An Axiomatic Approach to Failures in Contingent Reasoning*

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Abstract

The aim of this paper is to characterize *incomplete preferences with optimism and pessimism* (IPOP) over Anscombe-Aumann acts, a class of preference orders that may fail axioms that require the decision-maker to think contingently. The main result axiomatizes preferences $\succsim$ represented by the following rule:

$$f \succsim g \iff \inf_{\mu \in C^{\flat}} \int (u \circ f) d\mu \geq \sup_{\mu \in C^{\sharp}} \int (u \circ g) d\mu,$$

for any distinct acts f and g. Here, u is an affine utility function on lotteries, and $C^{\sharp}$ and $C^{\flat}$ are non-disjoint, closed, and convex sets of prior beliefs over states of the world. This representation has a natural interpretation as cautious attitudes of the decision-maker under contingencies. She concludes that f is superior to another act g if and only if the pessimistic expected utility from f is still greater than the optimistic expected utility from g. The class reduces to the standard SEU preferences when belief sets are minimal, while it encapsulates obvious dominance, the decision criterion proposed by Li (2017), as the polar case of when belief sets are maximal.

Keywords: Incomplete preferences, Anscombe-Aumann axioms, multiple prior models, obvious dominance, failures in contingent reasoning.

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1 Introduction

1.1 Background

Among several solution concepts in game theory, a dominant strategy equilibrium is particularly popular to make reasonable predictions about strategic consequences. Indeed, it yields predictions that are independent of how players perceive both strategic and structural uncertainty, and thus, robust against information structures held by them. For this reason, the optimality of dominant strategies is clear for players as well even in the presence of uncertainty. It is fair to say that a dominant strategy equilibrium only relies on weaker assumptions about the rationality of players than many other solution concepts.

Still, some implicit presumptions should be satisfied. One thing among others is that players should be rational enough to be able to imagine the possible consequences of their actions under different contingencies.[1] Specifically, for a player to conclude that one action dominates another action, the consequences of two actions should be compared conditional on each possible event at first, and subsequently, those conditional relations are synthesized to derive an (unconditional) dominance relation. This sort of deductive inference is often called *contingent reasoning*, and the rationality for any choice to follow it is originally referred to as the sure thing principle by Savage (1961). Thus, the popularity of dominant strategy equilibria is partly backed by the compelling sounds of decision-theoretic foundations.

On the other hand, ranging in a wide variety of economic environments, many pieces of experimental evidence have witnessed significantly many subjects who fail in taking their dominant actions.[2] Especially recently, Esponda and Vespa (2019) design laboratory experiments to detect the cause for these anomalistic outcomes in the laboratory, and find that most common anomalies are in large part explainable by the failures in contingent reasoning. In parallel with the empirical literature, there have been theoretical attempts to study how easy games or mechanisms are to be played by cognitively limited players. Among them, ever since the seminal work by Li (2017), *obvious dominance* and *obvious strategy-proofness* gather a lot of attention from both theoretical and practical standpoints, as benchmark criteria to evaluate

[1] Yet another, more primitive, presumption is that players must have properly had in their mind all possible consequences that can realize in the future. This may be arguably demanding, as people often experience unexpected events out of the blue. Perhaps due to tractability, however, the state space approach is still popularly used in economics to build a model of uncertainty. There are recent attempts by some researchers to construct decision models that do not start with the state space, including Karni and Vierø (2013) and Sadler (2020).

[2] To list a few, these environments include voting (Esponda and Vespa, 2014), auctions (Kagel et al. 1987; Kagel and Levin, 1993; Charness and Levin, 2009; Li, 2017), and matching problems (Rees-Jones, 2017). See the introductory section of Esponda and Vespa (2019) for further review of experimental literature.

the simplicity of mechanisms.

It is perhaps intuitive to some degree that an obviously strategy-proof mechanism assumes less about the cognitive abilities of participants than so does a (not obviously) strategy-proof mechanism. But, there seem to be no solid foundations that relate cognitive assumptions to actual behavior. Putting differently, the literature has not yet been provided behavioral axioms that motivate such boundedly rational decision rules.[3] The aim of this paper is to shed some light on these issues. To this end, we weaken the classical axioms of decision theory, so as to be satisfied even when the decision-maker may not be able to conduct contingent reasoning. As a sequel, we obtain a class of preferences that include subjective expected utility (SEU) preferences and obvious dominance in two polar extremes. Let us explain our approach and findings in more detail below.

1.2 Overview

We employ the decision-theoretic model by Anscombe and Aumann (1963) that has a weak preference relation $\succsim$ over acts as a primitive, but their axioms are weakened so as not to require the ability of contingent reasoning. Specifically, when stating axioms, we make clear distinctions between *constant* and *contingent* (i.e., non-constant) acts. The structure of a constant act is simple enough to understand without thinking outcomes brought by it state-by-state, while it may not be the case for contingent acts. Thus, $\succsim$ is supposed to satisfy the standard postulates only when constant acts are involved in comparisons, e.g., *C-completeness* by Bewley (2002) and *C-independence* by Gilboa and Schmeidler (1989) are adopted. Beside them, for instance, our weakened version of *C-monotonicity* posits that if an act f is better than another act g in every state and at least one of them is constant, then $f \succsim g$ should be confirmed.

Our key axiom is what we call *C-calibration*. Roughly speaking, the axiom posits that every strict ranking between contingent acts (complex objects) should have been calibrated via a constant act (simple object) that mediates the two. How can this axiom be interpreted? Perhaps, in reality, people may avoid comparing some complex objects in a direct way if they are unfamiliar with both. Instead, people tend to compare those objects in an indirect way with the help of something familiar that lies between the two. For example, suppose that the decision-maker is not skilled enough in comparing two items of interest, say x and y, due to the lack of direct experiences or the probabilistic sense to evaluate relevant events. Still, she may be able to rank them by invoking yet another item z that she understands well to use as

[3] Zhang and Levin (2017) provide a partial but not complete answer to this issue. See Section 5.2 for more details.

a reference point. Provided that z is surely ranked below x but above y, she can concatenate these propositions to conclude that x is ranked above y. In that sense, a familiar item is playing a role in "calibrating" the ranking between unfamiliar items.

Getting back to the context of decision making under uncertainty, we have presumed that the decision-maker is not enough fluent in contingent reasoning, and thus, evaluating contingent acts is a non-trivial task for her as their outcomes vary in states. On the other hand, C-completeness presumes that the decision-maker is skilled enough in making evaluations on constant acts. Thus, C-calibration posits that a strict ranking between contingent acts should have been calibrated via some constant act that mediates the two.

The main result is presented in Section 3. Specifically, our Theorem 1 axiomatizes incomplete preferences $\succsim$ that are represented as follows:

$$f \succsim g \Longleftrightarrow \left[\inf_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu \quad \text{or} \quad f = g \right],$$

for any acts f and g. Here, u is an affine utility function on lotteries, and $C^\sharp$ and $C^\flat$ ($\neq \emptyset$) are non-disjoint closed and convex sets of prior beliefs over states of the world. In the brackets above, the second rule is just for maintaining the reflexivity of $\succsim$, but rather, the essential part is the first rule. It may be naturally interpreted as capturing the decision-maker's conservative attitudes toward uncertainty. Namely, she concludes that f is superior to another act g if and only if the pessimistic expected utility from f is still greater than the optimistic expected utility from g. Following this interpretation, we shall name the above class the *incomplete preferences with optimism and pessimism* (IPOP). One feature of IPOP representations is the "dual-self" perspective on decisions under uncertainty: Upon making a decision, the decision-maker adopt possibly different beliefs driven by her optimism and pessimism.[4]

Then, we discuss some implications of having different belief sets. In some existing multiple prior models, the size of a belief set reflects the amount of uncertainty perceived by the decision-maker. Indeed, an analogous result is available in the current context as well: The larger the belief sets, the less complete the preference will be. Unlike existing models, however, there is no general need for $C^\sharp$ and $C^\flat$ to coincide with each other. Thus, the decision-maker may perceive different amounts of uncertainty depending on whether she is optimistic or pessimistic. At this point, two extra axioms are presented to characterize the relative sizes of $C^\sharp$ and $C^\flat$ in terms

[4] We remark that such a dual-self perspective is not entirely new in the literature, but also available in some different contexts. One example is the so-called *interval representations* which date back to Luce (1956) and Fishburn (1970). In the context of uncertainty, Frick et al. (2019) have recently introduced *Boolean representations*, in which the optimism and pessimism of the decision-maker interact differently.

of set-inclusion relationships, which jointly characterize when we have $C^\sharp = C^\flat$.

We can relate IPOP with several existing decision rules. In light of the comparative statics, SEU and obvious dominance correspond to the special cases of when belief sets are minimal and maximal, respectively. On the other hand, while the intersections of IPOP with MEU (Gilboa and Schmeidler, 1989) and unanimity rules (Bewley, 2002) are both solely consist of SEU preferences, these classes are obtained as compatible extensions of IPOP. In fact, there are many other compatible extensions beside them. We provide a full characterization of complete preferences that are extended from IPOP and that maintain our mild continuity requirement. Therefore, IPOP can be interpreted as giving novel behavioral foundations to the existing classes of multiple prior models: An analyst can understand the observed behavior of people who are following MEU or unanimity rules as being due to failures in contingent reasoning.

The rest of this paper is organized in a standard way. The next subsection discusses some recent works in decision theory that are related to the current paper. Section 2 introduces the setting, representations, and axioms. Section 3 presents the main axiomatization theorem, and Section 4 conducts comparative statics regarding belief sets. Section 5 provides further results in connection with the existing decision-theoretic models. Section 6 discusses the potential use of our representations in light of an application to second-price auctions. Lastly, Section 7 concludes our discussion. All omitted proofs are relegated to Appendix, where the main theorem is proved in Appendix A and B, the rest of propositions are proved in Appendix C, and a few more results omitted from the main text are provided in Appendix D.

1.3 Related Literature

Presumably, state-wise dominance has been regarded as one of the most basic rationality in almost all economic decisions, and its plausibility has been widely accepted. As is already mentioned, however, we have witnessed the violation of dominance in reality. We briefly discuss a few recent attempts in decision theory to make explanations for the violation of dominance.

Ellis and Piccione (2017) proceed toward this direction, motivated by anomalies in financial markets where investors sometimes reveal a strict preference between portfolios with exactly the same returns. To this end, they enrich the standard Anscombe-Aumann model by adding some structures. Specifically, given an expression of mixture $h = \alpha f + (1-\alpha)g$, their framework distinguishes the left-side and right-side even when the objective returns are the same; h is interpreted as buying a single portfolio, while $\alpha f + (1-\alpha)g$ is interpreted as buying α unit of asset f and $(1-\alpha)$ unit of asset g. Then, their *basic correlation representations* allow the decision-maker to strictly prefer h to $\alpha f + (1-\alpha)g$, which would have resulted from wrongly perceiving the correlation between f and g when computing the mixture. Although there are no formal relations since our analysis does not employ this sort of additional structures, this paper and Ellis and Piccione (2017) are complementable at the conceptual level, as the same phenomenon would be explained from different perspectives: While they attribute the source of anomalies to correlation misperception, this paper attributes it to failures in contingent reasoning.

In much the same spirit as us, some papers are built on the idea that alternatives with high contingencies may be difficult for the decision-maker to evaluate. A few examples include Puri (2020), Saponara (2020), and Valenzuela-Stookey (2020). These papers incorporate "cognitive costs" of contemplating complex objects into traditional decision models. Puri (2020) adopts the vNM framework to study preferences for simplicity, where the expected value of each lottery is subtracted by a cost increasing in lottery's support size. As such, her *simplicity representations* predict the violation of first order stochastic dominance, which is analogous to the violation of monotonicity in this paper. Since completeness is maintained, however, simplicity representations potentially predict dramatic preference reversal resulting from small change in support size. This does not occur in IPOP representations thanks to the existence of "thick" incomparable regions, which have been obtained by giving up completeness. More importantly, her main focus is choice under risk where objective distributions are available, while our main focus is choice under uncertainty that involves the state space and its subjective quantification. Hence, this paper and Puri (2020) have quite different empirical content and potential areas of applications.

On the other hand, Saponara (2020) and Valenzuela-Stookey (2020) adopt the Anscombe-Aumann framework as similar to this paper. They model cognitive costs as partitions of the state space that put constraints on the set of contingencies that the decision-maker can consider. As such, similar ideas may also be found in the literature on coarse reasoning, e.g., Jehiel (2005). There are many formal differences between this paper and their works. For instance, *revealed reasoning representations* by Saponara (2020) maintain both completeness and monotonicity, and *simple bound representations* by Valenzuela-Stookey (2020) maintain monotonicity, while we weaken both in light of modeling failures in contingent reasoning.

As another important difference from the papers mentioned in the previous paragraph, IPOP representations utilize no new primitives except for those that have already appeared in standard multiple prior models. This feature will be particularly useful when we study formal relationships with multiple preferences studied by several authors such as Gilboa and Schmeidler (1989), Bewley (2002), Ghirardato et al. (2004), and Frick et al. (2019). Specifically, all of these preferences are less complete than IPOP representations, thereby they would be observed as "reveled" preferences in actual choice when the decision-maker's underlying preference follows our behavioral axioms. In this regard, our analysis in Section 5.2 is closely related to Gilboa et al. (2005) and its generalization Frick et al. (2020). As in these papers, we start with incomplete preferences and show that they must be extended to a specific class of complete preferences to keep certain consistency requirements. While incomplete and complete preferences admit Bewley and MEU representations in Gilboa et al. (2005), and Bewley and α-MEU representations in Frick et al. (2020), we show that IPOP representations are consistently extended to what we call generalized α-MEU representations.

Finally, since this paper is motivated by the literature on experimental mechanism design, there are close relationships with Li (2017), as well as its follow up paper Zhang and Levin (2017). As is already mentioned, IPOP representations generalize the decision rules considered by them and add some new behavioral foundations. These points are more formally discussed in Section 5.1. In addition, the same application to second price auctions as Li (2017) is studied in Section 6, where we see that his insight as to the departure from truth-telling is not specific to obvious dominance, but it is generalized to IPOP representations.

2 The Setting

Let Z be a compact metric space of prizes, and $\Delta(Z)$ be the set of all probability measures on Z. A generic element $p \in \Delta(Z)$ is called a *lottery*.[5] As usual, let $\Delta(Z)$ be endowed with the weak-* topology.[6] There is a state space (Ω, Σ), where Ω is a compact metric space and Σ is a field on Ω. Denote by $\Delta(\Omega, \Sigma)$, or simply $\Delta(\Omega)$, the set of all finitely additive probability measure (a.k.a. probability charges) on (Ω, Σ). An *act* $f : \Omega \to \Delta(Z)$ is a Σ-measurable function that maps each state ω to a lottery $p \in \Delta(Z)$. Therefore, the set of all acts is

$$\mathcal{F} = \left\{ f \in (\Delta(Z))^{\Omega} : f \text{ is } \Sigma\text{-measurable} \right\}.$$

Note that $\mathcal{F}$ is an affine space if we define addition and scalar multiplication in a state-wise way, i.e., given two acts $f, g \in \mathcal{F}$ and a number $\alpha \in (0,1)$, the mixture $\alpha f + (1-\alpha)g$ is defined to be the act that carries a lottery $\alpha f(\omega) + (1-\alpha)g(\omega)$ in each state $\omega \in \Omega$.

[5] More generally, we can start our discussion with a compact convex subset X of a certain linear space, and let an act be any measurable function from Ω to X.

[6] That is, the topology defined by the following convergence notion: A sequence $\{p_n\}_{n=1}^{\infty}$ in $\Delta(Z)$ weakly converges to $p \in \Delta(Z)$ if $\mathbb{E}_{p_n}[\varphi] \to \mathbb{E}_p[\varphi]$ for all continuous function $\varphi : Z \to \mathbb{R}$. Note that every such expectation is well-defined since φ is bounded, as it is a continuous function on a compact domain. Also, it is well-known that $\Delta(Z)$ is compact and metrizable (and thus separable), provided that Z is a compact metric space. For reference, see Chapter 15 of Aliprantis and Border (2006).

Denote by $\mathbf{1}_A$ the indicator function of an event $A \in \Sigma$. Thus, given two acts $f, g \in \mathcal{F}$ and an event $A \in \Sigma$, the compound act $(f\mathbf{1}_A + g\mathbf{1}_{\Omega \setminus A}) \in \mathcal{F}$ is defined as a function that yields $f(\omega)$ for $\omega \in A$, and $g(\omega)$ for $\omega \in \Omega \setminus A$. In particular, a constant act that yields $p \in \Delta(Z)$ in every state is denoted by $p\mathbf{1}_\Omega$. Let $\mathcal{F}^c = \{p\mathbf{1}_\Omega \in \mathcal{F} : p \in \Delta(Z)\}$ be the set of all constant acts. Clearly, $\mathcal{F}^c$ is isomorphic to the compact topological space $\Delta(Z)$, so they are often identified. Thus, the indicator function $\mathbf{1}_\Omega$ can be abbreviated to write a constant act as p instead of $p\mathbf{1}_\Omega$.

2.1 Preferences and Representations

The primitive is a (weak) preference order $\succsim$ defined on $\mathcal{F}$. As usual, let $\succ$ and $\sim$ be the asymmetric and symmetric parts of $\succsim$, respectively, i.e., $f \succ g$ if and only if $f \succsim g$ and $g \not\succsim f$, and $f \sim g$ if and only if $f \succsim g$ and $g \succsim f$. We do not require $\succsim$ to be a complete order. Namely, there could exist two acts $f, g \in \mathcal{F}$ such that neither $f \succsim g$ nor $g \succsim f$. In such a case, we say f and g are *incomparable*. Otherwise, f and g are said to be *comparable*.

We are interested in preferences $\succsim$ that are represented as follows:

$$f \succsim g \Longleftrightarrow \left[\inf_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu \quad \text{or} \quad f = g \right], \tag{1}$$

where $u : \Delta(Z) \to \mathbb{R}$ is a utility function on lotteries, and $C^\sharp, C^\flat \subseteq \Delta(\Omega)$ are non-empty sets of prior beliefs over the state space. As we have discussed in Section 1, we can interpret the above representation as if the decision-maker has two different selves or identities inside her mind, which we refer to as the optimism and pessimism. Following this interpretation, we call the above class IPOP (incomplete preference with optimism and pessimism) representations. For a reason that will become apparent later, we assume that $C^\sharp \cap C^\flat \neq \emptyset$.[7]

Definition 1. *A preference order $\succsim$ admits an* IPOP representation *if there exists a tuple $(u, C^\sharp, C^\flat)$, where $u : \Delta(Z) \to \mathbb{R}$ is a continuous affine function, and $C^\sharp, C^\flat \subseteq \Delta(\Omega)$ are non-disjoint, closed, and convex sets, that represents $\succsim$ in a way described as in (1).*

Contour sets are useful to grasp the nature of IPOP representations. In Figure 1, we present the upper/lower-contour sets of several weak preferences $\succsim$ that admit IPOP representations. There are two states ω_1 and ω_2, and each axis represents the corresponding coordinate of utility acts $x \equiv u \circ f \in \mathbb{R}^2$. A reference point is marked with black bullet, at which the upper/lower-contour sets are painted in dark/light green, respectively. Note that the boundaries are included in contour sets, so that green regions are closed convex subsets of $\mathbb{R}^2$, whereby the rest region

[7] See the exposition after the main theorem, as well as Appendix D.1.

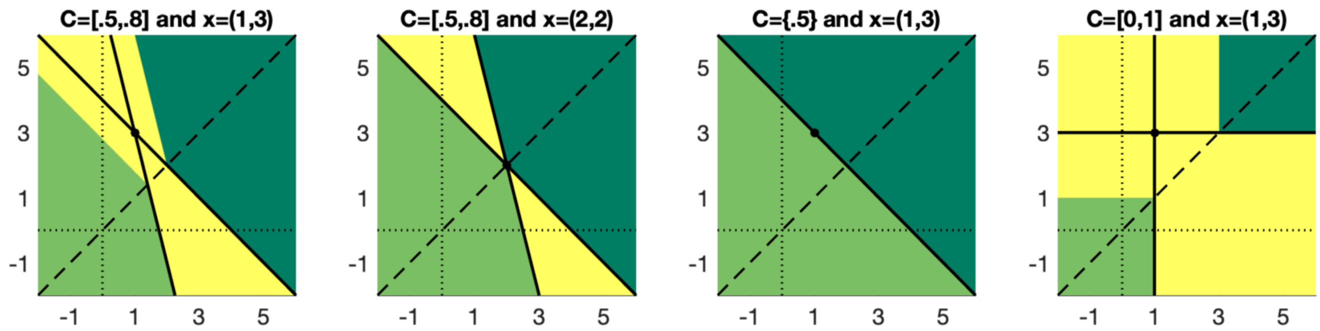

Figure 1: Upper contour-sets (Dark Green), lower contour-sets (Light Green) and incomparable sets (Yellow) are displayed for weak preference relations $\succsim$. All the boundaries are included in dark/light green regions.

painted in yellow becomes an open set of incomparable utility acts. Here, we focus on the case of $C \equiv C^\sharp = C^\flat$, so that the contour sets are symmetrically located around the reference points.

In general, when the reference point is not on the 45° line (leftmost), there appears a thick incomparable region due to the lack of monotonicity. On the other hand, when the reference point is on the 45° line (left middle), the thickness disappears near the neighborhood of the point, in which case the contour sets coincide with the ones generated by Bewley preferences. It is worth mentioning that the thickness of incomparable regions depends on the "degree of asymmetry" and expands as the reference point goes away from the 45° line. As to the effects of different belief sets C, the SEU preferences are captured as the extreme special case of when C is contains only one belief (right middle). As the other extreme case, when C contains all the beliefs on the binary state space (rightmost), the corresponding incomplete preference embodies the decision criterion so-called obvious dominance that ranks one act over another act if and only if the former's worst outcome is still greater than the latter's best outcome.

2.2 Axioms

A preference $\succsim$ is a *preorder* if it is reflexive and transitive. It is non-degenerate if there exist some $f, g \in \mathcal{F}$ such that $f \succ g$. For tractability, we assume that any act f is bounded by some lotteries $p, q \in \Delta(Z)$ with respect to $\succsim$. **A1** is a collection of these basic properties. **A2** and **A3** are the weakened versions of completeness and continuity. The first use of C-completeness can be found in Bewley (2002). He has also adopted a slightly different version of C-continuity.

A1 (Structural Assumptions)**.** $\succsim$ is a non-degenerate preorder such that for any $f \in \mathcal{F}$ there exist some $p, q \in \mathcal{F}^c$ for which $p \succsim f$ and $f \succsim q$.

A2 (C-Completeness)**.** For any $p, q \in \mathcal{F}^c$, either $p \succsim q$ or $q \succsim p$ holds.

A3 (C-Continuity)**.** For any $f \in \mathcal{F}$, the sets $\{p \in \mathcal{F}^c : p \succsim f\}$ and $\{p \in \mathcal{F}^c : f \succsim p\}$ are closed in $\Delta(Z)$.

The next two axioms weaken the standard independence. The first use of **A4** can be found in Gilboa and Schmeidler (1989) to model preferences under uncertainty, but it should be interpreted differently in the current context. Note that the contingency of any act does not increase when it is mixed with an arbitrary constant act. In other word, any mixture $\alpha f + (1-\alpha)p$ is f-measurable. Hence, if the decision-maker has managed to rank f and g despite the contingencies associated with both, then the same ranking should be preserved after mixing each with a constant act since there is no additional cognitive difficulty in delivering the conclusion.

The first half of **A5** corresponds to Uncertainty Aversion adopted by Gilboa and Schmeidler (1989). Our starting point to interpret it is the standard argument that a compound act must get closer to the 45° line than each of compounded acts. As such, fair mixing of any pair of acts reduces the original contingencies, which facilitates the comparison with others. Now, suppose that the decision-maker reveals her solid preferences that f and g are better than a constant act p. As such, f and g are understood to be at least as "secure" as the deterministic outcome p, thereby the mixture $\frac{1}{2}f + \frac{1}{2}g$ should be as secure as p. Furthermore, in the current context, the reverse of this argument is plausible as well. Namely, if f and g have "potential" at most the deterministic outcome p, then the potential of $\frac{1}{2}f + \frac{1}{2}g$ must be at most p.[8] This is the reason why we also impose the latter half of the axiom unlike Gilboa and Schmeidler (1989).

A4 (C-Independence)**.** For any $f, g \in \mathcal{F}$ and $p \in \mathcal{F}^c$, we have $f \succsim g$ if and only if $\alpha f + (1-\alpha)p \succsim \alpha g + (1-\alpha)p$ for all $\alpha \in (0, 1)$.

A5 (Secure-Potential Independence)**.** For any $f, g \in \mathcal{F}$ and $p \in \mathcal{F}^c$, if $f \succsim p$ and $g \succsim p$, then $\frac{1}{2}f + \frac{1}{2}g \succsim p$. Also, if $p \succsim f$ and $p \succsim g$, then $p \succsim \frac{1}{2}f + \frac{1}{2}g$.

Implementing the standard monotonicity requires highly sophisticated abilities of contingent reasoning, which may be lacked by our decision-maker. In light of this, we weaken monotonicity to the next axiom. **A6** says that if an act f state-wise dominates another act g, then f should be at least as secure as g, as well as f should have at most as much potential as g. Note that the direct ranking between f and g may not be available.

[8] We are borrowing expositional words such as "security" or "potential" from the original definitions by Kopylov (2009) to describe the qualities of acts in light of constant acts.

A6 (Secure-Potential Monotonicity)**.** For any $f, g \in \mathcal{F}$ and $p \in \mathcal{F}^c$, if $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $p \succsim f$ implies $p \succsim g$, and $g \succsim p$ implies $f \succsim p$.

Presumably, the last axiom is key to our analysis, whose motivation has been already explained in Section 1. To repeat it, **A7** says that transitive relations, which are mediated by constant acts, are the only inference rules that the decision-maker can use to derive the rankings between distinct non-constant acts.

A7 (C-Calibration)**.** For any distinct acts $f, g \in \mathcal{F}$ such that $f \succsim g$, there exists a constant act $p \in \mathcal{F}^c$ for which $f \succsim p$ and $p \succsim g$.

Meanwhile, the contrapositive of **A7** gives rise to yet another interpretation. We refer to this another form of the axiom as *contagion of incomparability* for the following reason. Perhaps, constant acts are those easiest to evaluate their values, thereby if neither f nor g is comparable with a constant act, then such a pair must be incomparable with one another. Hence, incomparability relations are contagious as they will be spreading via constant acts. **B7** formalizes this intuition.

B7 (Contagion of incomparability)**.** For any distinct atcs $f, g \in \mathcal{F}$, if there exists a constant act $p \in \mathcal{F}^c$ such that neither f nor g is comparable with p, then f and g are incomparable.

It is easy to see that **A7** and **B7** are equivalent in the presence of other axioms.

3 Main Result

Now, we are ready to present the main result of this paper.

Theorem 1. *A preference relation $\succsim$ satisfies* **A1–7** *if and only if there exist a non-constant continuous and affine function $u : \Delta(Z) \to \mathbb{R}$ and non-empty closed and convex sets $C^\sharp, C^\flat \subseteq \Delta(\Omega)$ with $C^\sharp \cap C^\flat \neq \emptyset$ such that*

$$f \succsim g \Longleftrightarrow \left[\inf_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu \quad \text{or} \quad f = g \right]. \tag{2}$$

Moreover, $C^\sharp$ and $C^\flat$ are unique, and u is unique up to positive affine transformations.

As usual, the closedness and convexity are without loss of generality, but only for establishing the uniqueness. Hence, $C^\sharp \cap C^\flat \neq \emptyset$ is the only essential restriction imposed on belief sets. This disjointness condition guarantees the transitivity of $\succsim$, while it has no role more than that. Appendix D.1 discusses more on these points.

Let us sketch the proof of the above theorem, while the formal arguments are relegated to Appendix A and B. As a matter of course, a crucial part is the sufficiency of the axioms, which proceeds in the following steps.

(1) Firstly, we can calibrate a utility function $u : \Delta(Z) \to \mathbb{R}$ on lotteries, as the restriction of $\succsim$ on $\mathcal{F}^c$ is a continuous weak order by **A1–3**. Then, for each general act f, utility functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ are, respectively, calibrated as the values of the "minimal" lottery that is superior to f and the "maximal" lottery that is inferior to f. As a direct implication of transitivity, $f \succsim g$ holds whenever $\underline{U}(f) \geq \overline{U}(g)$. It then remains for us to prove the reverse direction, i.e., $\succsim$ is *not* more complete than the joint representation given by $\overline{U}$ and $\underline{U}$. At this point, **A7** plays a key role: As we have discussed, it has an implication as contagious incomparability, which adjusts the level of $\succsim$'s completeness down to the joint representation by $\overline{U}$ and $\underline{U}$, so that $\underline{U}(f) \geq \overline{U}(g)$ whenever $f \succsim g$ with $f \neq g$, as well.

(2) Secondly, each act f is transformed into a *utility act* by the mapping $f \mapsto u \circ f$. Then, we naturally define functionals $\overline{T}$ and $\underline{T}$ over utility acts via the previous utility functions $\overline{U}$ and $\underline{U}$. Here, we note that **A6** assures the well-definedness of these functionals. Moreover, **A4–6** are used to yield the desirable properties of $\overline{T}$ and $\underline{T}$ such as *positive homogeneity*, *monotonicity*, and *C-additivity*, as well as they respectively satisfy *sub/super-additivity*. Having these properties established, we apply the Gilboa-Schmeidler's theorem to yield the desired integral representations.

The main feature of IPOP representations – the dual-self perspective – is obtained in the first step, where we only assume **A1–3** and **A7**. The rest of axioms are used to add some geometric properties to general utility functions $\overline{U}$ and $\underline{U}$. Note that **A7** plays a key role in enabling such joint representation by two utility functions. A similar technique is utilized by Valenzuela-Stookey (2020) to obtain his simple bound representations.[9]

Somewhat ad-hoc looks the way we incorporate reflexivity into IPOP representations. At this point, there is a tradeoff between reflexivity and transitivity. While reflexivity dictates $\underline{U} \geq \overline{U}$, we must have $\overline{U} \geq \underline{U}$ to maintain transitivity.[10] Thus, $\overline{U} = \underline{U}$ should hold to fulfill both, but then an IPOP representation reduces to a standard SEU preference. Since the reflexivity of a preference order may be such a basic condition, an extra rule has been added to the definition of IPOP representations.

[9] Specifically, the axiom called Uniform Comparability plays the same role as our C-calibration in his paper.
[10] This is due to Lemma 5 in Appendix D.1.

Finally, we point out the aforementioned dual-self perspective is closely related to Fishburn's (1970) interval representations.[11] Indeed, a strict preference $\succ$ admits an interval representation in his sense if and only if **A1–3** and **A7** are satisfied by its weak part $\succsim$. Indeed, since $\succsim$ is represented by $\overline{U}$ and $\underline{U}$, we can naturally construct an interval $I(f) = [\underline{U}(f), \overline{U}(f)]$ for each act $f \in \mathcal{F}$, while transitivity assures that $I(f)$ defined so becomes an interval. In this regard, we obtain some behavioral interpretations of interval preferences as a byproduct of the main theorem.

4 Comparative Statics

Throughout this section, we assume that every preference relation $\succsim$ of interest admits an IPOP representation by some non-constant continuous and affine function $u : \Delta(Z) \to \mathbb{R}$, and non-disjoint closed and convex sets $\emptyset \neq C^\sharp, C^\flat \subseteq \Delta(\Omega)$.

4.1 (Non-)Equalizable Belief Sets

Remark that the decision-maker's belief sets $C^\sharp$ and $C^\flat$ are uniquely identified, but there is no need for them to coincide with one another. That is, the decision-maker may perceive different amounts of uncertainty depending on whether she is optimistic or pessimistic. It is natural to ask when the optimism and pessimism have entirely the same beliefs. More generally, we may be interested in the situations such that one belief set is bigger than another, putting differently, the decision-maker's optimism perceives more uncertainty than the pessimism, or vice versa.

Definition 2. *Given an act $f \in \mathcal{F}$, an act g is a* perfect hedge against *f if there exists some $\alpha \in (0, 1)$ that makes $(\alpha f + (1 - \alpha)g)$ a constant act. In that case, we write as $f \overset{\alpha}{\smile} g$.*

It is clear that $f \overset{\alpha}{\smile} g$ if and only if $g \overset{1-\alpha}{\smile} f$, and $\smile$ defined as the union of $\{\overset{\alpha}{\smile}: \alpha \in (0, 1)\}$ is a symmetric binary relation on $\mathcal{F}$. Also, note that $f \smile f$ holds if and only if f is a constant act.

[11] A strict order $\succ$ on a domain X is an *interval order* if it is irreflexive and satisfies the condition: $x_1 \succ y_1$ and $x_2 \succ y_2$ imply that either $x_1 \succ y_2$ or $x_2 \succ y_1$. Fishburn (1970) shows that any interval relation $\succ$ is represented by a correspondence $I : X \twoheadrightarrow \mathbb{R}$, which outputs a closed interval $I(x) = [a(x), b(x)]$ for each $x \in X$, in the following manner:

- If $I(x) \cap I(y) \neq \emptyset$, then neither $x \succ y$ nor $y \succ x$ (incomparable);
- If $I(x) \cap I(y) = \emptyset$ and $I(x)$ is the right to $I(y)$, then $x \succ y$; and
- If $I(x) \cap I(y) = \emptyset$ and $I(x)$ is the left to $I(y)$, then $y \succ x$.

That is, an interval order ranks x higher than y if and only if the left-most point of the interval $I(x)$ is greater than the right-most point of $I(y)$.

Intuitively, g is a perfect hedge against f if the decision-maker can create a risk-free portfolio by mixing f and g at an appropriate ratio. Any pair of such acts that hedge against each other cannot be *co-monotonic* according to the definition of Schmeidler (1989), but rather, those acts must move toward "opposite" directions as functions of states. Hence, each act would have own preferable states in which one performs better than the other, and this makes the comparison of the two acts difficult. Thus, if the decision-maker could somehow manage to rank f higher than g even in such a difficult situation, then comparing one of the original acts with their mixture should be an easier task, as the mixture is expected to "approach" the 45° line. The next axioms postulate that such comparisons with the mixture are possible when we take the mixing ratio α to make g perfectly hedging against f.

A9 (a). For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \succsim g$ and $f \overset{\alpha}{\smile} g$, then f is comparable with $\alpha f + (1-\alpha)g$.

A9 (b). For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \succsim g$ and $f \overset{\alpha}{\smile} g$, then g is comparable with $\alpha f + (1-\alpha)g$.

Each of the above axioms characterizes the relative "size" of belief sets, by which we mean the relation of set inclusion between optimistic and pessimistic belief sets. As a consequence, **A9 (a-b)** jointly characterize the situations in which the two belief sets are equalized.

Proposition 1. *Let $\succsim$ be represented by $(u, C^{\sharp}, C^{\flat})$.*

(i) $C^{\sharp} \supseteq C^{\flat}$ *if and only if* $\succsim$ *satisfies* **A9 (a)**.

(ii) $C^{\sharp} \subseteq C^{\flat}$ *if and only if* $\succsim$ *satisfies* **A9 (b)**.

(iii) $C^{\sharp} = C^{\flat}$ *if and only if* $\succsim$ *satisfies both* **A9 (a-b)**.

The intuition behind the proof would be explained as follows. Let us say, $f \succsim g$ and $f \overset{\alpha}{\smile} g$ hold, so that $\alpha f + (1-\alpha)g$ becomes a constant act. Interpreting **A7** as the axiom of contagion incomparability, $\alpha f + (1-\alpha)g$ should be comparable with at least one of f or g. Our new axiom **A9 (a)** posits that it is indeed the case that the "greater" one f is comparable with the mixture. Putting differently, f is comparable with the mixture whenever so does g, meaning that the decision-maker is more decisive as to evaluating f than how much she is as to g. On the other hand, since $f \succsim g$, her pessimism is who evaluates f, while her optimism is who evaluates g. Therefore, we would conclude that the pessimism perceives less uncertainty than so does the optimism, thereby $C^{\sharp} \supseteq C^{\flat}$ is predicted. The implication of **A9 (b)** can be symmetrically understood.

4.2 The Amount of Uncertainty

Proposition 1 characterizes the relative sizes of optimistic and pessimistic belief sets. How can we relate these sizes with the attitudes of the decision-maker toward uncertainty? In some existing multiple prior models, the size of a belief set reflects the amount of uncertainty perceived by the decision-maker, which is further equivalent to the degree of how complete her preference is. These ideas have been formalized by Rigotti and Shannon (2005) for Bewley preferences, and by Ghirardato et al. (2004) for MEU preferences.

An analogous result is available in the current context as well.

Definition 3. *A preference $\succsim_1$ is an* extension *of $\succsim_2$ if $f \succsim_1 g$ whenever $f \succsim_2 g$, i.e., $\succsim_1 \supseteq \succsim_2$ when they are viewed as subsets of $\mathcal{F} \times \mathcal{F}$. Furthermore, $\succsim_1$ is called a* compatible extension *of $\succsim_2$ if $\succsim_1 \supseteq \succsim_2$, and $p \succ_1 q$ whenever $p \succ_2 q$ for any $p, q \in \mathcal{F}^c$.*

In words, if $\succsim_1$ is an extension of $\succsim_2$, then they agree on any ranking between a pair of acts whenever it is confirmed by $\succsim_2$, whereas $\succsim_1$ may potentially be able to confirm more rankings. In particular, if $\succsim_1$ is a compatible extension, then $\succsim_1$ preserves all the strict rankings among constant acts confirmed by $\succsim_2$. Note that if both $\succsim_1$ and $\succsim_2$ satisfy C-completeness, then $\succsim_1$ becomes a compatible extension of $\succsim_2$ if and only if both $\succsim_1 \supseteq \succsim_2$ and $\succsim_1 |_{\mathcal{F}^c} = \succsim_2 |_{\mathcal{F}^c}$ hold.[12]

Proposition 2. *For each $i \in \{1, 2\}$, let $\succsim_i$ admit an IPOP representation by $(u_i, C_i^\sharp, C_i^\flat)$. The followings are equivalent:*

(i) *$\succsim_1$ is a compatible extension of $\succsim_2$.*

(ii) *$C_1^\natural \subseteq C_2^\natural$ for each $\natural \in \{\sharp, \flat\}$, and $u_1 = au_2 + b$ for some $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$.*

Therefore, among the class of transitive preferences admitting IPOP representations, no preference is strictly less conservative than preferences with singleton belief sets such as $C^\sharp = C^\flat = \{\mu\}$, namely, SEU preferences.[13] On the other hand, no preference is more conservative than preferences with the universal belief set $C^\sharp = C^\flat = \Delta(\Omega)$. In that sense, provided that any singleton event is Σ-measurable, obvious dominance (Li, 2017) is characterized as the most conservative IPOP representations. Section 5.2 presents the exact axiom added to pin down such conservatism.

[12] Our definition of compatible extension is slightly different from the other popular one in mathematics, e.g., see Chapter 1 of Aliprantis and Border (2006). Specifically, we require the preservation of strict rankings only on constant acts. This treatment is essential when we characterize the class of all compatible extensions to complete orders.

[13] Note that if $C^\sharp$ and $C^\flat$ are singletons, then these must be the same to meet the condition $C^\sharp \cap C^\flat \neq \emptyset$.

5 Connections with Other Decision Models

5.1 Obvious Dominance

As we have mentioned, the class of IPOP representations encapsulate SEU (Anscombe and Aumann, 1963) and obvious dominance (Li, 2017) as the polar cases. While the former is pinned down by requiring the full completeness axiom, the latter will be obtained by adding the full "incompleteness" axiom that is formally presented below. Throughout this subsection, we assume that every singleton set is Σ-measurable, i.e., $\{\omega\} \in \Sigma$ for every $\omega \in \Omega$.

A10 (Strong Incomparability). For any $f, g \in \mathcal{F}$, if there exists some $\omega \in \Omega$ such that $g(\omega) \succ f(\omega)$, then $f \not\succsim g$.

The axiom embodies extremely conservative attitudes toward uncertainty and looks substantially demanding. Not surprisingly, **A10** reduces an IPOP to obvious dominance.[14]

Proposition 3. *A preference relation $\succsim$ satisfies* **A1–4**, **A6–7** *and* **A10** *if and only if there exists a non-constant continuous and affine utility function $u : \Delta(Z) \to \mathbb{R}$ such that*

$$f \succsim g \Longleftrightarrow \left[\inf_{\omega\in\Omega} u(f(\omega)) \geq \sup_{\omega\in\Omega} u(g(\omega)) \quad or \quad f = g\right]. \tag{3}$$

Moreover, u is unique up to positive affine transformations.

Let us mention some differences between the above result and another axiomatization of obvious dominance due to Zhang and Levin (2017). Based on the Anscombe-Aumann framework, they also weaken the standard monotonicity. Specifically, their *O-monotonicity* posits that having $f(\omega) \succsim g(\omega')$ for all $\omega, \omega' \in \Omega$ leads to $f \succsim g$, which essentially applies monotonicity when the worst outcome resulting from f is better than the best outcome resulting from g. Under **A1–3**, it is easy to show that C-monotonicity is equivalent to O-monotonicity. Beside O-monotonicity, they also weaken independence and continuity, while the standard transitivity and completeness are maintained. Their main theorem characterizes a class of preference orders $\succsim$ that are represented by the following utility function:

$$U_\alpha(f) = \alpha(f) \sup_{\omega\in\Omega} u(f(\omega)) + (1 - \alpha(f)) \inf_{\omega\in\Omega} u(f(\omega)),$$

[14] Note that we do not explicitly impose **A5** in the next proposition because it is implied by **A10** under C-independence and C-monotonicity.

where $\alpha : \mathcal{F} \to [0,1]$ is an arbitrary function, so no special structure is assumed.[15]

Given a utility index u, each preference is identified with a function $\alpha(\cdot)$, which collectively characterizes obvious dominance as the "intersection" of several complete preferences represented by U_α. Hence, their axiomatization is somewhat indirect as their axioms are pertaining to each $\succsim_\alpha$ but not to obvious dominance. This paper takes an alternative approach that takes an incomplete preference as the primitive, which has enabled us to directly axiomatize obvious dominance.

We also remark that the class considered by Zhang and Levin (2017) corresponds to the special case of "revealed preferences" that can emerge from IPOP representations. This point will be clear soon after we characterize the complete extensions of IPOP in the next subsection.[16]

5.2 Compatible Extensions of IPOP

5.2.1 Multiple Prior Representations

We shall clarify the connections among IPOP and some other popular classes of multiple prior preferences, emerging from various axioms pertaining to the decision-maker's behavior under uncertainty. On one hand, there are *MEU preferences* (Gilboa and Schmeidler, 1989), in which a pair of acts are compared in terms of minimal expected values:

$$f \succsim g \Longleftrightarrow \inf_{\mu \in C} \int (u \circ f) d\mu \geq \inf_{\mu \in C} \int (u \circ g) d\mu.$$

Here, u is an affine utility function on lotteries, and C is a non-empty closed and convex set of priors. The class relaxes independence to *C-independence* and *uncertainty aversion*, while the rest of Anscombe-Aumann axioms, including completeness, is maintained.

On the other hand, by employing *C-completeness* but restoring the full independence, there emerges the class of *Bewley preferences* (Bewley, 2002), in which a pair of acts is compared in a prior-wise way:

$$f \succsim g \Longleftrightarrow \int (u \circ f) d\mu \geq \int (u \circ g) d\mu, \forall \mu \in C.$$

Again, u is an affine utility function on lotteries, and C is a non-empty closed and convex set of priors. The intersection of MEU and Bewley preferences solely consists of *SEU preferences*

[15] For this reason, the above class is not the special case of invariant biseparable preferences by Ghirardato et al. (2004), as the latter imposes some measurability conditions on $\alpha(\cdot)$; See the next subsection.

[16] In particular, see Proposition 5.

(Anscombe and Aumann, 1963), corresponding to the case of when C is a singleton. As is pointed out by Ghirardato et al. (2004) and Gilboa et al. (2010) in different contexts, however, MEU preferences can be interpreted as the complete extensions of Bewley preferences.

We can find analogous relations between IPOP and Bewley preferences. Namely, the intersection solely consists of SEU preferences, but any transitive preference admitting an IPOP representation can be extended to some Bewley preference. To formalize the idea, we introduce the following binary relation over classes of preferences.

Definition 4. *A non-empty class of preferences $\mathcal{P}$ is said to be* more conservative *than another non-empty class $\mathcal{P}'$ if*

- *for any $\succsim \in \mathcal{P}$ there exists some $\succsim' \in \mathcal{P}'$ such that $\succsim'$ is a compatible extension of $\succsim$; and*
- *for any $\succsim' \in \mathcal{P}'$ there exists some $\succsim \in \mathcal{P}$ such that $\succsim'$ is a compatible extension of $\succsim$.*

Evidently, this more-conservative-relation is a partial order on $(\mathcal{F} \times \mathcal{F}) \setminus \{\emptyset\}$. The next proposition summarizes the relations among different multiple prior preferences. By weakening the standard Anscombe-Aumann axioms so as not to require the ability of contingent reasoning, we obtain more conservative preferences than most popular preferences in the literature of decision making under uncertainty.

Proposition 4. *Let $\mathcal{P}_{\text{SEU}}$, $\mathcal{P}_{\text{MEU}}$, and $\mathcal{P}_{\text{Bewley}}$ denote the classes of corresponding preferences. Also, let $\mathcal{P}_{\text{IPOP}}$ be the class of preferences admitting IPOP representations.*

(i) $\mathcal{P}_{\text{MEU}} \cap \mathcal{P}_{\text{Bewley}} = \mathcal{P}_{\text{SEU}}$*, while $\mathcal{P}_{\text{Bewley}}$ is more conservative than $\mathcal{P}_{\text{MEU}}$.*

(ii) $\mathcal{P}_{\text{Bewley}} \cap \mathcal{P}_{\text{IPOP}} = \mathcal{P}_{\text{SEU}}$*, while $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{Bewley}}$.*

5.2.2 Completions of IPOP

MEU and Bewley preferences can be obtained by considering compatible extensions of IPOP. Since a single incomplete preference could have multiple extensions, however, neither completely characterizes a whole family of compatible extensions. This subsection aims to provide the complete characterization of all complete preferences obtained by extending IPOP. To this end, let $\succsim$ be a transitive preference admitting an IPOP representation by $(u, C^\sharp, C^\flat)$. Recall that a preference relation $\succsim^*$ is a *compatible extension* of $\succsim$ if both $\succsim^* \supseteq \succsim$ and $\succsim^* |_{\mathcal{F}^c} = \succsim |_{\mathcal{F}^c}$ hold. In this subsection, we are particularly interested in extensions of $\succsim$ to complete preferences. Thus, we say that a compatible extension $\succsim^*$ is a *completion* of $\succsim$ if it is a weak order (i.e., complete and transitive).

The next proposition characterizes the class of weak orders obtained by considering completions that maintain C-continuity. The class admits the utility representation quite similar to the well-know α-maximin, or its generalization called *invariant biseparable preferences* (Frick et al., 2019; Ghirardato et al., 2004), but more general than either because $\alpha(\cdot)$ here can be any act-dependent function whose range is in $[0,1]$. In particular, since $\alpha(\cdot)$ need not satisfy the measurability condition in Ghirardato et al. (2004), some completions may still violate monotonicity.[17]

Proposition 5. *Let* $\succsim \in \mathcal{P}_{\text{IPOP}}$ *be a preference that admits an IPOP representation by* $(u, C^\sharp, C^\flat)$. *A preference* $\succsim^*$ *is a C-continuous completion of* $\succsim$ *if and only if* $\succsim^*$ *is represented by a utility function* $I : \mathcal{F} \to \mathbb{R}$ *taking the following form:*

$$I(f) = \alpha(f) \inf_{\mu \in C^\flat} \int (u \circ f) d\mu + (1 - \alpha(f)) \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu, \tag{4}$$

where $\alpha : \mathcal{F} \to [0,1]$ *is an arbitrary function.*

We refer to the class of preferences expressed by (4) as the *generalized α-maximin preferences*. The class is general in that $\alpha(\cdot)$ can vary across acts, which allows the possibility for $\succsim^*$ to violate independence or monotonicity. Also, we remark that no structures of $\alpha(\cdot)$ are needed to ensure the C-continuity of $\succsim^*$ since the expected utility of a constant act is independent of beliefs. Hence, $I(p)$ is invariant to the choice of $\alpha(\cdot)$, which means that $\alpha(\cdot)$ is not identified on $\mathcal{F}^c$.

The above proposition provides a bridge between IPOP and generalized α-maximin preferences. In many decision problems of interest, the solid preference of the decision-maker may be incomplete, but a certain choice eventually has to be singled out. In this regard, it is suggested that the actual process of choice formation proceeds in two steps. The decision-maker first applies her incomplete preference to obtain rankings over acts that she can confirm with confidence. Then, she applies a subjectively chosen completion if there is indeterminacy that should be broken.

Conceptually, the relation between an incomplete preference and its completion is well understood in line with Gilboa et al. (2010), who introduce the notion of objective and subjective rationality in choice formation processes. According to their definitions, a choice is "objectively rational" if the decision-maker has a proof that she is right in making it, whereas a choice is

[17] As a technical remark, the proposition restricts attention to C-continuous completions. Indeed, allowing for the violation of C-continuity, there emerge lexicographic-type completions that are not represented in generalized α-maximin forms. See Appendix D.2 for such an example.

"subjectively rational" if others do not have a proof that she is wrong in making it. If the decision-maker's objective and subjective rationality relations satisfy the axioms in Gilboa et al. (2010), then these relations admit representations by Bewley and MEU preferences, respectively.

The expositions made by Gilboa et al. (2010) can be slightly arranged to accommodate the current context. An IPOP reflects choices made by the decision-maker that are rational in an "obviously objective" sense: she has a proof that she is right in making them even when others may not share a common sense of the states of the world. Similarly, since our decision-maker is boundedly rational and cannot conduct contingent reasoning, a proof to convince her should not be so complicated that involves state-by-state considerations. Thus, a completion reflects choices that are rational in "obviously subjective" sense: there is no proof that convinces her that she is wrong in making them without employing contingent reasoning. Clearly, obvious subjectiveness is more admissible than subjectiveness of choice rationality, and hence, our incomplete preferences can be completed to a broader class of weak orders than Bewley preferences.

6 An Application to Second-price Auctions

6.1 Introduction

This section applies our decision-theoretic model to revisit second-price auctions, where economists have encountered a "gap" between theory and empirics. As is mentioned in Section 1, there are many pieces of empirical evidence that document subjects who are contradicting to the theoretical optimality to report their true valuations. To provide a clue to understanding this gap, Li (2017) proposes a new concept of optimality, that is, obvious dominance, which explains the departure from truth-telling in second-price auctions by assuming that subjects are "extremely" cautious.[18] In this section, we show that the same bottom line can be obtained even when subjects' preferences are captured by IPOP representations, unless representations are not collapsed in the sense that will be apparent later. In particular, this means that the empirical observations can be theoretically explained by parsimoniously assuming that subjects are "slightly" cautious.

[18] Recall that obvious dominance is characterized within the class of IPOP representations by the extra axiom **A10**, which postulates the decision-maker's extremely cautious attitudes. Meanwhile, it should be emphasized that Li's interest is not in explaining the departure from truth-telling itself, but rather in clarifying a practical difference in extensive mechanisms that share the same normal form. It is fair to say that Li's attempt has succeeded greatly in this regard.

In our analysis, the set of undominated bids $B^*(C^\sharp, C^\flat)$ will serve as theoretical predictions of how a bidder would play an auction when she conceives dispersed beliefs $(C^\sharp, C^\flat)$ about the strongest opponent. On one hand, it is shown that there are a continuum of undominated bids unless belief sets are not collapsed. On the other hand, while any strategy is undominated (thus plausible as a prediction) when obvious dominance is adopted, allowing for general belief sets can eliminate strategies far enough away from a true valuation. In this regard, our approach can suggest something informative about what sort of misreporting is likely to happen than others, which could not be answered by obvious dominance. In particular, we utilize this point to shed some light on an empirical question of why overbidding is more commonly observed in laboratories than underbidding.[19]

6.2 The Model

A single indivisible good is auctioned off among potential buyers. We fix an arbitrary buyer (DM) among them and study her optimal bidding in a second-price auction. The outcome space (for her) is given by $(x, t) \in [0, 1] \times [0, \bar{b}] \equiv Z$, where x is a probability of obtaining the good, t is payment to the auctioneer, and $\bar{b}$ is the maximal payment set by the auction platform.[20] Assume that the DM evaluates each outcome (x, t) by the ex-post utility function $u(x, t)$, which satisfies:

- $u(\cdot, t)$ is strictly increasing for each $t \in [0, \bar{b}]$;
- $u(x, \cdot)$ is strictly decreasing and continuous for each $x \in [0, 1]$; and
- $u(0, 0) = 0$, and there exists $v \in (0, \bar{b})$ such that $u(1, v) = u(0, 0)$.

Clearly, such a payment v can uniquely exist, and we refer to it as the DM's (individual) valuation. Note that $u(x, t) = vx - t$ when u is quasi-linear. Let us fix a function u throughout this section.

To model uncertainty faced by the DM, we consider the following reduced form. Since only the highest opponent's bid can affect the outcome of a second-price auction, all the payoff relevant uncertainty is summarized by a single-dimensional space $\Omega = [0, \bar{b}]$ endowed with the Borel algebra Σ. As such, the DM's ex-post payoff is determined by her action $b \in [0, \bar{b}]$

[19] For example, see Kagel et al. (1987) and Kagel and Levin (1993)

[20] Recall that our representation results can be obtained even when we replace the random outcomes $\Delta(Z)$ with an arbitrary compact convex subset of a linear space. Thus, we no longer think about randomization over (x, t), but regard $[0, 1] \times \mathbb{R}$ as the range of acts.

and realized state $\omega \in \Omega$. Specifically, a second-price auction induces the outcome functions $(x, t) : [0, \bar{b}]^2 \to Z$ given by

$$(x(b, \omega), t(b, \omega)) = \begin{cases} (1, \omega) & \text{if } b \geq \omega, \\ (0, 0) & \text{if } b < \omega. \end{cases}$$

Note that ties are broken in favor of the DM, but it is without loss of generality if the DM's beliefs are non-atomic.

Now, assume that the DM subjectively conceives that the highest opponent's bid ω is distributed according to $\mu \in \Delta(\Omega)$. Then, her expected utility is simply derived as follows:[21]

$$U(b, \mu) \equiv \int_0^{\bar{b}} u(x(b, \omega), t(b, \omega)) d\mu(\omega) = \int_0^b u(1, \omega) d\mu(\omega).$$

A prominent feature of second-price auctions is that $b^* = v$ maximizes $U(\cdot, \mu)$ regardless of μ, and it is a unique maximizer under a weak condition, e.g., μ is fully supported on a neighborhood of v. Putting differently, the truth-telling is, in general, a unique undominated bid when DM's uncertainty is captured by a single subjective belief.

On the other hand, this is no longer the case when the DM has multiple beliefs and is boundedly rational in the sense of our decision model. Specifically, assume that the DM is characterized by non-disjoint, closed, and convex belief sets $C^\sharp, C^\flat \subseteq \Delta(\Omega)$, and she evaluates a pair of different actions based on the IPOP representation. Then, the set of undominated actions can be naturally defined as

$$B^*(C^\sharp, C^\flat) = \left\{ b \in [0, \bar{b}] : \nexists b' \in [0, \bar{b}] \setminus \{b\} \text{ s.t. } \inf_{\mu \in C^\flat} \int_0^{b'} u(1, \omega) d\mu(\omega) \geq \sup_{\mu \in C^\flat} \int_0^b u(1, \omega) d\mu(\omega) \right\}.$$

We say b is *dominated* by b' if b has another b' with which the above relation holds.

6.3 Undominated Bids

This subsection uncovers the structures of undominated bids. A probability measure μ is *non-atomic* if for any event $A \in \Sigma$ with $\mu(A) > 0$, there exists an even $B \subseteq A$ such that $\mu(A) > \mu(B) > 0$. Clearly, if μ is non-atomic, every singleton receives zero probability from μ, so that we need not concern about tie cases. For the sake of tractability, we impose this

[21] Henceforth, we assume all integrations are performed on closed intervals unless otherwise specified. Also, we sometimes abuse notation to write $\mu([a, b]) = \mu[a, b]$, $\mu((a, b]) = \mu(a, b]$, and so forth.

regularity condition on the DM's belief sets.

Condition 1. *Every* $\mu \in C^\sharp \cup C^\flat$ *is a non-atomic probability measure on* (Ω, Σ).

Condition 2. *There exists some* $\mu \in C^\sharp \cap C^\flat$ *that is fully supported on a neighborhood of* v.

It is well-known that under Condition 1, each μ has a continuous probability density function, which we may write as ϕ_μ. Condition 2 assures the existence of a belief μ according to which truth-telling is a unique optimal action.

Truth-telling is *transparent* for the DM if the optimistic and pessimistic expected utilities are the same evaluated at v, namely,

$$\sup_{\mu \in C^\sharp} \int_0^v u(1, \omega) d\mu(\omega) = \inf_{\mu \in C^\flat} \int_0^v u(1, \omega) d\mu(\omega). \tag{5}$$

If truth-telling is not transparent, then the left-side is strictly greater than the right-side in the above equation since $C^\sharp \cap C^\flat \neq \emptyset$. We remark that (5) is a knife-edge condition, or truth-telling is not transparent for "almost every" pair of belief sets. This sort of genericity argument is formalized in Appendix D.3.

The next proposition reveals the structure of an undominated set. First, it is enough for us to check whether b is dominated by the truth-telling action v to determine whether b is in $B^*(C^\sharp, C^\flat)$. Second, an undominated set can become a singleton if and only if the above knife-edge condition is satisfied. Third, there almost surely emerge a continuum of deviating actions around v that are not dominated by any other actions.

Proposition 6. *Suppose that Condition 1 is satisfied.*

(i) *Any* $b \neq v$ *is dominated by some* b' *if and only if it is dominated by* v.

(ii) *If truth-telling is transparent, then* $B^*(C^\sharp, C^\flat) \subseteq \{v\}$. *In particular, if Condition 2 is satisfied, then* $B^*(C^\sharp, C^\flat) = \{v\}$.

(iii) *If truth-telling is not transparent, then* $B^*(C^\sharp, C^\flat)$ *is an open interval that contains* v.[22]

The intuition behind Proposition 6 is quite simple and can be easily understood from Figure 2, where the DM is assumed to have only three beliefs, say μ_1, μ_2, and μ_3, according to which her expected utility is drawn as a function of bids. These functions vary across beliefs, but we know that each expected utility function is increasing on $[0, v)$, while it is decreasing on $(v, \bar{b}]$.[23]

[22] The topological notion is relative to $[0, \bar{b}]$.

[23] This is due to the fact that v is a dominant strategy in the usual sense, and thus, optimal when the DM's uncertainty is governed by any single belief.

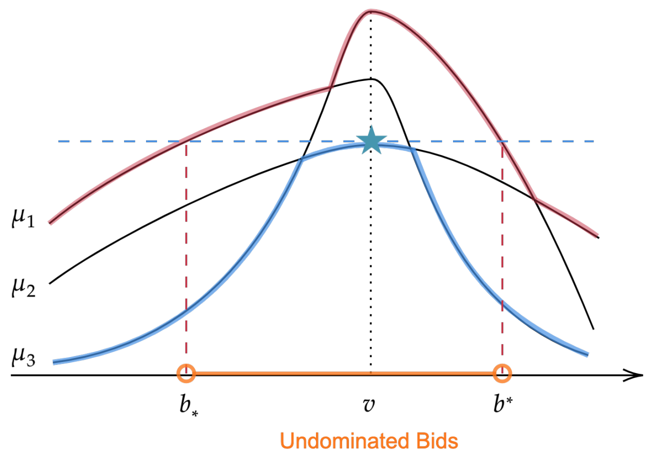

Figure 2: The intuition bihind Proposition 6

Namely, all of them are single-peaked around v. Importantly, the same property is inherited to the minimal expected utility, that is, the blue line in the figure again satisfies single-peakedness. Hence, in order to check whether a given bid b belongs to the undominated set, all we need is to check whether b's maximal expected utility exceeds v's minimal expected utility, marked with blue star above. Furthermore, the maximal expected utility inherits single-peakedness as well, so that the region in which the red line exceeds the peak of the blue line must take an interval form. In particular, the undominated interval would have a non-empty interior, provided that the peaks of these lines are different, i.e., the transparency condition is not met.

According to the previous proposition, the departure from truth-telling in second-price auctions can be theoretically explained by a parsimonious assumption that the participants are "slightly" bounded rational, rather than they are "extremely" bounded rational as is assumed by Li (2017). As such, obvious dominance may be assuming too much: Only the trivial prediction set would be available when the DM's belief sets are too large.

Proposition 7. *If both belief sets are consist of the all probability measures on* (Ω, Σ)*, then* $B^*(C^\sharp, C^\flat) = [0, \bar{b}]$*.*

Proof. Consider a belief μ_0 that assigns a mass to a point $0 \in \Omega$, according to which every bid b trivially achieves the maximal expected utility of $u(1, 0)$. (Note that $b = 0$ can also achieve $u(1, 0)$ because ties are broken in favor of the DM.) On the other hand, v's minimal expected utility is strictly less than $u(1, 0)$ since $C^\flat$ contains every probability measure. Thus, any b is undominated, thereby $B^*(C^\sharp, C^\flat)$ coincides with $[0, \bar{b}]$. □

We also remark that our comparative statics result (Proposition 2) is easily translated to undominated bids in second-price auctions.

Corollary 1. *Suppose that Condition 1 is satisfied by* $(C_i^\sharp, C_i^\flat)$ *for* $i \in \{1,2\}$*. If* $C_1^\natural \subseteq C_2^\natural$ *for each* $\natural \in \{\sharp, \flat\}$*, then* $B^*(C_1^\sharp, C_1^\flat) \subseteq B^*(C_2^\sharp, C_2^\flat)$*.*

Proof. By Proposition 2, we know that the incomplete preference $\succsim_1$ defined by $(C_1^\sharp, C_1^\flat)$ is a subset of $\succsim_2$, which is defined by $(C_2^\sharp, C_2^\flat)$. Thus, if $\succsim_1$ does not rank v higher than b, then so does $\succsim_2$. The proof is done with this and (i) of Proposition 6. $\square$

6.4 Overbidding vs Underbidding

In laboratory experiments, subjects are more likely to exhibit a consistent pattern of overbidding, rather than underbidding. For example, in Kagel et al. (1987), the actual bids submitted from subjects are on average 11% higher than the theoretical predictions. In Kagel and Levin (1993), while only 8% of all bids fall below the true valuations, about 62% of all bids exceed the true valuations.

Despite the robust empirical evidence of overbidding, there is little theory that accounts for why it happens. Among some candidate explanations available, Kagel et al. (1987) infer that overbidding is likely based on the illusion that it improves the chance of winning with no real cost to the winner because only the second-highest bid is paid, while underbidding substantially decreases the chance of winning with no real benefit of reducing the payment. Namely, they conjecture that overbidding can be partly attributed to the asymmetry in rules of second-price auctions.

In the current decision model, however, the structural asymmetry of second-price auctions *per se* is not enough to explain the empirical tendency toward overbidding. Specifically, the next proposition says that the undominated bids would be symmetric around the true valuation, provided that the DM's tastes and beliefs are symmetric.[24] Hence, in order to generate an asymmetric range of undominated bids, we must seek explanations for the DM's primitives, rather than a mechanism itself.

Proposition 8. *Suppose that Condition 1 is satisfied. Furthermore, suppose that* $u(x,t) = vx - t$ *is quasi-linear, and every* $\mu \in C^\sharp$ *satisfies* $\phi_\mu(v+t) = \phi_\mu(v-t)$ *whenever* $v \pm t \in [0, \bar{b}]$*. Then, for any such* t*, we have* $v + t \in B^*(C^\sharp, C^\flat)$ *if and only if* $v - t \in B^*(C^\sharp, C^\flat)$*.*

[24] Note that the below result does not speak about the "frequency" of misreporting in a specific direction, but it is just pertaining to the "magnitude".

Proof. Fix any t such that $v \pm t \in [0, \bar{b}]$. By Proposition 6, it is enough to show that $\overline{U}(v+t) = \overline{U}(v-t)$, where $\overline{U} = \max_{\mu \in C^\sharp} U(\cdot, \mu)$. Indeed, even a sufficient condition for this is true: We show that $\overline{U}(v+t, \mu) = \overline{U}(v-t, \mu)$ for every $\mu \in C^\sharp$. By the quasi-linearity, we have

$$\overline{U}(v+t, \mu) - \overline{U}(v-t, \mu) = \int_{v-t}^{v+t} (v - \omega)\phi_\mu(\omega) d\omega = -\int_{-t}^{t} \omega \phi_\mu(\omega - v) d\omega,$$

where we perform the change of variables $\omega \mapsto \omega - v$. Since a function $\omega \mapsto \omega$ is odd, and $\omega \mapsto \phi_\mu(\omega - v)$ is even due to the ϕ_μ's symmetry around v, the above integral is zero. □

There are at least two ways that we can depart from Proposition 8 to generate an asymmetric range of undominated bids. First, and most obvious, we could assume that $C^\sharp$ contains asymmetric beliefs. In particular, if more beliefs are focusing on large ω, then the undominated set becomes more inclined to overbidding. Second, we can depart from the assumption of quasi-linearity, which has generated the symmetric ex-post payoffs around the truth-telling action in the above proposition. This would be plausible in auctions for expensive items, whose income effects are not negligible.

7 Conclusion

In the presence of uncertainty, the decision-maker has to think contingently to properly understand the consequences of her decision. In some contexts, this type of reasoning is too difficult for an average agent, as is documented in a large body of empirical literature that witnesses significantly many subjects who have failed in playing dominant actions. Motivated by these empirical findings, the present paper has studied the implications of weakening the standard postulates in decision theory. Specifically, we have weakened the Anscombe-Aumann axioms to hold only when at least one of the acts involved in comparison takes a constant form. As a sequel, we have obtained more conservative preferences than most popular preferences in the literature of decision making under uncertainty.

How does this paper contribute to the development of economic theory? In our view, one major contribution is that the paper introduces novel representations that continuously fill the gap between two decision criteria – expected utility maximization and obvious dominance – that provide foundations for Bayesian incentive compatibility and obvious strategy-proofness in mechanism design. Presumably, the former is one of the least demanding incentive constraints, while the latter is the most demanding among those proposed in the literature thus far. The use of a stronger concept can ensure the robustness of mechanisms to misplays by boundedly rational participants, but at the same time non-existence issues may arise, i.e., there may not exist a mechanism that fulfills too strong incentive constraints. Indeed, it is known that obviously strategy-proof mechanisms exist only in a limited number of economic environments. In this regard, our axiomatic analysis of bounded rationality may be useful to indicate possible directions to which incentive constraints should be weakened to make a better compromise.

An interesting extension is the following generalized dual-self representations: Let us remember our Lemma 1, in which the relevant axioms are stated using only a few mathematical structures of model primitives. Specifically, the act space has not necessarily been an affine space. We conjecture that it would rather be possible to start our analysis with an abstract topological space X of acts, and its subset $C \subseteq X$ interpreted as the collection of acts somewhat "clearly" understood by the decision-maker, while these sets have been specified to be $\mathcal{F}$ and $\mathcal{F}^c$, respectively, in the present paper. Generalizing the model in this way opens a variety of new applications. For example, such modeling may be useful to study the *framing effect* (Tversky and Kahneman, 1981) by allowing us to make an explicit distinction between acts that are materially the same but different in connotations, expositions, or descriptions.[25] Given the recent focus on the simplicity of mechanism design, these issues await further inspection.

Appendix A General Representation Results

Before proving Theorem 1, we provide a benchmark result which are stated in terms of general representation forms. Specifically, we show that a part of our axioms suffice to represent $\succsim$ jointly by abstract utility functions $\overline{U}$ and $\underline{U}$.

Given any function $U : \mathcal{F} \to \mathbb{R}$, we denote its image by $\mathrm{Im}(U) = \{U(f) \in \mathbb{R} : f \in \mathcal{F}\}$.

Lemma 1. *A preference relation $\succsim$ satisfies* **A1–3** *and* **A7** *if and only if there exist non-constant functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ with $\overline{U} \geq \underline{U}$ such that*

(i) $\overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ *holds, and the restriction is continuous on $\mathcal{F}^c \simeq \Delta(Z)$;*

(ii) $\mathrm{Im}(\overline{U}|_{\mathcal{F}^c}) = \mathrm{Im}(\overline{U}) = \mathrm{Im}(\underline{U}) = \mathrm{Im}(\underline{U}|_{\mathcal{F}^c})$*; and*

(iii) $f \succsim g$ *if and only if* $\underline{U}(f) \geq \overline{U}(g)$ *or* $f = g$.

[25] On the other hand, in the current model, the decision-maker perceives two acts in the same way whenever they are identical as functions from states to outcomes. Thus, $f(\omega) = \frac{1}{\cos^2(\omega)} - \tan^2(\omega)$ and 1 are treated as the same object, but the former may be less likely to be recognized as being a constant act.

Moreover, a pair of functions $\overline{V}, \underline{V} : \mathcal{F} \to \mathbb{R}$ *satisfies (i)-(iii) for the same preference relation* $\succsim$ *if and only if there exists a continuous and strictly increasing function* $\phi : \mathbb{R} \to \mathbb{R}$ *such that* $\overline{V} = \phi \circ \overline{U}$ *and* $\underline{V} = \phi \circ \underline{U}$.

Proof. The crucial part is the sufficiency of axioms. Suppose that $\succsim$ satisfies **A1–3** and **A7**. By **A1–3**, the restriction $\succsim|_{\mathcal{F}^c}$ is a continuous weak order on $\mathcal{F}^c \simeq \Delta(Z)$, and hence, there exists a continuous function $u : \Delta(Z) \to \mathbb{R}$ that represents $\succsim|_{\mathcal{F}^c}$. Define $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ by

$$\overline{U}(f) = \inf \big\{u(p) : p \in \mathcal{U}^c(f)\big\}, \tag{6}$$

$$\underline{U}(f) = \sup\big\{u(p) : p \in \mathcal{L}^c(f)\big\}, \tag{7}$$

where

$$\mathcal{U}(f) = \{g \in \mathcal{F} : g \succsim f\}, \quad \mathcal{U}^c(f) = \mathcal{U}(f) \cap \mathcal{F}^c, \tag{8}$$

$$\mathcal{L}(f) = \{g \in \mathcal{F} : f \succsim g\}, \quad \mathcal{L}^c(f) = \mathcal{L}(f) \cap \mathcal{F}^c. \tag{9}$$

Indeed, $\overline{U}$ and $\underline{U}$ are well-defined because $\mathcal{U}^c(f) \neq \emptyset$ and $\mathcal{L}^c(f) \neq \emptyset$ by **A1**. It is clear that $u = \overline{U}|_{\mathcal{F}^c} = \overline{U}|_{\mathcal{F}^c}$, and the restriction u is weak-* continuous on the space $\mathcal{F}^c \simeq \Delta(Z)$. Also, **A3** implies that $\mathcal{U}^c(f)$ and $\mathcal{L}^c(f)$ are compact as being closed subsets of $\Delta(Z)$. Therefore, $\overline{U}(f)$ and $\underline{U}(f)$ are attained by some lotteries in $\mathcal{U}^c(f)$ and $\mathcal{L}^c(f)$, which in turn implies that $\mathrm{Im}(u) = \mathrm{Im}(\overline{U}) = \mathrm{Im}(\underline{U})$.

We want to show that $f \succsim g$ if and only if $\underline{U}(f) \geq \overline{U}(g)$ or $f = g$. Let us start with the if direction. Trivially, reflexivity yields $f \succsim g$ whenever $f = g$. Suppose that $\underline{U}(f) \geq \overline{U}(g)$. Recall that there are lotteries $p, q \in \Delta(Z)$ such that $u(p) = \underline{U}(f) \geq \overline{U}(g) = u(q)$. Since u represents $\succsim$, we have $p \succsim q$, while $f \succsim p$ and $q \succsim g$ hold by the constructions of p and q. Using transitivity twice, we get $f \succsim g$ as required. Conversely, suppose that $\underline{U}(f) < \overline{U}(g)$ and $f \neq g$. Again, let $p, q \in \Delta(Z)$ be lotteries that attain the values of $\underline{U}(f)$ and $\overline{U}(g)$, respectively. Suppose not, $f \succsim g$ holds. By **A7**, there exists $r \in \mathcal{F}^c$ for which $f \succsim r$ and $r \succsim g$, meaning that $r \in \mathcal{L}^c(f)$ and $r \in \mathcal{U}^c(g)$. Since p minimizes u in $\mathcal{L}^c(f)$, it follows that $u(p) \geq u(r)$. Similarly, by the maximality of q in $\mathcal{U}^c(g)$, we must have $u(r) \geq u(q)$. Therefore, it follows that $u(p) \geq u(q)$, but this is a contradiction to that $u(p) = \underline{U}(f) < \overline{U}(g) = u(q)$. Hence, $f \not\succsim g$ must hold.

Finally, we shall prove the uniqueness part. The if direction of the statement is easy to verify. Now, suppose that $(\overline{U}, \underline{U})$ and $(\overline{V}, \underline{V})$ satisfy (i)–(iii) of Lemma 1 for the same preference $\succsim$. In particular, the restrictions $u \equiv \overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ and $v \equiv \overline{V}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ represent the same weak order $\succsim|_{\mathcal{F}^c}$ on $\mathcal{F}^c \simeq \Delta(Z)$, and hence, each must be a monotonic transformation of one another, i.e., there exists a strictly increasing function $\phi : \text{Im}(u) \to \mathbb{R}$ for which $v = \phi \circ u$. Note that ϕ must be continuous to maintain continuity of u and v. Fix any $f \in \mathcal{F}$. By $\text{Im}(u) = \text{Im}(\overline{U})$, there exists $\overline{p}_f$ such that $u(\overline{p}_f) = \overline{U}(f)$. Since $(\overline{U}, \underline{U})$ represents $\succsim$,

$$\mathcal{U}(f) \setminus \{f\} = \{g \in \mathcal{F} : \underline{U}(g) \geq \overline{U}(f)\} = \{g \in \mathcal{F} : \underline{U}(g) \geq u(\overline{p}_f)\} = \mathcal{U}(\overline{p}_f),$$

where $\mathcal{U}(\cdot)$ is a contour set defined by $\succsim$. On the other hand, repeating the same argument for $(\overline{V}, \underline{V})$, we can there find $\overline{q}_f$ such that $u(\overline{p}_f) = \overline{U}(f)$, which in turn yields

$$\mathcal{U}(f) \setminus \{f\} = \{g \in \mathcal{F} : \underline{V}(g) \geq \overline{V}(f)\} = \{g \in \mathcal{F} : \underline{U}(g) \geq u(\overline{q}_f)\} = \mathcal{U}(\overline{q}_f).$$

Hence, $\mathcal{U}(\overline{p}_f) = \mathcal{U}(\overline{q}_f)$ holds. In particular, this implies $\mathcal{U}^c(\overline{p}_f) = \mathcal{U}^c(\overline{q}_f)$, which can be true only when $u(\overline{p}_f) = u(\overline{q}_f)$. Therefore, we have

$$\phi \circ \overline{U}(f) = \phi \circ u(\overline{p}_f) = \phi \circ u(\overline{q}_f) = v(\overline{q}_f) = \overline{V}(f),$$

as desired. The symmetric argument verifies that $\phi \circ \underline{U} = \underline{V}$. □

We say U is *C-affine* if $U(\alpha f + (1-\alpha)p) = \alpha U(f) + (1-\alpha)U(p)$ holds for all $f \in \mathcal{F}$, $p \in \mathcal{F}^c$, and $\alpha \in (0,1)$. Also, a real function u defined on $\Delta(Z)$, or any space which is isomorphic to $\Delta(Z)$, is *affine* if $u(\alpha p + (1-\alpha)q) = \alpha u(p) + (1-\alpha)u(q)$ for all $p, q \in \Delta(Z)$ and $\alpha \in (0,1)$. Note that if U is C-affine, then the restriction $U|_{\mathcal{F}^c}$ is affine on the domain $\mathcal{F}^c$ that is isomorphic to $\Delta(Z)$. Perhaps not surprisingly, **A4** adds C-affinity to $\overline{U}$ and $\underline{U}$.

Lemma 2. *Suppose that $\succsim$ satisfies* **A1–3** *and* **A7**. *Then, $\succsim$ satisfies* **A4** *if and only if there exist non-constant and C-affine functions $\overline{U}$ and $\underline{U}$ that represent $\succsim$ in the way of Lemma 1.*

Proof. It is easy to see that $\succsim$ satisfies **A4** whenever $\overline{U}$ and $\underline{U}$ are C-affine. Conversely, suppose that $\succsim$ satisfies **A4**. Since $\succsim|_{\mathcal{F}^c}$ maintains all the vNM axioms, there exists a continuous affine function $u : \Delta(Z) \to \mathbb{R}$ that represents $\succsim|_{\mathcal{F}^c}$. Again, let $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ be defined by (6) and (7), so that $\overline{U}$ and $\underline{U}$ jointly represent $\succsim$ in the way that Lemma 1 prescribes.

Since the argument is symmetric, we only prove that $\overline{U}$ is C-affine. Fix any $f \in \mathcal{F}$, $q \in \mathcal{F}^c$, and $\alpha \in (0,1)$. Let $p \in \mathcal{U}^c(f)$ be a constant act such that $\overline{U}(f) = u(p)$. By the construction, $p \succsim f$ holds, so that **A4** yields $\alpha p + (1-\alpha)q \succsim \alpha f + (1-\alpha)q$. Since $\overline{U}$ and $\underline{U}$ jointly represent $\succsim$, we have

$$\overline{U}(\alpha f + (1-\alpha)q) \leq \underline{U}(\alpha p + (1-\alpha)q) = u(\alpha p + (1-\alpha)q) = \alpha u(p) + (1-\alpha)u(q),$$

where the equalities follow from $\underline{U}|_{\mathcal{F}^c} = u$ and the affinity of u.

To show that the above inequality is tight, we first claim that p is on the boundary of $\mathcal{U}^c(f) \subseteq \Delta(Z)$. If not, we could have an open ball of p which is contained in $\mathcal{U}^c(f)$, but then, there must exist some $\tilde{p} \in \mathcal{U}^c(f)$ with $u(\tilde{p}) < u(p)$ because u is affine and non-constant. This leads to a contradiction to the minimality of p in $\mathcal{U}^c(f)$. Hence, by the continuity of u, for an arbitrarily small $\epsilon > 0$, we can pick $p_\epsilon \in \Delta(Z) \backslash \mathcal{U}^c(f)$ such that $|u(p) - u(p_\epsilon)| < \epsilon$. In particular, since $\overline{U}$ and $\underline{U}$ represent $\succsim$, we have $u(p) = \underline{U}(f) > u(p_\epsilon)$, from which $0 < u(p) - u(p_\epsilon) < \epsilon$. Moreover, **A4** and $p_\epsilon \not\succsim f$ together imply that $\alpha p_\epsilon + (1-\alpha)q \not\succsim \alpha f + (1-\alpha)q$. Then, again by the fact that $\overline{U}$ and $\underline{U}$ represent $\succsim$, we have $\overline{U}(\alpha f + (1-\alpha)q) \geq u(\alpha p_\epsilon + (1-\alpha)q)$. To sum up, we get

$$\begin{aligned}\alpha u(p) + (1-\alpha)u(q) &< \alpha u(p_\epsilon) + (1-\alpha)u(q) + \alpha\epsilon \\ &\leq u(\alpha p_\epsilon + (1-\alpha)q) + \alpha\epsilon \leq \overline{U}(\alpha f + (1-\alpha)q) + \alpha\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ yields the desired inequality. □

Appendix B Proof of Theorem 1

We omit the trivial proof of necessity. Step 1 is the crucial step which proves the sufficiency of the axioms. Step 2 verifies the uniqueness of belief sets.

Step 1: Sufficiency.

Suppose that $\succsim$ satisfies **A1–7**. By Lemma 1 and 2, the axioms **A1–4** and **A7** imply the existence of C-affine utility functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ that represent $\succsim$ in the way described by these lemmas. Specifically, there exists a non-constant continuous affine function $u : \Delta(Z) \to \mathbb{R}$ that calibrate $\overline{U}$ and $\underline{U}$ in the way of (6) and (7), respectively. Henceforth, we arbitrarily fix such a vNM function u, and assume without loss of generality that $[-1, 1] \subseteq \mathrm{Im}(u)$.

Define $\Xi = \{u \circ f : f \in \mathcal{F}\} \subseteq (\mathrm{Im}(u))^\Omega$. A generic element $\xi \in \Xi$ is a Σ-measurable real function on Ω and called a *utility act*, interpreted as a profile of utility values carried by an act. Since u is affine, Ξ is a convex subset of $\mathbb{R}^\Omega$. We have $\{\xi \in [-1, 1]^\Omega : \xi \text{ is } \Sigma\text{-measurebale}\} \subseteq \Xi$ since u is normalized so that $[-1, 1] \subseteq \mathrm{Im}(u)$. Then, define two functionals $\overline{T}, \underline{T} : \Xi \to \mathbb{R}$ by

$$\overline{T}(\xi) = \overline{U}(f) \;\text{ and }\; \underline{T}(\xi) = \underline{U}(f), \;\text{ where } \xi = u \circ f.$$

Remark that these functionals are well-defined, i.e., $\overline{T}(\xi)$ and $\underline{T}(\xi)$ are uniquely determined regardless of the choice of acts f and g such that $\xi = u \circ f = u \circ g$. This is so because **A6** implies that such f and g must have the same contour sets, and thus, $\overline{U}(f) = \overline{U}(g)$ and $\underline{U}(f) = \underline{U}(g)$ due to (6) and (7).

Let $\mathcal{B}(\Omega, \Sigma)$, or simply denoted by $\mathcal{B}$, be the set of all bounded Σ-measurable real functions, and let $\mathcal{B}^c = \{\xi \in \mathcal{B} : \xi(\omega) = \xi(\omega') \text{ for all } \omega, \omega' \in \Omega\}$ be the set of diagonal elements of $\mathcal{B}$. Also, let $\mathcal{B}_+ = \mathcal{B} \cap \mathbb{R}^\Omega_+$ and $\mathcal{B}_- = \mathcal{B} \cap \mathbb{R}^\Omega_-$. As usual, $(\mathcal{B}, \|\cdot\|_\infty)$ becomes a Banach space endowed with the sup norm $\|\xi\|_\infty = \sup_{\omega \in \Omega} |\xi(\omega)|$. Note that $\Xi \subseteq \mathcal{B}$ since u is continuous on the compact domain $\Delta(Z)$. Now, we shall show that $\overline{T}$ and $\underline{T}$ are positively homogeneous, and hence, these functionals can be uniquely extended to cover the whole space $\mathcal{B}$.

Claim 1. $\overline{T}(\lambda \mathbf{1}_\Omega) = \underline{T}(\lambda \mathbf{1}_\Omega) = \lambda$ *for any* $\lambda \in \mathrm{Im}(u)$.

Proof. Given any $\lambda \in \mathrm{Im}(u)$, let $p \in \Delta(Z)$ be a lottery such that $u(p) = \lambda$. By construction, $\lambda \mathbf{1}_\Omega = u \circ p\mathbf{1}_\Omega$, and hence, $\overline{T}(\lambda \mathbf{1}_\Omega) = \overline{U}(p) = u(p) = \lambda$. Similarly, we have $\underline{T}(\lambda \mathbf{1}_\Omega) = \lambda$. □

Claim 2. $\overline{T}$ *and* $\underline{T}$ *are positively homogeneous, i.e.,* $\overline{T}(\lambda\xi) = \lambda\overline{T}(\xi)$ *and* $\underline{T}(\lambda\xi) = \lambda\underline{T}(\xi)$ *for all* $\xi \in \Xi$ *and* $\lambda \geq 0$. *Consequently, they have unique extensions to the set of all* Σ*-measurable real functions on* Ω *that preserve positive homogeneity.*

Proof. Let us show that $\overline{T}$ is positively homogeneous. The claim is trivial when $\lambda = 1$, and it follows from Claim 1 when $\lambda = 0$. Moreover, the results for $\lambda > 1$ are implied by those for $\lambda \in (0, 1)$; Indeed, provided that positive homogeneity holds for $\lambda \in (0, 1)$, we would have $\frac{1}{\mu}\overline{T}(\mu\xi) = \overline{T}(\xi)$ for an arbitrary $\mu > 1$, from which $\overline{T}(\mu\xi) = \mu\overline{T}(\xi)$. Therefore, we assume that $\lambda \in (0, 1)$ henceforth. Let $f \in \mathcal{F}$ be an act such that $\xi = u \circ f$, and let $p \in \mathcal{U}^c(f)$ be a lottery such that $u(p) = \overline{U}(f)$. By $[-1, 1] \subseteq \mathrm{Im}(u)$, we can find a lottery $p_0 \in \Delta(Z)$ such that $u(p_0) = 0$. By the fact that u is affine, we have

$$\begin{aligned}
\overline{T}(\lambda\xi) = \overline{T}(\lambda u \circ f) &= \overline{T}(u \circ (\lambda f + (1-\lambda)p_0)) \\
&= \overline{U}(\lambda f + (1-\lambda)p_0) \\
&= \lambda\overline{U}(f) + (1-\lambda)u(p_0) = \lambda\overline{U}(f) = \lambda\overline{T}(\xi).
\end{aligned}$$

where the third line follows from the fact that $\overline{U}$ is C-affine. Hence, we have shown that $\overline{T}$ is positively homogeneous.

By the similar argument, $\underline{T}$ is shown to be positively homogeneous. Finally, we claim that these functionals can be uniquely extended to $\mathcal{B}$ by preserving positive homogeneity. Indeed, for any non-zero bounded ξ, we can consider the normalized functional $\tilde{\xi} = \frac{\xi}{\|\xi\|_\infty}$, but $\tilde{\xi} \in \Xi$ since $[-1, 1] \subseteq \mathrm{Im}(u)$. To maintain positive homogeneity, we must have $\overline{T}(\xi) = \|\xi\|_\infty T(\tilde{\xi})$ and $\underline{T}(\xi) = \|\xi\|_\infty T(\tilde{\xi})$, which uniquely define the extensions. □

Let us verify several properties of the functional $\overline{T}$ and $\underline{T}$. The next sequence of claims verify monotonicity, C-additivity, and sub/super-additive.[26]

Claim 3. *$\overline{T}$ and $\underline{T}$ are monotonic.*

Proof. Clear from **A6** and positive homogeneity. □

Claim 4. *$\overline{T}$ and $\underline{T}$ are C-additive.*

Proof. Recall that $\overline{U}$ is C-affine, and thus, $\overline{T}(\lambda\xi + (1-\lambda)c\mathbf{1}_\Omega) = \lambda\overline{T}(\xi) + c$ trivially holds whenever $\xi \in \Xi$ and $c \in \mathrm{Im}(u)$. To generalize this observation, given any $\xi \in \mathcal{B}$ and $\lambda \in \mathbb{R}$, let $K = \max\{\|\xi\|_\infty, |\lambda|\}$. Assume $K > 0$ for non-trivial arguments. By positive homogeneity, we see that

$$\begin{aligned}\overline{T}(\xi + \lambda\mathbf{1}_\Omega) &= \overline{T}\left(2K\left(\frac{\xi}{2K} + \frac{\lambda\mathbf{1}_\Omega}{2K}\right)\right)\\ &= 2K\overline{T}\left(\frac{1}{2}\underbrace{\frac{\xi}{K}}_{\in\Xi} + \frac{1}{2}\underbrace{\frac{\lambda\mathbf{1}_\Omega}{K}}_{\in\mathrm{Im}(u)}\right) = 2K\left(\frac{1}{2}\overline{T}\left(\frac{\xi}{K}\right) + \frac{1}{2}\overline{T}\left(\frac{\lambda\mathbf{1}_\Omega}{K}\right)\right) = \overline{T}(\xi) + \lambda,\end{aligned}$$

provided that $K > 0$. Therefore, we conclude that $\overline{T}$ is C-additive. Similarly, we can show that $\underline{T}$ is C-additive. □

Claim 5. *$\overline{T}$ is subadditive, and $\underline{T}$ is superadditive.*

Proof. As before, we assume $\xi, \zeta \in \Xi$ since the general case would be then implied by using homogeneity. So, let $\xi = u \circ f$ and $\zeta = u \circ g$ for some $f, g \in \mathcal{F}$. We shall show that $\overline{T}(\xi + \zeta) \leq \overline{T}(\xi) + \overline{T}(\zeta)$, or equivalently, $\frac{1}{2}\overline{T}(\xi + \zeta) \leq \frac{1}{2}\overline{T}(\xi) + \frac{1}{2}\overline{T}(\zeta)$ holds. Moreover, by positive homogeneity, the left-side is equal to $\overline{T}(\frac{1}{2}\xi + \frac{1}{2}\zeta)$, which is in turn equal to $\overline{U}(\frac{1}{2}f + \frac{1}{2}g)$ by the constructions of ξ and ζ. Hence, it is enough to show that

$$\overline{U}\left(\frac{1}{2}f + \frac{1}{2}g\right) \leq \frac{1}{2}\overline{U}(f) + \frac{1}{2}\overline{U}(g), \tag{10}$$

for arbitrary $f, g \in \mathcal{F}$.

To this end, we first consider the case where $\overline{U}(f) = \overline{U}(g)$. Let $p_f \in \mathcal{U}^c(f)$ and $p_g \in \mathcal{U}^c(g)$ be lotteries that attain these values, i.e., $u(p_f) = \overline{U}(f) = \overline{U}(g) = u(p_g)$. By constructions, we have $p_f \succsim f$, $p_g \succsim g$, and $p_f \sim p_g$. In particular, transitivity implies that $p_f \succsim f$ and

[26] A functional $T : \mathcal{B} \to \mathbb{R}$ is *monotonic* if $T(\xi) \geq T(\zeta)$ whenever $\xi \geq \zeta$; *C-additive* if $T(\xi+\lambda\mathbf{1}_\Omega) = T(\xi)+\lambda$ for all $\xi \in \mathcal{B}$ and $\lambda \in \mathbb{R}$; *super-additive* if $T(\xi+\zeta) \geq T(\xi)+T(\zeta)$ for all $\xi, \zeta \in \mathcal{B}$; *sub-additive* if $T(\xi+\zeta) \leq T(\xi)+T(\zeta)$ for all $\xi, \zeta \in \mathcal{B}$. Also, note that uniform continuity amounts to continuity in $\|\cdot\|_\infty$.

$p_f \succsim g$, thereby **A5** dictates $p_f \succsim \frac{1}{2}f + \frac{1}{2}g$, or $p_f \in \mathcal{U}^c(\frac{1}{2}f + \frac{1}{2}g)$. Hence, by the definition of $\overline{U}$, $\overline{U}(\frac{1}{2}f + \frac{1}{2}g) \leq u(p_f)$. Therefore, (10) is obtained when $\overline{U}(f) = \overline{U}(g)$.

Now, consider the case where $\overline{U}(f) \neq \overline{U}(g)$. Without loss of generality, assume that $\overline{U}(f) > \overline{U}(g)$, or equivalently, $\overline{T}(\xi) > \overline{T}(\zeta)$. Let $\lambda = \overline{T}(\xi) - \overline{T}(\zeta) > 0$, and let $\tilde{\zeta} = \zeta + \lambda \mathbf{1}_\Omega$. Then, the C-additivity of $\overline{T}$ implies that $\overline{T}(\tilde{\zeta}) = \overline{T}(\zeta) + \lambda = \overline{T}(\xi)$. Thus, applying the conclusion of the previous case, we get

$$\overline{T}(\xi + \tilde{\zeta}) \leq \overline{T}(\xi) + \overline{T}(\tilde{\zeta}) = \overline{T}(\xi) + \overline{T}(\zeta) + \lambda.$$

On the other hand, the C-additivity of $\overline{T}$ yields

$$\overline{T}(\xi + \tilde{\zeta}) = \overline{T}(\xi + \zeta + \lambda \mathbf{1}_\Omega) = \overline{T}(\xi + \zeta) + \lambda.$$

The above equations together imply $\overline{T}(\xi + \tilde{\zeta}) \leq \overline{T}(\xi) + \overline{T}(\zeta)$, which shows that $\overline{T}$ is subadditive. The symmetric argument proves that $\underline{T}$ is superadditive, whereas inequalities must be flipped due to the converse implication of **A5**. □

Now, we are ready to apply the integral representation theorem. The following results are due to Gilboa and Schmeidler (1989).

Lemma 3. *Suppose that a functional $T : \mathcal{B} \to \mathbb{R}$ satisfies $T(\mathbf{1}_\Omega) = 1$, positive homogeneity, monotonicity, C-additivity and superadditivity. Then, there exists a closed convex set $C \subseteq \Delta(\Omega)$ such that for all $\xi \in \mathcal{B}$,*

$$T(\xi) = \inf_{\mu \in C} \int \xi d\mu.$$

When we replace superadditivity by subadditivity and keep all the other assumptions, there exists a closed convex set $C \subseteq \Delta(\Omega)$ such that for all $\xi \in \mathcal{B}$,

$$T(\xi) = \sup_{\mu \in C} \int \xi d\mu.$$

Proof. The first part is due to Lemma 3.5 in Gilboa and schmeidler (1989). For the second part, assume that T satisfies all the assumptions of the second part, and let $\tilde{T} : \mathcal{B} \to \mathbb{R}$ be defined by $\tilde{T}(\xi) = -T(-\xi)$. It is straightforward to show that $\tilde{T}$ in turn satisfies all the assumptions of the first part. Hence, there exists a closed convex subset $C \subseteq \Delta(\Omega)$ such that for all $\xi \in \mathcal{B}$,

$$\tilde{T}(-\xi) = \inf_{\mu \in C} \int (-\xi) d\mu,$$

from which

$$T(\xi) = -\tilde{T}(-\xi) = -\inf_{\mu \in C} \int (-\xi) d\mu = \sup_{\mu \in C} \int \xi d\mu,$$

as desired. $\square$

Applying Lemma 3 to $\overline{T}$ and $\underline{T}$ yields closed convex sets $C^\sharp, C^\flat \subseteq \Delta(\Omega)$, where all the presumptions are satisfied due to Claim 1–5. Hence, together with Lemma 1,

$$\begin{aligned} f \succsim g &\Longleftrightarrow \underline{U}(f) \geq \overline{U}(g) \\ &\Longleftrightarrow \underline{T}(u \circ f) \ \geq \overline{T}(u \circ g) \\ &\Longleftrightarrow \inf_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu, \end{aligned} \tag{11}$$

for any $f \neq g$, so that the desired representation is obtained.

By the constructions, $C^\sharp$ and $C^\flat$ are non-empty, closed, and convex. To conclude the proof of sufficiency, we have to show they are non-disjoint. Indeed, $C^\sharp \cap C^\flat \neq \emptyset$ is solely derived from the transitivity of $\succsim$. This claim is obtained as a corollary to Proposition 9 in Appendix D.

Step 2: Uniqueness of belief sets.

We shall prove that $C^\sharp$ and $C^\flat$ are uniquely determined. Given a utility function $u : \Delta(Z) \to \mathbb{R}$, suppose by contradiction that there exist two different pairs of belief sets, say $(C^\sharp, C^\flat)$ and $(D^\sharp, D^\flat)$ that represent the same preference $\succsim$. Without loss of generality, assume that there exists $\mu^\sharp \in C^\sharp \setminus D^\sharp$. Applying the strong separating hyperplane theorem to $\{\mu^\sharp\}$ and $D^\sharp$, we obtain a non-zero bounded linear functional $\xi^{**} \in \mathcal{B}^{**}$ and a scalar $\lambda \in \mathbb{R}$ such that

$$\sup_{\mu \in D^\sharp} \langle \xi^{**}, \mu \rangle \leq \lambda < \langle \xi^{**}, \mu^\sharp \rangle. \tag{12}$$

In particular, we can let $\xi^{**} \equiv \xi \in \mathcal{B}$. Considering a constant function $\lambda \mathbf{1}_\Omega \in \mathcal{B}$, we trivially have

$$\inf_{\mu \in C^\flat} \langle \lambda \mathbf{1}_\Omega, \mu \rangle = \inf_{\mu \in D^\flat} \langle \lambda \mathbf{1}_\Omega, \mu \rangle = \lambda. \tag{13}$$

Then, let $K = \max\{\|\xi\|_\infty, |\lambda|\}$, which is strictly positive because ξ is non-zero. Combining (12) and (13), it follows that

$$\lambda = \inf_{\mu \in D^\flat} \left\langle \frac{\xi}{K}, \mu \right\rangle \leq \sup_{\mu \in D^\sharp} \left\langle \frac{\xi}{K}, \mu \right\rangle, \tag{14}$$

$$\lambda = \inf_{\mu \in C^\flat} \left\langle \frac{\xi}{K}, \mu \right\rangle > \langle \xi, \mu^\sharp \rangle \geq \sup_{\mu \in C^\sharp} \left\langle \frac{\xi}{K}, \mu \right\rangle, \tag{15}$$

while we have $\frac{\xi}{K}, \frac{\lambda \mathbf{1}_\Omega}{K} \in [-1,1]^\Omega \cap \mathcal{B} \subseteq \Xi$. Hence, there exist $f \in \mathcal{F}$ and $p \in \Delta(Z)$ for which $u \circ f = \frac{\xi}{K}$ and $u(p) = \frac{\lambda}{K}$. Therefore, we must have $p \succsim f$ according to (14), but $p \not\succsim f$ according to (15), a contradiction. *Q.E.D.*

Appendix C Proofs for Section 4 and 5

We adopt the same notation as the previous section. Let $\mathcal{B}(\Omega, \Sigma)$, or simply $\mathcal{B}$, denote the set of all bounded Σ-measurable real functions on Ω. Again, $\mathcal{B}$ is endowed with the sup norm $\|\cdot\|_\infty$. The norm dual of $\mathcal{B}$ is written as $\mathcal{B}^*$, and the double dual is written as $\mathcal{B}^{**}$. We sometimes use inner product notation: $\langle \xi^*, \xi \rangle$ stands for a functional $\xi^* \in \mathcal{B}^*$ acting on $\xi \in \mathcal{B}$ etc.

C.1 Proof of Proposition 1

The proof makes use of a couple of results in functional analysis. Recall that $\Delta(Z)$ is endowed with the weak-* topology. As Z is a compact metric space, it is well-known that $\Delta(Z)$ is weak-* compact and metrizable, e.g., by the Prokhorov metric, written as $d^{\mathrm{P}}(\cdot, \cdot)$. For a lottery $p \in \Delta(Z)$, the support of p is defined as the smallest closed set to which p assigns the probability one, written as $\mathrm{Supp}(p)$. We say p is *simple* if $\mathrm{Supp}(p)$ is finite. Let $\Delta_s(Z)$ denote the set of all simple probability measures. Then, $\Delta_s(Z)$ is dense in $\Delta(Z)$, provided that Z is separable (which is true because Z is a compact metric space). This is called the *density lemma*, c.f., Theorem 15.10 of Aliprantis and Border (2006).

Now, we are ready to prove the proposition. Clearly, the third statement is a corollary to the first two. Moreover, since the argument is symmetric, we shall prove only the first statement, namely, $C^\sharp \supseteq C^\flat$ if and only if **A9 (a)** is satisfied.

Step 1: Necessity.

Suppose that $C^\sharp \supseteq C^\flat$. Take any $f, g \in \mathcal{F}$ with $f \succsim g$ and $f \overset{\alpha}{\smile} g$. If $f = g$, then we must have $f \in \mathcal{F}^c$ to have $f \overset{\alpha}{\smile} f$ for $\alpha \in (0,1)$, but then **A9 (a)** is trivially satisfied. Thus, assume that $f \neq g$. By $f \overset{\alpha}{\smile} g$, there exists $\lambda \in \mathbb{R}$ for which

$$\begin{aligned}
&\alpha u \circ f + (1-\alpha) u \circ g = \lambda \mathbf{1}_\Omega \\
&\iff u \circ g = \frac{\lambda}{1-\alpha}\mathbf{1}_\Omega - \frac{\alpha}{1-\alpha} u \circ f.
\end{aligned} \tag{16}$$

Then, we see that

$$\begin{aligned}
\inf_{\mu \in C^\flat} \int \left(\frac{\lambda}{1-\alpha}\mathbf{1}_\Omega - \frac{\alpha}{1-\alpha} u \circ f \right) d\mu &= \inf_{\mu \in C^\flat} \int (u \circ g)\, d\mu \\
\leq \sup_{\mu \in C^\flat} \int (u \circ g)\, d\mu &\leq \sup_{\mu \in C^\sharp} \int (u \circ g)\, d\mu \leq \inf_{\mu \in C^\flat} \int (u \circ f)\, d\mu,
\end{aligned} \tag{17}$$

where the inequalities follow from $C^\sharp \supseteq C^\flat$ and $f \succsim g$. Moreover, by using (17)

$$\begin{aligned}
&\sup_{\mu \in C^\sharp} \int \big(u \circ (\alpha f + (1-\alpha) g)\big) d\mu = \sup_{\mu \in C^\sharp} \int \lambda \mathbf{1}_\Omega d\mu = \lambda \\
&= \alpha \inf_{\mu \in C^\flat} \int (u \circ f) d\mu - \alpha \inf_{\mu \in C^\flat} \int (u \circ f) d\mu + \lambda \\
&= \alpha \inf_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\alpha) \inf_{\mu \in C^\flat} \int \left(\frac{\lambda}{1-\alpha}\mathbf{1}_\Omega - \frac{\alpha}{1-\alpha} u \circ f \right) d\mu \\
&\leq \alpha \inf_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\alpha) \inf_{\mu \in C^\flat} \int (u \circ f) d\mu \\
&= \inf_{\mu \in C^\flat} \int (u \circ f) d\mu,
\end{aligned}$$

from which $f \succsim \alpha f + (1-\alpha) g$. Therefore, **A9 (a)** is satisfied.

Step 2: Sufficiency.

Suppose that there exists some $\mu^\flat \in C^\flat \setminus C^\sharp$. The separating hyperplane theorem yields a bounded linear functional $\xi : \Delta(\Omega) \to \mathbb{R}$ such that $\langle \mu^\flat, \xi \rangle < \inf_{\mu \in C^\sharp} \langle \mu, \xi \rangle$. Note that we can assume $\inf_{\mu \in C^\flat} \langle \mu, \xi \rangle \in \mathrm{Int}(\mathrm{Im}(u))$ by considering $\xi' = \frac{\xi}{K\|\xi\|_\infty}$ with large enough $K > 0$ if necessary. After this normalization, we can find some $f \in \mathcal{F}$ with $\xi = u \circ f$, so that

$$\inf_{\mu \in C^\flat} \int (u \circ f) d\mu \leq \int (u \circ f) d\mu^\flat < \inf_{\mu \in C^\sharp} \int (u \circ f) d\mu. \tag{18}$$

Now, we claim that it is without loss of generality to assume that f carries at most finitely many prizes while satisfying (18).

Claim 6. *There exists an act $\tilde{f}$ such that $|\bigcup_{\omega \in \Omega} \mathrm{Supp}(\tilde{f}(\omega))| < \infty$, and*

$$\inf_{\mu \in C^\flat} \int (u \circ \tilde{f}) d\mu \leq \int (u \circ \tilde{f}) d\mu^\flat < \inf_{\mu \in C^\sharp} \int (u \circ \tilde{f}) d\mu.$$

Proof. Let $c = \inf \langle \xi, \mu - \mu^\flat \rangle > 0$, and fix any $\epsilon < \frac{c}{2}$. Since $u : \Delta(Z) \to \mathbb{R}$ is weak-* continuous on the compact domain, it is uniformly weak-* continuous. Hence, there exists $\delta > 0$ such that for any $p, q \in \Delta(Z)$,

$$d^{\mathrm{P}}(p, q) < \delta \Longrightarrow |u(p) - u(q)| < \epsilon. \tag{19}$$

Since $\Delta(Z)$ is a compact metric space, there exist finitely many lotteries $p_1, \ldots, p_n \in \Delta(Z)$ such that

$$\min_{i=1,\ldots,n} d^{\mathrm{P}}(q, p_i) < \frac{\delta}{2},$$

fro every $q \in \Delta(Z)$. Moreover, by the density lemma, each p_i has some simple lottery $\tilde{p}_i \in \Delta_s(Z)$ for which $d^{\mathrm{P}}(p_i, \tilde{p}_i) < \frac{\delta}{2}$. Now, for each $\omega \in \Omega$, the triangle inequality implies that

$$\begin{aligned}
\min_{i=1,\ldots,n} d^{\mathrm{P}}(f(\omega), \tilde{p}_i) &\leq \min_{i=1,\ldots,n} \left(d^{\mathrm{P}}(f(\omega), p_i) + d^{\mathrm{P}}(p_i, \tilde{p}_i) \right) \\
&< \min_{i=1,\ldots,n} d^{\mathrm{P}}(f(\omega), p_i) + \frac{\delta}{2} \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned} \tag{20}$$

Therefore, we can define $\tilde{f} : \Omega \to \Delta(Z)$ by $\tilde{f}(\omega) = \tilde{p}_i$ where the index i is chosen so that $d^{\mathrm{P}}(f(\omega), \tilde{p}_i) < \delta$. In particular, we can let $\tilde{f}(\omega) = \tilde{f}(\omega')$ whenever $f(\omega) = f(\omega')$, thereby $\tilde{f}$ is f-measurable. Hence, $\tilde{f}$ is Σ-measurable, and thus, it is an act. By the construction of $\tilde{f}$,

$$\left| \bigcup_{\omega \in \Omega} \mathrm{Supp}(\tilde{f}(\omega)) \right| \leq \sum_{i=1}^{n} |\mathrm{Supp}(\tilde{p}_i)| < \infty$$

Moreover, (19) and (20) imply that $|u(f(\omega)) - u(\tilde{f}(\omega))| < \epsilon$, that is,

$$u(f(\omega)) - \epsilon < u(\tilde{f}(\omega)) < u(f(\omega)) + \epsilon,$$

for every $\omega \in \Omega$. Then, by the monotonicity of integral operators,

$$\int (u \circ \tilde{f}) d\mu^\flat \leq \int (u \circ f + \epsilon \mathbf{1}_\Omega) d\mu^\flat = \int (u \circ f) d\mu^\flat + \epsilon, \tag{21}$$

$$\int (u \circ \tilde{f}) d\mu^\sharp \geq \int (u \circ f - \epsilon \mathbf{1}_\Omega) d\mu^\sharp = \int (u \circ f) d\mu^\sharp - \epsilon, \tag{22}$$

where $\mu^\sharp \in C^\sharp$ is arbitrary. Therefore, combining (21) and (22) yields

$$\inf_{\mu \in C^\sharp} \int (u \circ \tilde{f}) d\mu - \int (u \circ \tilde{f}) d\mu^\flat \geq \inf_{\mu \in C^\sharp} \int (u \circ f) d\mu - \int (u \circ f) d\mu^\flat - 2\epsilon = c - 2\epsilon > 0,$$

from which we have constructed the desired act $\tilde{f}$. □

Let $D = \bigcup_{\omega \in \Omega} \mathrm{Supp}(f(\omega)) \subseteq Z$. Claim 6 implies that we can assume $|D| < \infty$, so that we denote by $f_z(\omega)$ the probability that a prize z realizes from a lottery $f(\omega)$. Then, define an act $h \in \mathcal{F}$ by

$$h_z(\omega) = \begin{cases} \frac{1 - f_z(\omega)}{|D| - 1} & \text{if } z \in D, \\ 0 & \text{if } z \notin D, \end{cases}$$

for each $\omega \in \Omega$ and $z \in Z$. Note that h is f-measurable, and thus, Σ-measurable. By the construction, for any $z \in D$,

$$\frac{1}{|D|} f_z(\omega) + \frac{|D| - 1}{|D|} h_z(\omega) = \frac{1}{|D|} f_z(\omega) + \frac{|D| - 1}{|D|} \cdot \frac{1 - f_z(\omega)}{|D| - 1} = \frac{1}{|D|},$$

meaning that a lottery $(\frac{1}{|D|} f + \frac{|D|-1}{|D|} h)(\omega)$ is the uniform distribution over D, regardless of ω. In particular, this implies that $\frac{1}{|D|} f + \frac{|D|-1}{|D|} h$ is a constant act.

Claim 7. *There exist $\alpha \in (0, 1)$ and $p \in \mathcal{F}^c$ such that if we define $g = \alpha h + (1 - \alpha) p$, then*

$$\inf_{\mu \in C^\flat} \int (u \circ f) d\mu = \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu. \tag{23}$$

Proof. For notational simplicity, let $v_1 = \inf_{\mu \in C^\flat} \int (u \circ f) d\mu$ and $v_2 = \sup_{\mu \in C^\sharp} \int (u \circ h) d\mu$. Given

$\alpha \in (0,1)$ and $p \in \mathcal{F}^c$, if we define $g = \alpha h + (1-\alpha)p$, then

$$\sup_{\mu \in C^\sharp} \int (u \circ g) d\mu = \alpha v_2 + (1-\alpha) u(p).$$

Thus, we want to specify α and p to solve $v_1 = \alpha v_2 + (1-\alpha)u(p)$. Recall that $v_1 \in \mathrm{Int}(\mathrm{Im}(u))$ due to normalization. If $v_1 = v_2$, then we can use p such that $u(p) = v_1$, and $\alpha \in (0,1)$ is arbitrary. If $v_1 \neq v_2$, then we shall set $\alpha = \frac{v_1 - u(p)}{v_2 - u(p)}$, while p is chosen to satisfy: $u(p) > v_1$ if $v_1 > v_2$, and $u(p) < v_1$ if $v_1 < v_2$. By doing so, the desired inequality is satisfied while ensuring $\alpha \in (0,1)$. $\square$

Let α, p, and g be given as in Claim 7, and set $\beta = \frac{\alpha}{\alpha + |D| - 1} \in (0,1)$. We shall show that **A9 (a)** is violated by f, g and their mixture $\beta f + (1-\beta) g$. Note that $f \succsim g$ holds by (23). Moreover, we have

$$\begin{aligned} \beta f + (1-\beta) g &= \beta f + \alpha(1-\beta) h + (1-\alpha)(1-\beta) p \\ &= \left(\frac{\alpha}{\alpha + |D| - 1} \right) f + \left(\frac{\alpha(|D|-1)}{\alpha + |D| - 1} \right) h + \left(\frac{(1-\alpha)(|D|-1)}{\alpha + |D| - 1} \right) p \\ &= \left(\frac{\alpha |D|}{\alpha + |D| - 1} \right) \underbrace{\left(\frac{1}{|D|} f + \frac{|D|-1}{|D|} h \right)}_{=\text{const.}} + \left(\frac{(1-\alpha)(|D|-1)}{\alpha + |D| - 1} \right) p, \end{aligned}$$

from which $f \overset{\beta}{\smile} g$.

Lastly, let us show that f and $\beta f + (1-\beta) g$ are incomparable. Let $\mu^\sharp \in C^\sharp$ be a belief that achieves $\sup_{\mu \in C^\sharp} \int (u \circ g) d\mu$. By using (23),

$$\begin{aligned} \inf_{\mu \in C^\flat} \int (u \circ f) d\mu &= \beta \inf_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\beta) \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu \\ &< \beta \int (u \circ f) d\mu^\sharp + (1-\beta) \int (u \circ g) d\mu^\sharp \\ &= \int \big(u \circ (\beta f + (1-\beta) g \big) d\mu^\sharp \\ &= u(\beta f + (1-\beta) g), \end{aligned}$$

where the strict inequality follows by $\mu^\sharp \in C^\sharp$ and the construction of f. Hence, we get

$f \not\succsim \beta f + (1-\beta)g$. Furthermore, again by (23) and the construction of f, we see that

$$\begin{aligned} u(\beta f + (1-\beta)g) &= \sup_{\mu \in C^\sharp} \int \big(u \circ (\beta f + (1-\beta)g\big) d\mu \\ &\leq \beta \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu + (1-\beta) \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu \\ &= \beta \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu + (1-\beta) \inf_{\mu \in C^\sharp} \int (u \circ f) d\mu \\ &< \beta \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu + (1-\beta) \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu \\ &= \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu, \end{aligned}$$

from which $\beta f + (1-\beta)g \not\succsim f$. Therefore, f and $\beta f + (1-\beta)g$ are incomparable. *Q.E.D.*

C.2 Proof of Proposition 2

Suppose that (ii) is true. We can let $u \equiv u_1 = u_2$ without loss of generality. Clearly, $\succsim_1 |_{\mathcal{F}^c} = \succsim_2 |_{\mathcal{F}^c}$. Fix any $f, g \in \mathcal{F}$ with $f \succsim_2 g$. If $f = g$, then both $f \sim_1 g$ and $f \sim_2 g$ hold by reflexivity, so there is nothing to prove. Otherwise, it holds that

$$\inf_{\mu \in C_1^\flat} \int (u \circ f) d\mu \geq \inf_{\mu \in C_2^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C_2^\sharp} \int (u \circ g) d\mu \geq \sup_{\mu \in C_1^\sharp} \int (u \circ g) d\mu,$$

where the first and third inequalities follow from $C_1^\flat \subseteq C_2^\flat$ and $C_1^\sharp \subseteq C_2^\sharp$, respectively. Hence, we obtain $f \succsim_1 g$, from which $\succsim_1$ is a compatible extension of $\succsim_2$.

Conversely, suppose that (ii) is false. If u_1 is not a positive affine transformation of u_2, the two preferences differ on $\mathcal{F}^c$, from which one cannot be a compatible extension of another. Thus, we can let $u \equiv u_1 = u_2$. Since (ii) fails, we have either $\mu \in C_1^\sharp \setminus C_2^\sharp$ or $\mu \in C_1^\flat \setminus C_2^\flat$. Assuming the former case, the separating hyperplane theorem yields $\xi^{**} \in \mathcal{B}^{**}$ such that

$$\big\langle \xi^{**}, \mu \big\rangle > c \equiv \sup_{\mu_2 \in C_2^\sharp} \big\langle \xi^{**}, \mu_2 \big\rangle. \tag{24}$$

After normalization, we can let $\xi^{**} = u \circ f$ for some $f \in \mathcal{F}$. Now, let $p \in \Delta(Z)$ be a lottery such that $u(p) = c$. By (24), it follows that $p\mathbf{1}_\Omega \succsim_2 f$. On the other hand, by $\mu \in C_1^\sharp$, (24)

Table 1: Acts in Proposition 3

	ω	$\Omega \setminus \{\omega\}$
$u \circ f$	γ	α
$u \circ g$	β	δ

implies that

$$\sup_{\mu_1 \in C_1^\sharp} \int (u \circ f) d\mu_1 > u(p),$$

whereas $f \neq p$ since f must not be constant to maintain (24). Thus, we have $p \not\succsim_1 f$, from which $\succsim_1$ is not a compatible extension of $\succsim_2$. The other case of when $\mu \in C_1^\flat \setminus C_2^\flat$ is similarly discussed. *Q.E.D.*

C.3 Proof of Proposition 3

Clearly, the representation in this proposition is the special case of Theorem 1, corresponding to the case of when $C^\sharp = C^\flat = \Delta(\Omega)$. In particular, it is easy to see that the representation satisfies **A10**. To prove the sufficiency, suppose that $\succsim$ satisfies all the listed axioms, and thus, admit an IPOP representation by some $(C^\sharp, C^\flat, u)$. Let $[0, 1] \subseteq \mathrm{Im}(u)$ without loss of generality. It is enough to show that $c^\sharp = c^\flat = 1$, where $c^\natural = \sup_{\mu \in C^\natural} \mu(\{\omega\})$ for each $\natural \in \{\sharp, \flat\}$. To this end, let $\alpha > \beta > \gamma > \delta$ be arbitrary numbers in $[0, 1]$, and consider acts $f, g \in \mathcal{F}$ which pay the utility values, in each event $\{\omega\}$ and $\Omega \setminus \{\omega\}$, as being summarized in Table 1. Note that f does not dominate g, and so, **A10** implies that $f \not\succsim g$. That is, we must have

$$\begin{aligned}\inf_{\mu \in C^\flat} \int (u \circ f) d\mu &= c^\flat \gamma + (1 - c^\flat)\alpha \\ &< c^\sharp \beta + (1 - c^\sharp)\delta = \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu.\end{aligned}$$

which is satisfied for *all* $\alpha > \beta > \gamma > \delta$ if and only if $c^\flat = c^\sharp = 1$. *Q.E.D.*

C.4 Proof of Proposition 4

As to (i), $\mathcal{P}_{\text{MEU}} \cap \mathcal{P}_{\text{Bewley}} = \mathcal{P}_{\text{SEU}}$ follows from the fact that a Bewley preference becomes complete if and only if the associated prior set is singleton. Note that any $\succsim_{\text{Bewley}} \in \mathcal{P}_{\text{Bewley}}$ can be extended to $\succsim^* \in \mathcal{P}_{\text{SEU}} \subseteq \mathcal{P}_{\text{MEU}}$ by letting $\succsim^*$ be a SEU preference defined by the same utility function u as $\succsim_{\text{Bewley}}$, and an arbitrary $\mu \in C$ in the belief set of $\succsim_{\text{Bewley}}$. Thus, the crucial part would be to show that any $\succsim_{\text{MEU}} \in \mathcal{P}_{\text{MEU}}$ has some $\succsim_{\text{Bewley}} \in \mathcal{P}_{\text{Bewley}}$ for which $\succsim_{\text{MEU}}$ is a compatible extension of $\succsim_{\text{Bewley}}$. Indeed, this claim follows from the combination of Definition 3 and Proposition 4 and 5 in Ghirardato et al. (2004).

Let us prove (ii). We first show that any $\succsim \in \mathcal{P}_{\text{IPOP}}$ satisfies independence if and only if $C^\sharp = C^\flat = \{\mu_0\}$ for some $\mu_0 \in \Delta(\Omega)$, i.e., $\succsim \in \mathcal{P}_{\text{SEU}}$. The if direction is trivial, so let us consider the only if direction. Since any $\succsim \in \mathcal{P}_{\text{IPOP}}$ is transitive, Proposition 9 implies that $C^\sharp \cap C^\flat \neq \emptyset$. Now, let $\mu_0 \in C^\sharp \cap C^\flat$, and suppose that there exists some $\mu_0 \neq \mu_1 \in C^\sharp$. After normalization, the separating hyperplane theorem yields an act $f \in \mathcal{F}$ such that

$$\sup_{\mu \in C^\sharp} \int (u \circ f) d\mu \geq \int (u \circ f) d\mu_1 > \int (u \circ f) d\mu_0 \geq \inf_{\mu \in C^\flat} \int (u \circ f) d\mu.$$

Since $2 \leq |C^\sharp| \leq |\Delta(\Omega)|$, we know $2 \leq |\Omega|$. Thus, we can pick a non-constant act $g \in \mathcal{F} \setminus \mathcal{F}^c$. Let $p \in \Delta(Z)$ be a lottery such that $u(p) = \inf_{\mu \in C^\flat} \int (u \circ f) d\mu$. By the constructions, $g \succsim p$. However, for an arbitrary $\lambda \in (0,1)$, we see that

$$\begin{aligned}
&\inf_{\mu \in C^\flat} \int \big(u \circ (\lambda g + (1-\lambda) f)\big) d\mu \\
&\leq \lambda \inf_{\mu \in C^\flat} \int (u \circ g) d\mu + (1-\lambda) \inf_{\mu \in C^\flat} \int (u \circ f) d\mu \\
&< \lambda u(p) + (1-\lambda) \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu = \sup_{\mu \in C^\sharp} \int \big(u \circ (\lambda p + (1-\lambda) f)\big) d\mu,
\end{aligned}$$

from which $\lambda g + (1-\lambda) f \not\succsim \lambda p + (1-\lambda) f$. Therefore, $\succsim$ violates independence. Consequently, since Bewley preferences satisfy independence, we confirm that $\mathcal{P}_{\text{Bewley}} \cap \mathcal{P}_{\text{IPOP}} = \mathcal{P}_{\text{SEU}}$.

Next, we shall show that $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{Bewley}}$. Again, note that any $\succsim_{\text{IPOP}} \in \mathcal{P}_{\text{IPOP}}$ can be extended to $\succsim^* \in \mathcal{P}_{\text{Bewley}}$; Since $\succsim_{\text{IPOP}}$ is transitive, we have $C^\sharp \cap C^\flat \neq \emptyset$, thereby the SEU preference $\succsim^*$ defined by u and an arbitrary $\mu \in C^\sharp \cap C^\flat$ does work. Conversely, fix any $\succsim_{\text{Bewley}} \in \mathcal{P}_{\text{Bewley}}$ that is represented by (u, C). Then, we define $\succsim_{\text{IPOP}} \in \mathcal{P}_{\text{IPOP}}$ by using the same utility function u, and setting $C = C^\sharp = C^\flat$. Clearly, $\succsim_{\text{Bewley}} |_{\mathcal{F}^c} = \succsim_{\text{IPOP}} |_{\mathcal{F}^c}$. Moreover, it is easy to see that $f \succsim_{\text{IPOP}} g$ whenever $f \succsim_{\text{Bewley}} g$, thereby $\succsim_{\text{Bewley}}$ is a compatible extension of $\succsim_{\text{IPOP}}$. Hence, $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{Bewley}}$. *Q.E.D.*

C.5 Proof of Proposition 5

Suppose that $\succsim^*$ is a C-continuous completion of $\succsim$. Given an act $f \in \mathcal{F}$, consider the non-empty sets $A = \{p \in \mathcal{F}^c : p \succsim^* f\}$ and $B = \{p \in \mathcal{F}^c : f \succsim^* p\}$. Since $\succsim^*$ maintains C-continuity, A and B are closed. Moreover, $A \cup B = \Delta(Z)$ by the completeness of $\succsim^*$, so that the connectedness of $\mathcal{F}^c \simeq \Delta(Z)$ gives $A \cap B \neq \emptyset$. Then, we fix an arbitrary $p_f \in A \cap B$ and set $I(f) = u(p_f)$. Note that $I(f)$ is independent of how p_f is chosen because $f \sim^* p$ for every $p \in A \cap B$.

Let us show that $I(f)$ takes the form as in the statement. Since $\succsim^*$ is a compatible extension, neither $f \succ p_f$ nor $p_f \succ f$ holds to assure that $f \sim^* p_f$. Note that $f \not\succ p_f$ gives rise to either $f \not\succsim p_f$ or $p_f \succsim f$. Similarly, $p_f \not\succ f$ gives rise to either $f \succsim p_f$ or $p_f \not\succsim f$. Hence, if $f \sim^* p_f$, then f and p must be either indifferent or incomparable with respect to $\succsim$. In the former case, the minimum and maximum expected utilities of f are both equal to $u(p_f)$, and thus, $I(f)$ takes the form (4) for an arbitrary $\alpha(f) \in (0,1)$. In the latter case, we see that

$$\sup_{\mu \in C^\sharp} \int (u \circ f) d\mu > u(p_f) > \inf_{\mu \in C^\flat} \int (u \circ f) d\mu,$$

from which we can find an $\alpha(f) \in (0,1)$ that provides (4).

Now, we shall show that $I(\cdot)$ represents $\succsim^*$. Assume that $I(f) \succsim I(g)$. By construction, there exist $p_f \sim^* f$ and $p_g \sim^* g$ such that $u(p_f) = I(f) \geq I(g) = u(p_g)$, meaning that $p_f \succsim p_g$. In particular, this implies $p_f \succsim^* p_g$, from which transitivity dictates $f \succsim^* g$. Furthermore, when $I(f) > I(g)$, we must have $f \sim^* p_f \succ^* p_g \sim^* g$, thereby $f \succ^* g$. Therefore, $f \succsim^* g$ if and only if $I(f) \geq I(g)$, as desired.

As to the converse direction, suppose that $\succsim^*$ is represented by $I : \mathcal{F} \to \mathbb{R}$ as in (4), where $\alpha : \mathcal{F} \to [0,1]$ is an arbitrary function. Clearly, $\succsim^*$ is a weak order, and it is C-continuous by the fact that $I(p) = u(p)$ for every $p \in \mathcal{F}^c$. Note that this implies that $\succsim |_{\mathcal{F}^c} = \succsim^* |_{\mathcal{F}^c}$. Now, take any $f, g \in \mathcal{F}$ with $f \succsim g$. If $f = g$ then $f \succsim^* g$, as $\succsim^*$ is complete. If $f \neq g$, by the transitivity of $\succsim$, we have

$$I(f) \geq \inf_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C^\sharp} \int (u \circ f) d\mu \geq I(g),$$

from which $f \succsim^* g$. Therefore, $\succsim^*$ is a compatible extension of $\succsim$. *Q.E.D.*

C.6 Proof of Proposition 6

Recall that $U(\cdot,\mu)$ is maximized at v for any μ, and so is $\inf_{\mu\in C^\flat} U(\cdot,\mu)$. This immediately implies (i). Similarly, note that $\sup_{\mu\in C^\flat} U(\cdot,\mu)$ is maximized at v, so that $b \notin B^*(C^\sharp, C^\flat)$ for any $b \neq v$ whenever truth-telling is transparent. Hence, in general, we have $B^*(C^\sharp, C^\flat) \subseteq \{v\}$. Moreover, it is easy to see Condition 2 assures that v is not dominated by any $b \neq v$, so that we would have $B^*(C^\sharp, C^\flat) = \{v\}$ under this condition.

Now, suppose that truth-telling is not transparent, so that the left-side is strictly greater than the right side in (5). In particular, this assures $v \in B^*(C^\sharp, C^\flat)$. Moreover, by this and (i), we have $b \in B^*(C^\sharp, C^\flat)$ if and only if $\overline{U}(b) > \underline{U}(v)$, where

$$\overline{U}(b) \equiv \sup_{\mu\in C^\sharp} \underbrace{\int_0^b u(1,\omega)d\mu(\omega)}_{\equiv U(b,\mu)} \quad \text{and} \quad \underline{U}(b) \equiv \inf_{\mu\in C^\sharp} \int_0^v u(1,\omega)d\mu(\omega). \tag{25}$$

Thus, to see that $B^*(C^\sharp, C^\flat)$ takes an interval form, it suffices to show that $\overline{U}$ is weakly increasing on $[0, v]$, and it is weakly decreasing on $[v, \bar{b}]$. Indeed, it is straightforward to show $U(\cdot,\mu)$ has these properties for every $\mu \in C^\sharp$, and thus, so does $\overline{U}$ defined as the maximum.[27] Finally, let us show that $B^*(C^\sharp, C^\flat)$ is open (relative to $[0, \bar{b}]$). Note that $B^*(C^\sharp, C^\flat)$ is the inverse image of an open set under $\overline{U}$, so that it is enough to prove that $\overline{U}$ is continuous. To this end, fix any $\mu \in C^\sharp$. By Condition 1, we know $\mu[0, b]$ is continuous in b. Furthermore, observe that $\nu[0, b] = \int_0^b u(1,\omega)d\mu(\omega)$ is continuous in b, as ν is obtained by transforming μ with the continuous Radon-Nikodym derivative $\frac{d\nu}{d\mu}(\omega) = u(1,\omega)$. Hence, each $U(\cdot,\mu)$ is continuous, thereby so is $\overline{U}$. *Q.E.D.*

Appendix D Additional Results

D.1 Weaker Transitivity and Disjointness of Belief Sets

Recall that we have allowed for the possibility that the decision-maker's optimism and pessimism possess different sets of beliefs, namely, $C^\sharp \neq C^\flat$. Indeed, $C^\sharp \cap C^\flat \neq \emptyset$ was the only restriction imposed on them in Theorem 1. We shall show that this disjointness condition guarantees the transitivity of $\succsim$, while it has no role more than that. Also, it will be shown

[27] More formally, recall that each $\mu \in C^\sharp$ has a probability density function ϕ by Condition 1, so that we can write $U(b,\mu) = \int_0^b u(1,\omega)\phi(\omega)d\omega$. By the Leibniz rule, it follows that $\frac{\partial U(b,\mu)}{\partial b} = u(1,b)\phi(b)$, which is weakly positive (resp. negative) if $b \leq v$ (resp. $b \geq v$) by the assumptions on u. Moreover, these properties are inherited to $\overline{U}$ because monotonicity is preserved by the max operator.

that the transitivity of $\succsim$ has further equivalent restatements that have natural interpretations in light of the decision-maker's rationality.

To this end, we consider the following alternative of **A1** that drops transitivity but replaces it with what we call *C-transitivity*. It is a weakening of transitivity, while the only difference is that C-transitivity always takes a middle act to be constant.[28]

B1. $\succsim$ is a non-degenerate and reflexive order such that for every $f \in \mathcal{F}$ there exist some $p, q \in \mathcal{F}^c$ for which $p \succsim f$ and $f \succsim q$. Moreover, $\succsim$ is *C-transitive*; for any $f, g \in \mathcal{F}$ and $p \in \mathcal{F}^c$, if $f \succsim p$ and $p \succsim g$, then $f \succsim g$.

D.1.1 The Role of Transitivity in General Representations

After replacing **A1** by **B1**, we continue to have our previous representation results with only minor modifications. For general representations, our Lemma 1 is restored by deleting the condition $\overline{U} \geq \underline{U}$ from its statement. The proof is essentially the same as before, so omitted.

Lemma 4. *A preference relation $\succsim$ satisfies* **B1**, **A2**, **A3**, *and* **A7** *if and only if there exist non-constant functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ such that*

(i) $\overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ *holds, and the restriction is continuous on $\mathcal{F}^c \simeq \Delta(Z)$;*

(ii) $\mathrm{Im}(\overline{U}|_{\mathcal{F}^c}) = \mathrm{Im}(\overline{U}) = \mathrm{Im}(\underline{U}) = \mathrm{Im}(\underline{U}|_{\mathcal{F}^c})$*; and*

(iii) $f \succsim g$ *if and only if* $\underline{U}(f) \geq \overline{U}(g)$ *or* $f = g$.

Moreover, a pair of functions $\overline{V}, \underline{V} : \mathcal{F} \to \mathbb{R}$ satisfies (i)-(iii) for the same preference relation $\succsim$ if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\overline{V} = \phi \circ \overline{U}$ and $\underline{V} = \phi \circ \underline{U}$.

The next result shows that the omitted condition $\overline{U} \geq \underline{U}$ is equivalent to the transitivity of $\succsim$. In that sense, transitivity guarantees that the optimism inside the decision-maker is actually more "optimistic" than her pessimism. Yet another interpretation is available. (iii) of the below lemma says that the optimism and pessimism rank a pair of acts in the same order whenever the decision-maker is decisive for the pair. Putting differently, the different selves of the decision-maker share the same ordinal preferences on the domain on which the decision-maker confirms solid rankings.

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[28] In this regard, C-transitivity is viewed as the reverse of C-calibration, and thus, it should be thought of as being more or less "minimal" in representation results. Fortunately, as stated in this Appendix, we have essentially the same results as before both for general and expected utility representations.

Lemma 5. *Suppose that $\succsim$ satisfies* **B1**, **A2**, **A3**, *and* **A7**. *Let $\overline{U}$ and $\underline{U}$ be arbitrary functions that represent $\succsim$ in the way of Lemma 4. The following conditions are equivalent:*

(i) *$\succsim$ satisfies transitivity;*

(ii) *$\overline{U}(f) \geq \underline{U}(f)$ for all $f \in \mathcal{F}$; and*

(iii) *$(\overline{U}(f) - \overline{U}(g))(\underline{U}(f) - \underline{U}(g)) \geq 0$ whenever f and g are comparable.*

Proof. It is easy to see (ii) ⇒ (i). Let us show (i) ⇒ (ii). Suppose not, there exists some $f \in \mathcal{F}$ for which $\overline{U}(f) < \underline{U}(f)$. Take lotteries $p, q \in \Delta$ such that $u(p) = \overline{U}(f)$ and $u(q) = \underline{U}(f)$, where u stands for the common restriction of $\overline{U}$ and $\underline{U}$ on $\mathcal{F}^c$.[29] Then, we have $\underline{U}(p) \geq \overline{U}(f)$ and $\underline{U}(f) \geq \overline{U}(q)$, each of which leads to $p \succsim f$ and $f \succsim q$, respectively. Hence, transitivity dictates $p \succsim q$. However, $\overline{U}(f) < \underline{U}(f)$ implies that $\overline{U}(p) < \underline{U}(q)$ and $\underline{U}(p) < \overline{U}(q)$, from which $q \succ p$, a contradiction.

Let us show that (iii) is equivalent to (ii). Note that (iii) is vacuously satisfied when $f = g$. So, take any $f, g \in \mathcal{F}$ such that $f \succsim g$ and $f \neq g$, that is to say, $\underline{U}(f) \geq \overline{U}(g)$. Assuming that (ii) is true, we see that $\overline{U}(f) \geq \underline{U}(f) \geq \overline{U}(g) \geq \underline{U}(g)$, from which (iv) is obtained. Conversely, suppose that (iii) is violated, i.e., there exists $f \in \mathcal{F}$ for which $\overline{U}(f) < \underline{U}(f)$. For an arbitrarily fixed number $c \in (\overline{U}(f), \underline{U}(f))$, let $r \in \Delta(Z)$ be a lottery for which $\overline{U}(r) = \underline{U}(r) = c$. Note that $r \succsim f$ holds. On the other hand, we see that

$$\underbrace{\left(\overline{U}(f) - \overline{U}(r)\right)}_{<0}\underbrace{\left(\underline{U}(f) - \underline{U}(r)\right)}_{>0} < 0,$$

from which (iii) is violated. Therefore, we have established all the desired equivalences. □

D.1.2 The Role of Transitivity in Expected Utility Representations

After replacing **A1** by **B1**, an alternative result is established for expected utility representations. In the below theorem, only the difference from Theorem 1 is that the condition $C^\sharp \cap C^\flat \neq \emptyset$ is deleted. The proof is essentially the same, so omitted.

Theorem 2. *A preference relation $\succsim$ satisfies* **B1** *and* **A2–7** *if and only if there exist a non-constant continuous and affine function $u : \Delta(Z) \to \mathbb{R}$ and non-empty closed and convex sets*

[29] Since the images of $\overline{U}$ and $\underline{U}$ are the same as the image of u by Lemma 1, we can find such lotteries.

$C^\sharp, C^\flat \subseteq \Delta(\Omega)$ *such that*

$$f \succsim g \Longleftrightarrow \left[\inf_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \sup_{\mu \in C^\sharp} \int (u \circ g) d\mu \quad or \quad f = g \right].$$

Moreover, $C^\sharp$ and $C^\flat$ are unique, and u is unique up to positive affine transformations.

The next result shows that the disjointness of $C^\sharp$ and $C^\flat$ is equivalent to the transitivity of $\succsim$. In particular, since Lemma 4 deals with a more general class of preferences than Theorem 2, the next proposition inherits the results of Lemma 5 as well.

Proposition 9. *Suppose that $\succsim$ satisfies* **B1** *and* **A2–7**. *Let $(u, C^\sharp, C^\flat)$ be an arbitrary profile that represents $\succsim$ in the way of Theorem 2. The following are equivalent:*

(i) $C^\sharp \cap C^\flat \neq \emptyset$*;*

(ii) $\succsim$ *satisfies transitivity;*

(iii) *For any $f \in \mathcal{F}$,*

$$\sup_{\mu \in C^\sharp} \int (u \circ f) d\mu \geq \inf_{\mu \in C^\flat} \int (u \circ f) d\mu; \text{ and}$$

(iv) *For any $f, g \in \mathcal{F}$ such that $f \succsim g$,*

$$\left(\sup_{\mu \in C^\sharp} \int (u \circ f) d\mu - \sup_{\mu \in C^\flat} \int (u \circ g) d\mu \right) \left(\inf_{\mu \in C^\sharp} \int (u \circ f) d\mu - \inf_{\mu \in C^\flat} \int (u \circ g) d\mu \right) \geq 0.$$

Proof. The statements (ii), (iii), and (iv) are equivalent due to Lemma 5. Also, it is straightforward to see (i) $\Rightarrow$ (iii). Finally, we can show (iii) $\Rightarrow$ (i) by appealing to the separating hyperplane theorem. □

D.2 An Example of Not C-continuous Completions

We show that C-continuity in Proposition 5 is indispensable. Indeed, there exists a completion $\succsim^*$ of an IPOP representation $\succsim$ that is not represented by a generalized α-maximin form, while violating C-continuity.

Let $\succsim$ be represented by $(u, C^\sharp, C^\flat)$. For simplicity, assume that $C \equiv C^\sharp = C^\flat$ and $|C| \neq 1$. Furthermore, we simplify $Z = \{z_H, z_L\}$, so that $\Delta(Z)$ is identified with $[0, 1]$ whose elements are read as probabilities of carrying the high prize z_H. Assume that $u : [0,1] \to \mathbb{R}$ is strictly increasing. Now, consider the following two utility functions on acts:

$$I(f) = \alpha \inf_{\mu \in C} \int (u \circ f) d\mu + (1-\alpha) \sup_{\mu \in C} \int (u \circ f) d\mu,$$
$$J(f) = \beta \inf_{\mu \in C} \int (u \circ f) d\mu + (1-\beta) \sup_{\mu \in C} \int (u \circ f) d\mu,$$

where $0 \leq \alpha < \beta \leq 1$. By Proposition 5, the complete preferences derived from I and J are completions of $\succsim$.

Beside these completions, we consider a weak order $\succsim^*$ defined as follows:

$$f \succsim^* g \Longleftrightarrow \begin{cases} I(f) > I(g); \text{ or} \\ I(f) = I(g) \text{ and } J(f) \geq J(g). \end{cases}$$

It is easy to check that $\succsim^*$ is a compatible extension of $\succsim$, and hence, it is also a completion of $\succsim$. On the other hand, $\succsim^*$ violates C-continuity. To see this, let f be a non-constant act such that $\inf_{\mu \in C} \int (u \circ f) d\mu < \sup_{\mu \in C} \int (u \circ f) d\mu$.[30] It holds that $p \in \mathcal{L}^c(f)$ whenever $u(p) < I(f)$, and $p \in \mathcal{U}^c(f)$ whenever $u(p) > I(f)$. What if $u(p) = I(f)$? Since $I(f) < J(f)$ and $u(p) = I(p) = J(p)$, we have $f \succsim p$ but $p \not\succsim f$. Therefore, $\mathcal{L}^c(f) = \big[0, u^{-1}(I(f))\big]$ and $\mathcal{U}^c(f) = \big(u^{-1}(I(f)), 1\big]$, from which we confirm the violation of C-continuity.

D.3 On the Genericity of Non-transparency

As in Section 6, let $\Omega = [0, \bar{b}]$ be an interval of bids in a second price auction, and Σ be its Borel algebra. Slightly changing the notation from the main text, we denote by $\Delta(\Omega)$ the set of all probability measures on (Ω, Σ).[31] As before, $\Delta(\Omega)$ is endowed with the weak-* topology, which is metrized by the Prokhorov metric d^{P}, c.f., Appendix C.1.

Let $\mathcal{C}$ be the family of non-empty, closed, and convex subsets of $\Delta(\Omega)$, and let $\mathcal{C}^*$ collect all pairs $(C^\sharp, C^\flat) \in \mathcal{C}$ satisfying $C^\sharp \cap C^\flat \neq \emptyset$. We endow $\mathcal{C}$ with the Hausdorff metric defined by

$$d^{\mathrm{H}}(C_1, C_2) = \max \left\{ \sup_{\mu_1 \in C_1} \inf_{\mu_2 \in C_2} d^{\mathrm{P}}(\mu_1, \mu_2), \sup_{\mu_2 \in C_2} \inf_{\mu_1 \in C_1} d^{\mathrm{P}}(\mu_1, \mu_2) \right\},$$

and endow $\mathcal{C}^*$ with the natural extension of d^{H} given by $\max\{d^{\mathrm{H}}(C_1^\sharp, C_2^\sharp), d^{\mathrm{H}}(C_1^\flat, C_2^\flat)\}$ to mea-

[30] Such an act can be found by applying the separating hyperplane theorem to any $\mu, \mu' \in C$ with $\mu \neq \mu'$.
[31] That is, each element of $\Delta(\Omega)$ is countably additive, not just finitely additive.

sure the distance between pairs $(C_1^\sharp, C_1^\flat)$ and $(C_2^\sharp, C_2^\flat)$. Slightly abusing the notation, the metric on $\mathcal{C}^*$ is again denoted by d^{H}.

Let $u : [0, \bar{b}]^2 \to \mathbb{R}$ be the DM's ex-post utility function that satisfies all the conditions presented in Section 6. Given $(C^\sharp, C^\flat)$, recall that truth-telling is said to be transparent for the DM if the maximal expected utility over $C^\sharp$ is equal to the minimal expected utility over $C^\flat$. Using this notion, we define the pairs of "collapsed" belief sets by

$$\mathcal{T} = \left\{ (C^\sharp, C^\flat) \in \mathcal{C}^* : \sup_{\mu \in C^\sharp} \int_0^v u(1,\omega) d\mu(\omega) = \inf_{\mu \in C^\flat} \int_0^v u(1,\omega) d\mu(\omega) \right\}$$

Now, we claim that transparency is a knife-edge condition that is almost surely violated. Formally, the next proposition shows that the complement of the collapsed belief set pairs constitutes a dense subset of $\mathcal{C}^*$.

Proposition 10. *$\mathcal{C}^* \setminus \mathcal{T}$ is dense in $\mathcal{C}^*$ with respect to d^{H}*

Proof. Fix any collapsed pair $(C^\sharp, C^\flat) \in \mathcal{T}$ and $\delta > 0$. We are done if there exists a pair $(D^\sharp, D^\flat) \in \mathcal{C}^* \setminus \mathcal{T}$ whose distance from $(C^\sharp, C^\flat)$ is bounded by δ. To this end, let μ_L be a probability measure that assigns a mass to $0 \in [0, \bar{b}]$, and let μ_H be an arbitrary probability measure supported on $(v, \bar{b}]$. Given an arbitrary $\epsilon > 0$, we set

$$D^\sharp = \left\{ (1-\epsilon)\mu + \epsilon_L \mu_L + \epsilon_H \mu_H : \mu \in C^\sharp \text{ and } \epsilon_L, \epsilon_H \geq 0 \text{ with } \epsilon_L + \epsilon_H = \epsilon \right\},$$
$$D^\flat = \left\{ (1-\epsilon)\mu + \epsilon_L \mu_L + \epsilon_H \mu_H : \mu \in C^\flat \text{ and } \epsilon_L, \epsilon_H \geq 0 \text{ with } \epsilon_L + \epsilon_H = \epsilon \right\}.$$

Clearly, $D^\sharp$ and $D^\flat$ are non-disjoint, closed, and convex, so that $(D^\sharp, D^\flat) \in \mathcal{C}^*$. Also, note that $\tilde{D}^\sharp \equiv (1-\epsilon)C^\sharp + \epsilon\mu_L \subseteq D^\sharp$ and $\tilde{D}^\flat \equiv (1-\epsilon)C^\flat + \epsilon\mu_H \subseteq D^\flat$ hold. Hence,

$$\begin{aligned}
\sup_{\mu \in D^\sharp} \int_0^v u(1,\omega) d\mu(\omega) &\geq \sup_{\mu \in \tilde{D}^\sharp} \int_0^v u(1,\omega) d\mu(\omega) \\
&= (1-\epsilon) \sup_{\mu \in C^\sharp} \int_0^v u(1,\omega) d\mu(\omega) + \epsilon u(1,0) \\
&> (1-\epsilon) \inf_{\mu \in C^\flat} \int_0^v u(1,\omega) d\mu(\omega) + \epsilon \times 0 \\
&= \inf_{\mu \in \tilde{D}^\sharp} \int_0^v u(1,\omega) d\mu(\omega) \geq \inf_{\mu \in D^\flat} \int_0^v u(1,\omega) d\mu(\omega),
\end{aligned}$$

from which $(D^\sharp, D^\flat) \in \mathcal{C}^* \setminus \mathcal{T}$. Let us show that $d^{\mathrm{H}}(C^\sharp, D^\sharp) < \delta$ when ϵ is small enough. To this end, fix any $\mu^\sharp \in C^\sharp$, and notice that $\mu_\epsilon = (1-\epsilon)\mu + \epsilon_L \mu_L + \epsilon_H \mu_H$ weakly converges to $\mu^\sharp$

as ϵ tends to 0. Hence, for sufficiently small $\epsilon > 0$, we have

$$\inf_{\mu \in D^\sharp} d^{\mathrm{P}}(\mu^\sharp, \mu) \le d^{\mathrm{P}}(\mu^\sharp, \mu_\epsilon) < \delta.$$

In particular, since $C^\sharp$ is weak-* compact, we can pick $\epsilon_1 > 0$ such that the above inequality uniformly holds for any $\mu^\sharp$. Thus, if we use such an ϵ_1 in the definition of $D^\sharp$, then

$$\sup_{\mu^\sharp \in C^\sharp} \inf_{\mu \in D^\sharp} d^{\mathrm{P}}(\mu^\sharp, \mu) < \delta.$$

Conversely, consider any belief of the form $\mu = (1-\epsilon)\mu^\sharp + \epsilon_L \mu_L + \epsilon_H \mu_H$, which weakly converges to $\mu^\sharp \in C^\sharp$ as ϵ tends to 0. For sufficiently small $\epsilon > 0$, we have

$$\inf_{\rho^\sharp \in C^\sharp} d^{\mathrm{P}}(\rho^\sharp, \mu) \le d^{\mathrm{P}}(\mu^\sharp, \mu) < \delta.$$

Note that the above inequality does hold no matter how ϵ is split into ϵ_L and ϵ_H, provided that $\epsilon > 0$ is small enough. Thus, each $\mu \in D^\sharp$ is parametrized by $\mu^\sharp$. Again, since $C^\sharp$ is weak-* compact, we can pick $\epsilon_2 > 0$ to be used in the definition of $D^\sharp$, so that

$$\sup_{\mu^\sharp \in D^\sharp} \inf_{\mu \in C^\sharp} d^{\mathrm{P}}(\mu^\sharp, \mu) < \delta.$$

Hence, by choosing $\epsilon < \min\{\epsilon_1, \epsilon_2\}$, we obtain $d^{\mathrm{H}}(C^\sharp, D^\sharp) < \delta$, as desired. Furthermore, the symmetric argument proves $d^{\mathrm{H}}(C^\flat, D^\flat) < \delta$, thereby $d^{\mathrm{H}}((C^\sharp, C^\flat), (D^\sharp, D^\flat)) < \delta$. □

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