz in $l$ on $(\underline{l}, \overline{l})$,
7. $0 < e^*_L, e^*_H \leq 1$,
8. $v_L \leq v_H$.

**Proof.** If $\underline{l} < \overline{l}$, then Eqs. (14) and (15) together are necessary and sufficient for $e^*_L, e^*_H, v_L$ and $v_H$ to solve the HJB equations and satisfy the equilibrium condition.

If $v_L, v_H$ are twice continuously differentiable on $(\underline{l}, \overline{l})$, continuous on $[\underline{l}, \overline{l}]$ and integrable in the probability law of $l$ given the starting state $l_0$, uniformly over Markov controls $e_H, e_L$, then by Theorem 11.2.2 of Øksendal (2010), $v_L$ and $v_H$ coincide with the value functions $V_L$ and $V_H$. In that case, $e^*_L, e^*_H$ are the optimal Markov controls for $V_L$ and $V_H$ and by Theorem 11.2.3 of Øksendal (2010), $e^*_L$ and $e^*_H$ maximize $V_L$ and $V_H$ in the class of all nonanticipating controls.

The Lipschitz condition on $e^*_L, e^*_H$ is sufficient for the belief process to be well-defined (Lemma 13).

The restrictions $0 < e^*_L, e^*_H \leq \overline{e}$ and $V_L \leq V_H$ come from first principles and Lemma 15.

**Proposition 18.** An interval equilibrium with signalling region $(\underline{l}, \overline{l})$ is formed by
• $V_H(l) = \frac{R(l)}{r}$,

• $V_L(l) = \begin{cases} \frac{A_L R(l)}{R(l)} - A_L \sigma^2 l & \text{if } l \in (\bar{l}, \bar{\bar{l}}), \\ \frac{R(l)}{r} & \text{if } l \notin (\bar{l}, \bar{\bar{l}}), \end{cases}$

• $e^*_H = 1 \{ (\bar{l}, \bar{\bar{l}}) \}$ and

• $e^*_L(l) = \begin{cases} 1 - \frac{A_H \sigma^2}{R(l)} & \text{if } l \in (\bar{l}, \bar{\bar{l}}), \\ 0 & \text{if } l \notin (\bar{l}, \bar{\bar{l}}), \end{cases}$

if the following hold

1. $\bar{\bar{\sigma}} > 1$,

2. $\bar{l} < \bar{\bar{l}}$ satisfy

\[ R(\bar{\bar{l}}) = \frac{A_L A_H r \sigma^2}{A_L - A_H}, \quad R(\bar{l}) = \frac{A_L A_H r \sigma^2}{A_L - A_H} l, \quad \text{(16)} \]

3. $\frac{R'(l)}{r} > A_H \sigma^2$ for all $l \in (\bar{l}, \bar{\bar{l}})$,

4. $\frac{R(l)}{r} \leq \frac{A_L A_H \sigma^2}{A_L - A_H} l$ for all $l \in (\bar{l}, \bar{\bar{l}})$.

Proof. For $V_H, V_L, e^*_H$ and $e^*_L$ to form an interval equilibrium with signalling region $(\bar{l}, \bar{\bar{l}})$, it is sufficient that they satisfy the assumptions of Lemma 17. Here by definition, $e^*_L, e^*_H, V_L$ satisfy Eq. (14) and $V_L$ and $V_H$ satisfy Eq. (15) for all $l \in (\bar{l}, \bar{\bar{l}})$.

Since $R$ is assumed bounded and twice continuously differentiable, $V_H, V_L$ are twice continuously differentiable on $(\bar{l}, \bar{\bar{l}})$, bounded and integrable in the probability law of $l$ given the starting state $l_0$, uniformly over Markov controls $e_H, e_L$.

The payoff in the pooling region provides the boundary conditions $V_\theta(l) = \frac{R(l)}{r}$ and $V_\theta(\bar{l}) = \frac{R(\bar{l})}{r}$. Since $V_\theta$ is twice continuously differentiable on $(\bar{l}, \bar{\bar{l}})$, for it to be continuous on $[\bar{l}, \bar{\bar{l}}]$ it is sufficient that $\lim_{l \to \bar{l}^+} V_\theta(l) = \frac{R(l)}{r}$ and $\lim_{l \to \bar{l}^-} V_\theta(l) = \frac{R(\bar{l})}{r}$. These conditions clearly hold for $V_H = \frac{R(l)}{r}$. For $V_L$, they hold iff Eqs. (16) hold. This may be seen by rearranging $\lim_{l \to \bar{l}^+} V_L(l) = \frac{A_L R(l)}{A_H r} - A_L \sigma^2 l = \frac{R(l)}{r} = V_L(l)$.

If $\frac{R'(l)}{r} > A_H \sigma^2$, then $e^*_L(l) > 0$. In the signalling region, $e^*_H > 0$ holds by definition. By the assumption $\bar{\bar{\sigma}} > 1$, we have $e^*_H, e^*_L < \bar{\bar{\sigma}}$. 

53
\[ V_H(l) \geq V_L(l) \] may be written as \( \frac{R(l)}{\tau} \geq \frac{A_L R(l)}{A_H \tau} - A_L \sigma^2 l \), or equivalently \( \frac{R(l)}{\tau} \leq \frac{A_L A_H \sigma^2 l}{A_L - A_H} \).

Slightly perturbing \( v_H \) and \( \epsilon_H^* \) (while ensuring \( \epsilon_H^* \leq \tau \)) in Proposition 18 and again deriving \( v_L \) and \( \epsilon_L^* \) from (15) and (14) results in a different interval equilibrium with the same signalling interval. Proposition 18 essentially says that in a certain parameter region, there is a degree of freedom in specifying \( \epsilon_L^*, \epsilon_H^* \) and also in specifying \( v_L, v_H \). In the signalling region, fixing one of \( \epsilon_L^*, \epsilon_H^* \) and one of the other three functions determines the remaining two via Eqs. (14) and (15). The degree of freedom is specific to the model with a quadratic cost of effort and does not have a natural interpretation.

Despite the quadratic cost of effort and the bounded benefit from belief, it would be possible to have arbitrarily large signalling efforts in equilibrium if effort was not restricted to \([0,1]\). The key is that as efforts grow, the difference in efforts also becomes large and the drift and volatility of the log likelihood ratio process are proportional to this difference. With large efforts, \( l \) quickly moves out of the region where the equilibrium prescribes the large efforts. As the cost of effort gets large, it is only paid for a very short time in expectation. This may be the case for many other cost functions, but only with quadratic cost is the increase in the effort cost exactly offset by the decrease in the duration of the effort.

References


Working paper, UC Berkeley.


Figure 4: Region where ICs hold (shaded) for $A_H = 0.15$, $A_L = 0.2$ and $r = \sigma^2 = 1$. 
Figure 5: Region where ICs hold (shaded) for $A_H = 0.1$, $A_L = 0.24$, $r = 1$ and $\sigma^2 = 0.5$. 
Figure 6: Region where ICs hold (shaded) for $A_H = 0.15$, $A_L = 0.28$, $r = 1$ and $\sigma^2 = 1$. 
Figure 7: $V_H$ for signalling region $(-1.5, 3)$ (the curve that is lower on the right), and $\frac{\exp(l)}{r(1+\exp(l))}$. The parameters are $A_H = 0.1$ and $r = \sigma^2 = 1$.

Figure 8: Equilibrium where $H$ mixes: payoff and strategy of $H$ for $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = -0.5$, $\bar{l} = 0.5$. 
Figure 9: Equilibrium where $L$ mixes: payoffs and $L$’s strategy for $A_H = 0.1$, $A_L = 0.2$, $r = \sigma^2 = 1$, $\underline{l} = -0.05$, $\bar{l} = 0.05$. 
Figure 10: Region where $IC_H$ and feasibility constraint hold (shaded) for $A_H = 0.1$, $A_L = 0.2$, $r = \sigma^2 = 1$. 