BACKWARD INDUCTION IN GAMES WITHOUT PERFECT RECALL

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Abstract. The equilibrium concepts that we now think of as various forms of backwards induction, namely subgame perfect equilibrium (Selten, 1965), perfect equilibrium (Selten, 1975), sequential equilibrium (Kreps and Wilson, 1982), and quasi-perfect equilibrium (van Damme, 1984), are explicitly restricted their analysis to games with perfect recall. In spite of this the concepts are well defined, exactly as they defined them, even in games without perfect recall. There is now a small literature examining the behaviour of these concepts in games without perfect recall.

We argue that in games without perfect recall the original definitions are inappropriate. Our reading of the original papers is that the authors were aware that their definitions did not require the assumption of perfect recall but they were also aware that without the assumption of perfect recall the definitions they gave were not the "correct" ones. We give definitions of two of these concepts, sequential equilibrium and quasi-perfect equilibrium, that identify the same equilibria in games with perfect recall and behave well in games without perfect recall.

1. Introduction

The game theorists who defined the equilibrium concepts that we now think of as various forms of backwards induction, namely subgame perfect equilibrium (Selten, 1965), perfect equilibrium (Selten, 1975), sequential equilibrium (Kreps and Wilson, 1982), and quasi-perfect equilibrium (van Damme, 1984), explicitly restricted their analysis to games with perfect recall. In spite of this the concepts are well defined, exactly as they defined them, even in games without perfect recall. There is now a small literature examining the behaviour of these concepts in games without perfect recall. Jeff Kline (2005) looks at what happens in games without perfect recall to solutions defined in exactly the same way as they were defined in games with perfect recall. Joe Halpern and Rafael Pass (2016) modify the definitions of Selten and Kreps and Wilson in a somewhat different manner than we do.

We shall argue that in games without perfect recall the original definitions are inappropriate. Our reading of the original papers is not that the authors were unaware that their definitions did not require the assumption of perfect recall, but rather that they were aware that without the assumption of perfect recall the definitions they gave were not the "correct" ones. In this paper we give definitions of two of these concepts, sequential equilibrium and quasi-perfect equilibrium that identify the same equilibria in games with perfect recall and behave well in games without perfect recall. We focus on these concepts because they seem to us to be the right concept for backward induction and a combination of backward induction

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and admissibility respectively. We find some support for this view in the fact that our definitions seem quite natural. It is possible to redefine the concept of extensive form perfect equilibrium for games without perfect recall, but the definition is less straightforward. See Hillas and Kvasov (2017b).

2. Some Notation and Informal Definitions

In this section we give informal definitions of some of the concepts we use in our discussions. More formal definitions can be found in the cited papers. The notation for extensive form games is quite standard, though the formal specification of the model takes a number of forms. We assume that the reader is familiar with the notation. The idea of perfect recall is that a player has perfect recall if, at each of his information sets he remembers what he knew and what he did in the past. This idea was introduced and formally defined by Kuhn (1950, 1953). Later Selten (1975) gave an equivalent definition that is perhaps closer to the intuitive idea. We give here the definition by Selten.

Definition 1. A player is said to have perfect recall if whenever that player has an information set containing nodes x and y and there is a node x' of that player that precedes node x there is also a nodes y' in the same information set as x' that precedes node y and the action of the player at y' on the path to y is the same as the action of the player at x' on the path to x.

An implication of a player having perfect recall is that any play of the game will cut each of the player's information sets at most once. Kuhn made this requirement part of his definition of an extensive form game.

This requirement was later relaxed by Isbell (1957), and this more general definition was later, under the name "repetitive games," considered by Alpern (1988), and more famously, under the name "absent-mindedness," by Piccione and Rubinstein (1997a), and following them by many others (Gilboa, 1997; Battigalli, 1997; Grove and Halpern, 1997; Halpern, 1997; Lipman, 1997; Aumann, Hart, and Perry, 1997a,b; Piccione and Rubinstein, 1997b). For the moment we shall be restricting attention to linear games. The issues are not so different in nonlinear games and we shall return to briefly discuss an extension of the concepts to nonlinear games later in the paper and more fully in Hillas and Kvasov (2017a).

We now give definitions of some of the concepts we shall be using.

Definition 2. A pure strategy in an extensive form game for Player n is a function that maps each of his information sets to one of the actions available at that information set.

Definition 3. A behaviour strategy in an extensive form game for Player n is a function that maps each of his information sets to a probability distribution on the actions available at that information set.

Definition 4. A mixed strategy in an extensive form game for Player n is a probability distribution over the player's pure strategies.

We shall denote the set of players by N, the set of Player n's pure strategies by S_n, the set of Player n's behaviour strategies by B_n with $B = \times_{n\in N} B_n$, and the set of Player n's mixed strategies by Σ_n with $\Sigma = \times_{n\in N}\Sigma_n$. We shall let $S = \times_{n\in\{0\}\cup N} \cup S_n$, that is, when we refer to profiles of pure strategies we shall specify also the "strategy" of Nature. For notational simplicity we shall assume, as is usually done, that all of Nature's "information sets" are singletons.

An immediate implication of these definitions is that the set of pure strategies is embedded in both the set of behaviour strategies and the set of mixed strategies. A pure strategy s is equivalent to the behaviour strategy that takes each information set to the probability distribution that puts weight 1 on the action that s selects at that information set. And s is equivalent to the mixed strategy that puts weight 1 on s.

We shall denote the collection of information sets by $\mathcal{H}$, with $\mathcal{H}_n$ the information sets of Player n. We shall also consider the collection of non-empty subsets of $\mathcal{H}_n$ which we shall denote $\bar{\mathcal{H}}_n$. An element of $\bar{\mathcal{H}}_n$ is a collection of information sets of Player n.

Kuhn (1953) showed that if a player has perfect recall then each of his mixed strategies has an equivalent behaviour strategy, that is, a behaviour strategy that, whatever the other player may choose, will induce the same distribution on terminal nodes as the mixed strategy. If the player does not have perfect recall this will not be true for all mixed strategies, that is, there will be some mixed strategies for which no equivalent behaviour strategy exists. Isbell (1957) showed that in a linear game for each behaviour strategy of a player there is an equivalent mixed strategy, though this result is almost implicit in Kuhn. In nonlinear games this is not true. There may exist behaviour strategies for which no equivalent mixed strategy exists.

3. Comments on Equilibrium in Games without Perfect Recall

An immediate implication of the definitions of strategies is that the set of pure strategies is embedded in both the set of behaviour strategies and the set of mixed strategies. A pure strategy s is equivalent to the behaviour strategy that takes each information set to the probability distribution that puts weight 1 on the action that s selects at that information set. And s is equivalent to the mixed strategy that puts weight 1 on s.

This means that in games in which there is a unique equilibrium in mixed strategies and that mixed strategy profile is not equivalent to a profile of behaviour strategies there is no equilibrium in behaviour strategies. For, if there were, then the mixed strategy profile equivalent to that profile would also be an equilibrium in mixed strategies.

We first consider a game without perfect recall given in Figure 1. (This is a very slight modification of a game considered by Kuhn (1953).) In this game there is a unique equilibrium in mixed strategies. For one of the players the equilibrium mixed strategy is not equivalent to any behaviour strategy.

Consider the normal form of the game given in Figure 2. Note that for Player 1, the strategies (In, D) and (Out, S) are strictly dominated. Once the dominated strategies are removed the game is similar to matching pennies and the unique equilibrium is

$$\left\{\left(\tfrac{1}{4}, 0, 0, \tfrac{3}{4}\right), \left(\tfrac{3}{4}, \tfrac{1}{4}\right)\right\},$$

a mixed strategy profile in which Player 1 is playing a strategy that is not equivalent to any behaviour strategy.

Among other things this implies that there is no equilibrium in behaviour strategies. For, if there were then the equivalent mixed strategies would be such that neither player had a behaviour strategy that he preferred—this is the definition

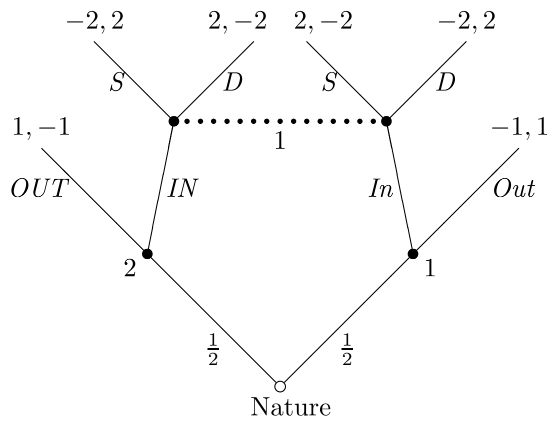

Figure 1. A game without perfect recall.

		Player 2	
		IN	OUT
	In, S	$0, 0$	$\frac{3}{2}, -\frac{3}{2}$
Player 1	In, D	$0, 0$	$-\frac{1}{2}, \frac{1}{2}$
	Out, S	$-\frac{3}{2}, \frac{3}{2}$	$0, 0$
	Out, D	$\frac{1}{2}, -\frac{1}{2}$	$0, 0$

Figure 2. The corresponding normal form game.

of an equilibrium in behaviour strategies—and thus no pure strategy that he preferred. Thus that profile of mixed strategies would be an equilibrium, contradicting the fact that this game has a unique equilibrium in mixed strategies in which one of the players plays a strategy that is not equivalent to any behaviour strategy.

Let us think for a moment why there is, in this example, no equilibrium in behaviour strategies. In the mixed equilibrium Player 1 at his first information set sometimes plays *In* and sometimes plays *Out* and at his second information set sometimes plays S and sometimes plays D. However he coordinates his choices so that if he plays *In* at his first information set he plays S at his second information set and if he plays *Out* at his first information set he plays D at his second information set. It is not possible to achieve such coordination using behaviour strategies.

Another candidate for an equilibrium in behaviour strategies is the profile of behaviour strategies corresponding to an equilibrium of the agent normal form. The equilibria of the agent normal form are (In, S, IN), (Out, D, OUT) and $\{((x, 1-x), (1/4.3/4), (x, 1-x)) \mid 0 \leq x \leq 1\}$. Why are these (or at least the equivalent behaviour strategies) not equilibria? Well, Player 1 could deviate to a different behaviour strategy that involves different behaviour at *both* of his information sets.

We should not be surprised at the nonexistence of equilibria in behaviour strategies. The definition of such equilibria allows coordination by the player in his deviations that he is not permitted in his equilibrium strategies.

If we want a solution where such coordination is not permitted then the appropriate solution is equilibria of the agent normal form, and the perfect equilibria—since each player will have only one information set there is no need to distinguish between extensive form perfect, normal form perfect, or quasi perfect—of the game will encompass whatever aspects of backward induction we want. In this paper we shall consider solutions in which such coordination among the choices of a player at his different information sets is possible.

In the game we've being considering there is a unique equilibrium and in that equilibrium all information sets are reached with positive probability. There is no need for backward induction arguments. Nevertheless it is instructive to look at the example to see what the nature of a backward induction requirement will be. In the example all information sets are reached so there should be no issue about what the beliefs of the players will be.

Lets consider the beliefs of Player 1 at his second information set. It's clear that his beliefs will differ depending on what pure strategy he is playing. If he is playing (In, S) then he will assess a probability 3/7 on the left node and 4/7 on the right node, while if he is playing the strategy (Out, D) then he will assess assess a probability 1 on the left node and 0 on the right node. Moreover his assessment of what the other player is playing is also different. When Player 1 is playing the strategy (Out, D) then if called upon to move at his second information set he will assess a probability of 1 on Player 2 having played IN rather than his prior probability of 3/4.

In games with perfect recall there is an equivalence, roughly corresponding to the equivalence between mixed and behaviour strategies shown by Kuhn, between beliefs on the nodes of the information set and beliefs on what the other players are playing. In games without perfect recall this is not so. It will be clear as we consider other examples that beliefs about which node of an information set a player is at are not rich enough to encompass what is needed in games without perfect recall. Thus we shall think of a Player's assessment as a belief about the strategies of the other players, including Nature. Here, when Player 1 is playing (In, S), his beliefs will be that with probability 4/7 Nature chose "Right" (and Player 2 chose IN with probability 3/4 and OUT with probability 1/4) and with probability 3/7 Nature chose "Left" and Player 2 choose IN. On the other hand, if he is playing (Out, D) then he will assess a probability 1 on the fact that Nature chose "Left" and Player 2 chose IN.

Let us summarise the general features illustrated by the analysis in the previous paragraph of our particular example. A player's beliefs at an information set may depend on which pure strategy he himself is playing. And, for each of his pure strategies, his beliefs about the strategies of the others may not be an independent product (across players) of probability distributions on the pure strategies of the other players. Of course, this latter fact is also true in games with perfect recall.

Before proceeding further with our consideration of how we should define beliefs we need to consider another aspect in which the situation differs between games

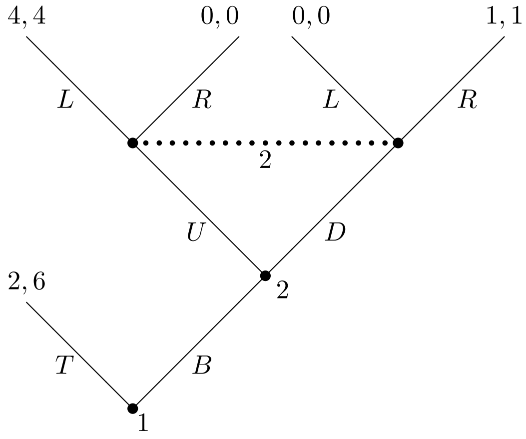

Figure 3. No One Deviation Principle.

with perfect recall and games without perfect recall. In games with perfect recall in order to show that a strategy is optimal it is enough to show that at each information set it is optimal taking as fixed the choice at the other information sets of the player. In games without perfect recall this is not true.

Consider the game of Figure 3. Player 1 decides whether to take an outside option or allow the two agents of Player 2 to play a coordination game in which Player 1 obtain the same payoff as Player 2.

Consider the strategies in which Player 1 chooses T and Player 2 DR. This is an equilibrium. Moreover, if we consider only deviations at one information set at a time, it appears to robustly satisfy backward induction type arguments. Nevertheless, in the one person subgame beginning with Player 2's first move there is only one equilibrium, namely UL, and hence the only subgame perfect equilibrium is (B, UL).

Thus there seems to be no hope that we can satisfy the "one deviation principle" that several backward induction concepts satisfy in games with perfect recall.

Nevertheless when we consider a player at a particular information set we cannot allow that player to deviate at an arbitrary collection of other information sets following that one, or, at least, that such a requirement is problematic. In the game of Figure 4 consider the equilibrium (T, RU). We claim that this is a reasonable equilibrium, and in particular that it intuitively satisfies a reasonable notion of backward induction. However, if we allow Player 2 at the node at which he chooses between L and R to deviate at both information sets that would upset this equilibrium.

One option would be to allow players to deviate at an information set and at information sets all of whose nodes follow some node in the information set under consideration. Such a requirement is however not strong enough. Consider again the game of Figure 1. Since neither information set of Player 1 follows the other in this sense we cannot at either information set have Player 1 consider deviating

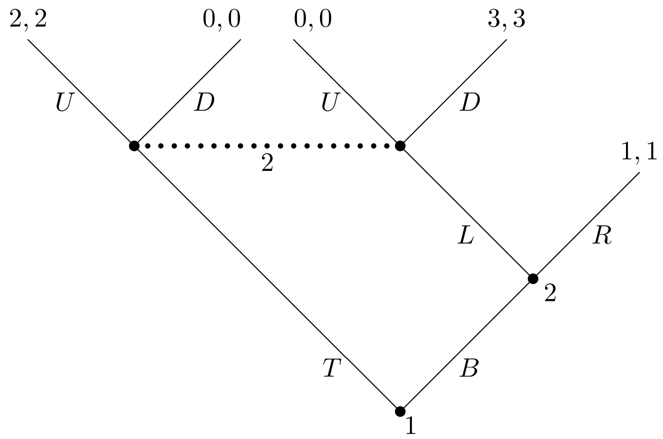

FIGURE 4. Cannot always deviate at multiple information sets.

at both of his information sets. Thus we would allow a strategy equivalent to the equilibrium of the agent normal form, which we have argued we do not wish to do. One could perhaps get around this by adding dummy moves at various points throughout the game. We take a somewhat different approach, which seems to us a little more attractive.

4. Definitions of the Central Concepts

We shall now define sequential equilibria and quasi-perfect equilibria. Since we have seen that we cannot hope to satisfy a one-deviation property and that it will be necessary to consider players deviating simultaneously at a number of information sets we shall define beliefs not at an information set but at a collection of information sets. In the original definition of sequential equilibrium beliefs were defined as a probability distribution over the nodes of an information set. Here we define beliefs as distributions over the pure strategies that are being played, including Nature's strategy.

Definition 5. A system of beliefs μ defines, for each n in N and each H in $\bar{\mathcal{H}}_n$ a distribution $\mu(s_0, s_1, \ldots, s_N \mid H)$ over the profiles of pure strategies that reach H. Given μ we also consider $\mu_{S_n}(s_n \mid H)$ and $\mu_{S_{-n}}(s_{-n} \mid s_n, H)$ the marginal distribution on S_n given H and the conditional distribution on S_{-n} conditional on s_n and H.

Recall that we have seen above that a player's beliefs at an information set about what strategies the other players are playing may differ depending on what pure strategy he himself is playing. Notice also that we include the (pure) strategy of Nature in the list of strategies over which Player n has beliefs.

We first define sequential equilibria.

Definition 6. Given a pair (σ, μ) we say that the pair is consistent (or is a consistent assessment) if there is a sequence of completely mixed strategy profiles

$\sigma^t \to \sigma$ with μ^t a system of beliefs obtained from μ^t as conditional probabilities and $\mu^t \to \mu$.

Definition 7. Given a pair (σ, μ) we say that the pair is sequentially rational if, for each n, for each H in $\bar{\mathcal{H}}_n$, and for each s_n in S_n if $\mu_{S_n}(s_n \mid H) > 0$ then s_n maximises

$$E_{\mu_{S_{-n}}(s_{-n}|s_n,H)} u_n(t_n, s_{-n})$$

over the set of all t_n in S_n such that t_n differs from s_n only at information sets in H.

Definition 8. Given a pair (σ, μ) we say that the pair is a sequential equilibrium if it is both consistent and sequentially rational.

Quasi-perfect equilibria are defined in a similar way.

Definition 9. A strategy profile σ is a quasi-perfect equilibrium if there is a sequence of completely mixed strategy profiles $\sigma^t \to \sigma$ with μ^t a system of beliefs obtained from μ^t as conditional probabilities and $\mu^t \to \mu$ and for each n, for each H in $\bar{\mathcal{H}}_n$, and for each s_n in S_n if $\mu_{S_n}(s_n \mid H) > 0$ then s_n maximises

$$E_{\mu^t_{S_{-n}}(s_{-n}|s_n,H)} u_n(t_n, s_{-n})$$

over the set of all t_n in S_n such that t_n differs from s_n only at information sets in H.

Observe that the definitions of sequential equilibrium and quasi-perfect equilibrium differ only in use of μ^t rather than μ in defining the expected utility that is maximised.

5. Results

We now give a number of results about the concepts we have defined. The proofs are straightforward but are not yet included.

The first two results say that in games with perfect recall we obtain the "same" equilibria as the original definitions.

Proposition 1. *If the game has perfect recall then if (σ, μ) is a sequential equilibrium then there is a behaviour strategy profile b, equivalent to σ in the sense of Kuhn, that is the strategy part of a sequential equilibrium according to the definition of Kreps and Wilson (1982). Moreover for any sequential equilibrium in the sense of Kreps and Wilson there is an equivalent mixed strategy σ and a system of beliefs μ such that (σ, μ) is a sequential equilibrium.*

Proof. □

Proposition 2. *If the game has perfect recall then if σ is a quasi-perfect equilibrium then there is a behaviour strategy profile b, equivalent to σ in the sense of Kuhn, that is a quasi-perfect equilibrium according to the definition of van Damme (1984). Moreover, for any quasi-perfect equilibrium in the sense of van Damme there is an equivalent mixed strategy profile that is a quasi perfect equilibrium.*

Proof. □

The next two results say that the relation between the concepts is as it was in games with perfect recall.

Proposition 3. *Every quasi-perfect equilibrium is a sequential equilibrium.*

Proof. □

Proposition 4. *For any extensive form, except for a semialgebraic set of payoffs of lower dimension than the set of all payoffs, every sequential equilibrium is a quasi-perfect equilibrium.*

Proof. The proof follows in a straightforward way similar to the proof of the generic equivalence of perfect and sequential equilibria in Blume and Zame (1994) and the proof, based on Blume and Zame, of the generic equivalence of quasi-perfect and sequential equilibria in Hillas, Kao, and Schiff (2016). □

Finally we have the result proved for games with perfect recall by van Damme (1984) and Kohlberg and Mertens (1986) relating quasi-perfect and sequential equilibria to proper equilibria (Myerson, 1978) of the normal form.

Proposition 5. *Every proper equilibrium is a quasi-perfect equilibrium (and hence a sequential equilibrium).*

Proof. □

Since every game has a proper equilibrium this result also implies the existence of sequential and quasi-perfect equilibria.

6. Extension to Nonlinear Games

We now turn to nonlinear games, that is, we remove the restriction that each play of the game cuts each information set at most once.

In nonlinear games we also need to consider randomisations over behaviour strategies. We *can* consider such strategies for linear games, but we do not need to do so.

Definition 10. A general strategy in an extensive form game for Player n is a probability distribution over the player's behaviour strategies. We denote the set of Player n's general strategies by G_n and the set of general strategy profiles by $G = \times_{n \in N} G_n$.

In nonlinear games a player may achieve outcomes using behaviour strategies that he cannot achieve using pure or mixed strategies. Further he may need to conceal the particular behaviour strategy he is using from his opponents. Thus we are led to consider mixtures of behaviour strategies which we call general strategies. Since there are an infinite number of behaviour strategies the space of general strategies is infinite dimensional. Fortunately, we do not need to consider all mixtures of behaviour strategies. The following result allows us to restrict ourselves to a finite dimensional subset of G_n. This result was proved by Alpern (1988).

Proposition 6 (Alpern 1988)**.** *For any Player n in N there is a finite finite number K_n such that for any general strategy g_n of Player n there is general strategy g'_n that puts weight on only on K_n elements of B_n that is equivalent to g_n in the sense that for any general strategies g_{-n} of the other players (g_n, g_{-n}) and (g'_n, g_{-n}) induce the same distribution on the terminal nodes.*

Proof. Each terminal node t in T, the set of terminal nodes, defines a set of decision nodes of Player n on the path from the initial node to t. For each of these nodes there is a branch $\{x, y\}$ from x with y also on the path to t. A behaviour strategy b_n of Player n induces a conditional probability on the branch $\{x, y\}$ conditional on x having been reached. Let $q_n(t, b_n)$ be the product of the conditional probabilities generated by b_n on the branches following nodes owned by Player n that occur on the path to t. If Player n has no nodes on the path to t we let $q_n(t, b_n) = 0$. Similarly define $q_0(t, b_0)$ for Nature, where b_0 is Nature's only strategy. Thus if the players play $b = (b_1, b_2, \dots, b_N)$the probability that terminal node t will be reached is $\prod_{n \in \hat{N}} q_n(t, b_n)$. Notice that b_n influences the distribution over terminal nodes only through $q_n(t, b_n)$.

Let

$$
\begin{aligned}
Q_n = \{(q_t)_{t \in T} \subset [0,1]^T \mid & \text{for some } g_n \text{ in } G_n \text{ for all } t \\
& q_t = \int_{B_n} q_n(t, b_n) dg_n(b_n)\}.
\end{aligned}
$$

It is clear that Q_n is the convex hull of those points $(q_t)_{t \in T}$ in Q_n with g_n putting weight only on one behaviour strategy, that is, of the set

$$
\begin{aligned}
C_n = \{(q_t)_{t \in T} \subset [0,1]^T \mid & \text{for some } b_n \text{ in } B_n \text{ for all } t \\
& q_t = q_n(t, b_n)\}.
\end{aligned}
$$

But since C_n (and Q_n) are subsets of $\mathbb{R}^T$, by Carathéodory's Theorem any q in Q_n can be written is a convex combination of at most $T+1$ elements of C_n. That is it is generated by a general strategy g_n that puts weight on at most $T+1$ elements of B_n.

We have shown that for any g_n we can find a g'_n that puts weight on only $T+1$ elements of B_n such that g_n and g'_n generate the same element of Q_n. But g_n will influence the probability of a final node only through the element of Q_n it generates and the result follows. □

Remark 1. In our proof we have given $T+1$ as the bound on the number of behaviour strategies that may receive positive weight. This can be substantially strengthened. In general many different terminal nodes may be associated with the same set of edges following nodes of Player n on the path to that terminal node. We require in Q_n only one dimension for each such set of edges.

Remark 2. It is not true that we can restrict attention to only a fixed finite subset of B_n. Consider the Absent-minded Driver example in Figure 5. We claim that there is no finite set of behaviour strategies such that the outcome from any behaviour strategy can be replicated by some mixture over the given set. Suppose that we have T behaviour strategies $b^1, b^2, \dots, b^T$ with $b^t = (x^t, 1 - x^t)$ with x^t being the probability that the player chooses L. So, if the player plays b^t he ends up with outcome a with probability x^t, with outcome b with probability $x^t(1 - x^t)$, and with outcome c with probability $(1 - x^t)^2$. Let $\bar{x}$ be the smallest value of x^t strictly greater than 0.

Consider the behaviour strategy $b^0 = (\bar{x}/2, 1 - (\bar{x}/2))$. This strategy gives outcome a with probability $\bar{x}/2$ and outcome b with probability $(\bar{x}/2)(1 - (\bar{x}/2))$. Now for any b^t which give strictly positive probability of outcome b we have

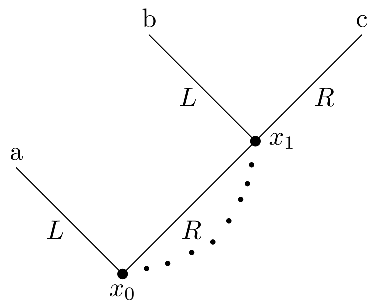

FIGURE 5. The absent-minded driver (Piccione and Rubinstein 1977).

that the ratio of the probability of outcome a to the probability of outcome b is $1/(1 - x^t) \geq 1/(1 - \bar{x})$.

Thus if we have a general strategy putting weight only on $b^1, b^2, \ldots, b^T$ that gives outcome b with the same probability as b^0, that is with probability $(\bar{x}/2)(1-(\bar{x}/2))$ it will give outcome a with probability at least

$$\left(\frac{1}{1-\bar{x}}\right)\left(\frac{\bar{x}}{2}\right)\left(1-\frac{\bar{x}}{2}\right) = \left(\frac{\bar{x}}{2}\right)\left(\frac{2-\bar{x}}{2-2\bar{x}}\right) > \frac{\bar{x}}{2},$$

and so it does not induce the same probabilities on outcomes as b^0.

As a consequence of Proposition 6, instead of working with the infinite dimensional space G_n we can instead work with the finite dimensional space

$$\hat{G}_n = \Delta_{K_n} \times {B_n}^{K_n},$$

the Cartesian product of the K_n-simplex with K_n copies of B_n. The typical element $(\alpha_1, \ldots, \alpha_k, \ldots, \alpha_{K_n}, b_n^1, \ldots, b_n^k, \ldots, b_n^{K_n}) \in \hat{G}_n$ means that for each k Player n plays his behaviour strategy b_n^k with probability α_k. For every element of G_n there is a Kuhn-equivalent element in the subset $\hat{G}_n$.

And this allows us to avoid certain technical issues and, more importantly, to use techniques of real algebraic geometry to prove the generic equivalence of sequential and quasi-perfect equilibria.

We claimed earlier that the issues in nonlinear games were similar to the issues in linear games without perfect recall. The idea is to replace pure strategies with behaviour strategies and mixed strategies with general strategies. If we simply did this we would obtain the following definitions.

Definition 11. A system of beliefs μ defines, for each n in N and each H in $\bar{\mathcal{H}}_n$ a finite distribution $\mu(s_0, b_1, \ldots, b_N \mid H)$ over the profiles of behaviour strategies. Given μ we also consider $\mu_{B_n}(b_n \mid H)$ and $\mu_{B_{-n}}(b_{-n} \mid b_n, H)$ the marginal distribution on B_n given H and the conditional distribution on B_{-n} conditional on b_n and H.

Definition 12. Given a pair (g, μ) we say that the pair is consistent (or is a consistent assessment) if there is a sequence of general strategies in $\hat{G}$ that put weight only on completely mixed behaviour strategies $g^t \to g$ with μ^t a system of beliefs obtained from g^t as conditional probabilities and $\mu^t \to \mu$.

Definition 13. Given a pair (g, μ) we say that the pair is sequentially rational if, for each n, for each H in $\bar{\mathcal{H}}_n$, and for each b_n in B_n if $\mu_{B_n}(b_n \mid H) > 0$ then b_n maximises

$$E_{\mu_{B_{-n}}(b_{-n}|b_n,H)} u_n(\beta_n, b_{-n})$$

over the set of all β_n in B_n such that β_n differs from b_n only at information sets in H.

Definition 14. Given a pair (g, μ) we say that the pair is a sequential equilibrium if it is both consistent and sequentially rational.

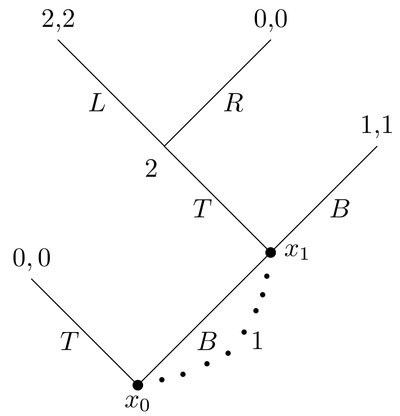

Figure 6. In a nonlinear game a consistent assessment may not reach H.

Consider the nonlinear game given in Figure 6. Suppose we take a sequence of general strategies g^t that put all weight on the single behaviour strategy profiles $b^t = ((p^t, 1-p^t), (q^t, 1-q^t))$ with $(p^t, q^t) \to (0, 0)$. In the limit this is equivalent to the pure strategy profile (B, R). This clearly should not be a sequential equilibrium—Player 2's behaviour is clearly not sequentially rational. And yet, with the suggested definition it is.

Unlike in the case of linear games the strategies that have positive probability conditional on H do not necessarily make H reachable. Or definition will have to be a little more delicate. One solution is to expand the definition of an assessment to include, for each $H \in \bar{\mathcal{H}}_n$, a distribution over both the nodes in H and the behaviour strategy profiles and to define sequential rationality using using expected utility conditional on being in H. This is done in more detail in Hillas and Kvasov (2017a).

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