Noisy signalling over time

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Abstract

Many signalling situations feature noisy observation of the sender’s effort and multiple opportunities to signal. Information release could be gradual or discrete. Gradual information revelation is modelled as the sender controlling the drift of a Brownian motion signal process, discrete information release as controlling the intensity of a Poisson process. If the intensity increases (decreases) in the effort, then the signal is called good (bad) news. The receivers observe the signals, but not the sender’s effort or type.

Under bad news, there are Markov stationary perfect Bayesian equilibria in which the bad type exerts strictly higher effort than the good type. In these equilibria, if no signal occurs, then the types eventually pool on zero effort. If a signal occurs, then the play transitions irreversibly to a regime in which a single bad signal reveals the bad type. In that regime, the bad type takes no effort and the good type takes maximal effort. The payoff is lower for the bad than the good type, which provides the incentive for the bad type to take higher effort initially.

Under good news, the bad type prefers pooling to all informative equilibria, even when the payoff is convex in the receivers’ posterior. In

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The author is greatly indebted to Johannes Hörner for many enlightening discussions about this research. Numerous conversations with Larry Samuelson have helped improve the paper. The author is grateful to Eduardo Faingold, Dirk Bergemann, Juuso Välimäki, Philipp Strack, Willemien Kets, Tadashi Sekiguchi, Flavio Toxvaerd, Françoise Forges, Jack Stecher, Daniel Barron, Benjamin Golub, Florian Ederer, Vijay Krishna, Sambuddha Ghosh, Sergiu Hart and participants of the 24th International Conference on Game Theory at Stony Brook University, EconCon 2013, the Yale micro theory breakfast and lunch and the Yale Graduate Summer Workshop for comments and suggestions. Any remaining errors are the author’s.
all the noisy games I study, the existence and payoffs of informative equilibria depend on the prior belief, addressing the concern raised about Spence (1973) by Kreps and Sobel (1994).

Keywords: Dynamic signalling, continuous time, repeated games, incomplete information

JEL classification: D82, D83, C73

1 Introduction

Most signalling situations are noisy. An employer looking at the diploma certifying the education of a job applicant cannot be sure the diploma is not forged, or that an educated applicant has not lost the diploma. A corrupt politician is not certain to be caught. A security guard sleeping on the job does not guarantee a theft.

Many signalling situations also feature effort and signals over time. A researcher may work decades before a breakthrough occurs. A smartphone producer may hire people to post positive reviews of its latest gadget, but the reviews had better be spread out over time to prevent discovery of the practice. An IT consultant can spend a lifetime accumulating certificates from courses and workshops on the latest technology.

Noise and multiple chances of exerting effort have long been recognized as realistic features of signalling. The connection of the current work to the literature is discussed in section 5.1.

The benefit of signalling may be received concurrently with exerting effort, unlike in traditional job market signalling where first the costly education is acquired and then the wage is received. For example, male deer grow and shed their antlers each year, as well as compete for herds of females. Success in one year affects the next, because antler growth depends on how well fed the animal is, which in turn depends on whether he is in a herd. Education signalling can also be interpreted as providing benefit simultaneously with the effort. Working hard in school leads to a better college, which may offer more amenities to its students. Effort in college leads to a better graduate school, which again provides some direct benefit. Effort there leads to a better job, at which research effort leads to a higher salary.

Signalling effort can be productive or unproductive. I focus on pure signalling, in which the sender’s effort and signal provide no direct benefit to the receivers. Pure signalling is a natural benchmark, as was recognized
by Spence (1973), and there are several environments in which it is realistic. Examples are advertising that provides no direct benefit to consumers, and a politician paying taxes honestly, which does not noticeably raise the amount of public goods the voters receive.

Information revelation may be gradual or discrete. An example of gradual information revelation is the appearance over time of hundreds of customer reviews of a product on a popular website. Discrete information release can take the form of scientific breakthroughs, environmental catastrophes or political scandals. In this paper, gradual information revelation is modelled as the sender controlling the drift of a Brownian motion. Discrete signals are modelled as a Poisson process, the intensity of which the sender controls. The different signal structures model different environments, so divergence of the results is to be expected.

With both the Poisson and the Brownian signal process, there is a long-lived strategic sender and a competitive market of receivers. The sender has two types. The type is known to the sender, but not to the receivers. The receivers offer higher compensation to the sender when they put higher probability on the high type. Other things equal, both types would like to pretend to be the high type. The sender has a continuum of signalling actions available, with higher actions more costly. The cost is linear and the marginal cost is higher for the low type. The sender’s action is not observable to the receivers, but it generates a public signal. A higher action by the sender leads to a higher signal distribution, with signal distributions ordered by first order stochastic dominance. Observing the signal, the receivers use Bayes’ rule to update their belief about the sender’s type. All signal histories are on the equilibrium path with a Brownian signal process, so refinements are not needed in that case. With a Poisson signal process, after zero-probability histories belief does not respond to the signal.

The Poisson signal can be good news (a breakthrough) or bad news (a breakdown). In the good news case, the intensity of the signal process is zero at zero signalling effort and rises linearly in effort. In the research breakthrough example, the harder a researcher works, the greater the probability of making a discovery. In the bad news case, the intensity is maximal at zero effort and falls linearly in effort. In the corruption scandal example, a politician refusing bribes is in effect paying a cost to lower the probability of being exposed.

The Brownian signal never resolves all uncertainty. Beliefs move as a diffusion process, the variance of which increases in the difference of the
efforts expected from the types of the sender. If the efforts are expected to be the same, the variance is zero and beliefs stay constant. If the efforts are expected to be different, then the cost structure leads the high type to take a higher effort than the low type.

In both the Poisson and the Brownian case, the focus is on stationary Markov strategies with the common posterior as the state variable. The Markov restriction eliminates history-dependent strategies that are well known from repeated games, but technically difficult to define in continuous time.

As benchmarks for the continuous time noisy signalling games, in section 4 a one-shot noisy signalling game is solved and an overview of a noiseless one-shot and a noiseless repeated signalling game is given. The repeated noiseless game gives results rather similar to Spence (1973): the existence and payoffs of a separating equilibrium do not depend on the prior and pooling equilibria with positive effort exist for all parameter values. These features are considered undesirable by Kreps and Sobel (1994). In the noisy signalling games studied in this paper, the existence and payoffs of informative equilibria depend on the prior.

All signalling models in this paper feature a tradeoff between the incentives of the low and the high type: increasing the benefit of signalling makes it easier to incentivize the high type to signal, but harder to motivate the low type not to signal. The dynamic models have in addition a link between present and future signalling. Since signalling is costly, if more of it is expected after some history, then reaching that history is less attractive. The type-dependent cost of signalling implies that histories involving more signalling are relatively less attractive for the low type. The expectation of signalling in the future can thus be used to deter the low type from signalling in the present, while maintaining the high type’s incentives.

Pooling on zero effort is an equilibrium for all parameter values in all the noisy signalling games in this paper, while pooling on positive effort is never an equilibrium. The reason is that if both types of the sender are expected to take the same effort, then the signal is uninformative about the type. Since the receivers do not respond to the signal, both types will choose zero effort to minimize the effort cost. In a noiseless game, a refinement requiring beliefs not to respond to the signal when both types are expected not to signal also leads to the result that pooling on zero effort is an equilibrium and pooling on positive effort is not.

In the dynamic noisy games, if a pure strategy nonpooling equilibrium
exists, then there is a continuum of such equilibria, unlike in the one-shot noisy games.

In the good news Poisson case, the good type always chooses a weakly higher action than the bad type. In the bad news model, there are equilibria in which the bad type initially takes strictly higher effort than the good type. In these equilibria, if no signal occurs, then the types eventually pool on zero effort. If a signal occurs, then the play transitions irreversibly to a regime in which a single bad signal reveals the bad type. In that regime, the bad type takes no effort and the good type takes maximal effort. The payoff is lower for the bad than the good type, which provides the incentive for the bad type to take higher effort initially.

With a Brownian signal, no information can be revealed for priors close to zero or one. In the good news Poisson model, information revelation is possible for priors close to zero, but not close to one. In the bad news case, information can be revealed for priors close to one, but not close to zero.

In the good news Poisson game, the bad type always prefers the pooling equilibrium to any informative equilibrium, even for payoffs convex in the receivers’ posterior. This differs from the other signalling models studied in this paper.

The set of beliefs at which both types of the sender take zero effort is called the pooling region and its complement the signalling region. The game with a Brownian signal process is distinguished from the Poisson games by the feature that for some parameter values there is no equilibrium with a signalling region containing the signalling regions of all other equilibria. In other words, the union of equilibrium signalling regions need not be an equilibrium signalling region.

2 Poisson signalling

This section turns to the main model where effort changes the intensity of a Poisson signal process. Both the good news and the bad news cases are considered, but first the setup of the model is described.

Time is continuous and the horizon is infinite. There is a strategic sender and a competitive market of receivers. The sender has two types, $H$ and $L$, with initial log likelihood ratio $l_0 \in \mathbb{R}$ that is common knowledge. The

\[^{1}\text{Throughout this paper, log likelihood ratio } l \text{ is used instead of belief } Pr(H) = \frac{e^{l}}{1 + e^{l}},\]
sender knows his type, the receivers do not. A generic log likelihood ratio \( l \) is an element of \( \mathbb{R} = \mathbb{R} \cup \{ \infty, -\infty \} \). The log likelihood ratio corresponding to \( \Pr(H) = 1 \) is \( l = \infty \) and corresponding to \( \Pr(H) = 0 \) is \( l = -\infty \).

The sender has action set \([0, 1]\) (endowed with the natural Borel \( \sigma \)-algebra) at each instant of time. The action 0 is interpreted as no effort of signalling and the action 1 as maximal effort. Effort \( e \) costs type \( \theta \) sender \( c_\theta e \), with \( c_L > c_H > 0 \). Effort benefits the sender via its effect on the signal process, which drives the market’s log likelihood ratio \( l \). This in turn determines the flow payoff. Before describing this benefit, the signal process, strategies and market expectations must be defined.

The signal is binary, with values in \([0, 1]\). The signal 1 occurs at a Poisson rate, and in its absence the signal is 0. In the good news (breakthrough) case, the rate of signal 1 is \( e_t \lambda \) at time \( t \). The parameter \( \lambda \in (0, \infty) \) is interpreted as the informativeness of effort and \( e_t \) denotes the effort at \( t \). The intensity increases in the sender’s effort, so the occurrence of the signal is good news about the sender. In the bad news (breakdown) case, the rate of 1 is \((1 - e_t)\lambda\), which decreases in effort. Note that zero effort in the good news case or maximal effort \( e = 1 \) in the bad news case ensures no signals occur. Given that the values of the signal are fixed, a realization of the signal process is described by the times at which 1 occurs. The receivers observe the public signals, but not the sender’s effort. Since the signal is public, the it leads to is common to all receivers.

A signal sequence is a sequence \((\tau_k)_{k=1}^\infty\) of signal times satisfying \( 0 = \tau_0 < \tau_1 < \tau_2 < \cdots \) and having no accumulation points. The set of signal sequences \( H_\infty \) is endowed with the \( \sigma \)-algebra generated by cylinders. An \( n \)-signal public history is a finite sequence \((\tau_1, \ldots, \tau_n, t)\) satisfying \( \tau_1 < \cdots < \tau_n < t \), with \( t \in (0, \infty) \). The set of \( n \)-signal histories is \( H_n \). It inherits the \( \sigma \)-algebra from \( H_\infty \). The set of nonterminal public histories is \( H = \bigcup_{n \in \mathbb{N} \cup \{0\}} H_n \). \( H \) inherits a \( \sigma \)-algebra from \( \{ H_n \} \) and is a Borel space (Yushkevich, 1980). The terminal public histories are the signal sequences. The truncation of a history \( h = (\tau_1, \ldots, \tau_n, t) \) to time \( s \leq t \) is \( h_s = (\tau_1, \ldots, \tau_m, s) \), with \( \tau_m < s \). The notation \( \tau_k \in h \) means that under history \( h \), a signal occurs at time \( \tau_k \).

A pure public strategy is a pair of measurable maps \( e = (e_H, e_L) \) from \( H \) into the action set \([0, 1]\). Throughout the paper, only public strategies are considered. In this section, the focus is on pure strategies. Denote the

as formulas simplify significantly in the dynamic models to follow. There is a one-to-one map from log likelihood ratio to belief, so all results can be stated in terms of beliefs.
strategy the market expects by \( e^* = (e^*_H, e^*_L) \). This notation is also used for equilibrium strategies.

Some sets of histories that are used in defining the updating rule for the log likelihood ratio are defined next. These consist of histories in which a signal occurs at a time at which the strategy expected from the sender is such that the signal is perfectly informative about the type. In the good news case,

\[
H^g_{\text{max}}(e^*) = \{ h \in H : \exists \tau_k \in h, e^*_H(h_{\tau_k}) > 0, e^*_L(h_{\tau_k}) = 0 \},
\]

\[
H^g_{\text{min}}(e^*) = \{ h \in H : \exists \tau_k \in h, e^*_H(h_{\tau_k}) = 0, e^*_L(h_{\tau_k}) > 0 \},
\]

and in the bad news case,

\[
H^b_{\text{max}}(e^*) = \{ h \in H : \exists \tau_k \in h, e^*_H(h_{\tau_k}) < 1, e^*_L(h_{\tau_k}) = 1 \},
\]

\[
H^b_{\text{min}}(e^*) = \{ h \in H : \exists \tau_k \in h, e^*_H(h_{\tau_k}) = 1, e^*_L(h_{\tau_k}) < 1 \}.
\]

The updating of the market’s log likelihood ratio is described next, starting with the response to signal 1. If the strategy the market expects is \( e^* \), signal 1 occurs at time \( t \), and at \( t \) the log likelihood ratio is \( l_t \), then in the good news case the log likelihood ratio jumps to

\[
j^g(l_t) = \begin{cases} 
    l_t + \ln \left( \frac{e^*_H(h_t)}{e^*_L(h_t)} \right) & \text{(with } 0 = 1) \\
    l_t & \text{if } h_t \notin H^g_{\text{max}}(e^*) \cup H^g_{\text{min}}(e^*) \text{,} \\
    l_t & \text{if } h_t \in H^g_{\text{max}}(e^*) \cup H^g_{\text{min}}(e^*) \text{.}
\end{cases}
\]

In the bad news case the log likelihood ratio jumps to

\[
j^b(l_t) = \begin{cases} 
    l_t + \ln \left( \frac{1-e^*_H(h_t)}{1-e^*_L(h_t)} \right) & \text{(with } 0 = 1) \\
    l_t & \text{if } h_t \notin H^b_{\text{max}}(e^*) \cup H^b_{\text{min}}(e^*) \text{,} \\
    l_t & \text{if } h_t \in H^b_{\text{max}}(e^*) \cup H^b_{\text{min}}(e^*) \text{.}
\end{cases}
\]

Two refinements are built into the \( j(l) \) formulas. First, if the efforts expected from the types at the time a signal occurs are such that the signal rate is zero, then by the \( 0 = 1 \) assumption, the log likelihood ratio does not respond to the signal. Second, signals are ignored when there is certainty about the type (a perfectly informative signal occurred in the past).

**Definition 1.** Under good news, the log likelihood ratio at \( t \) is

\[
l_t = l_0 - \lambda \int_0^t e^*_H(h_s) - e^*_L(h_s)ds + \sum_{k=1}^n j^g(l_{\tau_k}), \quad (1)
\]
and under bad news, it is

\[ l_t = l_0 + \lambda \int_0^t e_H^*(h_s) - e_L^*(h_s) ds + \sum_{k=1}^n j^b(l_{\tau_k}), \quad (2) \]

where \((e_H^*, e_L^*)\) is the strategy the market expects and \(h = (\tau_1, \ldots, \tau_n, t)\) is the history up to \(t\).

Given a signal sequence, the solution to (1) is the log likelihood ratio process \((l_t)_{t \geq 0}\) under good news, and the solution to (2) is the process under bad news.

The integrals in (1) and (2) are uniquely defined, because \(e_L, e_H\) are bounded and measurable in the \(\sigma\)-algebra of histories, which contains singletons. Fixing a signal sequence, a history is determined by its length \(t\), so \(e_L, e_H\) are measurable functions from time to actions.

A Markov stationary strategy is a public strategy measurable w.r.t. the log likelihood ratio process. A Markov stationary strategy can be written as a pair of functions \((e_L, e_H) : \mathbb{R} \to [0, 1]^2\). Subsequently only pure Markov stationary strategies are considered. To simplify the statements to follow, attention is restricted to \(e_L, e_H\) piecewise continuous\(^2\) and at every discontinuity, continuous from the left or the right. The state variable \(l\) is the left limit \(l_t^-\) of the log likelihood ratio (with the convention \(l_0^- = l_0\)), so jumps are not anticipated by a strategy.

In this paper, \(l_0\) is treated as a parameter, not a variable, so the strategies are a function of \(l_0\) in addition to \(l\). This is to avoid discussion of log likelihood ratios unreached in equilibrium after any deviation. Define the reachable set of log likelihood ratios

\[ \mathcal{L}(e^*) = \{ l \in \mathbb{R} : \exists h \in H \exists t \in \mathbb{R}_+ \text{ s.t. } l_t = l \}, \]

where \(l_t\) is given in Def. 1. The reachable set depends on the strategy the market expects and on \(l_0\). Only behaviour on the reachable set is discussed subsequently. Since no deviation can take \(l\) outside the reachable set, behaviour there can be arbitrary. Some \(l \in \mathcal{L}(e^*)\) can only be reached by deviating, not by following the equilibrium strategy. In particular, \(L\) cannot reach \(l = \infty\) in the good news case by following \(e_L^*(l) = 0 \ \forall l \in \mathcal{L}(e^*)\) and

\(^2\)A piecewise continuous strategy is understood to have at most finitely many discontinuities on \(\mathbb{R}\).
$H$ cannot reach $l = -\infty$ in the bad news case by following $e^*_H(l) = 1$ in the interior of the reachable set. To specify behaviour in these cases, assume $e^*_L(\infty) = 0$ and $e^*_H(-\infty) = 0$.

Now that strategies and the log likelihood ratio process have been described, the sender’s payoff can be defined. The sender is assumed to derive flow benefit $\beta(l)$ directly from the market’s log likelihood ratio $l$.\(^3\)

The sender’s flow utility from effort $e$ and the market’s log likelihood ratio $l$ is $\beta(l) - c_\theta e$, where $\beta$ is assumed strictly increasing, bounded and continuously differentiable. Denote the flow benefit from $l = \infty$ (corresponding to $\Pr(H) = 1$) by $\beta_{\text{max}}$ and from $l = -\infty$ by $\beta_{\text{min}}$.

Given the strategy $e^* = (e^*_L, e^*_H)$ the market expects, the payoff of type $\theta$ from the effort function $e_{\theta}(\cdot)$ and the log likelihood ratio process $(l_t)_{t \geq 0}$ is the expected discounted sum of flow payoffs

$$J^e_{l_0}(e^*) = \mathbb{E}^{e_\theta} \left[ \int_0^\infty \exp(-rt) [\beta(l_t) - c_\theta e_{\theta}(l_t)] dt | l_t = l_0 \right],$$

where the expectation is over the stochastic process $(l_t)_{t \geq 0}$, given $e_\theta$. The discount rate is $r > 0$. Except for jumps, $l$ evolves deterministically given the market expectations $(e^*_L, e^*_H)$. The jumps occur at Poisson times. Given $l$ at the time of a jump, the size of the jump is deterministic. The expectation in (3) is thus over the jump times of the Poisson signal process induced by $e_\theta$.\(^4\)

Since $l_0$ is a parameter, the payoff starting at $l_0$ need not in general equal the continuation value from $l_0$ on when starting at some $\hat{l}_0 \neq l_0$. However, if the strategy the market expects is Markov stationary, then every time $l$ is reached, the continuation value of type $\theta$ from $l$ is well defined and is denoted $V_\theta(l)$.\(^4\)

**Lemma 1.** $V_H(l) \geq V_L(l) \forall l \in \mathbb{R} \forall e^* \forall l \in L(e^*)$, with strict inequality if under the optimal $(e_L, e_H)$ starting at $l$, there is a positive probability of reaching some $\hat{l}$ with $e_L(\hat{l}) > 0$. $\frac{\beta_{\text{min}}}{r} \leq V_\theta(l) \leq \frac{\beta_{\text{max}}}{r}$, with strict inequalities if $l \in \mathbb{R}$.

\(^3\)This can be microfounded by assuming that each receiver has a unique one-shot best response $a^*(l)$ to each log likelihood ratio $l \in \mathbb{R}$. Since each receiver is infinitesimal, their current action does not influence the future, so in any equilibrium each receiver must play the one-shot best response. The sender is then assumed to derive flow benefit $\beta_\alpha(a^*(l))$ from the receivers’ action $a^*(l)$.

\(^4\)The dependence of $V_\theta(l)$ on $e^*$ and $l_0$ is suppressed in the notation.
All proofs omitted from the text are in Appendix A.

**Definition 2.** A Markov stationary equilibrium consists of a Markov stationary strategy $e^* = (e^*_H, e^*_L)$ of the sender and a log likelihood ratio process $(l_t)_{t \geq 0}$ s.t.

1. given $(l_t)_{t \geq 0}$, $e^*_\theta$ maximizes (3) over $e_\theta$,

2. given $e^*$, $(l_t)_{t \geq 0}$ is derived from (1) under good news and (2) under bad news.

The definition implies that on the reachable set, behaviour is optimal from any point on. Therefore the equilibrium concept could also be called Markov perfect. Henceforth equilibrium means a pure Markov stationary equilibrium.

Call an equilibrium simple when the equilibrium efforts only take values in $\{0, 1\}$. A pooling equilibrium is one where $e^*_L(l) = e^*_H(l) \forall l \in \mathcal{L}(e^*)$.

**Lemma 2.** Fix $l_0 \in \mathbb{R}$. In any equilibrium, $\exists l \in \mathcal{L}(e^*)$ satisfying $e^*_L(l) = e^*_H(l) > 0$. A simple equilibrium always exists.

**Proof.** If $e^*_L(l) = e^*_H(l)$, then $l$ stays constant regardless of the history by Def. 1. Then both types best respond with $e(l) = 0$ to avoid the signalling cost. This unique best response rules out $e^*_L(l) = e^*_H(l) > 0$ for $l \in \mathcal{L}(e^*)$ and justifies the conjecture $e^*_L(l) = e^*_H(l) = 0$. Therefore the pooling equilibrium, where $e^*_L(l_0) = e^*_H(l_0) = 0$, always exists. Pooling on zero effort is a simple equilibrium.

Pooling is the unique equilibrium if the benefit of signalling is low enough relative to the cost. It is proved below (Propositions 5 and 9) that if pooling is not the unique equilibrium, then there exists a continuum of nonpooling simple equilibria.

**Proposition 3.** Pooling is the unique equilibrium if $l_0 \in \mathbb{R}$ if $\frac{\beta_{\text{max}} - \beta_{\text{min}}}{r} \leq \frac{c_H}{\lambda}$.

The conditions for the existence of an informative equilibrium have the same intuition in the Brownian and the one-shot cases as in the Poisson model: for some initial log likelihood ratio, the benefit to signalling when the receivers expect $H$ to signal and $L$ not to signal has to be high enough to incentivize $H$ to signal. Then an $l_0$ can be found at which the benefit of signalling is low enough for $L$ not to imitate $H$, but high enough for $H$ to signal.
Given $e^*$, the pooling region is defined as

$$\mathcal{P}(e^*) = \{ l \in \mathcal{L}(e^*) : e^*_H(l) = e^*_L(l) = 0 \}.$$ 

Due to the assumptions that at $l = -\infty$ and $\infty$, the log likelihood ratio does not respond to signals and $e^*_L(\infty) = e^*_H(-\infty) = 0$, we have $-\infty, \infty \in \mathcal{P}(e^*) \forall e^*$.

It is intuitive that when the sender’s benefit is concave in the receivers’ posterior belief,\(^5\) then $L$ always prefers the pooling equilibrium to any other equilibrium. This is because for $L$ in an informative equilibrium the expected posterior is lower than the prior, and with a concave benefit, the variance in the posterior is not beneficial. Lemma 18 in Appendix A states this formally. The result holds in all the signalling models discussed in this paper.

### 2.1 The bad news case

In the bad news model, there exist equilibria in which for some beliefs of the receivers the $L$ type exerts higher effort than $H$, despite the uniformly higher marginal cost of effort. This distinguishes the bad news case from the previous literature on signalling.

Some effort patterns can be ruled out in equilibrium. Lemma 2 ruled out $e^*_L(l) > e^*_H(l) > 0$. Lemma 4 rules out $e^*_L(l) > e^*_H(l)$ under some conditions, in particular when the jump $j(l)$ lands in the pooling region.

**Lemma 4.** In equilibrium, the following are ruled out $\forall l_0 \in \mathbb{R} \forall l \in \mathcal{L}(e^*)$:

1. $e^*_L(l) > e^*_H(l)$ and pooling at $j(l)$,
2. $0 < e^*_L(l) < e^*_H(l) = 1$.

The intuition of the proof of (a) is as follows. The value after a jump is the same for both types, so due to $V_H \geq V_L$, the benefit of the jump is larger (or the loss is smaller) for $L$ than for $H$, regardless of whether $V_H(j(l)) > V_H(l)$ or not. The cost of avoiding the jump is larger for $L$, so it cannot be that $L$ chooses a greater effort to avoid jumps than $H$. For (b), it is proved that $V_L$ is strictly increasing in the region where $0 < e^*_L(l) < e^*_H(l) = 1$. If $L$ is

\(^5\) $\beta(l)$ is linear in the posterior belief if it has the form $k_1 \frac{\exp(l)}{1+\exp(l)} + k_2$, with $k_1 > 0$, because the probability corresponding to log likelihood ratio $l$ is $\frac{\exp(l)}{1+\exp(l)}$. The function $k_1 \frac{\exp(l)}{1+\exp(l)} + k_2$ is convex for $l < 0$ and concave for $l > 0$. 

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taking interior effort and \( H \) is taking maximal effort, then the jumps go to \( l = -\infty \). Given that \( V_L \) is strictly increasing, \( L \) is indifferent to the jumps at most at a single point, so cannot be taking interior effort over a range of \( l \).

Lemma 4 does not rule out \( e^*_{L} > e^*_{H} \) occurring in equilibrium, as shown in Example 1. Both \( j(l) > l \) and \( j(l) < l \) are possible, because if \( e^*_{L}(l) > e^*_H(l) \), then \( j(l) > l \). The effort pattern \( e^*_{L} > e^*_{H} \) is counterintuitive, because the benefit from a higher log likelihood ratio is the same for the types, but \( L \) has a higher marginal cost of signalling.

After the example, the set of simple equilibria is characterized. Equilibria in which \( e^*_H \in (0,1) \) and \( e^*_L = 0 \) exist (by Lemma 19 in Appendix A) and can be found numerically, but cannot be solved for in closed form.

**Example 1.** Take \( c_H = 0.1, c_L = 1.14, r = 1, \lambda = 2, \beta(l) = \frac{\exp(l)}{1+\exp(l)}, l_0 = 2 - \epsilon^2, \epsilon \in (0,1) \). An equilibrium in which \( e^*_L \in (0,1) \) and \( e^*_H = 0 \) over the interval \((2 - \epsilon, l_0]\) is shown in Figure 1. In the interval \([2, \infty)\), the efforts are \( e^*_L = 0 \) and \( e^*_H = 1 \). The log likelihood ratios immediately above 2 are not reachable from \( l_0 \), but the lowest reachable \( l \) in \([2, \infty)\) cannot be calculated explicitly. Efforts \( e_L = 0 \) and \( e_H = 1 \) in the unreachable region of \([2, \infty)\) are the unique best responses to expectations \( e^*_L = 0 \) and \( e^*_H = 1 \) in that region.

Figure 1 depicts \( V_H \) and \( V_L \) in the region \((2, \infty)\) as the solid lines, the higher (blue) line for \( V_H \) and the lower (purple) for \( V_L \). The dashed green line is the pooling payoff \( \beta(l) r \), and the values of the types in \((2 - \epsilon, l_0]\) are indistinguishable from \( \beta(l) r \). The value functions in \((2 - \epsilon, l_0]\) cannot be found in closed form, but they are continuous on \((2 - \epsilon, l_0]\) and value matching holds at \( 2 - \epsilon \) due to the downward drift of \( l \). These properties are used in Proposition 20 in Appendix A to provide sufficient conditions for equilibria similar to the one in Figure 1 to exist. The sufficient conditions hold for the parameters used in Figure 1.

The incentives for the types to signal or not in the appropriate regions are provided by the payoff differences. The jump of \( l \) from \((2 - \epsilon, l_0]\) to \( j(l) \in [2, \infty) \) lowers the payoff of \( L \) enough to make \( L \) indifferent to effort. The fall in \( V_H \) from the same jump is not large enough to incentivize \( H \) to take positive effort. The fall from \( V_L \) on \([2, \infty)\) to \( \frac{\beta_{\min}}{r} \) is small enough for \( L \) not to take effort to avoid signals. The difference between \( V_H \) on \([2, \infty)\) and \( \frac{\beta_{\min}}{r} \) is large enough for \( H \) to take maximal effort.
2.1.1 Simple equilibria

The focus in this section is on simple equilibria. In that class of equilibria, it is w.l.o.g. to consider only the ones in which $e^*_L \equiv 0$ and $e^*_H(l) = 1$ if $l \in [l_0, \bar{l})$ for some $\bar{l} \in \mathbb{R}$, with $e^*_H(l) = 0$ otherwise. The interval $[l_0, \bar{l})$ is called the signalling region. The log likelihood ratio cannot drift across a region on which $e^*_L = e^*_H = 0$ and if $e^*_H(l) = 1$, $e^*_L(l) = 0$, then $j(l) = -\infty$, so there are no jumps into another region where $e^*_L = 0$ and $e^*_H = 1$. The interval $[l_0, \bar{l})$ is assumed open at $l_0$, because at $l = \infty$, beliefs do not respond to signals, so both types will choose $e = 0$. Singleton intervals are ruled out by the assumption that $e^*_L, e^*_H$ are piecewise continuous and continuous from the left or the right at jumps. Nonsingleton closed intervals of finite length can be replaced with intervals open on the right without changing the results.

Next, the value functions of the types in simple equilibria are calculated. After that, bounds on the set of signalling regions are provided, the existence of nonpooling simple equilibria is proved if the condition in Proposition 3 fails, and finally comparative statics are reported.

At $\bar{l}$, the value functions of both types are $V_\theta(l) = \frac{\beta(l)}{r}$. The boundary $\bar{l}$ can be infinite. In $[l_0, \bar{l})$, the value functions are solved for using Hamilton-Jacobi-Bellman (HJB) equations and a verification theorem (Theorem 4.6 in Presman, Sonin, Medova-Dempster, and Dempster (1990) as modified for the discounted case in Yushkevich (1988)) is used to check that the solutions
coincide with the value functions. The HJB equation of type $\theta$ is

$$rV_\theta(l) = \beta(l) + \lambda V_\theta'(l) + \max_e \left\{ \lambda(1-e) \left[ \frac{\beta_{\min}}{r} - V_\theta(l) \right] - c_\theta e \right\}.$$  

After reaching $\bar{l}$, incentives are trivial. In the signalling region, type $\theta$ chooses $e_\theta = 1$ if $-\lambda[\frac{\beta_{\min}}{r} - V_\theta(l)] - c_\theta \geq 0$. Rearranging this, one obtains the incentive constraints (ICs)

$$\frac{c_H}{\lambda} + \frac{\beta_{\min}}{r} \leq V_H(l), \quad \frac{c_L}{\lambda} + \frac{\beta_{\min}}{r} \geq V_L(l),$$  

which must hold for every $l$ in the signalling region. These restrict the set of possible signalling regions and must be checked after solving for the candidate value functions.

To solve for the candidate value functions, substitute the equilibrium strategies $e^*_H = 1$ and $e^*_L = 0$ into the HJB equations of $H$ and $L$. The HJB equations become the ordinary differential equations (ODEs)

$$rV_H(l) = \beta(l) - c_H + \lambda V'_H(l)$$

and

$$rV_L(l) = \beta(l) + \lambda V'_L(l) + \lambda \beta_{\min} - \lambda V_L(l).$$  

In the absence of a signal, the log likelihood ratio rises continuously to $l$. Assume $l$ is finite (the case $l = \infty$ is discussed after solving the $l < \infty$ case). Then value matching gives $\lim_{l \to l} V_\theta(l) = V_\theta(\bar{l}) = \frac{\beta(\bar{l})}{r}$, which provides the boundary condition for the ODEs. The solutions of the ODEs are

$$V_H(l) = \exp \left( -r \frac{\bar{l} - l}{\lambda} \right) \frac{\beta(\bar{l})}{r} + \int_l^{\bar{l}} \frac{\beta(z) - c_H}{\lambda} \exp \left( -r \frac{z - l}{\lambda} \right) dz,$$

$$V_L(l) = \exp \left( -(r + \lambda) \frac{\bar{l} - l}{\lambda} \right) \frac{\beta(\bar{l})}{r} + \int_l^{\bar{l}} \left[ \frac{\beta(z)}{\lambda} + \frac{\beta_{\min}}{r} \right] \exp \left( -(r + \lambda) \frac{z - l}{\lambda} \right) dz.$$  

These are continuously differentiable on $(l_0, \bar{l})$, with a right derivative at $l_0$ and a left derivative at $\bar{l}$, so by the verification theorem in Yushkevich (1988), they coincide with the candidate value functions. The ICs must be checked to confirm that the candidate value functions are indeed the value functions.

If $\bar{l} = \infty$, then both types get benefit $\beta(l)$ forever. In addition, $L$ has a flow rate $\lambda$ of jumps to $\frac{\beta_{\min}}{r}$ and $H$ pays a flow cost $c_H$ forever. The candidate
value functions are
\[
V_H(l) = \int_l^\infty \frac{\beta(z) - c_H}{\lambda} \exp\left(-\frac{r}{\lambda}z - \frac{l}{\lambda}\right) \, dz,
\]
\[
V_L(l) = \int_l^\infty \left[ \frac{\beta(z)}{\lambda} + \frac{\beta_{\min}}{r}\right] \exp\left(-\frac{(r + \lambda)z - l}{\lambda}\right) \, dz.
\]

For every \(\epsilon > 0\) there exists \(\hat{l} \in \mathbb{R}\) s.t. \(\beta_{\max} - \beta(\hat{l}) < \epsilon\), which implies \(|\frac{\beta_{\max} - c_H}{r} - V_H(\hat{l})| < \epsilon\) and \(|\frac{\beta(\hat{l})}{r + \lambda} + \frac{\lambda\beta_{\min}}{r(r+\lambda)} - V_L(\hat{l})| < \epsilon\).

Bounds on the set of signalling regions \([l_0, l]\) for which the incentive constraints (4) are satisfied are provided next. The maximal upper boundary \(\max l\) that an equilibrium signalling region can have must satisfy the incentive constraint \(\lim_{l \to \max} V_L(l) \leq \frac{\beta_{\min}}{r} + \frac{c_L}{\lambda}\). The limit equals \(\frac{\beta(\hat{l})}{r + \lambda}\) for \(l\) finite and equals \(\frac{\beta_{\max}}{r + \lambda} + \frac{\lambda\beta_{\min}}{r(r+\lambda)}\) for \(l = \infty\). The benefit to signalling is avoiding the bad signal, so the larger the difference between \(\frac{\beta_{\min}}{r}\) and \(V_L(l)\), the greater the incentive of \(L\) to imitate \(H\). Since \(V_L\) is increasing, the \(L\) type incentive constraint determines the log likelihood ratio above which signalling cannot continue.

The minimal \(l_0\) at which signalling can start is determined by \(V_H\), which depends on \(\hat{l}\). Denote the minimal lower boundary given \(\hat{l}\) by \(l_0(\hat{l})\). This is finite, because \(-\infty\) is in the pooling region by Def. 1. The incentive constraint \(\lim_{l \to l_0(\hat{l})} V_H(l) \geq \frac{\beta_{\min}}{r} + \frac{c_H}{\lambda}\) for \(l_0(\hat{l})\). Since \(V_H\) is strictly increasing, if the IC for \(H\) holds at \(l\), then it holds at all \(\hat{l} > l\).

There exists an informative simple equilibrium if the condition in Proposition 3 for pooling to be the unique equilibrium fails.

**Proposition 5.** Suppose \(\frac{\beta_{\max} - \beta_{\min}}{r} > \frac{c_H}{\lambda}\). Then \(\exists l_0 \in \mathbb{R}\), \(\exists \epsilon > 0\) s.t. there exists a simple equilibrium with signalling region \([l_0, l_0 + \epsilon]\).

The intuition for Proposition 5 is that if the jumps from \(\infty\) to \(-\infty\) strictly incentivize \(H\) to signal, then there is an \(l_0\) large enough s.t. jumps from \(l_0\) to \(-\infty\) do so as well. In that case there exists a simple equilibrium with signalling region \([l_0, l_0 + \epsilon]\).

If there exists one simple equilibrium with a nonempty signalling region \([l_0, \hat{l}]\), then there is a continuum of such equilibria, each corresponding to a particular \(\tilde{l} \in (l_0, \hat{l}]\).

Comparative statics are the final item discussed in this section. Welfare is defined as \(W(l) = \frac{\exp(l)}{1 + \exp(l)} V_H(l) + \frac{1}{1 + \exp(l)} V_L(l)\), because the receivers form a
competitive market, so their payoff is zero in any equilibrium. Based on (5), \( V_L(l) \) and \( V_H(l) \) are infinitely differentiable in \( l, r, \lambda, c_H, c_L \) for any \( l \in \mathcal{L}(e^*) \), so derivatives can be used for comparative statics. Changing \( l \) changes \( e^* \) and therefore \( \mathcal{L}(e^*) \). In that case, the comparison is between payoffs at an \( l \) that is in the reachable set both before and after changing \( l \).

Proposition 6. If \( l \in \mathbb{R} \), then

(a) \( \frac{\partial V_L(l)}{\partial l} > 0 \) iff \( \beta'(\bar{l}) - \beta(\bar{l}) + \beta_{\min} > 0 \),

(b) \( \frac{\partial V_H(l)}{\partial l} > 0 \) iff \( \frac{\beta'(l)}{r} \geq \frac{c_H}{\lambda} > 0 \),

(c) \( \frac{\partial V(l)}{\partial l} > 0 \) iff \( \exp(l) \left[ \beta'(\bar{l}) - \frac{c_H}{\lambda} \right] + \beta'(\bar{l}) - \beta(\bar{l}) + \beta_{\min} > 0 \).

Proof. The proof is by taking the appropriate derivatives. \( \Box \)

The condition for \( \frac{\partial V_L(l)}{\partial l} > 0 \) holds when \( \beta(l) = \left( \frac{\exp(l)}{1+\exp(l)} \right)^n \), \( n \in \mathbb{N} \), \( n \geq 2 \), \( \bar{l} < \ln(n-1) \). The effects of raising \( \bar{l} \) on \( V_L \) are a higher payoff upon reaching \( \bar{l} \) (the \( \beta'(\bar{l}) \) term), but a lower chance of reaching it (the \( -\beta(\bar{l}) \) term) and a higher chance of jumping to \( l = -\infty \). If \( \beta'(\bar{l}) - \beta(\bar{l}) + \beta_{\min} > 0 \) as \( \bar{l} \to l_0 \), then an informative equilibrium yields a higher \( V_L(l_0) \) than pooling. If \( \beta'(\bar{l}) - \beta(\bar{l}) + \beta_{\min} \leq 0 \) for all \( \bar{l} > l_0 \), then pooling maximizes the payoff of \( L \). This is the case if \( \beta(l) = \frac{\exp(l)}{1+\exp(l)} \) (the benefit from the receivers’ belief is linear in the belief).

If the condition for \( \frac{\partial V_H(l)}{\partial l} > 0 \) holds as \( \bar{l} \to l_0 \), then an informative equilibrium gives \( H \) a higher payoff than pooling. If the condition fails at all \( \bar{l} > l_0 \), then pooling maximizes the payoff of \( H \). The interpretation of the condition is that increasing \( \bar{l} \) increases the payoff upon reaching \( \bar{l} \) at a rate \( \beta'(\bar{l}) \) and increases the time during which the signalling cost is paid. Whether \( V_H \) increases or decreases in \( \bar{l} \) depends on which effect dominates.

The effect of increasing \( \bar{l} \) on welfare is a combination of the effects on \( H \) and \( L \), with a weight \( \exp(l) \) on the payoff of \( H \) and a weight 1 on \( L \).

2.2 The good news case

The results about the good news model are presented next. The \( L \) type always prefers the pooling equilibrium under good news, even when the flow benefit from the receivers’ log likelihood ratio is convex. This is reminiscent
of \( L \) preferring pooling to any separating equilibrium in noiseless models, but differs from the other noisy models considered in this paper.

Some preliminary observations about the equilibrium efforts are collected in the following lemma. Since the lemma rules out \( e^*_L > e^*_H \), the log likelihood ratio can only jump up: \( j(l) \geq l \) in the good news case.

**Lemma 7.** \( \forall l_0 \in \mathbb{R} \forall l \in \mathcal{L}(e^*) \), the equilibrium efforts satisfy \( e^*_L(l) = e^*_H(l) = 0 \) or \( e^*_L(l) < e^*_H(l) \). Moreover, if \( e^*_L(l) = 0 < e^*_H(l) \), then \( e^*_H(l) = 1 \).

Some equilibria with \( e^*_L(l), e^*_H(l) \in (0, 1) \) at some \( l \in \mathcal{L}(e^*) \) can also be ruled out, as shown in Lemma 21 in Appendix A.

Restricting attention to simple equilibria,\(^6\) Lemma 7 implies that in equilibrium the only possible effort combinations are \( e^*_H = e^*_L = 0 \) and \( e^*_H = 1, e^*_L = 0 \). A simple equilibrium must therefore be an interval of log likelihood ratios \((l, l_0]\) on which \( e^*_H = 1, e^*_L = 0 \) and outside which \( e^*_H = e^*_L = 0 \).\(^7\) The lower boundary of the interval can be infinite. The upper boundary is finite, as shown in Lemma 8.

**Lemma 8.** \( \forall l_0 \in \mathbb{R} \exists \hat{l} \in \mathbb{R} \) s.t. in any equilibrium \( \forall l \in \mathcal{L}(e^*) \cap [\hat{l}, \infty), e^*_H(l) = e^*_L(l) = 0 \).

As in the bad news case, the value functions are calculated first. Then bounds on the set of signalling regions are discussed, followed by the existence of informative simple equilibria. Comparative statics are derived at the end of the section.

Outside \((l, l_0]\), the value functions of both types are \( V_\theta(l) = \beta(l) \). In \([l, l_0]\), the value functions are solved for using the HJB equation and a verification theorem. The HJB equation for type \( \theta \) is

\[
rV_\theta(l) = \beta(l) - \lambda V_\theta(l) + \max_e \left\{ \lambda \left[ \frac{\beta_{\max}}{r} - V_\theta(l) \right] - c_\theta \right\}.
\]

\(^6\)Focussing on simple equilibria is a restriction. There exist equilibria where \( e^*_L \in (0, 1) \)
and \( e^*_H = 1 \), as shown in Lemma 22 in Appendix A.

\(^7\)It is w.l.o.g. to consider only one interval, because \( l \) cannot drift across a region where \( e^*_H = e^*_L = 0 \) and \( e^*_H = 1, e^*_L = 0 \), then jumps are to \( l = \infty \). The interval is open at \( l \) because \( e^*_H, e^*_L \) are piecewise continuous and continuous from the left or the right at jumps. This rules out singleton intervals. The interval \([-\infty, l_0]\) is ruled out because beliefs do not respond to signals at \( l = -\infty \), so both types will choose zero effort there.
In the pooling region, incentives are trivial. At every $l$ in the signalling region, the incentive constraints
\[
\lambda \left[ \frac{\beta_{\text{max}}}{r} - V_H(l) \right] - c_H \geq 0, \quad \lambda \left[ \frac{\beta_{\text{max}}}{r} - V_L(l) \right] - c_L \leq 0
\]
must be satisfied in order for $H$ to choose $e_H(l) = 1$ and $L$ to choose $e_L(l) = 0$. These incentive constraints restrict the set of possible signalling regions and must be checked after solving for the candidate value functions.

The constraints have a simple interpretation: the marginal benefit of an increase in effort is the increased probability of jumping to $l = \infty$ and getting $\beta_{\text{max}}$ forever instead of the current value $V_\theta(l)$. The probability increases with effort at rate $\lambda$. The marginal cost of effort is $c_\theta$. If marginal cost minus marginal benefit is positive, type $\theta$ chooses $e = 1$, otherwise $e = 0$.

To solve for the candidate value functions, substitute the equilibrium strategies $e^*_H = 1$ and $e^*_L = 0$ into the HJB equations of $H$ and $L$. The HJB equations become the ODEs
\[
\lambda V'_H(l) + (\lambda + r) V_H(l) = \beta(l) + \frac{\beta_{\text{max}}}{r} - c_H
\]
and
\[
\lambda V'_L(l) + r V_L(l) = \beta(l).
\]
In the absence of a signal, the log likelihood ratio falls continuously to $l$. For $l > -\infty$, the value matching condition $\lim_{l \to -\infty} V_\theta(l) = V_\theta(l) = \frac{\beta(l)}{r}$ holds, because close to $l$, reaching it is likely and a jump to another value unlikely. The limit gives the boundary condition $V_\theta(l) = \frac{\beta(l)}{r}$ for the ODEs. The case where $l = -\infty$ is discussed after solving the $l > -\infty$ case.

The solutions of the ODEs are
\[
V_H(l) = \exp \left( - (r + \lambda) \frac{l - l_0}{\lambda} \right) \frac{\beta(l)}{r} + \int_{l_0}^l \left[ \frac{\beta(z) - c_H}{\lambda} + \frac{\beta_{\text{max}}}{r} \right] \exp \left( - (r + \lambda) \frac{l - z}{\lambda} \right) dz, \quad (7)
\]
\[
V_L(l) = \exp \left( -r \frac{l - l_0}{\lambda} \right) \frac{\beta(l)}{r} + \int_{l_0}^l \frac{\beta(z)}{\lambda} \exp \left( -r \frac{l - z}{\lambda} \right) dz.
\]
The solutions to the HJB equations of the types are continuously differentiable on $(l, l_0)$, with a right derivative at $l$ and a left derivative at $l_0$, so by the verification theorem in Yushkevich (1988), the solutions are the candidate value functions. If the ICs are satisfied in $(L, l_0]$, then $V_H, V_L$ are the value functions.

Next, the signalling regions with $l = -\infty$ are discussed. Type $L$ has no chance of a jump and gets the discounted flow benefit forever, so $L$'s
candidate value is

\[ V_L(l) = \int_{-\infty}^{l} \frac{\beta(z)}{\lambda} \exp \left( -\frac{r}{\lambda} z \right) dz. \]

Since \( \beta \) is bounded, for any \( \epsilon > 0 \) there exists \( \hat{l} \in \mathbb{R} \) s.t. \( \beta(\hat{l}) - \beta_{\min} < \epsilon \), which implies \( V_L(\hat{l}) - \beta_{\min} < \epsilon \). Therefore \( \lim_{l \to -\infty} V_L(l) = \beta_{\min} \). The term \( \beta(l) = \beta_{\min} \) does not appear in the \( V_L(l) \) expression, because it takes an infinite time to reach \( l = -\infty \) and, as can be seen from (7), the discounting then makes the (finite) \( \beta(l) \) vanish.

Type \( H \) has a constant rate \( \lambda \) of jumps to \( l = \infty \) and pays a flow cost \( c_H \) forever, in addition to getting the flow benefit. The candidate value of \( H \) is

\[ V_H(l) = \int_{-\infty}^{l} \left[ \frac{\beta(z) - c_H}{\lambda} + \frac{\beta_{\max}}{r} \right] \exp \left( -(r + \lambda) \frac{l - z}{\lambda} \right) dz. \]

The limiting value is \( \lim_{l \to -\infty} V_H(l) = \frac{\beta_{\min} - c_H}{r + \lambda} + \frac{\lambda \beta_{\max}}{r(r + \lambda)} \). The RHS is greater than \( \frac{\beta_{\min}}{r} \) iff \( \frac{\beta_{\max} - \beta_{\min}}{r} \geq \frac{c_H}{\lambda} \).

The signalling regions \([l, l_0]\) for which the incentive constraints are satisfied can now be characterized. Due to \( V'_L > 0 \), if \( V_L(l) \geq \frac{\beta_{\max}}{r} - \frac{c_H}{\lambda} \) holds at \( l \), then it holds for all \( \hat{l} > l \). If \( L \) is deterred from imitating \( H \) at \( l \), then \( L \) is also deterred at all \( \hat{l} > l \), because the benefit to imitation is the difference between the payoff of being believed to be the \( H \) type and the current value. The higher \( l \), the higher the current value, so the lower the incentive to exert effort. Since \( l \) drifts down in the absence of a signal, the \( L \) type incentive constraint determines the log likelihood ratio below which signalling must stop. The minimal lower boundary \( \min l \geq -\infty \) that an equilibrium signalling region can have must satisfy the incentive constraint \( \lim_{l \to \min l} V_L(l) \geq \frac{\beta_{\min}}{r} - \frac{c_H}{\lambda} \), where \( \lim_{l \to \min l} V_L(l) = \beta(\hat{l}) \) and \( \beta(-\infty) = \beta_{\min} \). If the expectation was for signalling to continue at \( \min l \), then \( L \) would deviate to \( e = 1 \).

The maximal initial log likelihood ratio at which signalling can start is determined by \( V_H \), which depends on \( l \). Due to \( V'_H > 0 \), if the incentive constraint \( V_H(l) \leq \frac{\beta_{\max}}{r} - \frac{c_H}{\lambda} \) holds at \( l \), then for all \( \hat{l} < l \) we have \( V_H(\hat{l}) \leq \frac{\beta_{\max}}{r} - \frac{c_H}{\lambda} \). If \( H \) is incentivized to signal at \( l \), then \( H \) is also incentivized at all \( \hat{l} < l \), because the benefit to signalling is the difference between the payoff of being believed to be the \( H \) type and the current value. Since \( l \) drifts down in the absence of a signal, the \( H \) type incentive constraint determines the log likelihood ratio \( l_0(l) \) above which signalling cannot start. If the expectation
was for signalling to start above \(l_0(l)\), then \(H\) would deviate to \(e = 0\). By Lemma 8, \(l_0(l) < \infty\) for any \(l\).

The parameter values for which there exists an informative simple equilibrium are given in Proposition 9.

**Proposition 9.** Suppose \(\frac{\beta_{\text{max}} - \beta_{\text{min}}}{r} > \frac{cu}{\lambda}\). Then \(\exists l_0 \in \mathbb{R} \exists \epsilon > 0\) s.t. there exists a simple equilibrium with signalling region \([l_0, l_0 + \epsilon]\).

If there exists one simple equilibrium with a nonempty signalling region \((l_0, l_0]\), then there is a continuum of such equilibria, each corresponding to a particular \(l' \in [l_0, l_0]\).

Comparative statics of simple equilibria are presented next. Based on (7), \(V_H(l)\) and \(V_L(l)\) are infinitely differentiable in \(l, r, \lambda, c_H, c_L\) for all \(l \in L(e^*)\), so derivatives can be used for comparative statics. Changing \(l\) changes \(e^*\) and therefore \(L(e^*)\). In that case, the comparison is between payoffs at an \(l\) that is in the reachable set both before and after changing \(l\).

**Proposition 10.** If \(l \in \mathbb{R}\), then

(a) \(\frac{\partial V_L(l)}{\partial l} > 0\),

(b) \(\frac{\partial V_H(l)}{\partial l} > 0\) iff \(\beta'(l) - \beta_{\text{max}} + \beta(l) + \frac{cuv}{\lambda} > 0\),

(c) \(\frac{\partial W(l)}{\partial l} > 0\) iff \(\exp(l) \left[\beta'(l) - \beta_{\text{max}} + \beta(l) + \frac{cuv}{\lambda}\right] + \beta'(l) > 0\).

**Proof.** The proof is by taking the relevant derivatives.

Comparing simple equilibria, \(\frac{\partial V_L(l)}{\partial l} > 0\), so pooling always gives \(L\) the highest payoff. This holds even when the benefit from the receivers’ log likelihood ratio is arbitrarily convex. The reason is that \(e^*_L = 0\) in simple equilibria, so \(L\) never receives good signals. In informative equilibria, there is a downward drift in \(l\), which lowers \(V_L\) below pooling.

The result that pooling always gives \(L\) the highest payoff in the good news model holds not just for simple equilibria. By Lemma 7, a non-simple equilibrium must feature \(e^*_L \in (0, 1)\) in the signalling region, i.e. \(L\) is indifferent to receiving good signals and paying the signalling cost. In that case, \(e = 0\) is still a best response for \(L\), so \(V_L\) is unchanged if the chosen action of \(L\) is switched to 0, keeping expectations \(e^*_L, e^*_H\) equal to the equilibrium strategies. In other words, the payoff of \(L\) in an informative equilibrium is
If the condition for \( \frac{\partial V_H(l)}{\partial l} > 0 \) holds at every \( l < l_0 \), then pooling gives \( H \) the highest payoff. If the condition fails as \( l \to l_0 \), then an informative equilibrium gives \( H \) a higher payoff than pooling. The condition has a straightforward interpretation. Increasing \( l \) increases the payoff upon reaching \( l \) at a rate \( \beta'(l) \), lowers the chance of jumping to \( l = \infty \) (the \(-\beta_{\text{max}}\) term), increases the chance of reaching \( l \) (the \( \beta(l) \) term) and reduces the time during which the signalling cost is paid. The balance of these effects determines whether \( V_H \) increases or decreases in \( l \).

The condition for welfare to increase in \( l \) holds when \( \beta(l) = \frac{\exp(l)}{1+\exp(l)} \), so pooling gives the highest welfare when the sender’s benefit from the receivers’ belief is linear. The condition for welfare to increase in \( l \) is a combination of the effects on the payoffs of \( H \) and \( L \). The initial \( \exp(l) \) multiplier is a weight on the payoff of \( H \). The weight increases in the probability that the sender’s type is \( H \). The term in the square brackets is the effect of increasing \( l \) on the payoff of \( H \), which is discussed above. The final term \( \beta'(l) \) (with a weight of 1) is positive and describes \( L \)'s benefit from an increase in \( l \), namely that the payoff upon reaching \( l \) is larger.

### 2.3 Final remarks on the Poisson model

Some of the restrictive assumptions in the Poisson games considered above are relaxed in the appendix. Including both good and bad news in the game, with good signals occurring at rate \( \lambda_g e \) and bad signals at rate \( \lambda_b (1 - e) \), the set of simple equilibria is similar to the good news case when \( \lambda_g > \lambda_b \) and similar to the bad news case when \( \lambda_g < \lambda_b \). If \( \lambda_g = \lambda_b \), then the log likelihood ratio stays constant in the absence of signals and jumps when a signal occurs. The game with both good and bad news is discussed in Appendix C.1. Adding a small positive rate of bad news to the good news model does not change the result that \( L \) always prefers pooling. A small positive rate of good news in the bad news model leaves the results unchanged. In particular, the equilibrium where \( e_L^* > e_H^* \) survives.

The assumption that zero effort in the good news case and maximal effort in the bad news case ensure the absence of signals is rather stark. However, as shown in Appendix C.2, a low Poisson intensity \( \epsilon > 0 \) of signals even with zero effort in the good news case or with maximal effort in the bad news case does not qualitatively change the set of simple equilibria. The positive lower
bound on the signal rate in the good news model does overturn the result that $L$ always prefers pooling.

3 Signalling with Brownian noise

The Poisson signalling game corresponds to an environment where information is revealed by rare and significant events. There are situations in which a gradual and continuous information revelation is more realistic. This section turns to the gradual information revelation case and models the signal process as a Brownian motion. In this model, the union of equilibrium signalling regions need not be an equilibrium signalling region. The union would mean the receivers expect ‘too much’ signalling, which induces one of the types to deviate. An analogous result holds in the one-shot model in section 4.2, where for some parameter values an equilibrium with $e^*_L = 0, e^*_H = 1$ exists iff the initial log likelihood ratio belongs to one of two disjoint finite intervals. The expectation of ‘too much’ signalling between these intervals induces $L$ to imitate.

The setup is similar to the Poisson model. The sender’s effort process $(e_t)_{t \geq 0}$ now controls the drift of a signal process $(X_t)_{t \geq 0}$ given by

$$dX_t = e_t dt + \sigma dB_t,$$

where $X_0 \in \mathbb{R}$ is a given parameter, $B_t$ is standard Brownian motion and $\sigma > 0$. Denote the filtration generated by $(X_t)_{t \geq 0}$ by $(\mathcal{F}_t)_{t \geq 0}$. The receivers at time $t$ observe $(X_{\tau})_{\tau \in [0,t]}$, but not the sender’s type or present or past actions. Based on the signal, the receivers update their log likelihood ratio. The log likelihood ratio process $(l_t)_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$.

The flow utility of a sender of type $\theta$ from action $e \in [0,1]$ and the receivers’ log likelihood ratio $l$ is $\beta(l) - c_\theta e$. Assume $\beta$ is bounded, Lipschitz, strictly increasing and twice continuously differentiable on $\mathbb{R}$.

A pure public strategy of the sender is a pair of random processes $(e_L, e_H)$, each taking values in $[0,1]$ and adapted to $(\mathcal{F}_t)_{t \geq 0}$. A Markov stationary strategy of the sender is a pure public strategy measurable w.r.t. the log likelihood ratio process $(l_t)_{t \geq 0}$. It can be written as a pair of measurable functions $(e_L, e_H) : \mathbb{R} \to [0,1]^2$. The state variable is the receivers’ log likelihood ratio $l$. Henceforth, strategy is understood as a Markov stationary strategy.
Given the initial log likelihood ratio $l_0 \in \mathbb{R}$ and the strategy $(e^*_L, e^*_H)$ expected from the sender, the receivers update the log likelihood ratio

$$dl_t = \sigma^{-2}(e^*_H(l_t) - e^*_L(l_t))[dX_t - \frac{1}{2}e^*_H(l_t)dt] - \frac{1}{2}e^*_L(l_t)dt]. \quad (8)$$

The initial log likelihood ratio $l_0$ is a parameter, so a strategy is a function of $l_0$ in addition to $l$. The reachable set of log likelihood ratios is

$$L(e^*) = \{ l \in \mathbb{R} : \exists t \in \mathbb{R}_+ \exists \text{ a path of } l_t \text{ s.t. } l_t = l \},$$

where $l_t$ is defined in (8). Only behaviour on the reachable set is discussed subsequently. Behaviour outside $L(e^*)$ can be arbitrary, because no deviation can take $l$ there.

**Definition 3.** A Markov stationary equilibrium consists of a strategy $(e^*_H, e^*_L)$ and a log likelihood ratio process $(l_t)_{t \geq 0}$ s.t.

1. given $(l_t)_{t \geq 0}$, $e^*_\theta$ solves

   $$\sup_{e^*_\theta} \mathbb{E}^e_{\theta} \left[ \int_0^\infty \exp(-rt) [\beta(l_t) - c_\theta e^*_\theta(l_t)] \, dt \mid l_{t=0} = l_0 \right],$$

   where the expectation is over the process $(l_t)_{t \geq 0}$,

2. given $(e^*_H, e^*_L)$ and $l_0$, the process $(l_t)_{t \geq 0}$ is derived from Bayes’ rule (8).

Given $e^*$, the pooling region is

$$P(e^*) = \{ l \in L(e^*) : e^*_H(l) = e^*_L(l) \}.$$  

The complement of the pooling region in $L(e^*)$ is called the signalling region. The pooling equilibrium where $e^*_H = e^*_L \equiv 0$ always exists, because if $e^*_H = e^*_L$, then $l$ does not respond to signals, so both types optimally choose zero effort to avoid the signalling cost. A nonpooling equilibrium exists if the sender is patient enough. The proof is postponed to Proposition 15.

Since $l_0$ is a parameter, the payoff starting at $l_0$ need not in general equal the continuation value from $l_0$ after starting at $l \neq l_0$. However, if the strategy the market expects is Markov stationary, then every time $l$ is reached, the continuation value $V_\theta(l)$ of type $\theta$ from $l$ on is well defined. Some observations about the value functions are formalized in the following lemma.

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8 The updating rule for the log likelihood ratio is derived from the continuous time Bayes’ rule for probability (Liptser and Shiryaev (1977) Theorem 9.1) using Itô’s formula.
Lemma 11. \( V_H(l) \geq V_L(l) \) \( \forall l_0 \in \mathbb{R} \) \( \forall e^* \forall l \in \mathcal{L}(e^*) \), with strict inequality if under the optimal \((e_L, e_H)\) starting at \( l \), there is a positive probability of reaching some \( \hat{l} \) with \( e_L(\hat{l}) > 0 \). \( \frac{\beta_{\max}}{r} \leq V_0(l) \leq \frac{\beta_{\max}}{r} \), with strict inequalities if \( l \in \mathbb{R} \). \( V_0 \) is strictly increasing.

Equilibrium strategies are partially characterized in the following lemma. If \( L \) is expected to take higher effort than \( H \), then a higher signal lowers the log likelihood ratio and both types optimally choose 0.

Lemma 12. Fix \( l_0 \in \mathbb{R} \). In any equilibrium, \( \nexists \ l \in \mathcal{L}(e^*) \) satisfying \( e^*_L(l) = e^*_H(l) > 0 \) or \( e^*_L(l) > e^*_H(l) \).

It is clear from (8) that once the log likelihood ratio process reaches the pooling region, \( l \) stays constant forever, so signalling must occur in an interval of \( l \) containing \( l_0 \). Starting inside the interval, if the \( l \) process reaches the boundary, then in the next instant it enters the pooling region due to the rapidly varying Brownian motion driving \( l \). For this reason, it is w.l.o.g. to consider only open signalling regions. In light of this and Lemma 12, it is w.l.o.g. to consider only equilibria in which outside an interval of log likelihood ratios \((l, \bar{l})\), both types choose action 0 and inside that interval, \( e^*_L < e^*_H \).

The subsequent focus of this section is on simple equilibria, in which outside an interval of log likelihood ratios \((l, \bar{l}) \ni l_0\), both types choose action 0 and inside that interval, \( e^*_L = 0 \), \( e^*_H = 1 \). In \((l, \bar{l})\), the \( l \) process is a simple Brownian motion with drift either \( \frac{1}{2} \) or \( -\frac{1}{2} \), depending on whether the sender’s chosen action is \( e = 1 \) or 0.

The value functions are calculated next, using the HJB equations and a verification theorem. Payoff comparisons readily apparent from the value functions are noted while deriving the value functions. Necessary conditions for equilibrium are derived. After that, the existence of informative simple equilibria is proved, followed by comparative statics, which are mostly numerical.

It is clear that for \( \hat{\beta}(\mu) = \beta(\ln \frac{\mu}{1-\mu}) \) concave in \( \mu \) (the receivers’ belief), \( L \) prefers pooling to any simple equilibrium, because the expected posterior is lower than the prior and the variance in the posterior does not increase the payoff. The concavity of \( \hat{\beta} \) in \( \mu \) implies the condition in Proposition 13 that is sufficient for \( L \) to prefer pooling. If pooling gives the highest payoff to \( L \), then among simple equilibria, \( L \)’s payoff is higher in an equilibrium with a smaller signalling region.
Proposition 13. If $\beta(\ln(z))$ is concave in $z$, then $V_L$ is higher in pooling than in any simple equilibrium. In that case if $V_{L2}$ is the value of $L$ in an equilibrium with signalling region $(\bar{l}_2, \bar{l}_2)$ and $V_{L1}$ is the value of $L$ in an equilibrium with signalling region $(\bar{l}_1, \bar{l}_1) \subset (\bar{l}_2, \bar{l}_2)$, then $V_{L1}(l) \geq V_{L2}(l)$ for all $l \in (\bar{l}_1, \bar{l}_1)$. If $\beta(\ln(z))$ is convex in $z$, then $V_L$ is lower in pooling than in any interval equilibrium and $V_{L1}(l) \leq V_{L2}(l)$ for all $l \in (\bar{l}_1, \bar{l}_1)$.

The idea of the proof is to transform the $l$ process into a zero-drift process $f(l)$ by an increasing transformation $f$ using Itô’s lemma. The benefit function $\beta$ is simultaneously transformed by the inverse of $f$. Pooling gives $L$ a higher payoff than an informative simple equilibrium iff the transformed benefit function is concave in $f(l)$. It turns out $f(l)$ is the likelihood ratio $\exp(l)$.

An example where $L$ prefers pooling has $\beta(l) = \left(\frac{\exp(l)}{1+\exp(l)}\right)^n$, $n \geq 2$ and $\bar{l} \leq \ln \frac{n-1}{2}$.

Next, the HJB equations are solved and a verification theorem is used to check that the solutions of the HJB equations coincide with the value functions. The HJB equation of type $\theta$ is

$$rV_\theta(l) = \beta(l) + \max \left\{ -c_\theta e + V_\theta'(l)\sigma^{-2} \left(e - \frac{1}{2}\right) + \frac{1}{2}V_\theta''(l)\sigma^{-2}, \right\} + \frac{1}{2}V_\theta''(l)\sigma^{-2}.$$ 

Given the signalling region $(\bar{l}, \bar{l})$ the receivers expect, the optimal strategy of type $\theta$ is to choose

$$e_\theta(l) = \begin{cases} 1 \{ -c_\theta + V_\theta'(l)\sigma^{-2} \geq 0 \} & \text{if } l \in (\bar{l}, \bar{l}), \\ 0 & \text{if } l \notin (\bar{l}, \bar{l}). \end{cases}$$

The incentive constraints for $H$ to choose $e_H(l) = 1$ and $L$ to choose $e_L(l) = 0$ in the signalling region are

$$V_H'(l) \geq c_H\sigma^2, \quad V_L'(l) \leq c_L\sigma^2. \quad (9)$$

Call these $\text{IC}_H$ and $\text{IC}_L$. After finding the candidate equilibrium, it must be verified that the ICs hold at every point in the signalling region.

Set the chosen actions equal to the equilibrium actions. The HJB equations become the pair of linear second-order ODEs

$$rV_H(l) = \beta(l) - c_H + \frac{1}{2}V_H'(l)\sigma^{-2} + \frac{1}{2}V_H''(l)\sigma^{-2},$$

$$rV_L(l) = \beta(l) - \frac{1}{2}V_L'(l)\sigma^{-2} + \frac{1}{2}V_L''(l)\sigma^{-2}.$$
This is where using the log likelihood ratio instead of the belief is helpful—with belief, the ODEs do not have constant coefficients. After solving the ODEs for $V_L, V_H$, the ICs as well as the smoothness conditions for the verification theorem must be checked at every point in the signalling region.

The solutions $V_\theta$ to the ODEs are the sum of the general solution $C_{\theta 1}y_{\theta 1} + C_{\theta 2}y_{\theta 2}$ of the homogeneous equation and a particular solution $y_{\theta p}$ of the inhomogeneous equation. The general solutions for $H$ and $L$ respectively are

$$C_{H1} \exp \left( \frac{l-1-\sqrt{1+8r\sigma^2}}{2} \right) + C_{H2} \exp \left( \frac{l-1+\sqrt{1+8r\sigma^2}}{2} \right),$$

$$C_{L1} \exp \left( \frac{l-1+\sqrt{1+8r\sigma^2}}{2} \right) + C_{L2} \exp \left( \frac{l+\sqrt{1+8r\sigma^2}}{2} \right).$$

Using d’Alembert’s method, the particular solutions are

$$y_{H p} = -\frac{c_H}{r} + \frac{2\sigma^2}{\sqrt{1+8r\sigma^2}} \exp \left( \frac{l-1-\sqrt{1+8r\sigma^2}}{2} \right) \int \beta(l) \exp \left( \frac{l+\sqrt{1+8r\sigma^2}}{2} \right) dl$$

$$- \frac{2\sigma^2}{\sqrt{1+8r\sigma^2}} \exp \left( \frac{l-1+\sqrt{1+8r\sigma^2}}{2} \right) \int \beta(l) \exp \left( \frac{l-1-\sqrt{1+8r\sigma^2}}{2} \right) dl,$$

$$y_{L p} =$$

$$\frac{2\sigma^2}{\sqrt{1+8r\sigma^2}} \exp \left( \frac{l-1+\sqrt{1+8r\sigma^2}}{2} \right) \int \beta(l) \exp \left( \frac{l-1-\sqrt{1+8r\sigma^2}}{2} \right) dl$$

$$- \frac{2\sigma^2}{\sqrt{1+8r\sigma^2}} \exp \left( \frac{l+\sqrt{1+8r\sigma^2}}{2} \right) \int \beta(l) \exp \left( \frac{l-1-\sqrt{1+8r\sigma^2}}{2} \right) dl,$$

where the integrals are nonelementary even for simple functional forms of $\beta$, e.g. for $\beta(l) = \frac{\exp(l)}{1+\exp(l)}$, which describes linear benefit from the receivers’ belief.

Imposing the boundary conditions $V_\theta(l) = \frac{\beta(l)}{r}$ and $V_\theta(I) = \frac{\beta(I)}{r}$, the con-
The constants for $H$ are

\[ C_{H1} = \frac{y_{H1}(\bar{t})[\frac{\beta(l)}{r} - y_{Hp}(\bar{t})] - y_{H2}(\bar{t})[\frac{\beta(l)}{r} - y_{Hp}(\bar{t})]}{y_{H1}(\bar{t})y_{H2}(\bar{t}) - y_{H2}(\bar{t})y_{H1}(\bar{t})}, \]

\[ C_{H2} = \frac{-y_{H1}(\bar{t})[\frac{\beta(l)}{r} - y_{Hp}(\bar{t})] + y_{H1}(\bar{t})[\frac{\beta(l)}{r} - y_{Hp}(\bar{t})]}{y_{H1}(\bar{t})y_{H2}(\bar{t}) - y_{H2}(\bar{t})y_{H1}(\bar{t})}. \]

The constants for $L$ are determined by a similar expression, replacing the $H$ subscripts with $L$.

Now that all components of the solutions of the HJB equations have been found, it can be verified that they coincide with the candidate value functions. The ICs remain to be checked.

**Lemma 14.** The solutions $V_H, V_L$ of the HJB equations equal the candidate value functions in the signalling region. The Markov controls for the HJB equations maximize the candidate value functions.

Based on the candidate value function expressions, the signalling region depends on $r$ and $\sigma^2$ only through their product $r\sigma^2$. Closed form comparative statics results are not available for parameters other than $c_L$ due to the complexity of the $V_\theta$ expressions. Numerical simulations will be used instead.

As to $c_L$, the LHS of IC$_L$ in (9) does not contain $c_L$, so there exists $\hat{c}_L$ s.t. for $c_L < \hat{c}_L$, IC$_L$ fails and for $c_L \geq \hat{c}_L$, IC$_L$ holds.

Before turning to numerics, the conditions for the existence of nontrivial interval equilibria are provided.

**Proposition 15.** If $\exists l, \bar{l} \in \mathbb{R}$ satisfying $l < \bar{l}$, $c_H r \sigma^2 < \frac{\beta(l) - \beta(\bar{l})}{l - \bar{l}} < c_L r \sigma^2$, then $\exists l_0 \in (l, \bar{l})$ contained in the signalling region of an interval equilibrium. If $\not{\exists} l, \bar{l} \in \mathbb{R}$ satisfying $l < \bar{l}$, $c_H r \sigma^2 \leq \frac{\beta(l) - \beta(\bar{l})}{l - \bar{l}} \leq c_L r \sigma^2$, then pooling is the unique equilibrium.

The idea of the proof is that if the conditions hold on $(l, \bar{l})$, then there is a small subinterval of $(l, \bar{l})$ on which they also hold and on which $V'_H(l), V'_L(l)$ are close to $\frac{\beta(l)}{r}$. The latter is close to $\frac{\beta(l) - \beta(\bar{l})}{r(l - \bar{l})}$ for an appropriately chosen $l \in (l, \bar{l})$. Therefore $V'_H(l), V'_L(l)$ satisfy the ICs in (9).

It is clear that for any $l, \bar{l} \in \mathbb{R}$, $\sigma^2 > 0$, $c_L > c_H > 0$ and strictly increasing $\beta(l)$, there exists $r \in (0, \infty)$ that makes the sufficient condition in
Proposition 15 hold. One can always find a level of patience for a nontrivial interval equilibrium to exist.

Simple necessary conditions for the ICs are presented in Proposition 16. These provide bounds for the boundaries of equilibrium signalling regions. The region where the necessary conditions fail is a subset of the region where pooling is the unique equilibrium.

**Proposition 16.** For \((\underline{l}, \overline{l})\) to be an equilibrium signalling region, \(\underline{l}, \overline{l}\) must satisfy

\[
\frac{\beta(\overline{l}) - \beta(\underline{l})}{\overline{l} - \underline{l}} \geq c_H \sigma^2 r, \quad \frac{\beta(\overline{l}) - \beta(\underline{l})}{\overline{l} - \underline{l}} \leq c_L \sigma^2 r.
\]

This implies \(\underline{l}, \overline{l} \in [-K, K]\), where \(K \in [0, \infty)\) depends on the parameters.

The necessary conditions in Proposition 16 are illustrated in Figure 2. The figures to follow depict signalling intervals \((\underline{l}, \overline{l})\) as points on a plane, with the \(x\)-coordinate of the point equalling \(\underline{l}\) and the \(y\)-coordinate equalling \(\overline{l}\).

For \(\beta(l) = \frac{\exp(l)}{1+\exp(l)}\), \(c_H = 0.15\), \(c_L = 0.2\) and \(r = \sigma^2 = 1\), the white area in both panels of Figure 2 depicts points \((\underline{l}, \overline{l})\) such that the interval \((\underline{l}, \overline{l})\) cannot be the signalling region of an interval equilibrium. The points \((\underline{l}, \overline{l})\) forming the inner curve of the horseshoe shape satisfy \(\frac{\beta(\overline{l}) - \beta(\underline{l})}{\overline{l} - \underline{l}} = c_H \sigma^2 r\) and the outer curve satisfies \(\frac{\beta(\overline{l}) - \beta(\underline{l})}{\overline{l} - \underline{l}} = c_L \sigma^2 r\).

As \(c_L\) rises, the inner curve of the horseshoe will shrink towards the origin and as \(c_H\) falls, the outer curve of the horseshoe will expand away from the origin (compare the panels of Figure 2). Raising \(r\) or \(\sigma^2\) has the same effect as increasing both \(c_L\) and \(c_H\), i.e. both boundaries of the horseshoe move towards the origin. There exists \(\tau\) depending on the other parameters s.t. for all \(r \geq \tau\), pooling is the unique equilibrium. The reason is that with low patience, \(H\) cannot be incentivized to signal even when \(e^*_H = 1\), \(e^*_L = 0\) is expected. For other expectations, the benefit from signalling is lower, so for \(r \geq \tau\), equilibria with interior effort levels also fail to exist.

Numerical comparative statics on the set of signalling regions are presented next. As in the Poisson signalling game, for some initial log likelihood ratios there is a continuum of informative equilibria. Until the end of this section, it is assumed that \(\beta(l) = \frac{\exp(l)}{1+\exp(l)}\), so the sender’s benefit from the receivers’ belief equals the belief.
For $c_H = 0.1$, $c_L = 0.24$ and $r = \sigma^2 = 1$, the region where the ICs hold is depicted in panel (c) of Figure 3 as the shaded area. Panel (a) shows the area where IC_H holds and panel (b) the area where IC_L holds. The shaded area on panel (c) is the intersection of panels (a) and (b). The disconnectedness of the set of equilibrium signalling regions is in part due to restricting attention to simple equilibria.

The effect of increased patience or reduced noise on the ICs is shown in Figure 4, where $c_H = 0.1$, $c_L = 0.24$, $r = 1$ and $\sigma^2 = 0.5$. Note the different scale of the axes compared to Figure 3. Since $r$ and $\sigma^2$ affect the ICs only through their product, reducing $\sigma^2$ by half has the same effect as reducing $r$ by half.

There need not exist a signalling region containing all others. Such a signalling region is the point at the upper left corner of the shaded area of panel (c), i.e. a point that is simultaneously at maximal horizontal and vertical distance from the diagonal. Figure 5 shows that for $c_H = 0.15$, $c_L = 0.28$ and $r = \sigma^2 = 1$, a higher $l$ permits a higher $\tilde{l}$ for a signalling region. Therefore the union of two equilibrium signalling regions need not be an equilibrium signalling region. This distinguishes the game with Brownian noise from the Poisson signalling game, the repeated noiseless game and the one-shot noisy and noiseless games.

In a given informative equilibrium, the $H$ type payoff can be higher or
Figure 3: Region where ICs hold (shaded) for $c_H = 0.1$, $c_L = 0.24$ and $r = \sigma^2 = 1$. 
Figure 4: Region where ICs hold (shaded) for $c_H = 0.1$, $c_L = 0.24$, $r = 1$ and $\sigma^2 = 0.5$. 
Figure 5: Region where ICs hold (shaded) for $c_H = 0.15$, $c_L = 0.28$, $r = 1$ and $\sigma^2 = 1$. 

(a) $IC_H$ holds

(b) $IC_L$ holds

(c) Both ICs hold
lower than the pooling payoff \( \frac{\exp(l)}{r(1+\exp(l))} \) for different log likelihood ratios. This is illustrated in Figure 6, where \( V_H \) is strictly higher than \( \frac{\exp(l)}{r(1+\exp(l))} \) for \( l \in (-1.5, -0.2) \) and strictly lower for \( l \in (-0.2, 3) \). In this equilibrium, the comparison of informative equilibrium and pooling payoffs of \( H \) accords well with Spence (1973), where for a higher fraction of \( H \) in the population, the payoff difference (separating minus pooling) for \( H \) is lower. In Spence’s model, the reason is that for a higher \( l_0 \) there is less scope for the log likelihood ratio to rise \( (l = \infty \text{ \text{after the high action})}. \) In the present model this mechanism does not work, because the rise in belief after a good signal is highest for intermediate \( l_0 \). Correspondingly the Spence intuition does not always hold in the continuous time model. Close to the upper bound of the signalling region, the payoff from the informative equilibrium rises to the pooling one as \( l_0 \) increases, so in that region, the informative equilibrium payoff minus the pooling payoff rises in \( l \).

![Figure 6: \( V_H \) for signalling region (-1.5, 3) (the curve that is lower on the right), and \( \frac{\exp(l)}{r(1+\exp(l))} \). The parameters are \( c_H = 0.1 \) and \( r = \sigma^2 = 1 \).](image)

For the signalling region \((0, 3)\), with \( c_H = 0.1 \) and \( r = \sigma^2 = 1 \), the informative equilibrium payoff of \( H \) is below pooling in the whole signalling region. Close to the upper bound, the informative equilibrium payoff minus the pooling payoff rises in \( l \), while close to the lower bound, it falls in \( l \). This pattern is reversed in the informative equilibrium with signalling region \((-3, 0)\), \( c_H = 0.1 \) and \( r = \sigma^2 = 1 \). In that case, the informative equilibrium payoff is above pooling in the whole signalling region. Within the signalling region of a given informative equilibrium there is always a region where the payoff difference with pooling moves in the opposite direction to the pre-
diction of Spence (1973). Comparing equilibria with \( \tilde{l} \leq 0 \) to those with \( \tilde{l} \geq 0 \), the Spence pattern holds—shifting the signalling region up raises the informative equilibrium payoff minus the pooling payoff.

In numerical simulations, as \( r \) or \( \sigma^2 \) increases, the payoff of \( H \) from an informative equilibrium minus the pooling payoff falls. Intuitively, patience favours signalling and noise favours pooling. Across values of \( r \) and \( \sigma^2 \), the payoff difference between an informative equilibrium and pooling can be positive or negative.

For the \( L \) type, as \( r \) increases, the payoff from an informative equilibrium minus the pooling payoff rises. Since in the signalling region \( L \) expects the receivers’ log likelihood ratio (and \( L \)’s own future payoff) to fall, the more the future payoff matters, the worse off the occurrence of signalling makes \( L \). As \( \sigma^2 \) increases, \( L \)’s payoff from an informative equilibrium increases—noise is good for \( L \), since the receivers learn about the types more slowly.

So far, only equilibria with effort zero or one have been considered. Other kinds of equilibria also exist: one class is where \( e^*_H \in (0, 1) \) and \( e^*_L = 0 \), another class has \( e^*_H = 1 \) and \( e^*_L \in (0, 1) \). These interior effort equilibria are more difficult to work with than the equilibria where \( e^*_\theta \in \{0, 1\} \) and fewer results are available. They are discussed in Appendix D. A continuum of interior effort equilibria in each class exists for some parameter values.

As in the one-shot noisy signalling game, a natural question arising in the above Brownian model is whether the results are driven by the linear cost. Appendix E solves a Brownian signalling game with quadratic cost. There is a continuum of equilibrium signalling intervals and on each signalling interval, a continuum of equilibrium effort profiles. The reason why many effort profiles on a given interval constitute equilibria is a linear dependence in the first order conditions. This is a feature of the quadratic cost and unlikely to generalize to other convex cost functions.

4 Benchmarks

This section studies the benchmark cases of one-shot noiseless and noisy signalling and repeated noiseless signalling, in order to provide comparisons to the main models of dynamic noisy signalling.

Both the one-shot and repeated noiseless models always have a continuum of separating equilibria in which the payoffs of the types are independent of the initial log likelihood ratio. In the dynamic noisy models of this paper, the
existence and payoffs of nonpooling equilibria depend on \( l_0 \). The noiseless models feature a continuum of pooling equilibria with positive effort, which are absent from the dynamic noisy models.

The one-shot noisy model has at most four equilibria. The existence and payoffs of the nonpooling equilibria depend on \( l_0 \). Pooling on positive effort is never an equilibrium. The effort of \( L \) is lower than that of \( H \) in any equilibrium. This differs from the bad news Poisson model. Unlike in the good news Poisson model, the utility of \( L \) can be higher in an informative equilibrium than in pooling.

4.1 One-shot noiseless signalling

In this section, we recall the game of Spence (1973). Modifications to Spence’s model are pointed out as they are made.

The players are a sender and a competitive market of receivers. The sender is one of two types, \( H \) or \( L \), with initial log likelihood ratio \( l_0 \in \mathbb{R} \).

Both types of sender have action set \([0, \infty)\). A mixed action of type \( \theta \) is a cdf \( F_{\theta} \) on \([0, \infty)\). A pure action is a cdf \( F_{\theta} \) that is constant everywhere except at some \( e \in [0, \infty) \). Such a pure action is written as \( e \) for simplicity. Effort \( e \) costs \( c_{\theta}e \) to type \( \theta \in \{H, L\} \), with \( c_L > c_H > 0 \).

Upon observing an action \( e \), the receivers update the log likelihood ratio from \( l_0 \) to \( l(e) \), using Bayes’ rule where possible. Type \( \theta \) sender’s utility from effort \( e \) and the receivers’ log likelihood ratio \( l \) is \( u_{\theta} = \beta(l) - c_{\theta}e \). In Spence (1973), the sender’s benefit from the receivers’ belief is one plus the belief, i.e. \( \beta(l) = 1 + \frac{\exp(l)}{1+\exp(l)} \).

**Definition 4.** A perfect Bayesian equilibrium consists of (possibly mixed) actions \( F_{L}^* \), \( F_{H}^* \) for types \( L, H \) of the sender and, for each action \( e \) of the sender, a log likelihood ratio \( l(e) \) s.t.

1. if \( e_{\theta} \) is in the support of \( F_{\theta}^* \), then \( e_{\theta} \in \arg \max_{e} \beta(l(e)) - c_{\theta}e \),

2. \( l(e) \) is derived from Bayes’ rule after efforts \( e \) that are in the support of \( F_{L}^* \) or \( F_{H}^* \) and is arbitrary elsewhere.

There is a continuum of separating equilibria with \( e_L^* = 0 \) and \( e_H^* = e^* > 0 \). In them, \( e^* \) satisfies \( \beta_{\min} \geq \beta_{\max} - c_L e^* \) and \( \beta_{\min} \leq \beta_{\max} - c_H e^* \). For \( e < e^* \), the log likelihood ratio is \( l(e) = -\infty \) and for \( e \geq e^* \), \( l(e) = \infty \). The expected utility of \( L \) is the same from all separating equilibria, while
the utility of $H$ decreases in $e^*$. Neither the utilities of the types nor the minimal separating effort $\frac{\beta_{\text{max}} - \beta_{\text{min}}}{c_L}$ depends on $l_0$, which Kreps and Sobel (1994) consider unintuitive.

There is also a continuum of pooling equilibria with $e^*_L = e^*_H = \hat{e} > 0$, where $\beta_{\text{min}} \leq \beta(l_0) - c_L \hat{e}$. For $e < \hat{e} \leq \frac{\beta(l_0)}{c_L}$, $l(e) = -\infty$ and for $e \geq \hat{e}$, $l(e) = l_0$. These equilibria exist even if $c_L = c_H$.

A pooling equilibrium where $e^*_L = e^*_H = 0$ exists for all parameter values, supported by the updating rule $l(e) = l_0 \ \forall e$. The $L$ type has a higher expected utility in the equilibrium with pooling on $e = 0$ than in all the equilibria mentioned previously.

The utility of $H$ is higher in a separating equilibrium prescribing $e^*$ than under pooling on $e = 0$ iff $\beta_{\text{max}} - c_H e^* > \beta(l_0)$. A separating equilibrium in which $H$ has higher utility than in pooling on $e = 0$ exists iff

$$\beta_{\text{max}} - c_H \frac{\beta_{\text{max}} - \beta_{\text{min}}}{c_L} > \beta(l_0).$$

The receivers get zero profit from any equilibrium due to competition, so welfare is $\frac{\exp(l_0)}{1 + \exp(l_0)} u_H + \frac{1}{1 + \exp(l_0)} u_L$. Pooling on zero effort yields lower welfare than the least costly separating equilibrium iff

$$\beta(l_0) \leq \frac{\exp(l_0)}{1 + \exp(l_0)} \left[ \beta_{\text{max}} - c_H \frac{\beta_{\text{max}} - \beta_{\text{min}}}{c_L} \right] + \frac{1}{1 + \exp(l_0)} \beta_{\text{min}}.$$ 

This fails if $\beta(l) = \frac{\exp(l)}{1 + \exp(l)}$, so pooling on zero gives the highest welfare in that case.

If in the game described above the sender’s action is restricted to $[0, 1]$, then a separating equilibrium exists iff $c_L \geq \beta_{\text{max}} - \beta_{\text{min}}$. The other results are unchanged.

### 4.2 One-shot noisy signalling

Consider the game described in the previous section, with the action set of both types being $[0, 1]$. Assume the receivers do not observe the sender’s effort, but only see one of two signals, $g$ or $b$, interpreted as good and bad respectively. Effort increases the probability of $g$, specifically $\Pr(g|e) = \lambda e + \frac{1-\lambda}{2}$, with $\lambda \in (0, 1)$ being the precision of the signal and $e$ the chosen effort.

Both signals have positive probability after any $e$. Denote the log likelihood ratio after signal $x \in \{g, b\}$ by $l(x)$. The strategy the receivers expect
from type $\theta$ is the cdf $F^*_\theta$ on $[0,1]$. The updating rule is
\[
l(x) = l_0 + \ln \frac{\int_0^1 \Pr(x|e) dF_H^*(e)}{\int_0^1 \Pr(x|e) dF_L^*(e)}.
\] (10)

Type $\theta$ sender’s expected utility from pure action $e$ is
\[
-c_\theta e + \left[\lambda e + \frac{1-\lambda}{2}\right] \beta(l(g)) + \left[-\lambda e + \frac{1+\lambda}{2}\right] \beta(l(b)).
\] (11)

**Definition 5.** A perfect Bayesian equilibrium of the one-shot noisy signalling game consists of (possibly mixed) actions $F^*_L, F^*_H$ for types $L,H$ of the sender and log likelihood ratios $l(g), l(b)$ after signals $g, b$ s.t.

1. if $F^*_\theta$ puts positive probability on $e$, then $e$ maximizes (11),
2. given $F^*_L, F^*_H$, the log likelihood ratios $l(g), l(b)$ are derived from (10).

For any mixed strategy of the sender there is an indistinguishable pure strategy yielding the same signal distribution and the same cost. The signal distribution determines the distribution of the benefit. It is therefore without loss to consider only pure strategies. Henceforth ‘strategy’ and ‘equilibrium’ refer to pure strategy and pure equilibrium.

If the two types are expected to take actions leading to the same signal distribution, then $l(g) = l(b) = l_0$. In that case both types would choose action 0 to avoid the signalling cost. Unlike in Spence (1973), there can be no equilibrium with both types taking the same positive effort.

If the receivers expect actions $e^*_L > e^*_H$, then $l(g) < l(b)$ and therefore $\beta(l(g)) < \beta(l(b))$. Then both types would switch to $e = 0$ to avoid the signalling cost and reduce the probability of the $g$ signal.

The previous two paragraphs rule out $e^*_L = e^*_H > 0$ and $e^*_L > e^*_H$, so in equilibrium either $e^*_L = e^*_H = 0$ or $e^*_L > e^*_H$.

It cannot be that both types are taking $e \in (0,1)$ in equilibrium. The benefit from a good signal is the same for both types and the cost of effort is lower for $H$, so when $H$ is indifferent between two different efforts, $L$ strictly prefers $e = 0$.

The set of equilibria is characterized in Proposition 17. Define the maximal and minimal log likelihood ratios reachable from $l_0$ by
\[
\bar{l}(g) = l_0 + \ln \frac{\Pr(g|1)}{\Pr(g|0)}, \quad \bar{l}(b) = l_0 + \ln \frac{\Pr(b|1)}{\Pr(b|0)}.
\]
These are the log likelihood ratios after signals $g, b$ when type $L$ is expected to take the minimal effort $e^*_L = 0$ and type $H$ the maximal effort $e^*_H = 1$. The marginal cost of effort to type $\theta$ is $c_\theta$ at any $e$. The marginal benefit to both types under the expectation $e^*_L = 0$, $e^*_H = 1$ is $\lambda[\beta(\bar{l}(g)) - \beta(\bar{l}(b))]$, as can be seen from (11). This is the maximal feasible marginal benefit of effort.

**Proposition 17.**

1. If $c_H > \lambda[\beta(\bar{l}(g)) - \beta(\bar{l}(b))]$, then the unique equilibrium is $e^*_L = e^*_H = 0$, called pooling.

2. If $c_H = \lambda[\beta(\bar{l}(g)) - \beta(\bar{l}(b))]$, then the equilibria are pooling and $e^*_L = 0$, $e^*_H = 1$, called the max info equilibrium.

3. If $c_H < \lambda[\beta(\bar{l}(g)) - \beta(\bar{l}(b))] \leq c_L$, then the equilibria are pooling, max info and $e^*_L = 0$, $0 < e^*_H < 1$, called interior $e^*_H$.

4. If $c_L < \lambda[\beta(\bar{l}(g)) - \beta(\bar{l}(b))]$, then the equilibria are pooling, interior $e^*_H$ and $0 < e^*_L < 1$, $e^*_H = 1$, called interior $e^*_L$.

The parameter regions for the equilibria are illustrated in Fig. 7 for $\beta(l) = \exp(l) / (1 + \exp(l))$ (the sender’s benefit from the receivers’ belief equals the belief). The prior $\Pr(\mathcal{H}) = \exp(l_0) / (1 + \exp(l_0))$ is on the horizontal axis and the signal precision $\lambda$ on the vertical axis. The equilibria in the top (unshaded) region are pooling, interior $e^*_H$ and interior $e^*_L$. The equilibria in the middle (blue) region are pooling, max info and interior $e^*_H$. Pooling is the unique equilibrium in the bottom (purple) region. As the comparison of the panels indicates, decreasing $c_H$ expands the region where the max info equilibrium exists at the expense of the region where only pooling exists. Decreasing $c_L$ expands the region where the interior $e^*_L$ equilibrium exists and correspondingly shrinks the region where max info exists.

Figure 7 also shows that when $l_0 \to -\infty$ or $l_0 \to \infty$, pooling is eventually the unique equilibrium. This is in contrast to Spence (1973) where separating equilibria exist for all parameter values. Restricting the sender to action set $[0, 1]$ in the Spence game, the existence of separating equilibria depends on $c_L$, $\beta_{\max}$ and $\beta_{\min}$, but not on $l_0$.

An interesting feature of the interior $e^*_L$ equilibrium is that the expected posterior variance decreases in the signal precision. This is because $L$ must be indifferent: $c_L = \lambda[\beta(l(g)) - \beta(l(b))]$. If $\lambda$ rises, $\beta(l(g)) - \beta(l(b))$ must fall to keep equality, therefore $l(g) - l(b)$ must fall. It turns out that the expected
Figure 7: Equilibrium regions for $\beta(l) = \frac{\exp(l)}{1+\exp(l)}$
posterior variance increases in \( l(g) - l(b) \). The fall in \( l(g) - l(b) \) comes from \( e^*_L \) rising to become closer to \( 1 = e^*_H \). Expected posterior variance is a measure of the informativeness of an equilibrium, so in the interior \( e^*_L \) equilibrium, learning increases in the amount of noise.

With \( \beta \) concave over the range \( [l(b), l(g)] \), the \( L \) type expected utility is higher in pooling than in the other three equilibria. This is because \( L \) pays no signalling cost in the pooling equilibrium and expects the receivers’ belief to go down if there is positive signalling. The concavity of \( \beta \) implies that randomness in the posterior also lowers \( L \)’s expected utility. This utility comparison for \( L \) across equilibria fails with \( \beta \) sufficiently convex over the range \( [l(b), l(g)] \), as proved in the following example.

**Example 2.** If \( l_0 = 0, \lambda = \frac{1}{2} \) and \( \beta(l) = \left( \frac{\exp(l)}{1+\exp(l)} \right)^n \), then for \( n \geq 4 \), the utility of \( L \) from pooling is lower than from separating. With these parameters, \( l(g) = \ln 3 \) and \( l(b) = \ln \frac{1}{3} \), so the pooling utility is \( 2^{-n} \) and the separating utility is \( \frac{3}{4} \cdot \frac{1}{4^n} + \frac{1}{4} \cdot \frac{3^n}{4^n} \).

The \( H \) type utility comparison of equilibria depends on the parameters, because if there is any signalling, then \( H \) expects the receivers’ log likelihood ratio to go up, but it has greater variance and \( H \) must pay a signalling cost.

Since the receivers are a competitive market, they get zero profit from any equilibrium and do not affect welfare calculations. Welfare from max info decreases in \( c_H \), is independent of \( c_L \), increases in \( l_0 \). For \( \hat{\beta}(\mu) = \beta(\ln \frac{\mu}{1-\mu}) \) concave in \( \mu \) over the range \( [l(b), l(g)] \), welfare from max info is always lower than from pooling, due to the signalling cost and the variance of the posterior. If \( \beta(l) = \frac{\exp(l)}{1+\exp(l)} \), welfare increases in the signal precision \( \lambda \) iff \( \frac{\exp(l_0)}{1+\exp(l_0)} \geq \frac{(1+\lambda)^2}{2(1+\lambda)^2-2\lambda} \).

One may well wonder whether the results in this section are driven by the linear cost. With a strictly convex cost, the interior \( e^*_H \) and interior \( e^*_L \) equilibria disappear, because a signal distribution from a mixed action can be replicated by a pure action at strictly lower cost. Appendix B solves a game with type \( \theta \) sender’s utility \( \frac{\exp(l)}{1+\exp(l)} - \frac{c_0}{2} \epsilon^2 \), with other elements the same as in this section. There are at most two equilibria: pooling, and for some parameter values an informative equilibrium with \( e^*_H > e^*_L \). The efforts of both types in the informative equilibrium increase in \( \lambda \), the precision of the signal. In addition, \( e^*_H \) increases in \( c_L \) and decreases in \( c_H \). For \( l_0 \geq 0, e^*_H \) and \( e^*_L \) decrease in \( l_0 \), so the efforts in an informative equilibrium, as well as its existence, depend on the prior.
4.3 Repeated noiseless signalling

An overview of the game of Kaya (2009) is given in this section. The notation is changed to match the present paper.

Time is discrete, the horizon infinite. The sender’s action set is a closed real interval. The receiver at time $t$ observes the sender’s actions up to $t$ and updates the log likelihood ratio to $l_t$. Type $\theta$ sender’s per-period utility is $u_\theta(l,e)$, which increases in $l$ and is quasiconcave in $e$ (see Kaya (2009) for details). The discount factor is $\delta \in (0,1)$.

The time $t$ history is defined as the sender’s actions up to $t$. A pure strategy of the sender is a map from types and histories to the action set. A log likelihood ratio system is a map from histories to $[-\infty, \infty]$. A perfect Bayesian equilibrium consists of a strategy and a log likelihood ratio system s.t. given the log likelihood ratio system, the strategy maximizes the discounted future payoff of each type after every history and the log likelihood ratios are derived from the strategy using Bayes’ rule when possible. A separating equilibrium is an equilibrium in which the first period actions of the types differ. In a separating equilibrium, the log likelihood ratio is $\infty$ or $-\infty$ after every on-path history. The focus of Kaya (2009) is on separating pure strategy perfect Bayesian equilibria.

There is a continuum of separating equilibria. In these, the signalling cost can be distributed arbitrarily over time, provided the total cost up to each time $t$ is high enough to deter $L$ from imitating $H$ up to $t$. Formally, the equilibrium effort sequence $(e_\tau)_{\tau=1}^\infty$ of $H$ must satisfy

$$
\sum_{\tau=1}^t \delta^{t-\tau-1}[u_L(\infty,e_\tau) - u_L(-\infty,e_L(-\infty))] \leq 0 \quad \forall t \in \mathbb{R}_+.
$$

The model also admits a continuum of pooling equilibria with positive signalling cost in all or some periods, provided the total cost up to each time $t$ is low enough to deter $L$ from switching to minimal effort. Formally, the effort sequence of both types $(e'_\tau)_{\tau=1}^\infty$ must satisfy

$$
\sum_{\tau=1}^t \delta^{t-\tau-1}[u_L(l_0,e'_\tau) - u_L(-\infty,e_L(-\infty))] \geq 0 \quad \forall t \in \mathbb{R}_+.
$$

The log likelihood ratio system assigns $l_0$ to on-path histories and $-\infty$ to off-path histories.
As in Spence (1973), the existence and payoffs of a separating equilibrium do not depend on the initial log likelihood ratio and pooling equilibria with positive effort exist for all parameter values.

5 Discussion

5.1 Literature

There are many works with some aspects similar to the current paper, which is not surprising, because the fact that signalling takes time was pointed out already in Weiss (1983); Admati and Perry (1987) and noisy signalling was studied already in Matthews and Mirman (1983).

The repeated noiseless signalling games of Kaya (2009); Roddie (2012) differ from the present paper by the noiseless observation of the sender’s action. Their focus is on the existence and structure of least cost separating equilibria, while the current paper endeavours to characterize the equilibrium set. In Kaya (2009), the existence and payoffs of separating equilibria do not depend on the prior (an overview of Kaya (2009) is given in section 4.3). Making the informative equilibria depend on the prior is one of the motivations for using noise in the present paper. Noiseless repeated signalling describes situations in which the market observes the receiver’s actions perfectly. An example is the employers viewing the degree certificates of a job applicant. Noisy signalling models environments in which the effort does not translate deterministically into observables, e.g. a researcher’s work hours do not perfectly correlate with the number of patents.

Continuous time signalling with Brownian noise is considered in Daley and Green (2012b); Gryglewicz (2009); Dilme (2012). In Dilme (2012), the sender decides how much costly effort to exert over time, as well as when to stop the game and receive a final benefit. The leading example is an entrepreneur working to improve the financial indicators of his firm prior to selling it. In Daley and Green (2012b) the uninformed traders receive information (observations of a diffusion process) exogeneously over time and the informed trader decides when to stop the game (execute the trade) and receive a final payoff. Gryglewicz (2009) looks at limit pricing over time. The low-cost incumbent is a commitment type and the high-cost incumbent decides when to stop imitating the low-cost type. In the Brownian signalling model of the present paper, the benefit is received continuously during the
signalling activity. One may think of a hired CEO whose pay comes primarily from stock options and who exerts effort to improve the firm’s financial results. Unlike in Daley and Green (2012b), the signal is endogeneous, and unlike Gryglewicz (2009), both types are strategic.

More distantly related works on repeated noiseless signalling are Nöldeke and Van Damme (1990); Swinkels (1999) where the sender pays the signalling cost first, and receives the benefit only upon deciding to stop signalling forever. An example is completing a traditional education—the salary is received only after graduating. In the current paper, the benefit is received concurrently with the payment of the cost, as when a worker takes continuing education courses while being employed, or a firm advertises while selling its product. Nöldeke and Van Damme (1990) find a unique informative equilibrium and Swinkels (1999) finds a unique pooling equilibrium. The models in the current paper have many informative equilibria and one pooling equilibrium.

The benefit of signalling accrues at the end also in the models of Kremer and Skrzypacz (2007); Hörner and Vieille (2009), where the signalling action is delaying trade. The signaller could be selling a house or a car, and the buyers can interpret quick agreement as a signal of low quality. In the current paper, the sender does not choose whether to trade or not, but exerts a signalling effort, e.g. a firm advertising a product that is already on the shelves of retailers.

One-shot noisy signalling has been studied by Matthews and Mirman (1983); Carlsson and Dasgupta (1997); Daley and Green (2012a). The models in these works describe one-shot interactions, for example a seller of a used car offering a warranty to a buyer. The current paper addresses long-term relationships, such as a politician deciding each year how much to cheat on taxes, and voters remembering all past scandals involving the politician. The motivation for adding noise in Matthews and Mirman (1983); Daley and Green (2012a) is to better describe real-life signalling situations. Carlsson and Dasgupta (1997) use noise to eliminate unintuitive equilibria. Another motivation for adding noise in the present paper is to remove equilibria featuring pooling on positive effort, which exist in Spence (1973).

Noise interfering with the inference process of the receivers is reminiscent of the signal-jamming literature following Fudenberg and Tirole (1986). In signal-jamming, the incumbent tries to prevent the entrant from learning the entrant’s profitability. The present paper describes a situation in which the incumbent tries to convince the entrant that the incumbent is the low-cost
type.

Career concerns models (starting with Holmström (1999)) feature noisy effort over time, similarly to repeated signalling models. However, in the career concerns environment, the sender does not know his own type and the receivers are concerned with the sender’s future actions, not with the type. The present paper focuses on pure signalling, in which the sender knows his type and the action is unproductive. The receivers’ utility depends only on the sender’s type, not the action. Career concerns describe the situation of a manager working hard at his job to convince the employer that he will continue to exert high effort in the future. Signalling models depict the situation of a manager who completes an (unproductive) MBA in order to prove his talent to employers.

A variety of reputation models, starting with Kreps and Wilson (1982); Milgrom and Roberts (1982), share features with dynamic signalling models. Two prominent features of reputations are behaviour that does not occur in a one-shot interaction and a link between past behaviour and expectations about future behaviour (Mailath and Samuelson, 2006). Both features are present in the dynamic models of this paper. An important difference is that reputation models describe situations in which the receivers care about the future actions of the sender, not about the type directly. This is the opposite of signalling, where type matters to the receivers, but future actions do not. The present paper does not make use of commitment types as many reputation models do.

The simple equilibria of the dynamic signalling models of this paper can be interpreted as $H$ paying a cost to build a reputation for being $H$. If the type is $H$, then in expectation the receivers attach greater probability to $H$ over time, so reputation-building is on average successful. Once signalling stops, the belief stops changing, so reputations are in a sense permanent. On the other hand, for beliefs in the signalling region the reputation must constantly be supported at a cost, otherwise it is likely to deteriorate. Therefore in the signalling region, the reputation is transitory. Cripps, Mailath, and Samuelson (2004) show that in a wide class of repeated games, reputation is temporary and the type must eventually be learned. In the Brownian signalling model of the present paper, both types have positive probability of acquiring a ‘false’ permanent reputation, in the sense that when signalling stops, belief about $H$ may be lower than the prior and belief about $L$ may be higher. In expectation, beliefs move in the direction of the sender’s type, but mistakes have positive probability.
5.2 Extensions

The environment this paper focusses on is pure signalling, in which effort has no direct benefit. A natural question is how the results would change with productive effort. Formal models of productive effort in the frameworks used in this paper are left for future research, but this section discusses some anticipated results.

If the receivers value the signal the effort generates (e.g. work results) in addition to the type, then there is a benefit to signalling even when pooling on no effort is expected. If this benefit is small, the equilibrium set is similar to the case where it is zero. The only change is that signalling can be sustained for a slightly larger set of log likelihood ratios. If the reward the receivers offer the sender for a high signal is large enough, then both types are induced to signal and pooling on positive effort results. The receivers valuing current effort instead of the signal leads to the same conclusions as when the signal is valued directly, provided the effort is unobserved and the signal observed.

If the receivers value the future effort they expect from the sender, as in career concerns models, then the flow benefit to the sender depends not only on the current log likelihood ratio, but also on the strategy the receivers expect. Suppose the receivers expect higher future effort from $H$ than from $L$. Then under good news the log likelihood ratio drifts down in the absence of a signal. The effort expected from the sender and the expected type fall in $l$. The sender then has a lower benefit from a lower log likelihood ratio, so the qualitative properties of the pure signalling model are preserved. In the bad news case, the future effort expected from either type may fall in $l$ if pooling is expected at high $l$. If the weight the receivers place on future effort is large enough, the flow benefit of the sender may decrease as $l$ rises towards the pooling region. This does not incentivize $H$ to signal, so pooling on zero effort is the unique equilibrium. The same effect operates in the Brownian model close to the upper boundary of the signalling region, so the same result obtains.

A Proofs omitted from the text

Proof of Lemma 1. $V_\theta(l)$ is bounded above by $\int_0^\infty \exp(-rt)\beta_{\text{max}} dt = \frac{\beta_{\text{min}}}{r} \in \mathbb{R}$ and below by $\frac{\beta_{\text{min}}}{r} \in \mathbb{R}$.

$V_H(l)$ is greater than the payoff to $H$ from imitating an optimal strategy
of $L$ after reaching $l$ for the first time. An optimal strategy gives $L$ the continuation value $V_L(l)$ after $l$. $H$ can imitate an optimal strategy of $L$ at a lower cost, getting the same benefit, so the imitation payoff to $H$ is greater than $V_L(l)$.

Due to the piecewise continuity of the strategies, if $\exists \hat{l}$ s.t. $e_L(\hat{l}) > 0$, then there exists an open interval $(l_1, l_2) \ni \hat{l}$ s.t. $e_L(l') > 0 \forall l' \in (l_1, l_2)$. If the set of histories where reaching some $l$ satisfying $e_L(l) > 0$ has positive probability after $l$ under the optimal strategy, then $H$ can imitate $L$ at a strictly lower cost, getting the same benefit.

**Proof.** In the pooling equilibrium, the receivers’ posterior probability $\mu$ that the sender’s type is $H$ drifts down in expectation for $L$. The flow benefit is increasing in the posterior: $\hat{\beta}'(\mu) > 0$. The posterior has positive variance, which for a concave $\hat{\beta}$ does not raise the payoff of $L$. Therefore the continuation payoff from $l$ on in an informative equilibrium starting is below the pooling payoff $\frac{\beta(l)}{r}$ when starting at $l$.

**Lemma 18.** If $\hat{\beta}(\mu) = \beta \left( \ln \frac{\mu}{1-\mu} \right)$ is concave in $\mu$, then for any equilibrium $e^*$ and for all $l \in L(e^*)$, $V_L(l) \leq \frac{\beta(l)}{r}$ in the good or the bad news model.

**Proof.** In the pooling equilibrium, $V_L(l) = \frac{\beta(l)}{r} \forall l \in L(e^*)$. In an informative equilibrium, the receivers’ posterior probability $\mu$ that the sender’s type is $H$ drifts down in expectation for $L$. The flow benefit is increasing in the posterior: $\hat{\beta}'(\mu) > 0$. The posterior has positive variance, which for a concave $\hat{\beta}$ does not raise the payoff of $L$. Therefore the continuation payoff from $l$ on in an informative equilibrium starting is below the pooling payoff $\frac{\beta(l)}{r}$ when starting at $l$.

**Proof of Lemma 4.** If $e_L^*(l) > e_H^*(l)$, then $L$ weakly prefers $e = 1$ and $H$ weakly prefers $e = 0$. If $V_H(j(l)) = V_L(j(l)) = k$, then due to $V_H \geq V_L$, the jump in value $k - V_H(l)$ after a signal is higher for $L$. The cost of avoiding jumps is strictly larger for $L$, because $c_L > c_H$. It cannot be that $L$ prefers to avoid jumps and $H$ prefers to allow them. When pooling occurs after the jump, then $V_H(j(l)) = V_L(j(l)) = \frac{\beta(l)}{r}$.

If $0 < e_L^*(l) < e_H^*(l) = 1$, then $j(l) = -\infty$. It is enough to show $V_L$ is strictly increasing in the region in which $0 < e_L^* < e_H^* = 1$, because then indifference to the jump at one $l$ implies the absence of indifference at any $\hat{l} \neq l$ in the region. Efforts are continuous from the left or right, which rules
out the situation where $0 < e^*_L(l) < 1$ at one point, with $e^*_L = 0$ or 1 at neighbouring points.

Since $e^*_L < 1$, $V_L$ is unchanged by switching $e_L$ to 0 throughout the region $(\underline{l}, \bar{l})$ in which $0 < e^*_L < e^*_H = 1$. Then the flow rate of jumps and the flow cost are constant in the region, while the flow benefit is strictly increasing. The only influence that might make $V_L$ decreasing is the continuation value at $\bar{l}$. If $\bar{l} = \infty$, then $V_L$ is strictly increasing in the $0 < e^*_L < e^*_H = 1$ region.

With $\bar{l}$ finite and pooling at $\underline{l}$, $V_L(\bar{l}) > V_L(l)$ for $l$ in the $0 < e^*_L < e^*_H = 1$ region, because at $\bar{l}$ the probability of jumping to $l = -\infty$ is zero, the flow benefit is strictly higher and the flow cost is the same as at $l$. Again, $V_L$ is strictly increasing in the $0 < e^*_L < e^*_H = 1$ region. If at $\bar{l}$, a region where $e^*_L > e^*_H$ starts, then $l$ cannot drift into that region, because $l$ drifts down when $e^*_L > e^*_H$. Therefore $l$ must stay at the boundary $\bar{l}$. As shown previously, it cannot be that $e^*_L(l) = 1$ at some $l$ in equilibrium, so $e^*_L(\bar{l}) < 1$. Then $e_L$ can be switched to 0 at $\bar{l}$ without changing $V_L$. The flow benefit is higher than in the $0 < e^*_L < e^*_H = 1$ region, the flow cost and jump rate are the same and the jumps go to a value higher than $\frac{\beta_{\min}}{r}$. A higher $l$ means a shorter time until reaching $V_L(\bar{l})$, so again $V_L$ is strictly increasing in the $0 < e^*_L < e^*_H = 1$ region.

If at $\bar{l}$, a region where $e^*_L = 0$, $e^*_H > 0$ starts, then the union of that and the $0 < e^*_L < e^*_H = 1$ region can be taken and the preceding reasoning can be applied at the upper boundary of the union. \qed

**Lemma 19.** In the bad news model, if $\frac{\beta_{\max} - \beta_{\min}}{r} > \frac{c_H}{\lambda}$, then there exist equilibria in which for some $l \in \mathcal{L}(e^*)$, $e^*_H(l) \in (0, 1)$ and $e^*_L(l) = 0$.

**Proof.** Take $l_0 \in \mathbb{R}$ such that $\frac{\beta(l_0) - \beta_{\min}}{r} > \frac{c_H}{\lambda}$. An equilibrium in which $e^*_H(l) \in (0, 1)$ and $e^*_L(l) = 0$ on $[l_0, \bar{l}) \subset \mathbb{R}$ will be constructed. Assume $\bar{l} - l_0 = \epsilon > 0$ for $\epsilon$ small. The probability of reaching $\bar{l}$ from $[l_0, \bar{l})$ is close to 1, so the payoffs of the types on $[l_0, \bar{l})$ are close to $\frac{\beta(l_0)}{r} > \frac{\beta(l_0)}{r}$.

If the market expects $e^*_H(l) = 1$, $e^*_L(l) = 0$ (which implies that $l$ jumps to $j(l) = -\infty$ when a signal occurs at $l \in [l_0, \bar{l})$), then $H$ has the unique best response $e = 1$ at $l$. If the market expects $e^*_H(l) = e^*_L(l) = 0$ (which implies that $j(l) = l$ when a signal occurs at $l \in [l_0, \bar{l})$), then $H$ has the unique best response $e = 0$ at $l$. By the continuity of $\beta(\cdot)$ and $j(\cdot)$, there exists $\hat{e} \in (0, 1)$ s.t. when the market expects $e^*_H(l) = \hat{e}$, $e^*_L(l) = 0$, then $H$ is indifferent between $e = 1$ and $e = 0$ at $l$. The same reasoning holds for all points in $[l_0, \bar{l})$, with slightly different $\hat{e}$. 

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If $H$ is indifferent between $e = 1$ and $e = 0$, then $L$ strictly prefers $e = 0$. \hfill \square

**Proposition 20.** If the following hold

(a) $\lim_{l \to l_1} V_H(l) - \frac{\beta_{\min}}{r} \geq \frac{c_H}{\lambda}$,

(b) $\lim_{l \to \infty} V_L(l) - \frac{\beta_{\min}}{r} \leq \frac{c_L}{\lambda}$,

(c) $\frac{\beta(l)}{r} - \lim_{l' \to l_1} V_H(l') < \frac{c_H}{\lambda}$,

(d) $\frac{\beta(l)}{r} - \lim_{l' \to l_1} V_L(l') > \frac{c_L}{\lambda}$,

(e) $\frac{\beta(l)}{r} - \lim_{l' \to \infty} V_L(l') < \frac{c_L}{\lambda}$,

where $-\infty < l < l_0 < l_1$ and $V_H, V_L$ are given in terms of primitives in (6), then there exists an equilibrium in which

- $e^*_L(l) > e^*_H(l)$ if $l \in (l_1, l_0]$,
- $e^*_L(l) = 0$, $e^*_H(l) = 1$ if $l \in [l_1, \infty)$,
- $e^*_L(l) = e^*_H(l) = 0$ if $l \notin (l_1, l_0] \cup [l_1, \infty)$.

**Proof of Proposition 20.** The candidate value functions of the types in $[l_1, \infty)$ are continuous and strictly increasing according to (6).

For $H$ to optimally choose $e = 1$ in $[l_1, \infty)$, it is necessary and sufficient that $V_H(l) - \frac{\beta_{\min}}{r} \geq \frac{c_H}{\lambda}$ for all $l \in [l_1, \infty)$. Since $V_H$ is strictly increasing, the inequality holds for all $l \in [l_1, \infty)$ iff $\lim_{l \to l_1} V_H(l) - \frac{\beta_{\min}}{r} \geq \frac{c_H}{\lambda}$. A lower bound on $\lim_{l \to l_1} V_H(l)$ is $\frac{\beta(l_1) - c_H}{r}$.

For $L$ to optimally choose $e = 0$ in $[l_1, \infty)$, it is necessary and sufficient that $V_L(l) - \frac{\beta_{\min}}{r} \leq \frac{c_L}{\lambda}$ for all $l \in [l_1, \infty)$. Since $V_L$ is strictly increasing, the inequality holds for all $l \in [l_1, \infty)$ iff $\lim_{l \to \infty} V_L(l) - \frac{\beta_{\min}}{r} \leq \frac{c_L}{\lambda}$. The limit is $\lim_{l \to \infty} V_L(l) = \frac{\beta_{\max}}{r+\lambda} + \frac{\lambda \beta_{\min}}{r(r+\lambda)}$.

In $(l_1, l_0]$ in the absence of a signal $l$ drifts down and eventually reaches $l$ with positive probability. For any $l, l' \in (l_1, l_0]$ s.t. $l' < l$, the probability of reaching $l'$ from $l$ approaches 1 as $l - l' \to 0$, therefore for any $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ s.t. $l_0 - l < \epsilon_2$ implies $|V_0(l) - \frac{\beta(l)}{r}| < \epsilon_1$ for all $l \in (l_1, l_0]$ and $\theta = 1, 2$. 

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For all \( l \in (l, l_0] \), take \( e^*_H(l) = 0 \) and \( e^*_L(l) \in (0, 1) \) such that \( j(l) \in [l_1, \infty) \), where \( j(l) = l + \ln \frac{1-e^*_H(l)}{1-e^*_L(l)} \).

For \( H \) to optimally choose \( e = 0 \) at \( l \in (l, l_0] \), it is sufficient that \( V_H(l) - V_H(j(l)) < \frac{c_H}{\lambda} \). Sufficient for this is \( V_H(l) - \lim_{l' \to l_1} V_H(l') < \frac{c_H}{\lambda} \). Choose \( \varepsilon_2 \) s.t. \( |\frac{\beta(l)}{r} - V_H(l)| \leq \varepsilon_1 < \frac{c_H}{\lambda} - V_H(l) + \lim_{l' \to l_1} V_H(l') \). Then sufficient conditions for \( H \) to choose \( e = 0 \) are \( l_0 - l < \varepsilon_2 \) and \( \frac{\beta(l)}{r} - \lim_{l' \to l_1} V_H(l') < \frac{c_H}{\lambda} \).

For \( L \) to optimally choose \( e_L \in (0, 1) \) at \( l \in (l, l_0] \), it is necessary and sufficient that \( V_L(l) - V_L(j(l)) = \frac{c_L}{\lambda} \). By continuity of \( V_L \) on \([l_1, \infty)\), sufficient for this are \( V_L(l) - \lim_{l' \to l_1} V_L(l') = \frac{c_L}{\lambda} \) and \( V_L(l) - \lim_{l' \to \infty} V_L(l') < \frac{c_L}{\lambda} \). Then by reasoning similar to the \( H \) case in the previous paragraph, sufficient for \( e_L \in (0, 1) \) are \( l_0 - l < \varepsilon_2 \), \( \frac{\beta(l)}{r} - \lim_{l' \to l_1} V_L(l') = \frac{c_L}{\lambda} \) and \( \frac{\beta(l)}{r} - \lim_{l' \to \infty} V_L(l') < \frac{c_L}{\lambda} \).

**Proof of Proposition 5.** If \( \frac{\beta_{\max} - \beta_{\min}}{r} > \frac{c_H}{\lambda} \), then \( \exists l_0 \) for which an informative simple equilibrium can be constructed. Define \( y \) by

\[
y = \begin{cases} 
\infty & \text{if } \frac{\beta_{\max} - \beta_{\min}}{r} \leq \frac{c_H}{\lambda}, \\
\beta^{-1} \left( \frac{\beta_{\min}}{r} + \frac{c_L}{\lambda} \right) & \text{if } \frac{\beta_{\max} - \beta_{\min}}{r} > \frac{c_H}{\lambda}.
\end{cases}
\]

Take \( l \in (\beta^{-1} \left( \frac{\beta_{\min}}{r} + \frac{c_H}{\lambda} \right), y) \) in the simple equilibrium, so that \( H \) has a strict incentive to signal at \( \bar{l} \) and \( L \) has a strict incentive not to signal (recall that \( V_H(\bar{l}) = V_L(\bar{l}) = \frac{\beta(\bar{l})}{r} \)). By continuity and strict increasingness of \( V_H, V_L \), \( \exists \epsilon > 0 \) s.t. \( V_H(\bar{l} - \epsilon) \geq \frac{\beta_{\min}}{r} + \frac{c_H}{\lambda} \) and \( V_L(\bar{l} - \epsilon) \leq \frac{\beta_{\min}}{r} + \frac{c_L}{\lambda} \), so \( H \) has an incentive to signal at \( \bar{l} - \epsilon \) and \( L \) has an incentive not to signal. Take \( l_0 = \bar{l} - \epsilon \). This completes the construction of the simple equilibrium. \( \square \)

**Proof of Lemma 7.** If \( e^*_L = e^*_H \), then the log likelihood ratio stays constant regardless of the occurrence or absence of signals. Then both types optimally choose \( e_0 = 0 \) to avoid the effort cost. This rules out \( e^*_L = e^*_H > 0 \) occurring in equilibrium.

Three steps are needed to rule out \( e^*_L > e^*_H \). First, in equilibrium \( L \) always has a best response that avoids jumps up. Second, in the region where \( e^*_L > e^*_H \), \( V_L(l) \) is bounded below by \( \frac{\beta(l)}{r} \). Third, in the absence of jumps, \( V_L \) is increasing. If \( V_L \) is increasing, then \( L \) optimally does not exert effort to make \( l \) jump down.

Step 1. If \( e^*_L > e^*_H \), then jumps go down. If \( e^*_L < e^*_H \), then jumps go up, but \( e^*_L < 1 \), so \( e = 0 \) is a best response for \( L \). No jumps occur with \( e = 0 \).
Step 2. In the region where \( e_L^* > e_H^* \), the log likelihood ratio drifts up and jumps down. Taking \( e = 0 \) avoids jumps and the flow cost. Starting at \( l \), the flow benefit is at least \( \beta(l) \) due to the upward drift.

Step 3. Consider \( \hat{l} < l' \), with \( l' \) in the region where \( e_L^* > e_H^* \). At \( \hat{l} \), \( L \) has a best response that avoids jumps. At \( l' \), \( V_L \) is at least \( \frac{\beta(l')}{r} \), which is the payoff to a strategy that avoids jumps. If the paths of \( l \) starting at \( \hat{l} \) and \( l' \) never cross, then the flow benefit starting from \( \hat{l} \) is always strictly below \( \beta(l') \) and the cost is weakly higher. In that case \( V_L(\hat{l}) < V(l') \). If the paths of \( l \) starting at \( \hat{l} \) and \( l' \) cross, then starting from \( l' \), the strategy that takes \( e = 0 \) until the paths cross and reverts to the optimal strategy thereafter yields a strictly higher payoff than the optimal strategy starting from \( \hat{l} \). Before the crossing, the flow benefit starting from \( l' \) is strictly higher and the flow cost weakly lower than starting from \( \hat{l} \). After the crossing, the payoffs are the same. As before, \( V_L(\hat{l}) < V(l') \).

Consider an interval \( (\hat{l}_1, l_0) \) in which \( e_L^* = 0 < e_H^* < 1 \). Type \( H \) must be indifferent, so switching type \( H' \)'s choice from \( e_H^* \) to 0 in the whole \( (\hat{l}_1, l_0) \) does not change \( V_H \). If \( e = 0 \), then \( l \) drifts down deterministically to \( \hat{l}_1 \), and if \( \hat{l}_1 > -\infty \), reaches it and stops there forever. Consider \( l', l'' \in (\hat{l}_1, l_0) \), with \( l' > l'' \). Starting at \( l' \) or \( l'' \) yields flow cost zero. Starting at \( l' \) yields initially a strictly higher flow benefit than starting at \( l'' \), and later (when \( l_0 \) is reached) a weakly higher flow benefit. So \( V_H \) is strictly increasing in \( (\hat{l}_1, l_0) \).

The jumps from \( (\hat{l}_1, l_0) \) go to \( l = \infty \), due to \( e_L^* > e_H^* = 0 \). So if \( H \) is indifferent between \( e > 0 \) and \( e = 0 \) at some \( l^* \) \( (\hat{l}_1, l_0) \), he is not indifferent at any \( l \neq l^* \) in \( (\hat{l}_1, l_0) \). This rules out \( e_L^* = 0 < e_H^* < 1 \) occurring over intervals of positive length in equilibrium. Efforts are continuous from the left or right in \( l \), so the situation where \( e_L^*(l) = 0 < e_H^*(l) < 1 \) at one point, with \( e_H^* \) either 0 or 1 at neighbouring points is ruled out. \( \square \)

**Lemma 21.** \( \forall l_0 \in \mathbb{R} \Rightarrow \forall l \in L(e^*) \), the following are ruled out:

(a) \( e_L^*(l), e_H^*(l) \in (0, 1) \forall l \in (\hat{l}, l_0) \), with \( \hat{l} > -\infty \) and \( e_L^*(\hat{l}) = e_H^*(\hat{l}) = 0 \),

(b) \( e_L^*(l), e_H^*(l) \in (0, 1) \forall l \in (\hat{l}, l_0) \), with \( j(l) \in \mathcal{P}(e^*) \).

**Proof.** Lemma 7 ruled out \( e_L^*(l) = e_H^*(l) > 0 \) and \( e_L^*(l) > e_H^*(l) \), so \( e_L^*(l), e_H^*(l) \in (0, 1) \) implies \( e_L^*(l) < e_H^*(l) \) and \( j(l) > l \). By Lemma 1, \( V_H(j(l)) \geq V_L(j(l)) \).

For every \( \epsilon > 0 \), there exists \( \epsilon_2 > 0 \) s.t. \( |\hat{l} - l| < \epsilon_2 \) implies that the probability of reaching \( \hat{l} \) from \( l \) is \( 1 - \epsilon \). For every \( \epsilon_3 > 0 \), there exists \( \epsilon > 0 \) s.t. if the probability of reaching \( \hat{l} \) from \( l \) is \( 1 - \epsilon \), then \( |V_\theta(l) - V_\theta(\hat{l})| < \epsilon_3 \). Due to
pooling at \( l \). \( V_H(l) = V_L(l) = \frac{\beta(l)}{\lambda} \). Combining this with \( V_H(j(l)) \geq V_L(j(l)) \) and \( c_H < c_L \), it is clear that \( V_H(j(l)) - V_H(l) = \frac{c_H}{\lambda} \) implies \( V_L(j(l)) - V_L(l) < \frac{c_L}{\lambda} \). Both types cannot simultaneously be indifferent to signalling at \( l \) close to \( l \).

If \( j(l) \in P(e^*) \) and \( e_H^*(l), e_L^*(l) \in (0, 1) \), then the indifference condition at \( l \) is \( \frac{\beta(j(l))}{r} - V_\theta(l) = \frac{\theta}{\lambda} \) for \( \theta = H, L \). Take \( l_1, l_2 \in (l, l_0) \) with \( l_1 < l_2 \). Since \( l \) drifts down in the absence of signals and \( l_1 \) is reachable from \( l_2 \), \( V_\theta(l_1) \) is a continuation value from the perspective of \( l_2 \). We have \( V_\theta(l_2) = \alpha V_\theta(l_1) + k \), with \( \alpha \in (0, 1) \) and \( k \in \mathbb{R} \). If \( l > -\infty \), the constants \( \alpha, k \) are derived from

\[
V_\theta(l) = \exp \left( -\int_l^l \frac{rdz}{\lambda(e^*_H(z) - e^*_L(z))} \right) V_\theta(l)
+ \int_l^l \frac{\beta(x)}{\lambda(e^*_H(x) - e^*_L(x))} \exp \left( -\int_x^l \frac{rdz}{\lambda(e^*_H(z) - e^*_L(z))} \right) dx,
\]

and if \( l = -\infty \), then from

\[
V_\theta(l) = \int_{-\infty}^l \frac{\beta(x)}{\lambda(e^*_H(x) - e^*_L(x))} \exp \left( -\int_x^l \frac{rdz}{\lambda(e^*_H(z) - e^*_L(z))} \right) dx.
\]

The expressions given for \( V_\theta(l) \) satisfy the ODE \( rV_\theta(l) = \beta(l) - \lambda(e^*_H(l) - e^*_L(l))V_\theta'(l) \) derived from the HJB equation using the indifference condition \( \lambda\left[\frac{\beta(j(l))}{r} - V_\theta(l)\right] - c_\theta = 0 \). The HJB equation and the indifference condition are necessary for equilibrium.

The indifference conditions for both types for \( l_1, l_2 \) contradict each other: subtracting \( \lambda\left[\frac{\beta(j(l_1))}{r} - V_\theta(l_1)\right] - c_L = 0 \) from \( \lambda\left[\frac{\beta(j(l_1))}{r} - V_H(l_1)\right] - c_H = 0 \), we get \( \lambda[-V_H(l_1) + V_L(l_1)] = c_H - c_L \). Subtracting \( \lambda\left[\frac{\beta(j(l_1))}{r} - \alpha V_L(l_1) - k\right] - c_L = 0 \) from \( \lambda\left[\frac{\beta(j(l_1))}{r} - \alpha V_H(l_1) - k\right] - c_H = 0 \), we get \( \lambda\alpha[-V_H(l_1) + V_L(l_1)] = c_H - c_L \). This contradicts \( \alpha \in (0, 1), \lambda > 0, c_L > c_H > 0 \).

**Lemma 22.** In the good news model, if \( \frac{\beta_{\max} - \beta_{\min}}{r} > \frac{c_L}{\lambda} \), there exist equilibria in which for some \( l \in L(e^*) \), \( e_H^*(l) = 1 \) and \( e_L^*(l) \in (0, 1) \).

**Proof.** Take \( l_0 \in \mathbb{R} \) such that \( \frac{\beta_{\max} - \beta(l_0)}{r} > \frac{c_L}{\lambda} \). An equilibrium in which \( e_H^*(l) = 1 \) and \( e_L^*(l) \in (0, 1) \) on \((l, l_0) \subset \mathbb{R} \) will be constructed. Assume \( l_0 - l = \epsilon > 0 \) for \( \epsilon \) small. The probability of reaching \( l \) from \((l, l_0) \) is close to 1, so the payoffs of the types on \((l, l_0) \) are close to \( \frac{\beta(l)}{r} - \frac{\beta(l_0)}{r} \).

If the market expects \( e_H^*(l) = 1 \), \( e_L^*(l) = 0 \) (which implies that \( l \) jumps to \( j(l) = \infty \) when a signal occurs at \( l \in (l, l_0) \)), then \( L \) has the unique best
response \( e = 1 \) at \( l \). If the market expects \( e^*_H(l) = e^*_L(l) = 1 \) (which implies that \( j(l) = l \) when a signal occurs at \( l \in (l_0, l_1) \)), then \( L \) has the unique best response \( e = 0 \) at \( l \). By the continuity of \( \beta(\cdot) \) and \( j(\cdot) \), there exists \( \hat{e} \in (0, 1) \) s.t. when the market expects \( e^*_H(l) = \hat{e}, \ e^*_H(l) = 1 \), then \( L \) is indifferent between \( e = 1 \) and \( e = 0 \) at \( l \). The same reasoning holds for all points in \([l_0, l_1)\), with slightly different \( \hat{e} \).

If \( L \) is indifferent between \( e = 1 \) and \( e = 0 \), then \( H \) strictly prefers \( e = 1 \).

**Proof of Lemma 8.** If \( l = \infty \) or \( l = -\infty \), then \( l \) does not respond to signals, so clearly neither type will take positive effort.

The drift in \( l \) is finite and the discount rate \( r \) is positive, so for any \( \epsilon > 0 \) \( \exists T > 0 \) s.t. the flow payoff after time \( T \) contributes less than \( \epsilon \) to total payoff. For any \( \epsilon > 0 \) and \( T > 0 \) \( \exists \hat{l} \in \mathbb{R} \) s.t. starting at \( l \), after drifting down at rate \( \lambda \) for length of time \( T \), the log likelihood ratio \( l \) reached satisfies \( |\beta(l) - \beta_{\text{max}}| < \epsilon \).

The quantity \( V_H(\hat{l}) \) is bounded below by the payoff from taking \( e = 0 \) forever, which makes the rate of jumps zero. The payoff from taking \( e = 0 \) forever starting at \( \hat{l} \) is bounded below by \( \frac{\beta(l)}{r} - \epsilon \), where \( l \) is reached from \( \hat{l} \) after length of time \( T \). Therefore \( |V_H(\hat{l}) - \frac{\beta_{\text{max}}}{r}| < 2\epsilon \).

Type \( H \)'s cost of choosing \( e = 1 \) over a time interval \( \Delta \) is \( c_H \Delta \) and, starting at \( \hat{l} \), the benefit is bounded above by \( |V_H(\hat{l}) - \frac{\beta_{\text{max}}}{r}||1 - \exp(-\lambda \Delta)| \).

Thus at \( \hat{l} \) there exists \( \epsilon > 0 \) s.t. the optimal choice of \( H \) is \( e = 0 \). If \( e^*_H(l) > 0 \) is expected for \( l \geq \hat{l} \), then \( H \) will deviate to \( e = 0 \).

If the expectations are \( e^*_L \in (0, 1), e^*_H = 1 \), then jumps end at some \( j(l) < \infty \), which implies a smaller benefit to signalling than in the \( e^*_L = 0, e^*_H = 1 \) case. The previous reasoning still holds, with an even stronger incentive not to signal above \( \hat{l} \).

**Proof of Proposition 9.** \( V_H \) is bounded above by \( \frac{\beta_{\text{max}}}{r} \) and below by \( \frac{\beta_{\text{min}}}{r} \). If \( \frac{\beta_{\text{max}} - \beta_{\text{min}}}{r} \leq \frac{c_H}{\lambda} \), then even jumps from \( l = -\infty \) to \( l = \infty \) at rate \( \lambda \) do not provide enough benefit to outweigh the cost for \( H \). Thus \( H \) will not signal in this case. For any \( l \in (-\infty, \infty) \), jumps from \( V_H(l) \) to \( \frac{\beta_{\text{max}}}{r} \) are smaller than the jumps from \( \frac{\beta_{\text{min}}}{r} \) to \( \frac{\beta_{\text{max}}}{r} \).

If \( \frac{\beta_{\text{max}} - \beta_{\text{min}}}{r} > \frac{c_H}{\lambda} \), then for some \( l_0 \) an interval equilibrium can be constructed. Define \( y \) as follows. If \( \frac{\beta_{\text{max}} - \beta_{\text{min}}}{r} \leq \frac{c_H}{\lambda} \), then set \( y = -\infty \), otherwise set \( y = \beta^{-1} \left( \frac{\beta_{\text{max}}}{r} - \frac{c_H}{\lambda} \right) \). Take \( l \in (y, \beta^{-1} \left( \frac{\beta_{\text{max}}}{r} - \frac{c_H}{\lambda} \right)) \) in the interval equilibrium, so that \( H \) has a strict incentive to signal at \( l \) and \( L \) has a strict
incentive not to signal (recall that $V_H(l) = V_L(l) = \frac{\beta(l)}{\tau}$). By continuity and strict increasingness of $V_H, V_L$, $\exists \epsilon > 0$ s.t. $V_H(l + \epsilon) \leq \frac{\beta_{\max}}{\tau} - \frac{c_H}{\lambda}$ and $V_L(l + \epsilon) \geq \frac{\beta_{\max}}{\tau} - \frac{c_L}{\lambda}$, so $H$ has an incentive to signal at $l + \epsilon$ and $L$ has an incentive not to signal. Take $l_0 = l + \epsilon$. This completes the construction of an interval equilibrium.

Proof of Lemma 11. Due to the boundedness of $\beta(l)$ and $e_\theta$, discounting ensures that $V_\theta$ is finite—even without the expectation, the integral in the definition of $V_\theta$ is finite for any path of $l$ and any control $e_\theta$.

It is clear that $V_H \geq V_L$ in the signalling region, because $H$ can follow $L$’s strategy at a lower cost than $L$. Outside the signalling region, $V_H(l) = V_L(l) = \frac{\beta(l)}{\tau}$.

If there is positive probability of reaching $\hat{l}$ with the optimal $e_L(\hat{l}) > 0$, then $H$ can follow $L$’s strategy at a strictly lower cost.

To prove $V_\theta$ is strictly increasing, a standard coupling argument is used. Consider two diffusion processes: the $l$ process with optimal effort starting from $l_1$ and the $\hat{l}$ process under zero effort starting from $l_2 > l_1$. Call the former process $l^e$ and the latter $l^0$. Define the stopping time $\tau^* = \inf \{t > 0 : l^0_t = l^e_t = 0\}$. The receivers expect the optimal strategy in both cases.

Starting at $l_2$, the strategy $s = \text{“play 0 until } \tau^* \text{ and the optimal strategy thereafter”}$ yields a weakly lower payoff than $V_\theta(l_2)$, the payoff to the optimal stationary strategy starting from $l_2$. This holds even though $s$ is not stationary, because if the receivers expect a stationary strategy, then among the optimal strategies for the sender there is a stationary one. The argument is standard—the competitive receivers always play a static best response, which depends on their belief about the type, but not the sender’s strategy, so if at some $l$, a sender action $\hat{e}$ is optimal at one point in time, then $\hat{e}$ is optimal at that $l$ at another point in time.

Starting at $l_2$, the strategy $s$ yields a strictly higher payoff than $V_\theta(l_1)$, the payoff to the optimal strategy starting from $l_1$. This is because the revenue $\beta(l^0)$ is strictly higher than $\beta(l^e)$ before $\tau^*$ and the same in expectation after $\tau^*$. The cost of $l^0$ is zero while the cost of $l^e$ is positive before $\tau^*$. The costs of the two strategies are the same in expectation after $\tau^*$. Overall, $V_\theta(l_2) > V_\theta(l_1)$ for $l_1, l_2$ in the signalling region.

If both $l_1, l_2$ are outside the signalling region, then since $\beta$ was assumed strictly increasing, the payoffs are ordered $V_\theta(l_2) = \frac{\beta(l_2)}{\tau} > \frac{\beta(l_1)}{\tau} = V_\theta(l_1)$.
If \( l_2 \) is above the signalling region while \( l_1 \) is in the signalling region, then the expected benefit is strictly higher from \( l_2 \) onwards and the expected cost is the lowest possible from \( l_2 \) onwards, so \( V_\theta(l_2) > V_\theta(l_1) \). If \( l_2 \) is in the signalling region while \( l_1 \) is below the signalling region, then \( V_\theta(l_2) \) is higher than the payoff to the strategy of taking zero effort forever starting from \( l_2 \). The cost of this strategy is the same as the cost of the optimal strategy from \( l_1 \) onwards, while the benefit is strictly greater, so again \( V_\theta(l_2) > V_\theta(l_1) \).

**Proof of Lemma 12.** If \( e_H^*(l) = e_L^*(l) \), then the signal is statistically uninformative about the type, so the log likelihood ratio does not respond to the signal. Given this, both types will optimally choose \( e^\theta_l(l) = 0 \). Therefore in equilibrium, it cannot be that \( e_L^*(l) = e_H^*(l) > 0 \) for some \( l \).

A higher expected signal is more costly to both types. If \( e_L^*(l) > e_H^*(l) \), then based on (8), \( l \) falls in response to a higher signal. The benefit of signalling only depends on the log likelihood ratio, with a higher \( l \) giving a higher benefit. So if \( e_L^*(l) > e_H^*(l) \) is expected, then both types optimally choose \( e^\theta_l(l) = 0 \).

**Proof of Proposition 13.** \( L \) takes no effort in any simple equilibrium, including pooling, so the flow cost is the same in both cases. The flow benefit comparison is unaffected if \( \beta(l) \) is written as \( \beta(f^{-1}(f(l))) \) for some strictly increasing smooth \( f \). Use Itô’s rule to derive the process \( f(l) \):

\[
  df = \left[ \sigma^{-2}(e_H^* - e_L^*) \left( e_L - \frac{1}{2}(e_H^* + e_L^*) \right) \frac{df}{dl} + \frac{1}{2\sigma^2}(e_H^* - e_L^*)^2 \frac{d^2f}{dl^2} \right] dt + \frac{e_H^* - e_L^*}{\sigma} \frac{df}{dl} dB_t.
\]

The drift of \( f \) is zero iff \( \frac{d^2f}{dl^2} = -2\frac{e_L - \frac{1}{2}(e_H^* + e_L^*)}{e_H^* - e_L^*} \frac{df}{dl} \). Impose the equilibrium condition \( e_L = e_L^* \) and recall that in interval equilibria, \( e_H^* = 1 \) and \( e_L^* = 0 \). This leads to \( \frac{d^2f}{dl^2} = \frac{df}{dl} \). Using the normalization \( f(0) = 1, f'(0) = 1 \), we get \( f(l) = \exp(l) \) and \( f^{-1}(z) = \ln(z) \).

If \( \beta(\ln(z)) \) is concave in \( z \), which has zero drift, then the expectation of \( \beta(\ln(z)) \) decreases in the variance of \( z \). The variance of \( z \) is strictly increasing in the variance of \( l \). In the pooling equilibrium, \( l \) is constant, but in informative equilibria, it has positive variance. A similar reasoning establishes that if \( \beta(\ln(z)) \) is convex in \( z \), then the payoff of \( L \) in any informative equilibrium is above the pooling payoff.
For \( l \in \{ \tilde{l}_1, \tilde{l}_1 \} \), it follows from the above that \( V_{L1}(l) = \frac{\beta(l)}{r} \geq V_{L2}(l) \).

From any point in \((\tilde{l}_1, \tilde{l}_1)\), the log likelihood ratio process has positive probability of hitting \( \tilde{l}_1 \) and positive probability of hitting \( \tilde{l}_1 \). The flow cost to \( L \) is zero in all simple equilibria for all \( l \). For the same \( l \), the flow benefit to \( L \) is the same in all simple equilibria. The distribution over paths of \( l \) up to hitting \( \tilde{l}_1 \) or \( \tilde{l}_1 \) starting from \( l_0 \in (\tilde{l}_1, \tilde{l}_1) \) is the same in the two equilibria with signalling regions \((\tilde{l}_1, \tilde{l}_1)\) and \((\tilde{l}_2, \tilde{l}_2)\), because in both equilibria in the region \((\tilde{l}_1, \tilde{l}_1)\), \( H \) takes action 1 and \( L \) takes 0. Therefore the continuation value comparisons \( V_{L1}(\tilde{l}_1) \geq V_{L2}(\tilde{l}_1) \) and \( V_{L1}(\tilde{l}_1) \geq V_{L2}(\tilde{l}_1) \) determine the payoff comparison \( V_{L1}(l) \geq V_{L2}(l) \) for any \( l \in (\tilde{l}_1, \tilde{l}_1) \). \( \blacksquare \)

**Proof of Lemma 14.** For any signalling region \((l, \tilde{l})\), the solutions of the ODEs are differentiable at least as many times as \( \beta \) on \((l, \tilde{l})\) and continuous on \([l, \tilde{l}]\).

Since \( \beta \) was assumed twice continuously differentiable, \( V_L \) and \( V_H \) are as well. Given the signalling region, \( V_H, V_L \) are bounded for any path of \( l \) and control \( e_\theta \). Therefore \( V_H(l), V_L(l) \) are integrable in the probability law of the \( l \) process that starts from \( l_0 \) and is controlled by \( e_\theta \), uniformly over Markov controls \( e_\theta \). So by Theorem 11.2.2 of Øksendal (2010), \( V_L, V_H \) coincide with the value functions \( V_L, V_H \).

Under the previous conditions, Theorem 11.2.3 of Øksendal (2010) shows that the optimal Markov control does as well as the optimal nonanticipating control, so if the receivers expect Markov strategies, then both types of the sender have a Markov best response among their best responses. This does not imply that the payoffs of all non-Markov equilibria can be attained with Markov equilibria, since in a non-Markov equilibrium the receivers expect non-Markov strategies. \( \blacksquare \)

**Proof of Proposition 15.** \( \beta \in C^2 \), so by the Mean Value Theorem there exists \( l_0 \in (\beta, \tilde{l}) \) s.t. \( c_H r \sigma^2 < \beta'(l_0) < c_L r \sigma^2 \). By continuity of \( \beta'(l) \) there exist \( c_H r \sigma^2 + \delta < \frac{\beta(\tilde{l}_1) - \beta(l_0)}{\tilde{l}_1 - l_0} < c_L r \sigma^2 - \delta \).

The candidate value functions \( V_L, V_H \) calculated in the main text are infinitely differentiable in the signalling interval and satisfy the boundary conditions \( V_H(\tilde{l}_1) = V_L(\tilde{l}_1) = \frac{\beta(\tilde{l}_1)}{r} \) and \( V_H(\tilde{l}_1) = V_L(\tilde{l}_1) = \frac{\beta(\tilde{l}_1)}{r} \), so by the Mean Value Theorem there exist \( \tilde{l}_1, \tilde{l}_1 \in (\tilde{l}_1, \tilde{l}_1) \) satisfying \( V_L(\tilde{l}_1) - \beta(\tilde{l}_1) = \frac{\beta(\tilde{l}_1) - \beta(l_0)}{\tilde{l}_1 - l_0} \) and \( V_H(\tilde{l}_1) - \beta(\tilde{l}_1) = \frac{\beta(\tilde{l}_1) - \beta(l_0)}{\tilde{l}_1 - l_0} \). By the smoothness of \( V_L, V_H \), for all \( \epsilon_2 > 0 \) there
exists \( \epsilon_3 > 0 \) small enough s.t. if \( \bar{l}_1 - l_1 < \epsilon_3 \), then

\[
\max_{l \in (l_1, \bar{l}_1)} \left| V_L'(l) - \frac{\beta(l_1) - \beta(l)}{\bar{l}_1 - l_1} \right| + \left| V_L'(l) - \frac{\beta(l_1) - \beta(l)}{\bar{l}_1 - l_1} \right| < \epsilon_2.
\]

Therefore for all \( l \in (l_1, \bar{l}_1) \) we have \( c_H \sigma^2 < V_H'(l), V_L'(l) < c_L \sigma^2 \). The ICs are satisfied and \( (l_1, \bar{l}_1) \) is the signalling region of an interval equilibrium, with \( l_0 \in (l_1, \bar{l}_1) \).

Due to the boundedness of \( \beta \), \( \lim_{l \to \infty} \beta'(l) = \lim_{l \to -\infty} \beta'(l) = 0 \). If \( \nexists l, \bar{l} \in \mathbb{R} \) satisfying \( l < \bar{l} \), \( c_H r \sigma^2 \leq \frac{\beta(\bar{l}) - \beta(l)}{\bar{l} - l} \leq c_L r \sigma^2 \), then it must be that \( \forall l, \bar{l} \in \mathbb{R} \) satisfying \( l < \bar{l} \), \( c_H r \sigma^2 \leq \frac{\beta(\bar{l}) - \beta(l)}{\bar{l} - l} \). The maximal benefit to signalling at \( l \) occurs when the expectations of the market are \( e^*_H(l) = 1, e^*_L(l) = 0 \). If \( c_H r \sigma^2 \leq \frac{\beta(\bar{l}) - \beta(l)}{\bar{l} - l} \), then even \( e^*_H(l) = 1, e^*_L(l) = 0 \forall l \in (l, \bar{l}) \) does not incentivize \( H \) to signal in \( (l, \bar{l}) \). Then no other expectations incentivize \( H \) to signal either. Applying this reasoning to all \( (l, \bar{l}) \subseteq \mathbb{R} \), it is clear that pooling is the unique equilibrium. \( \square \)

**Proof of Proposition 16.** The necessary conditions are obtained by integrating the ICs over the signalling region:

\[
\int_{l}^{\bar{l}} V_H'(l)dl \geq c_H \sigma^2 (\bar{l} - l), \quad \int_{l}^{\bar{l}} V_L'(l)dl \leq c_L \sigma^2 (\bar{l} - l).
\]

Using \( \int_{l}^{\bar{l}} V_H'(l)dl = V_\theta(\bar{l}) - V_\theta(l) \) and the boundary conditions, the necessary conditions become \( c_H \sigma^2 (\bar{l} - l) \leq \frac{\beta(\bar{l}) - \beta(l)}{\bar{l} - l} \leq c_L \sigma^2 (\bar{l} - l) \). These conditions bound the average slope of \( \beta(l) \) over the signalling region from below and above. The slope of \( \beta(l) \) must go to zero as \( l \) approaches plus or minus infinity, because \( \beta \) is bounded. So for any \( c_H, \sigma^2, r \), the boundaries \( l, \bar{l} \) of equilibrium signalling regions are bounded above and below by the \( H \) type necessary condition. \( \square \)

**Proof of Proposition 17.** Given the expectations of the receivers, which determine \( l(g), l(b) \) via (10), action 1 is a best response for type \( \theta \) if

\[
-c_\theta + \frac{1 + \lambda}{2} \beta(l(g)) + \frac{1 - \lambda}{2} \beta(l(b)) \geq \frac{1 - \lambda}{2} \beta(l(g)) + \frac{1 + \lambda}{2} \beta(l(b)),
\]

which is equivalent to \( c_\theta \leq \lambda[\beta(l(g)) - \beta(l(b))] \). Action 0 is a best response if \( c_\theta \geq \lambda[\beta(l(g)) - \beta(l(b))] \).

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If pooling is expected, then \( l(g) = l(b) \) and \( \lambda[\beta(l(g)) - \beta(l(b))] = 0 \). In that case 0 is the unique best response for both types, so pooling is an equilibrium. This observation is independent of the parameters.

Maximizing \( \beta(l(g)) - \beta(l(b)) \) by choosing \( e^*_H, e^*_L \), we get \( e^*_L = 0 \) and \( e^*_H = 1 \) as the unique solution, because \( \beta \) is strictly increasing. If \( c_H > \lambda[\beta(l(g)) - \beta(l(b))] \), then for any expectations of the receivers, 0 is the unique best response for both types.

If \( c_H = \lambda[\beta(l(g)) - \beta(l(b))] \) and the receivers expect \( e^*_L = 0 \) and \( e^*_H = 1 \), then \( H \) is indifferent between actions 0 and 1, while \( L \) strictly prefers 0. Max info is thus an equilibrium. There are no other equilibria, because if the receivers expect \( e^*_H < 1 \) or \( e^*_L > 0 \), then \( \beta(l(g)) - \beta(l(b)) < \beta(l(g)) - \beta(l(b)) \), so 0 is the unique best response for both types.

If \( c_H < \lambda[\beta(l(g)) - \beta(l(b))] \leq c_L \) and the receivers expect \( e^*_L = 0 \) and \( e^*_H = 1 \), then \( H \) strictly prefers 1 and \( L \) weakly prefers 0, so max info is an equilibrium. Due to the continuity of \( \beta \) and the continuity of \( l(g), l(b) \) in \( e^*_H, e^*_L \), by the Mean Value Theorem there exists \( \hat{e}_H \in (0, 1) \) such that \( c_H = \lambda[\beta(l(g)) - \beta(l(b))] \) when the receivers expect \( e_L = 0, e_H = \hat{e}_H \). Therefore there exists an equilibrium where \( e^*_L = 0, e^*_H = \hat{e}_H \). \( \hat{e}_H \) is unique, because \( \beta \) is strictly increasing, \( l(g) \) is strictly increasing in \( e^*_L \) and strictly decreasing in \( e^*_H \) and \( l(b) \) is strictly decreasing in \( e^*_H \) and strictly increasing in \( e^*_L \).

If \( c_L < \lambda[\beta(l(g)) - \beta(l(b))] \), then under expectations \( e^*_H = 1, e^*_L = 0 \), \( L \) strictly prefers \( e = 1 \). Under expectations \( e^*_H = 1, e^*_L = 1 \), \( L \) strictly prefers \( e = 0 \). By the Mean Value Theorem, there exists \( \hat{e}_L \in (0, 1) \) such that \( c_L = \lambda[\beta(l(g)) - \beta(l(b))] \) when the receivers expect \( e^*_L = \hat{e}_L, e^*_H = 1 \). Again, \( \hat{e}_L \) is unique, so there is a unique interior \( e^*_L \) equilibrium. The reasoning in the previous paragraph showing the existence of a unique interior \( e^*_H \) equilibrium still holds.

\[ \square \]

B One-shot noisy signalling with quadratic cost

Some of the features of the one-shot noisy signalling model are driven by the linear cost of effort, which for some applications is unrealistic. Signalling may feature increasing marginal cost for the usual reason: first the easiest ways to signal are used, then if these are exhausted, more costly methods must be employed. The most tractable convex cost is quadratic, which is
assumed in this section. The sender’s action $e \in [0, 1]$ generates a signal $g$ or $b$, with probability of $g$ being

$$\Pr(g|e) = \lambda e + \frac{1 - \lambda}{2},$$

with $\lambda \in (0, 1)$. Denote the pure action the receivers expect type $\theta$ to take by $e^*_\theta$. Mixed actions add nothing, since the signal distribution from a mixed action can be replicated at strictly lower cost by a pure action. The best response to any beliefs by the receivers is pure.

The receivers observe the signal, but not the effort and update their log likelihood ratio using Bayes’ rule. Regardless of the action taken, both signals have positive probability, so Bayes’ rule applies after both signals and there are no off-path information sets. Denote by $l(x)$ the updated log likelihood ratio after signal $x$. Then

$$l(x) = l_0 + \ln \frac{\Pr(x|e^*_H)}{\Pr(x|e^*_L)}. \quad (12)$$

Type $\theta$ sender’s utility from action $e$ and receivers’ log likelihood ratio $l$ is

$$u_{\theta}(l, e) = \frac{\exp(l)}{1 + \exp(l)} - \frac{c_{\theta} e^2}{2},$$

with $c_L > c_H$. The sender’s benefit from the receivers’ belief is thus equal to the belief. Given a best response by the receivers to the realized signal and the sender’s equilibrium play, the sender’s expected utility from action $e$ is

$$u_{\theta} = -\frac{c_{\theta} e^2}{2} + \lambda e \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right) + \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} + \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))}. \quad (13)$$

**Definition 6.** A perfect Bayesian equilibrium is $(e^*_H, e^*_L, l(g), l(b))$ such that

(a) given $e^*_L, e^*_H$, the log likelihood ratios $l(g), l(b)$ are obtained from (12),

(b) given $l(g), l(b)$, for $\theta = H, L$, $e^*_\theta$ maximizes (13) over $e$.  

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If $e^*_H \leq e^*_L$, then both types of sender will choose $e_\theta = 0$, because there is no benefit to signalling, but there is a cost. Therefore pooling at $e^*_L = e^*_H = 0$ is an equilibrium for all parameter values.

If the efforts expected from the types satisfy $e^*_H > e^*_L$, then $l(g) > l(b)$. The marginal cost of signalling is zero at $e = 0$, the marginal benefit is $\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)$ everywhere. Due to this, both types will choose $e_\theta > 0$. Given the strategy expected in equilibrium, the chosen action of type $\theta$ must satisfy the first order condition (FOC)

$$c_\theta e_\theta = \lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right).$$

The FOC is necessary and sufficient for a global maximum due to the linear benefit and the quadratic cost. The FOC already allows the comparison of the signalling efforts and expected utilities of the types, formalized in the following proposition.

**Proposition 23.** The equilibrium signalling efforts of types $H$ and $L$ satisfy $e^*_H = \frac{c_L}{c_H} e^*_L$ and the expected utilities $u_H, u_L$ satisfy

$$u_H = \frac{c_L}{c_H} u_L \left[ \frac{(1 - \lambda) \exp(l(g))}{2(1 + \exp(l(g)))} - \frac{(1 + \lambda) \exp(l(b))}{2(1 + \exp(l(b)))} \right].$$

**Proof.** From the FOCs,

$$e^*_H = \frac{1}{c_H} \lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right) = \frac{c_L}{c_H} \frac{1}{c_L} \lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right) = \frac{c_L}{c_H} e^*_L.$$
Substituting this relation into the expected utilities, we get

\[ u_H = -\frac{c_H}{2} \left( \frac{\lambda \left( \frac{\exp(l(g))}{1+\exp(l(g))} - \frac{\exp(l(b))}{1+\exp(l(b))} \right)}{c_H} \right)^2 \]

\[ + \frac{\lambda}{c_H} \left( \frac{\exp(l(g))}{1+\exp(l(g))} - \frac{\exp(l(b))}{1+\exp(l(b))} \right) \left( \frac{\exp(l(g))}{1+\exp(l(g))} - \frac{\exp(l(b))}{1+\exp(l(b))} \right) \]

\[ + \frac{(1-\lambda) \exp(l(g)) + (1+\lambda) \exp(l(b))}{2(1+\exp(l(g)))} \frac{(1-\lambda) \exp(l(g)) + (1+\lambda) \exp(l(b))}{2(1+\exp(l(b)))} \]

and \( u_L = \frac{\lambda \left( \frac{\exp(l(g))}{1+\exp(l(g))} - \frac{\exp(l(b))}{1+\exp(l(b))} \right)}{2c_H} \left( \frac{\exp(l(g))}{1+\exp(l(g))} + \frac{(1-\lambda) \exp(l(g)) + (1+\lambda) \exp(l(b))}{2(1+\exp(l(g)))} \right), \) which gives the desired relation between the utilities of the types.

Proposition 23 holds for both informative and pooling equilibria. Under pooling, \( e_H^* = e_L^* = 0 \) and \( l(g) = l(b) = l_0, \) so \( u_H = u_L = \frac{\exp(l_0)}{1+\exp(l_0)}. \) Under separation, it is clear that \( u_H \geq u_L. \)

To find the equilibrium actions, equate the chosen action to the one expected by setting \( e_\theta = e_\theta^* \) in the FOCs. One solution is \( e_H^* = e_L^* = 0. \) There are two other solutions. The one for which there exist parameter values such that \( e_H^*, e_L^* \in (0, 1) \) is

\[ e_H^* = \frac{c_L c_H \lambda - \sqrt{c_L c_H \lambda^2}}{2 c_H \left[ c_H \frac{1}{1+\exp(l_0)} + c_L \frac{\exp(l_0)}{1+\exp(l_0)} \right]} \]

\[ e_L^* = \frac{c_L c_H \lambda - \sqrt{c_L c_H \lambda^2}}{2 c_L \left[ c_L \frac{1}{1+\exp(l_0)} + c_H \frac{\exp(l_0)}{1+\exp(l_0)} \right]} \]

Sufficient conditions for the existence of an informative equilibrium with interior \( e_L^*, e_H^* \) are presented in the following proposition.

**Proposition 24.** A separating equilibrium with \( e_H^*, e_L^* \in (0, 1) \) exists if

\[ \frac{4 c_L \exp(l_0) \lambda^2}{c_L \left( 1+\exp(l_0) \right)^2 + 4 \lambda^2 \exp(l_0)} \leq c_H \leq \frac{c_L \lambda^2 \exp(l_0)}{c_L \left[ 1-\frac{\lambda^2}{4} (1+\exp(l_0))^2 + \lambda^2 \exp(l_0) \right]}. \]
Proof. If $e^*_H, e^*_L$ satisfy (14) and $1 \geq e^*_H > e^*_L \geq 0$, then they are equilibrium strategies, because given the strategies the market expects, both types are optimizing and the chosen strategies equal the market’s expectation.

In (14), the term under the square root is nonnegative iff

$$c_H(c_L + 4 \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2) \geq 4c_L \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2,$$

which holds iff $c_H \geq \frac{4c_L \exp(l_0)}{c_L + 4 \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2}$. The fraction is always less than one and positive, because $\lambda \in (0, 1)$ and for any $l \in \mathbb{R}$, $\frac{\exp(l_0)}{(1 + \exp(l_0))^2} \in (0, \frac{1}{4})$.

Using Mathematica to simplify (14), it turns out that $e^*_H < 1$ always holds. Since $e^*_H = \frac{c_H}{c_L} e^*_L$, the inequality $e^*_L < e^*_H$ is implied by $e^*_L > 0$. Again using Mathematica to simplify (14), it is found that $e^*_L > 0$ iff

$$\frac{4c_L \exp(l_0) \lambda^2}{c_L(1 + \exp(l_0))^2 + 4\lambda^2 \exp(l_0)} \leq c_H < \frac{c_L \lambda^2 \exp(l_0)}{c_L \frac{1 - \lambda^2}{4} (1 + \exp(l_0))^2 + \lambda^2 \exp(l_0)}.$$

The comparative statics results are given in Proposition 25 below. The derivatives of $e^*_L$ with respect to $c_L$ or $c_H$ do not have clear signs.

**Proposition 25.** If the condition in Proposition 24 is satisfied, then the efforts of both types in an informative equilibrium increase in $\lambda$, the precision of the signal. $e^*_H$ increases in $c_L$ and decreases in $c_H$. For $l_0 \geq 0$, $e^*_H$ and $e^*_L$ decrease in $l_0$.

Proof. $\frac{\partial e^*_L}{\partial \lambda} = \frac{c_H e^*_L}{2[c_H \frac{1}{1 + \exp(l_0)} + c_L \frac{\exp(l_0)}{(1 + \exp(l_0))^2}] \lambda^2 \sqrt{c_H c_L [c_H - 4(c_L - c_H) \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2]}} > 0$. Based on Proposition 23, $\frac{\partial e^*_L}{\partial \lambda} = \frac{c_H}{c_L} \frac{\partial e^*_H}{\partial \lambda}$, which is also positive.

Rewriting the $e^*_H$ expression as

$$e^*_H = \frac{c_H \lambda - \sqrt{c_H^2 - 4c_H(1 - \frac{c_H}{c_L}) \frac{\exp(l_0)}{(1 + \exp(l_0))^2} \lambda^2}}{2c_H \left[\frac{c_H}{c_L (1 + \exp(l_0))} + \frac{\exp(l_0)}{1 + \exp(l_0)}\right] \lambda},$$

it is clear that the denominator is decreasing in $c_L$. In the numerator, $(1 - \frac{c_H}{c_L})$ is increasing in $c_L$, so the square root is decreasing in $c_L$. The numerator is increasing in $c_L$ and the denominator decreasing, so $e^*_H$ is increasing in $c_L$. 

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Rewriting the $e^*_H$ expression as

$$e^*_H = \frac{c_L \lambda - \sqrt{c_L^2 - 4 c_L (c_L - 1) \frac{\exp(l_0)}{1 + \exp(l_0)} \lambda^2}}{2 [c_H \frac{1}{1 + \exp(l_0)} + c_L \frac{\exp(l_0)}{1 + \exp(l_0)}] \lambda},$$

the denominator is increasing in $c_H$. In the numerator, the square root is increasing in $c_H$, so the numerator and therefore $e^*_H$ are decreasing in $c_H$.

Increasing $l_0$ increases the denominator of the $e^*_H, e^*_L$ expressions, since $c_L > c_H$. Increasing $|l_0|$ decreases the numerator, since the numerator is increasing in $\frac{\exp(l_0)}{1 + \exp(l_0)}$ and $\frac{\exp(l_0)}{1 + \exp(l_0)}$ is globally strictly concave with the maximum at $l_0 = 0$. So for $l_0 \geq \frac{1}{2}$, it is clear that $e^*_H$ and $e^*_L$ are decreasing in $l_0$. □

The expected utility of $L$ is higher in the pooling equilibrium than in the informative, because in an informative equilibrium $L$ pays an effort cost and expects $l$ to go down. The expected utility of $H$ may be higher in pooling or the informative equilibrium, but the expressions defining the region where $H$’s pooling payoff is higher are too long to present here.

Changing the game to incorporate a type-dependent drift $d$ in the signal, so that $H$ is more likely to generate signal $g$, eliminates the pooling equilibrium. If pooling is expected, then seeing $g$, the receivers update the probability of the $H$ type to $l(g) = l_0 + \ln \frac{1 - \lambda + 2d}{1 - \lambda} > l_0$ and seeing $b$, they update to $l(b) < l_0$. To ensure the probabilities are in $[0, 1]$ after any action expected from the sender, it must be that $d \leq \frac{1 - \lambda}{2}$. The marginal cost of signalling is zero at $e = 0$, the marginal benefit is $\lambda \left( \frac{\exp(l(g))}{1 + \exp(l(g))} - \frac{\exp(l(b))}{1 + \exp(l(b))} \right)$, which is positive for any $d > 0$. Due to this, both types will choose $e_\theta > 0$ even when pooling is expected. For some parameters, e.g. $c_L < \epsilon$, pooling at $e^*_L = e^*_H = 1$ occurs.

An arbitrarily small positive type-dependent drift destroying the pooling equilibrium is not just a feature of the quadratic cost. When the signal structure is such that the marginal benefit to effort is linear and positive, any convex cost with zero derivative at zero effort exhibits the same effect. There need not be a unique separating equilibrium.
C Extensions of the Poisson game

C.1 Both good and bad news in the Poisson signalling model

If the Poisson rate of good news signals occurring is \( \lambda g \), the rate of bad news is \( \lambda b (1 - e) \) and the efforts the market expects from the types are \( e^*_L \) and \( e^*_H \), then the HJB equation for type \( \theta \) is

\[
rv_\theta(l) = \beta(l) + (\lambda_b - \lambda_g)(e^*_H(l) - e^*_L(l))V_\theta(l)
\]

\[
+ \max_e \left\{ \lambda_g e \left[ V_\theta \left( l + \ln \frac{e^*_H(l)}{e^*_L(l)} \right) - V_\theta(l) \right] \right.
\]

\[
+ \lambda_b (1 - e) \left[ V_\theta \left( l + \ln \frac{1 - e^*_H(l)}{1 - e^*_L(l)} \right) - V_\theta(l) \right] - c_\theta e \}.
\]

Type \( \theta \) chooses \( e_\theta = 1 \) if

\[
\lambda_g V_\theta \left( l + \ln \frac{e^*_H(l)}{e^*_L(l)} \right) - \lambda_g V_\theta(l) - \lambda_b V_\theta \left( l + \ln \frac{1 - e^*_H(l)}{1 - e^*_L(l)} \right) + \lambda_b V_\theta(l) - c_\theta > 0.
\]

As in the main text, the focus is on simple equilibria in which type \( L \) always chooses \( e = 0 \) and type \( H \) chooses 1 in an interval of log likelihood ratios and 0 elsewhere. For simplicity, assume \( \beta_{\text{max}} > \frac{\lambda_b \beta_{\text{min}} - \lambda_g \beta_{\text{max}}}{\lambda_b - \lambda_g} + \frac{c_L r}{\lambda_b - \lambda_g} \) and \( \beta_{\text{min}} < \frac{\lambda_g \beta_{\text{max}} - \lambda_b \beta_{\text{min}}}{\lambda_g - \lambda_b} - \frac{c_r r}{\lambda_g - \lambda_b} \), which ensures \( l > -\infty \) and \( \ell < \infty \). In the signalling region, the jump after a good signal is to \( l = \infty \) and the jump after a bad signal to \( l = -\infty \). When \( e^*_H = 1, e^*_L = 0 \) is expected by the receivers, then the HJB equation becomes

\[
rV_\theta(l) = \beta(l) + (\lambda_b - \lambda_g)V_\theta(l) + \lambda_b \frac{\beta_{\text{min}}}{r} - \lambda_b V_\theta(l)
\]

\[
+ \max_e \left\{ \lambda_g \frac{\beta_{\text{max}}}{r} - \lambda_b \frac{\beta_{\text{min}}}{r} + (\lambda_b - \lambda_g)V_\theta(l) - c_\theta \right\}.
\]

Type \( \theta \) chooses \( e_\theta = 1 \) if \( \lambda_g \frac{\beta_{\text{max}}}{r} - \lambda_b \frac{\beta_{\text{min}}}{r} + (\lambda_b - \lambda_g)V_\theta(l) - c_\theta > 0 \).

If \( \lambda_b > \lambda_g \), then in the absence of a signal, \( l \) drifts up and eventually reaches \( \ell \). The signalling region has the form \( [l_0, \ell] \). This implies \( \lim_{l \to \ell} V_\theta(l) = \frac{\beta(\ell)}{r} \), as in the bad news case.

If \( \lambda_b < \lambda_g \), then in the absence of a signal, \( l \) drifts down and \( \lim_{l \to l_0} V_\theta(l) = \frac{\beta(l_0)}{r} \), as in the good news case. The signalling region has the form \( (\ell, l_0] \).
Substituting $e_H = 1$ and $e_L = 0$ into the HJB equations of the types, these become the ODEs
\[ rV_H = \beta(l) + (\lambda_b - \lambda_g)V'_H(l) + \lambda_g \frac{\beta_{\text{max}}}{r} - \lambda_g V_H(l) - c_H, \]
\[ rV_L = \beta(l) + (\lambda_b - \lambda_g)V'_L(l) + \lambda_b \frac{\beta_{\text{min}}}{r} - \lambda_b V_L(l). \]
The boundary condition depends on whether $\lambda_g > \lambda_b$ or vice versa. If $\lambda_b > \lambda_g$, then the boundary condition is $V_\theta(\tilde{t}) = \frac{\beta(\tilde{t})}{r}$ and the solutions are
\[ V_H(l) = \exp\left(\left(-(r + \lambda_g) \frac{l - \tilde{t}}{\lambda_b - \lambda_g}\right) \frac{\beta(\tilde{t})}{r}\right) \]
\[ + \int_{\tilde{t}}^l \left[ \frac{\beta(z) - c_H}{\lambda_b - \lambda_g} + \frac{\lambda_g \beta_{\text{max}}}{r(\lambda_b - \lambda_g)} \right] \exp\left(-(r + \lambda_g) \frac{z - \tilde{t}}{\lambda_b - \lambda_g}\right) dz, \]
\[ V_L(l) = \exp\left(\left(-(r + \lambda_b) \frac{l - \tilde{t}}{\lambda_b - \lambda_g}\right) \frac{\beta(\tilde{t})}{r}\right) \]
\[ + \int_{\tilde{t}}^l \left[ \frac{\beta(z)}{\lambda_b - \lambda_g} + \frac{\lambda_b \beta_{\text{min}}}{r(\lambda_g - \lambda_b)} \right] \exp\left(-(r + \lambda_b) \frac{z - \tilde{t}}{\lambda_b - \lambda_g}\right) dz. \]

If $\lambda_b < \lambda_g$, the boundary condition is $V_\theta(\tilde{l}) = \frac{\beta(\tilde{l})}{r}$ and the solutions are
\[ V_H(l) = \exp\left(\left(-(r + \lambda_g) \frac{l - \tilde{l}}{\lambda_b - \lambda_b}\right) \frac{\beta(\tilde{l})}{r}\right) \]
\[ + \int_{\tilde{l}}^l \left[ \frac{\beta(z) - c_H}{\lambda_g - \lambda_b} + \frac{\lambda_g \beta_{\text{max}}}{r(\lambda_g - \lambda_b)} \right] \exp\left(-(r + \lambda_g) \frac{l - z}{\lambda_g - \lambda_b}\right) dz, \]
\[ V_L(l) = \exp\left(\left(-(r + \lambda_b) \frac{l - \tilde{l}}{\lambda_g - \lambda_b}\right) \frac{\beta(\tilde{l})}{r}\right) \]
\[ + \int_{\tilde{l}}^l \left[ \frac{\beta(z)}{\lambda_g - \lambda_b} + \frac{\lambda_b \beta_{\text{min}}}{r(\lambda_g - \lambda_b)} \right] \exp\left(-(r + \lambda_b) \frac{l - z}{\lambda_g - \lambda_b}\right) dz. \]

The set of simple equilibria with finite signalling regions is similar to the case of only good news when $\lambda_g > \lambda_b$ and to the case of only bad news when $\lambda_g < \lambda_b$. 
In the knife-edge case of $\lambda_g = \lambda_b$, the log likelihood ratio stays constant at $l_0$ in the absence of signals, so in the signalling region, $V_H(l_0) = \frac{\beta(l_0) - c_H}{r + \lambda_g} + \frac{\lambda_g\beta_{\max}}{r(\tau + \lambda_g)}$ and $V_L(l_0) = \frac{\beta(l_0)}{r + \lambda_b} + \frac{\lambda_b\beta_{\min}}{r(\tau + \lambda_b)}$.

The result that $L$ always prefers pooling in the good news model is not sensitive to the rate of bad news being positive, provided it is less than the rate of good news. The intuition of why pooling gives $L$ the highest payoff is that the rate of good signals is zero for $L$, because $L$ takes zero effort. From the viewpoint of $L$, in an informative simple equilibrium, the log likelihood ratio can only decrease. This decreases the payoff to $L$, regardless of the curvature of $\beta$.

### C.2 Signal rate uniformly bounded below

This section looks at the case where even with zero effort in the good news case and even with maximal effort in the bad news case, the Poisson rate of signals is positive. At effort level $e$ the signal intensity is now $\lambda e + \epsilon$ in the good news case and $\lambda(1 - e) + \epsilon$ in the bad news case, with $e \in [0, 1]$ and $\epsilon > 0$ small.

Examining equations (1) and (2) that define the log likelihood ratio processes in the good and bad news cases, the continuous component of the process does not change when $\epsilon$ is added to the signal rate. However, the jump sizes change in the sums in (1) and (2), and if efforts $e^*_L = 0, e^*_H = 1$ are expected, the jumps no longer go to $\infty$ or $-\infty$.

In the good news case, the HJB equation of type $\theta$ under the expectations of the market $e^*_L, e^*_H$ is

$$rV_{\theta} = \beta(l) - \lambda(e_H^*(l) - e_L^*(l))V'_{\theta}(l) + \max_e \left\{ (\lambda e + \epsilon) \left[ V_{\theta} \left( l + \ln \frac{\lambda e^*_H(l) + \epsilon}{\lambda e^*_L(l) + \epsilon} \right) - V_{\theta}(l) \right] - c_{\theta} e \right\}.$$ 

Focus on interval equilibria in which the signalling interval is finite ($l > -\infty$ and $\bar{l} < \infty$). Assume $\epsilon$ is small enough for the jumps in the log likelihood ratio to go to the pooling region. The HJB equation becomes

$$rV_{\theta} = \beta(l) - \lambda V'_{\theta}(l) + \max_e \left\{ (\lambda e + \epsilon) \left[ \frac{\beta(l)}{r} - V_{\theta}(l) \right] - c_{\theta} e \right\},$$
which can be rearranged as
\[(r + \epsilon)V_\theta = \beta(l) + \frac{\epsilon \beta(j(l))}{r} - \lambda V_\theta(l) + \max_e \left\{ \lambda e \left( \frac{\beta(j(l))}{r} - V_\theta(l) \right) - c_\theta e \right\} .\]

This is the same as the HJB equation for a game with \(\epsilon = 0\), discount rate \(r + \epsilon\) and flow benefit function \(\beta(l) + \frac{\epsilon \beta(j(l))}{r}\). A similar reasoning can be carried out for the bad news case, showing that a small positive rate of signals at extreme effort levels does not qualitatively change interval equilibria.

The payoff comparison of pooling and informative equilibria for \(L\) in the good news case changes. It is no longer the case that \(L\) always prefers pooling, because of the \(\frac{\epsilon \beta(j(l))}{r}\) term in the flow payoff. If \(e^*_H > e^*_L\) is expected, then \(j(l) > l\) and \(\frac{\beta(j(l))}{r} > \frac{\beta(l)}{r}\). For a convex enough \(\beta\), the payoff increase from the jumps to \(j(l)\) in an informative equilibrium outweighs the payoff decrease from the downward drift of \(l\), making \(L\)'s payoff in the informative equilibrium higher than the pooling payoff \(\frac{(1 + \epsilon)\beta(l)}{r}\).

### D Interior effort equilibria in the Brownian signalling model

In the one-shot game interior effort equilibria exist for some parameter values, so a natural question is whether this is also the case in the Brownian signalling model.

Given efforts \(e^*_L, e^*_H\) that the market expects and the chosen strategy \(e_\theta\), the log likelihood ratio process satisfies
\[dl_t = \sigma^{-2} (e^*_H - e^*_L) \left( e_\theta - \frac{1}{2} e^*_H - \frac{1}{2} e^*_L \right) dt + \frac{e^*_H - e^*_L}{\sigma} dB_t, \tag{15}\]
due to \(dX_t = e_t dt + \sigma dB_t\) and (8).

If the strategy \((e^*_L, e^*_H)\) the market expects is Lipschitz in \(l\), then by Theorem 3.1 of Touzi (2013), Eq. (15) has a unique strong solution. In that case the log likelihood ratio process is well defined. The Lipschitz condition must be verified after solving for the optimal strategies.

Since \(H\) has a lower cost for any \(e > 0\) than \(L\), while the benefit from the future path of the receivers’ log likelihood ratio is the same, only one of the types can be taking interior effort at a given \(l\). If \(H\) takes \(e \in (0, 1)\) and is
therefore indifferent between $e = 0$ and $e = 1$, $L$ strictly prefers $e = 0$. If $L$ takes interior effort, then $H$ strictly prefers $e = 1$.

Following the same solution procedure as with pure strategies, the HJB equation of type $\theta$ is

$$rV_\theta(l) = \beta(l) + \frac{1}{2}V_\theta''(l)\sigma^{-2}(e^*_H(l) - e^*_L(l))^2$$

$$+ \max_{e \in [0,1]} \left\{ -c_\theta e + V_\theta'(l)\sigma^{-2}(e^*_H(l) - e^*_L(l))[e(l) - \frac{1}{2}e^*_H(l) - \frac{1}{2}e^*_L(l)] \right\}.$$ 

Outside the signalling region $(\underline{L}, \overline{L})$, both types always take action 0. If the equilibrium features $e^*_H \in (0, 1)$ and $e^*_L = 0$, then it must be that $e^*_H(l) - e^*_L(l) = \frac{c_H\sigma^2}{V_\theta''(l)}$ (for $H$ to be indifferent between $e = 0$ and 1) and $V_\theta'(l)(e^*_H - e^*_L) \leq c_L\sigma^2$ (the IC$_L$ constraint). Using the indifference condition and $e^*_L = 0$, IC$_L$ reduces to $\frac{V_\theta'(l)}{V_\theta''(l)} \leq \frac{c_L}{c_H}$. IC$_L$, as well as the feasibility constraint $e^*_H(l) = \frac{c_H\sigma^2}{V_\theta''(l)} \in [0, 1]$ must be checked after solving for candidate equilibrium strategies and value functions. The feasibility constraint is equivalent to $V_\theta'(l) \geq c_H\sigma^2$.

Equating $e_\theta$ and $e^*_\theta$ and substituting $\frac{c_H\sigma^2}{V_\theta''(l)}$ for $e^*_H$ and 0 for $e^*_L$, the following pair of second order ODEs obtains. The first ODE is nonlinear. The second is linear, but with variable coefficients.

$$rV_H(l) = \beta(l) - \frac{c_H^2\sigma^2}{2V_H''(l)} + \frac{c_H^2\sigma^2V_H''(l)}{2(V_H'(l))^2};$$

$$rV_L(l) = \beta(l) - \frac{c_H^2\sigma^2V_L''(l)}{2(V_L'(l))^2} + \frac{c_H^2\sigma^2V_L''(l)}{2(V_H'(l))^2}.$$ 

The boundary conditions are $V_\theta(\underline{L}) = \frac{\beta(l)}{r}$ and $V_\theta(\overline{L}) = \frac{\beta(l)}{r}$.

Since the $H$ type equation does not depend on the variables of $L$, it can be solved first. After that, $V_H'$ can be substituted into the $L$ type equation. It is clear from the similarity of the equations that the solution to the $L$ equation is $V_L = V_H$. If $\beta$ is concave on the signalling region, then both types are worse off in this interior effort equilibrium than in pooling. In both equilibria, $V_L = V_H$, but with interior effort, a signalling cost is paid. The concavity of $\beta$ implies that the variance in the posterior of the receivers does not benefit the sender.
Due to $V_L = V_H$, the IC$_L$ constraint $\frac{V'_L(l)}{V'_H(l)} \leq \frac{c_L}{c_H}$ is always satisfied. The set of possible signalling regions is limited only by the feasibility constraint $V'_H(l) \geq c_H \sigma^2$.

Unfortunately the ODEs resulting from mixed equilibria do not belong to a standard class for which simple solution methods exist. They must be solved numerically. Figure 8 shows $V_H$ and $e_H^*$ for $c_H = 0.1$, $r = \sigma^2 = 1$ and signalling region $l = -0.5$, $\bar{l} = 0.5$.

Figure 8: Equilibrium where $H$ mixes: payoff and strategy of $H$ for $c_H = 0.1$, $r = \sigma^2 = 1$, $l = -0.5$, $\bar{l} = 0.5$.

If the equilibrium features $e_L^* \in (0, 1)$ and $H$ choosing 1, then the indifference condition of $L$ is $e_H^* - e_L^* = \frac{c_L \sigma^2}{V'_L(l)}$ and IC$_H$ is $V'_H(l)(e_H^* - e_L^*) \geq c_H \sigma^2$.

Using the indifference condition, the latter reduces to $\frac{V'_H(l)}{V'_L(l)} \geq \frac{c_H}{c_L}$, which is slack near $l$ due to $V_H \geq V_L$, but may bind elsewhere. IC$_H$ as well as the
feasibility constraint $c_L \sigma^2 \frac{e_\theta}{V'_L(l)} \in [0,1]$ must be checked after solving for candidate equilibrium strategies and value functions. The feasibility constraint is equivalent to $V'_L \geq c_L \sigma^2$.

Equating $e_\theta$ and $e^*_\theta$ and substituting $1 - \frac{c_L \sigma^2}{V'_L(l)}$ for $e^*_L$ and 1 for $e^*_H$, the following pair of second order ODEs obtains.

$$rV_H(l) = \beta(l) - c_H + \frac{c^2 \sigma^2 V'_H(l)}{2(V'_L(l))^2} + \frac{c^2 \sigma^2 V''_H(l)}{2(V_L(l))^2},$$

$$rV_L(l) = \beta(l) - c_L + \frac{c^2 \sigma^2 V'_L(l)}{2V'_L(l)} + \frac{c^2 \sigma^2 V''_L(l)}{2(V_L(l))^2}.$$

The boundary conditions are the usual $V_\theta(l) = \frac{\beta(l)}{r}$, and $V_\theta(\tilde{l}) = \frac{\beta(\tilde{l})}{r}$. An example of the solution of the ODEs and the optimal strategy for $L$ is shown in Figure 9. The shapes of $V_H$, $V_L$ and $e^*_L$ for other equilibrium signalling regions are similar. Widening the signalling region will eventually violate the feasibility constraint $V'_L \geq c_L \sigma^2$ at $l$. This means $L$ does not want to signal close to the lower end of the signalling region, where reaching $\tilde{l}$ is unlikely due to the downward drift of $l$ that $L$ expects.

The payoff of $H$ in the signalling region is strictly greater than that of $L$, because $L$ is indifferent between mixing and setting $e^*_L = 1$ to imitate $H$, but the cost of $e = 1$ is lower for $H$ than for $L$.

The signalling regions that satisfy IC$_H$ and the feasibility constraint for the equilibrium where $L$ takes interior effort are depicted in Figure 10. The binding constraint is feasibility—the set of regions satisfying it is a proper subset of the set of regions satisfying IC$_H$. Compared to equilibria in which $e_\theta \in \{0,1\}$ at the same parameter values, the signalling regions sustainable in equilibria where $e^*_L \in (0,1)$ are much narrower. This is because the difference between the efforts expected from the types is smaller, so the benefit to signalling is smaller, and the cost for $L$ to signal is higher than that for $H$.

### E Quadratic cost in the Brownian signalling model

The setup in this section resembles the linear-cost Brownian model, but the results differ in an important way due to unusual mathematical behaviour of the model. Given a signalling region, a continuum of effort profiles of the
Figure 9: Equilibrium where $L$ mixes: payoffs and $L$’s strategy for $c_H = 0.1$, $c_L = 0.2$, $r = \sigma^2 = 1$, $\underline{a} = -0.05$, $\bar{a} = 0.05$. 
Figure 10: Region where IC$_H$ and feasibility constraint hold (shaded) for $c_H = 0.1$, $c_L = 0.2$, $r = \sigma^2 = 1$.

types constitute equilibria. Perturbing the quadratic cost model by changing the cost function slightly or by making the signal depend on the type as well as the action removes the linear dependence of the FOCs that makes the continuum of equilibria arise for a fixed signalling region.

The actions, costs, signal process and strategies are the same as with linear cost. The flow utility of a sender of type $\theta$ is

$$u_\theta(l, e) = \beta(l) - \frac{c_\theta}{2} e_\theta,$$

so the cost of effort is quadratic.

Overall, the game consists of two stochastic control problems, one for each type, related by an equilibrium condition. The solution procedure can be divided in two parts. First, given the equilibrium efforts the receivers expect from the two types of sender, a standard stochastic control problem is solved for each type to find the best response. Second, the chosen strategy is set equal to the strategy expected from the sender and solved for. The unusual mathematical features arise in the second part, where many strategies satisfy the equilibrium condition.
The log likelihood ratio process is defined by

\[
dl_t = \sigma^{-2}(e_H^* - e_L^*)(e_{\theta} - \frac{1}{2}e_H^* - \frac{1}{2}e_L^*)dt + \frac{e_H^* - e_L^*}{\sigma}dB_t,
\]

(16)
due to \(dX_t = e_t dt + \sigma dB_t\) and (8). In the signalling region it must be that \(e_H^* > e_L^*\), otherwise belief would fall or remain constant in the costly signal, which would make both types deviate to \(e_\theta = 0\). As a sufficient condition for the belief process to be well-defined and unique, assume the receivers expect strategies \(e_H^*, e_L^*\) that are Lipschitz in \(l\). It will turn out that the sender has a best response that is Lipschitz.

**Lemma 26.** If the receivers expect strategy \((e_H^*, e_L^*)\) that is Lipschitz in \(l\), then there is a unique belief process satisfying (16).

**Proof.** If for any control \(e_\theta\), the drift and variance in (16) are bounded and Lipschitz in belief, then by Theorem 3.1 of Touzi (2013), Eq. (16) has a unique strong solution. Since \(e_H^*, e_L^*\) are Lipschitz in \(l\) by assumption and the action space \([0,1]\) is bounded, the drift of the belief process \(\sigma^{-2}(e_H^*(l) - e_L^*(l))(e_\theta(l) - \frac{1}{2}e_H^*(l) - \frac{1}{2}e_L^*(l))\) and the variance \(\frac{(e_H^*(l) - e_L^*(l))^2}{\sigma^2}\) are both Lipschitz in \(l\) and bounded. \(\square\)

**Definition 7.** A Markov stationary equilibrium consists of a strategy \((e_H^*, e_L^*)\) and a log likelihood ratio process \((l_t)_{t \geq 0}\) s.t.

1. given \((l_t)_{t \geq 0}\), \(e_\theta^*\) solves

\[
\sup_{e_\theta(\cdot)} \mathbb{E}^{e_\theta} \left[ \int_{l_t}^{\infty} \exp(-rs) \left[ \beta(l_s) - \frac{e_\theta^2}{2}e_\theta^2(l_s) \right] ds \mid l_{t=0} = l_0 \right],
\]

where the expectation is over the process \((l_t)_{t \geq 0}\).

2. given \((e_H^*, e_L^*), (l_t)_{t \geq 0}\) is derived from Bayes’ rule (16).

The focus is on pure-strategy Markov stationary equilibria, where in an interval of log likelihood ratios \((\underline{l}, \overline{l}) \ni l_0\) at least one type chooses \(e_\theta > 0\). Once \(l\) reaches \(\underline{l}\) or \(\overline{l}\), both types choose \(e_\theta = 0\) forever. Such equilibria are called interval equilibria. The interval \((\underline{l}, \overline{l})\) is called the signalling region and its complement the pooling region. The reachable set \(\mathcal{L}(e^*)\) given a strategy \(e^*\) the receivers expect is defined as with linear cost.
Lemma 27. Fix \( l_0 \in \mathbb{R} \). For any interval equilibrium \( e^* \) and \( l \in C(e^*) \), it cannot be the case that \( e^*_L(l) = e^*_H(l) > 0 \) or that \( e^*_L(l) > e^*_H(l) \).

The proof is the same as for Lemma 12.

The same observations as in the linear cost case can be made about the value functions.

Lemma 28. \( V_H(l) \geq V_L(l) \forall l_0 \in \mathbb{R} \forall e^* \forall l \in C(e^*) \), with strict inequality if under the optimal \( (e_L, e_H) \) starting at \( l \), there is a positive probability of reaching some \( \hat{l} \) with \( e_L(\hat{l}) > 0 \). \( \frac{\beta_{\min}}{r} \leq V_\theta(l) \leq \frac{\beta_{\max}}{r} \), with strict inequalities if \( l \in \mathbb{R} \). \( V_\theta \) is strictly increasing.

The proof is the same as for Lemma 11.

With \( \beta \) concave in the signalling region, the \( L \) type strictly prefers pooling to an informative equilibrium, in the sense that for any log likelihood ratio of the receivers, the payoff of \( L \) is higher in a pooling equilibrium than in an informative equilibrium.

Proposition 29. If \( \beta \) is concave in the signalling region of an interval equilibrium, then \( V_L(l) < \frac{\beta(l)}{r} \) for all \( l \) in the signalling region.

Proof. \( L \) takes no effort in pooling, but takes positive effort in an informative equilibrium, so the flow cost is greater in the informative equilibrium. The flow benefit \( \beta \) is increasing in the log likelihood ratio \( l \). In pooling, \( l \) stays constant at \( l_0 \) forever, while in an equilibrium with a nonempty signalling region, \( L \) expects \( l \) to strictly decrease. With a concave \( \beta \), there is no benefit from the noise in the \( l \) process. This establishes \( V_L(l) < \frac{\beta(l)}{r} \).

To solve the control problems of the types, the HJB equations are solved and a verification theorem is used to check that the solutions of the HJB equations coincide with the value functions. To use Theorem 11.2.2 of Øksendal (2010) to prove that the solutions of the HJB equations equal the value functions, it is sufficient that the solutions are twice continuously differentiable on \( (l, \bar{l}) \), continuous on \( [l, \bar{l}] \) and integrable in the probability law of \( l \) given the starting state \( l_0 \), uniformly over Markov controls \( e_H, e_L \). As will be seen, these conditions are satisfied by the solutions of the HJB equations.

Under these conditions, Theorem 11.2.3 of Øksendal (2010) shows that the optimal Markov control does as well as the optimal nonanticipating control, so if the receivers expect Markov strategies, then both types of the sender have a Markov best response. This does not imply that the payoffs of
all non-Markov equilibria can be attained with Markov equilibria, since in a non-Markov equilibrium the receivers expect non-Markov strategies.

The HJB equation of type $\theta$ is

$$rv(l) = \beta(l) + \frac{(e_H^*(l) - e_L^*(l))^2}{2\sigma^2}\frac{rv''(l)}{rV''(l)} + \max \left\{ -\frac{c_\theta}{2}e^2 + V'_\theta(l)\sigma^{-2}(e_H^*(l) - e_L^*(l)) \left[ e_\theta - \frac{1}{2}e_H^*(l) - \frac{1}{2}e_L^*(l) \right] \right\}.$$  

The FOC of the type $\theta$ HJB equation is $-c_\theta e_\theta + V'_\theta(l)\sigma^{-2}(e_H^*(l) - e_L^*(l)) = 0$, so given the efforts expected by the market, both types have a unique optimal action $e_\theta = e_\theta^*$. The second order condition is $-c_\theta < 0$ for all $e_\theta$, so the FOCs are necessary and sufficient for a strict global maximum. This is not surprising, because the cost is quadratic and the benefit is linear in the control variable $e_\theta$.

Thus far, the control problems of the two types of the sender were solved for a given strategy expected by the receivers. For the second part of the solution of the signalling game, the equilibrium condition $e_\theta = e_\theta^*$ is imposed. The $L$ type FOC then becomes

$$e_L^*(l) = \frac{V'_L(l)}{c_L\sigma^2 + V'_L(l)}e_H^*(l).$$  

Substituting for $e_L^*$ in the $H$ type FOC gives

$$c_He_H^*(l) = V'_H(l)\sigma^{-2}\left[ 1 - \frac{V'_L(l)\sigma^{-2}}{c_L + V'_L(l)\sigma^{-2}} \right]e_H^*(l).$$

Therefore for any $l$, either $e_H^*(l) = 0 = e_L^*(l)$ or $c_Hc_L + c_HV'_L(l)\sigma^{-2} = c_LV'_H(l)\sigma^{-2}$. The latter is equivalent to

$$V'_H(l) = c_H\sigma^2 + \frac{c_H}{c_L}V'_L(l).$$  

The only corner solution is $e_H^* = e_L^* \equiv 0$. Other corner solutions would involve $e_H^*(l) > 0$ and $e_L^*(l) = 0$ for some $l$ in the signalling region. The $H$ type control problem has an interior solution if $e_H^*(l) > 0$ and the $H$ type FOC is then satisfied. This implies an equation similar to (17), except derived from the $H$ type FOC, requiring both $e_H^*$ and $e_L^*$ to be positive or both zero.
The relationship between the efforts and payoffs of the two types given by Eqs. (17) and (18) is similar to the one-shot model with quadratic cost, where \( e^*_L = \frac{c_H}{c_L} e^*_H \) and the expected utilities of the types are linearly related. This is not surprising, as in both cases the conditions are derived from the FOCs of quadratic problems with a similar structure.

The following lemma gives conditions on solutions of the HJB equations that are sufficient for these solutions to form an interval equilibrium with a nonempty signalling region (an informative equilibrium). Subsequently, Proposition 31 provides restrictions on parameters that are sufficient for the existence of a particular kind of interval equilibrium.

**Lemma 30.** \( e^*_L, e^*_H, V_L \) and \( V_H \) constitute an interval equilibrium with signalling region \((\underline{l}, \bar{l})\), where \( \underline{l} < \bar{l} \), if all of the following hold

1. \( e^*_L, e^*_H, V_L \) satisfy Eq. (17) for all \( l \in (\underline{l}, \bar{l}) \),
2. \( V_L \) and \( V_H \) satisfy Eq. (18) for all \( l \in (\underline{l}, \bar{l}) \),
3. \( V_L, V_H \) are twice continuously differentiable on \((\underline{l}, \bar{l})\),
4. \( V_L, V_H \) are continuous on \([\underline{l}, \bar{l}]\),
5. \( V_L, V_H \) are integrable in the probability law of \( l \) given the starting state \( l_0 \), uniformly over Markov controls \( e_H, e_L \),
6. \( e^*_L, e^*_H \) are Lipschitz in \( l \) on \((\underline{l}, \bar{l})\),
7. \( 0 < e^*_L, e^*_H \leq 1 \),
8. \( V_L \leq V_H \).

**Proof.** If \( \underline{l} < \bar{l} \), then Eqs. (17) and (18) together are necessary and sufficient for \( e^*_L, e^*_H, V_L \) and \( V_H \) to solve the HJB equations and satisfy the equilibrium condition.

If \( V_L, V_H \) are twice continuously differentiable on \((\underline{l}, \bar{l})\), continuous on \([\underline{l}, \bar{l}]\) and integrable in the probability law of \( l \) given the starting state \( l_0 \), uniformly over Markov controls \( e_H, e_L \), then by Theorem 11.2.2 of Øksendal (2010), they coincide with the value functions. In that case, \( e^*_L, e^*_H \) are the optimal Markov controls for \( V_L \) and \( V_H \) and by Theorem 11.2.3 of Øksendal (2010), \( e^*_L \) and \( e^*_H \) maximize \( V_L \) and \( V_H \) in the class of all nonanticipating controls.
The Lipschitz condition on $e^*_L, e^*_H$ is sufficient for the belief process to be well-defined (Lemma 26).

The restrictions $0 < e^*_L, e^*_H \leq \bar{e}$ and $V_L \leq V_H$ come from first principles and Lemma 28.

**Proposition 31.** An interval equilibrium with signalling region $(\underline{l}, \overline{l})$ is formed by

- $V_H(l) = \frac{\beta(l)}{r}$,
- $V_L(l) = \begin{cases} \frac{c_L\beta(l)}{c_Hr} - c_L\sigma^2 l & \text{if } l \in (\underline{l}, \overline{l}), \\ \frac{\beta(l)}{r} & \text{if } l \notin (\underline{l}, \overline{l}) \end{cases}$,
- $e^*_H = 1 \{(\underline{l}, \overline{l})\}$ and
- $e^*_L(l) = \begin{cases} 1 - \frac{c_Hr\sigma^2}{\beta(l)} & \text{if } l \in (\underline{l}, \overline{l}), \\ 0 & \text{if } l \notin (\underline{l}, \overline{l}) \end{cases}$

if the following hold

1. $\bar{e} > 1$,
2. $\underline{l} < \overline{l}$ satisfy

$$\beta(\overline{l}) = \frac{c_L c_H r \sigma^2}{c_L - c_H} \overline{l}, \quad \beta(\underline{l}) = \frac{c_L c_H r \sigma^2}{c_L - c_H} \underline{l}. \quad \text{(19)}$$

3. $\frac{\beta'(l)}{r} > c_H \sigma^2$ for all $l \in (\underline{l}, \overline{l})$,
4. $\frac{\beta(l)}{r} \leq \frac{c_L c_H r \sigma^2}{c_L - c_H} l$ for all $l \in (\underline{l}, \overline{l})$.

**Proof.** For $V_H, V_L, e^*_H$ and $e^*_L$ to form an interval equilibrium with signalling region $(\underline{l}, \overline{l})$, it is sufficient that they satisfy the assumptions of Lemma 30. Here by definition, $e^*_L, e^*_H, V_L$ satisfy Eq. (17) and $V_L$ and $V_H$ satisfy Eq. (18) for all $l \in (\underline{l}, \overline{l})$.

Since $\beta$ is assumed bounded and twice continuously differentiable, $V_H, V_L$ are twice continuously differentiable on $(\underline{l}, \overline{l})$, bounded and integrable in the probability law of $l$ given the starting state $l_0$, uniformly over Markov controls $e_H, e_L$. 

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The payoff in the pooling region provides the boundary conditions \( V_\theta(l) = \frac{\beta(l)}{r} \) and \( V_\theta(l) = \frac{\beta(l)}{r} \). Since \( V_\theta \) is twice continuously differentiable on \((l, \bar{l})\), for it to be continuous on \([l, \bar{l}]\) it is sufficient that \( \lim_{l \to l^+} V_\theta(l) = \frac{\beta(l)}{r} \) and \( \lim_{l \to l^-} V_\theta(l) = \frac{\beta(l)}{r} \). These conditions clearly hold for \( V_H = \frac{\beta(l)}{r} \). For \( V_L \), they hold iff Eqs. (19) hold. This may be seen by rearranging \( \lim_{l \to l^+} V_L(l) = \frac{c_L \beta(l)}{c_H r} - c_L \sigma^2 \frac{l}{r} = \frac{\beta(l)}{r} = V_L(l) \).

If \( \frac{\beta(l)}{r} > c_H \sigma^2 \), then \( e_H^*(l) > 0 \). In the signalling region, \( e_H^* > 0 \) holds by definition. By the assumption \( \bar{e} > 1 \), we have \( e_H^*, e_L^* < \bar{e} \).

\( V_H(l) \geq V_L(l) \) may be written as \( \frac{\beta(l)}{r} \geq \frac{c_L \beta(l)}{c_H r} - c_L \sigma^2 l \), or equivalently

\[
\frac{\beta(l)}{r} \leq \frac{c_L c_H \sigma^2}{c_L - c_H} l.
\]

Slightly perturbing \( V_H \) and \( e_H^* \) (while ensuring \( e_H^* \leq \bar{e} \)) in Proposition 31 and again deriving \( V_L \) and \( e_L^* \) from (18) and (17) results in a different interval equilibrium with the same signalling interval. Proposition 31 essentially says that in a certain parameter region, there is a degree of freedom in specifying \( e_L^*, e_H^* \) and also in specifying \( V_L, V_H \). In the signalling region, fixing one of \( e_L^*, e_H^* \) and one of the other three functions determines the remaining two via Eqs. (17) and (18). The degree of freedom is specific to the model with a quadratic cost of effort and does not have a natural interpretation.

Despite the quadratic cost of effort and the bounded benefit from belief, it would be possible to have arbitrarily large signalling efforts in equilibrium if effort was not restricted to \([0, 1]\). The key is that as efforts grow, the difference in efforts also becomes large and the drift and volatility of the log likelihood ratio process are proportional to this difference. With large efforts, \( l \) quickly moves out of the region where the equilibrium prescribes the large efforts. As the cost of effort gets large, it is only paid for a very short time in expectation. This may be the case for many other cost functions, but only with quadratic cost is the increase in the effort cost exactly offset by the decrease in the duration of the effort.

References


