# Information acquisition and strategic investment timing*

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**Abstract**

Motivated by stylized facts in the market for entrepreneurial fundraising, we build a two-firm model of investment timing with endogenous information acquisition to study the interaction between investment delay and free-riding. There are three perfect Bayesian equilibria of the model. In the unique symmetric equilibrium, both firms investigate the project simultaneously, while in the remaining asymmetric equilibria one firm leads the other in obtaining information and investing. Investment delay and free-riding arise only in the asymmetric equilibria, but nonetheless these equilibria improve aggregate welfare over or even Pareto-dominate the symmetric equilibrium when firms are patient.

JEL Classification: C73, D82, D83

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# 1 Introduction

Economists have long recognized that irreversible economic actions - for instance investment, product adoption, and market entry - may be undertaken inefficiently slowly in the presence of social learning. This phenomenon of *strategic delay* arises when information about profitability is dispersed, opportunities are non-rival, and decision-makers may time their actions in order to observe and learn from the actions of others. Existing theoretical models

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of this behavior have tended to focus on the consequences of social learning for the rate of aggregation of private information about a common fundamental state. Less attention has been given to the source of such information, which is typically modeled as exogenously granted.

Contrary to conventional models, in reality much of a decision maker's private information is plausibly the result of endogenous, costly information-acquisition activities. A leading example is the market for entrepreneurial fundraising, in which venture capitalists evaluate startups for potential early-stage investment. Many modern startups raise capital from multiple investors simultaneously, and are flexible as to the amount of capital raised, suggesting an environment of non-rival investment with significant risk and a common payoffs.[1] Further, evaluating startups is a costly, time-consuming process involving steps such as multiple presentations by startup founders; review of slides and documents detailing the company's strategy, product, and financials; and in-depth due diligence regarding the management team and the market in which the startup operates. There is also substantial anecdotal evidence that venture capitalists view startups as more appealing after they have received offers of funding from other investors. (In section 1.1 we provide evidence supporting these claims about the structure of the market for entrepreneurial fundraising.)

Our goal is to understand the dynamics of information production and aggregation in such markets featuring timing of both investment and information acquisition.[2] While the literature on investment timing suggests that firms should be slow to act on information they have acquired, the literature on free-riding in teams indicates that firms may alternatively *free-ride*, or reduce their rate of evaluation of the project in order to exploit the effort of others. When both avenues of delay are available, whether each emerges in equilibrium and their implications for welfare are not clear ex ante. In this paper we build a parsimonious model to study the equilibrium interplay of incentives to delay information acquisition versus investment.

In our model two firms have the opportunity to invest in a nonrival risky project. Ex ante the project has negative expected returns. Firms may exert variable costly effort, a process we denote "prospecting", at each moment in time for the chance of receiving a binary signal which is informative about the project's value. Each firm can acquire at most one signal,

[1]The non-rival assumption may be inexact in some applications - for instance, funding rounds for very attractive startups may become oversubscribed and generate rivalry; and conversely firms in capital-intensive markets may exhibit increasing returns to scale from further investment. Still, we view the non-rival assumption as a good first approximation for initial analysis.

[2]Our model could also be applied to study product adoption by consumers or firm entry into a new geographic market.

and signals are conditionally i.i.d. As a result, aggregating signals from multiple firms yields additional information about the profitability of investment beyond what any one firm could learn. Any information a firm acquires is private while investment is public, as in a standard strategic investment model. A key feature of our model is that effort yields information only stochastically. This creates a motive for acquiring information pre-emptively, because waiting until information becomes pivotal delays action. This benefit must be weighed against the cost of acquiring information that might be useful only far in the future, or never.

We fully characterize the set of perfect Bayesian equilibria of this game under a mild upper bound on prospecting costs.[3] There are exactly three equilibria, all in pure strategies. In the unique symmetric equilibrium, each firm prospects at the maximal rate until a cutoff time, after which it quits forever if it has not seen investment by the other firm. If at any time before the cutoff a firm receives a positive signal, it invests without delay. If it receives a negative signal it never invests. This equilibrium exhibits no free-riding or investment delay.[4] However, a novel discouragement effect arises - absent observing the other firm invest, each firm eventually becomes so pessimistic about the project that it abandons prospecting. This effect generates an ex post inefficiency whenever the other firm has failed to act due to lack of information rather than a negative signal.

There are also two asymmetric "leader-follower" equilibria. In these equilibria, one firm takes the role of a leader, prospecting at the maximal rate until acquiring a signal and then investing immediately if and only if the signal is positive. Meanwhile the other firm, the follower, either shirks from acquiring a signal, delays investment after acquiring a signal, or both.[5] The mix of free-riding and investment delay is controlled by the cost of prospecting. When prospecting costs are high (and firms are sufficiently patient), the follower free-rides on the leader and does not work to acquire a signal, but does not delay investment on the equillibrium path. When costs are intermediate, the follower initially works to acquire a signal but delays investment until the leader acts if it obtains one, and eventually shirks if it does not. And when costs are sufficiently low, the follower waits for the leader to act after acquiring a signal but never shirks from acquiring one. In particular, when costs are low it

[3]Essentially, we assume that prospecting costs are not too high relative to the information gained. When the condition is violated, we prove that investment delay never arises and that at least one firm shirks at some point in any equilibrium.

[4]By "free-riding" or "shirking" we mean that a firm does not prospect when it would be strictly optimal to do so in a one-player setting. "Investment delay" is defined similarly with respect to investment, with the qualifier that such delay arises on the equilibrium path.

[5]Depending on parameters, there may also be an initial period in which both firms prospect at the maximal rate and invest immediately, as in the symmetric equilibrium. The leader and follower roles therefore apply after this initial period, when multiple equilibrium behaviors are possible.

is possible for the follower to acquire more information than it would have in the symmetric equilibrium.

Our results indicate that free-riding and investment delay are both possible in equilibrium when information acquisition is endogenous, but are inhibited by symmetric play. Intuitively, symmetric play makes delay unsustainable because each firm's opponent must then delay as well, throttling information release and undercutting the rationale for delay. Meanwhile the mix of free-riding and investment delay tilts toward free-riding as costs rise. This is because high costs make early acquisition of a signal prohibitively costly. This substitutability of free-riding for investment delay suggests that when information is expensive, inefficiently slow action in settings of irreversible investment may owe as much to slow information production as to slow aggregation of existing information.

Finally, we compare firm welfare in the symmetric and leader-follower equilibrium. Given the efficiency concerns associated with free-riding and delayed information aggregation, it is natural to guess that firms might benefit by coordinating on the symmetric equilibrium, which does not exhibit these behaviors. Surprisingly, for patient firms this intuition is incorrect. We find that regardless of prospecting costs, as firms become patient, the leader-follower equilibrium first improves aggregate welfare over and eventually Pareto-dominates the symmetric equilibrium. This result arises because while the leader-follower exhibits slow production and revelation of a first signal about the project, the symmetric equilibrium by contrast suffers from occasional discouragement and failure to produce any signals at all. When firms are patient, the latter inefficiency dominates.

This result suggests that venture capitalists may benefit from coordinating on market leaders who investigate and invest in startups in particular categories ahead of their peers, even when investors are relatively homogeneous, despite the incentives this structure gives non-leaders to delay information production and aggregation. In reality, venture capitalists may possess expertise in particular industries lending them naturally to a leadership role when evaluating some startups. In such cases the efficiency advantages of equilibrium behavior and heterogeneous expertise reinforce one another and provide additional reason to expect coordination on the leader-follower equilibrium, with the expert firm selected as a the leader.

The remainder of the paper is organized as follows. Section 1.1 marshals evidence on the structure of the entrepreneurial fundraising market, while Section 1.2 briefly surveys related literature. Section 2 describes the model. Sections 3 and 4 analyze the model under alternative assumptions on the signal structure. Section 5 concludes.

## 1.1 Stylized facts about entrepreneurial fundraising

Here we provide evidence that the market for entrepreneurial fundraising exhibits the key economic forces motivating our model: endogenous information acquisition; non-rival payoffs; and social learning through observation of investment.

*Information acquisition is endogenous:* Fried and Hisrich (1994) document that evaluation of a potential investment is an elaborate, multistage process requiring substantial active input by investment partners and analysts. Based on interviews with a small sample of venture capital firms, they estimate that due diligence takes on average 97 days, and 130 hours of cumulative effort by the lead investor, to complete for investments which are ultimately funded. A recent study by Gompers et al. (2019) surveying a much larger number of venture capitalists reaffirms these numbers, finding that venture capitalists take on average 83 days and 118 hours of effort to close a deal. Further, they report strong evidence that this evaluation process yields important information on the profitability of investment - only 1 in 20 startups evaluated are ultimately offered capital.

*Payoffs are non-rival:* Evidence from empirical analysis of venture capital contracts indicates that venture-backed startups typically accept capital from more than one firm during fundraising periods. Lerner (1994) analyzes a sample of biotech startups which obtained funding in the 1980s and found that on average between 2 and 4 venture capital firms participated in each of the first several rounds of fundraising for each startup. Kaplan and Strömberg (2003) study a larger sample of more recent startups, including biotech as well as IT and software firms, and find even larger rounds of between 3-6 firms in early rounds, up to 10 firms in later rounds. The survey evidence of Gompers et al. (2019) reinforces these numbers, with 65% of reported investments by venture capitalists taking place as part of a syndicated (i.e. multi-investor) round. Further, a large fraction (75%) of investors cited capital constraints as an important reason for syndication, with 39% citing it as the single most important reason. This evidence suggests that many venture capitalists do not face significant congestion concerns when deciding whether to invest in a startup that has not yet secured capital from another fund.

Anecdotal evidence from practitioners reinforces this picture of non-rival investment. Paul Graham, a prominent entrepreneur and venture capitalist, advises entrepreneurs seeking funding that "It's a mistake to have fixed plans in an undertaking as unpredictable as fundraising... When you reach your initial target and you still have investor interest, you can just decide to raise more. Startups do that all the time. In fact, most startups that are very successful at fundraising end up raising more than they originally intended" (Graham (2013a)). This advice suggests that startups are often able to efficiently accept variable amounts of capital to meet unexpected demand.

*Social learning is important:* Lerner (1994), in his study of syndicated investing by venture capitalists, argues that social learning is so important that venture capitalists actively seek investing partners for the purpose of validating their investment decisions: "Another venture capitalist's willingness to invest in a potentially promising firm may be an important factor in the lead venture capitalist's decision to invest" (Lerner (1994, p. 16)). This assertion is reinforced by the survey evidence of Gompers et al. (2019), who report that 77% of venture capitalists surveyed cite "complementary expertise" of other investors as an important factor in deciding to join a syndicated round with multiple investors. (33% of respondents cited it as the single most important factor.)

Practitioners confirm the importance of social learning for investment decisions. Paul Graham writes that "The biggest component in most investors' opinion of you is the opinion of other investors... When one investor wants to invest in you, that makes other investors want to, which makes others want to, and so on." He elaborates that the cause of this phenomenon is social learning: "Judging startups is hard even for the best investors. The mediocre ones might as well be flipping coins. So when mediocre investors see that lots of other people want to invest in you, they assume there must be a reason" (Graham (2013b)). A handbook for fundraising, written by co-founders of a major venture capital fund, groups venture capitalists into "leader" and follower" categories, with leaders actively seeking to close a deal as a first investor while followers by contrast "han[g] around, waiting to see if there's any interest in your deal." Consistent with a social learning motive, they observe that followers are "not going to catalyze your investment. However, as your deal comes together with a lead, this VC is a great one to bring into the mix if you want to put a syndicate of several firms together" (Feld and Mendelson (2016, p. 25)).[6]

## 1.2 Related literature

Our paper builds on two distinct literatures studying strategic investment timing and collective experimentation. Existing papers have typically studied only one of these forces in isolation. However, in reality decision-makers often have the flexibility to time both their investment and their information-gathering efforts. Our paper bridges this gap by modeling the two decisions jointly. We will briefly overview each literature, highlighting how the

[6]The dichotomy between leaders and followers is echoed by Fred Wilson, another successful venture capitalist, who similarly describes leaders as venture capitalists who "have conviction" and "raise their hand and say 'we are in'" while followers "sit on the sidelines, wait until someone with conviction shows up, and then try to get in alongside the investor with conviction" (Wilson (2007)).

analyses and conclusions of existing work compare to ours.

Our paper builds most directly on the investment timing literature, in which multiple players decide when to make an irreversible investment in a risky project in the presence of social learning. The timing of investment is unrestricted and endogenous (in contrast to the herding literature), and information about the project's profitability is dispersed among the players, who can observe other players' investment decisions but not their information. This literature focuses on the strategic nature of the investment timing choice when agents learn from other agents' actions. Papers in this literature include Chamley and Gale (1994), Gul and Lundholm (1995), Chari and Kehoe (2004), Rosenberg, Solan, and Vieille (2007), and Murto and Välimäki (2011, 2013). All of these papers feature exogenous information arrival, and so abstract from the choice of when and whether to expend effort to learn about the project. A recent paper by Aghamolla and Hashimoto (2018) endogenize the precision of the time-zero signal in the model studied by Chamley and Gale (1994) and Murto and Välimäki (2013), but do not allow agents to dynamically acquire any further information over the course of the game.

Our analyses and results differ in several ways from existing strategic investment timing models. First, these models almost exclusively analyze properties of symmetric equilibria,[7] while we explore behavior across all equilibria. Second, the literature commonly emphasizes delays in information aggregation and clustering of investment due to hoarding of private information. By contrast, in our model clustering (when it occurs) does not necessarily reflect delays in either information aggregation or production. In particular, in the symmetric equilibrium, clustering occurs only when the follow-on investor has attempted to acquire information but has not yet succeeded. And information-hoarding is not necessarily harmful to welfare. On the contrary, when signal acquisition costs aren't too high, equilibria in which information is hoarded improve aggregate welfare or even Pareto-dominate equilibria in which it is not.

Our paper is also closely related to models of collective experimentation, in which multiple agents engage in social learning by observing the outcome of experimentation by other agents.[8] Experimentation is typically modeled via an exponential or Poisson bandit framework. Agents have the opportunity to repeatedly pull the arm of a slot machine with unknown average payout, and must decide whether to abandon the machine by learning from past pulls of the arm. The profitability of the slot machine is partially or perfectly

[7]One exception is Gul and Lundholm (1995), who characterize asymmetric equilibria and compare their welfare properties to those of a symmetric equilibrium. Nonetheless, the bulk of their analysis focuses on properties of the symmetric equilibrium.

[8]See Hörner and Skrzypacz (2017) for an excellent survey of this literature.

correlated across players, opening a channel for social learning. Bolton and Harris (1999), Keller, Rady, and Cripps (2005), and Keller and Rady (2010, 2015) adopt this framework under the assumption that the actions of each agent and payoffs from the slot machine are publicly observable. Bonatti and Hörner (2011, 2017) also use this formulation, but with private actions, so that only payoffs are observed.[9]

This literature focuses on the information spillovers of each agent's information acquisition to other agents' experimentation. A universal effect is the free-rider problem, where costly experimentation by one agent acts as a substitute for experimentation by others. Depending on the learning process, there may also be encouragement effects, where good news acquired by one agent spurs other agents to experiment more. These papers abstract from the endogenous timing of investment, as agents need never commit irreversibly to pulling the arm of the slot machine forever.

Our results depart from these papers in several ways. We find that adding strategic investment timing reduces the multiplicity of equilibria common in the literature. Our model yields exactly one symmetric and one asymmetric equilibrium, up to relabeling of players, which can be sharply characterized. In contrast, the typical experimentation model suffers from a large number of equilibria, even with only two players and private actions. As a result, predictions for welfare and free-riding are ambiguous. We also find patterns of effort sharing which contrast with common outcomes in the literature. Our symmetric equilibrium exhibits a bang-bang structure with both firms exerting maximal effort until abandoning the project, whereas interior effort is typical of symmetric equilibria in the literature. And in our asymmetric equilibrium, if the follower exerts effort at all prior to investment by the leader, it does so only early in the game. In the literature, for instance in Bonatti and Hörner (2011), followers typically shirk initially and only jump in later on.

Our paper is also linked to several others which incorporate related assumptions on information acquisition or the strategic timing of investment. Guo and Roesler (2018) examine a strategic experimentation model with a mixture of public and private signals and an option to quit permanently to secure an outside option. Klein and Wagner (2018) build a model of strategic investment timing in which investment is reversible, and players observe the outcomes of investments made by other players. Frick and Ishii (2016) study a model of investment timing in which investment boosts the arrival rate of public signals rather than signaling an agent's private information. Akcigit and Liu (2016) analyze a patent race with two possible innovations, in which firms may privately and irreversibly switch lines of re-

[9]Bonatti and Hörner (2011) also incorporate a payoff externality, in that a successful pull of the arm yields not only an informational benefit but also an immediate payout to all players in the game.

search after privately observing bad news about the viability of one line. Ali (2018) studies the consequences of endogenous information acquisition in the context of a classic herding model, where players act in a pre-specified order. And Campbell, Ederer, and Spinnewijn (2014) consider a setting of dynamic moral hazard in provision of a public good when each agent may unilaterally halt further effort by all agents and realize the current value of the good.

# 2 The model

Two firms have the opportunity to invest in a nonrival risky project of unknown quality. The project has underlying type $\theta$ and is either Good ($\theta = G$) or Bad ($\theta = B$). The payoff of the project is $R$ if $\theta = G$, and 0 otherwise, with $R > 1$. Each firm is risk-neutral with discount rate $r > 0$. The project is indivisible, investment in the project is irreversible, and project outcomes are not observed until the end of the game. Each firm is free to invest in the project at any time $t \in \mathbb{R}_+$.

Both firms begin with common prior belief $\pi_0$ that the project is Good. Each firm can additionally exert costly effort to search for an informative signal about the project's type, an activity we will refer to as *prospecting*. A signal, when it arrives, is binary with $S \in \{H, L\}$, i.e. High and Low, and is distributed as $\Pr(S = H \mid \theta = G) = q^H$ and $\Pr(S = L \mid \theta = B) = q^L$ with $q^H, q^L \in (1/2, 1)$. For a given belief $\mu \in [0, 1]$ that $\theta = G$, let

$$h(\mu) \equiv q^H \mu + (1 - q^L)(1 - \mu)$$

be the total probability that an arriving signal is High, and similarly

$$l(\mu) \equiv (1 - q^H)\mu + q^L(1 - \mu) = 1 - h(\mu)$$

be the total probability that an arriving signal is Low. The values $h(\mu)$ and $l(\mu)$ are the transition probabilities that a firm's posterior belief jumps up or down upon receiving a signal.

Each firm can obtain at most one signal, which we will denote $S^i$ for firm $i$, and firms observe conditionally IID signals. We will denote the posterior beliefs induced by one or more signals as follows: $\pi_+$ and $\pi_{++}$ are the posteriors induced by one and two High signals, respectively; similarly $\pi_-$ and $\pi_{--}$ are the posteriors induced by one and two Low signals. Finally, $\pi_{+-}$ is the posterior induced by one High and one Low signal. (Exchangeability implies that posterior beliefs are independent of the order of receipt of signals.) Given that

High signals are more likely when the state is Good, and conversely for Low signals when the state is Bad, $\pi_{++} > \pi_{+} > \pi_0, \pi_{+-} > \pi_{-} > \pi_{--}$. Note that in general $\pi_{+-} \neq \pi_0$, except in the special case when $q^H = q^L$. If $q^H > q^L$ then $\pi_{+-} < \pi_0$, and if $q^H < q^L$ then $\pi_{+-} > \pi_0$.

**Assumption 1.** $\pi_0 < 1/R < \pi_+$.

Under this assumption, investment in the project is ex ante unprofitable, but becomes profitable conditional on observation of a High signal.[10] Note that $1/R < \pi_+$ holds so long as $q^H$ is sufficiently large, i.e. a High signal is sufficiently correlated with a Good state.

**Assumption 2.** $\pi_{+-} < 1/R$.

This assumption is satisfied so long as $q^L$ is not too much smaller than $q^H$. Under this assumption, even after observing a High signal making investment profitable, observation of a Low signal would push beliefs back below the breakeven threshold. Without this assumption no equilibrium would exhibit "wait and see" behavior, since the optimality of investment following receipt of a High signal would not depend on the information obtained by the other firm. (Note that $\pi_{+-} < 1/R$ does not inevitably imply waiting to see, and indeed we will construct an equilibrium in which such behavior does not arise.)

Prospecting is a dynamic process unfolding in continuous time. At each instant $dt$, firm $i$'s signal arrives with probability $\lambda\, dt$ when firm $i$ exerts effort $C(\lambda)\, dt$. We will maintain the assumption of a linear cost structure:

$$C(\lambda) = \begin{cases} c\lambda, & \lambda \in [0, \overline{\lambda}], \\ \infty, & \lambda \in (\overline{\lambda}, \infty) \end{cases}$$

for some constant marginal cost $c > 0$ and maximum prospecting rate $\overline{\lambda}$, both of which are symmetric across firms. Conditional on prospecting rates, signal arrival times are independent across firms and independent of the type of the project.

Firms cannot observe each other's signals or prospecting intensities, nor can they observe whether another firm has received a signal. There are also no communication channels between firms. However, all investment decisions are public, introducing a channel for social learning.

[10] The case $\pi_+ < 1/R$ is uninteresting, as the unique equilibrium involves no prospecting and no investment by either firm. To see this, note that any firm investing first must have posterior beliefs weakly below $\pi_+$, meaning investment would be unprofitable. Hence no firm ever invests, and so never acquires a signal.

## 2.1 Strategies and payoffs

For each $i \in \{1, 2\}$, let $s^i$ be the process tracking what signal, if any, firm $i$ has received at each moment in time. That is, $s^i_t \in \{\emptyset, H, L\}$ for each $t$, with $s^i_0 = \emptyset$ and $s^i$ jumping at most once to either H or L at the time a signal is received. We will use $\nu^i = \inf\{t \ : \ s^i_t \neq \emptyset\}$ to denote the first time firm $i$ receives a signal. Also let $\mathbb{F}^i$ be the filtration generated by $s^i$ and a randomization device privately observed by $i$, with the latter allowing for mixed strategies.

**Definition 1.** *A strategy $\sigma^i$ for firm $i \in \{1, 2\}$ is a tuple $\sigma^i = (\lambda^i(T), \iota^i(T))_{T\in\{\emptyset\}\cup\mathbb{R}_+}$, where each $\lambda^i(T)$ is a $[0, \overline{\lambda}]$-valued $\mathbb{F}^i$-adapted process and each $\iota^i(T)$ is a $\{0, 1\}$-valued $\mathbb{F}^i$-adapted process.*

A strategy $\sigma^i$ consists of a prospecting process $\lambda^i$ and an investment decision process $\iota^i$ which may condition on the timing of any past investment by the other firm. So long as firm $i$ has not observed investment by firm $-i$, it prospects at rate $\lambda^i(\emptyset)_t$ until a signal is observed. And it invests at time $\tau^i(\emptyset) = \inf\{t \ : \ \iota^i(\emptyset) = 1\}$. After $i$ has observed $-i$ invest at some time $T$, the firm prospects at rate $\lambda^i(T)_t$ until a signal is observed, and it invests at time $\tau^i(T) = \inf\{t \geq T \ : \ \iota^i(T) = 1\}$. This construction allows for the possibility that firm $i$, upon observing investment by firm $-i$, immediately follows and invests "afterward at the same time".[11] In particular, consider a strategy and state of the world in which firm 1 invests at time $T$. It will be important to allow for strategies for firm 2 under which $\tau^2(\emptyset) > T$, so that firm 2 would not invest at time $T$ on its own, but under which $\tau^2(T) = T$, so that investment by firm 1 spurs firm 2 to act immediately.

For each firm, $\lambda^i$ and $\iota^i$ are adapted to the history of its own signal as well as its randomization device. Because prospecting does not occur after a signal has arrived, the conditioning of $\lambda^i$ on the signal history is redundant; however, it is important that the firm be allowed to condition on the randomization device to allow for mixed strategies. Additionally, $\iota^i$ contains information beyond what is necessary to construct the investment time $\tau^i$. This is because we will be interested in characterizing perfect Bayesian equilibria, which require a notion of optimality off the equilibrium path. Supposing that firm $i$ has deviated and failed to invest at time $\tau^i$, then at time $t > \tau^i$ firm $i$'s strategy induces the continuation investment time $\widetilde{\tau}^i = \inf\{t' \geq t \ : \ \iota^i = 1\}$. This allows for the important possibility that a firm who initially finds investment profitable may eventually become pessimistic and prefer not to invest

[11]In this respect, we follow the construction of strategy profiles used by Murto and Valimaki (2011), who model "exit waves" of firms who follow others out of the market with no delay. This model timing is necessary in continuous time to ensure existence of best replies. Otherwise a firm observing another investing/exiting might want to follow "as soon as possible", meaning any strategy of delaying a finite amount of time could be improved upon by delaying a bit less.

immediately.

Fix a strategy profile $\sigma = (\sigma^1, \sigma^2)$. Firm $i$'s expected payoff under $\sigma$ is then

$$U^i(\sigma) = \mathbb{E}\left[\left(R\,\mathbf{1}_{\{\theta=G\}} - 1\right)e^{-r\tau^i(\sigma)} - c\int_0^{\min\{\nu^i,\tau^i(\sigma)\}} e^{-rt}\lambda^i(\sigma)_t\,dt\right].$$

where

$$\lambda^i(\sigma)_t = \begin{cases} \lambda^i(\emptyset)_t, & t < \tau^{-i}(\emptyset), \\ \lambda^i(\tau^{-i}(\emptyset))_t, & t \geq \tau^{-i}(\emptyset) \end{cases}$$

and

$$\tau^i(\sigma) = \begin{cases} \tau^i(\emptyset), & \tau^i(\emptyset) \leq \tau^{-i}(\emptyset), \\ \tau^i(\tau^{-i}(\emptyset)), & \tau^i(\emptyset) > \tau^{-i}(\emptyset). \end{cases}$$

The first term in $U^i(\sigma)$ is the discounted payoff from investing in the project at time $\tau^i(\sigma)$. The second term is the cumulative discounted cost of prospecting according to $\lambda^i(\sigma)$. The upper limit of integration reflects the fact that prospecting stops whenever either a signal arrives (at time $\nu^i$) or the firm invests (at time $\tau^i(\sigma)$).

## 2.2 Beliefs

Given a strategy profile, let

$$\mu^i(t) \equiv \Pr(\theta = G \mid s^i_t = \emptyset, \tau^{-i}(\emptyset) \geq t)$$

be firm $i$'s posterior beliefs at time $t$ conditional on having seen no signal so far, with

$$\mu^i_+(t) \equiv \Pr(\theta = G \mid s^i_t = H, \tau^{-i}(\emptyset) \geq t) = \frac{q^H\mu^i(t)}{h(\mu^i(t))}$$

and

$$\mu^i_-(t) \equiv \Pr(\theta = G \mid s^i_t = L, \tau^{-i}(\emptyset) \geq t) = \frac{(1-q^L)\mu^i(t)}{l(\mu^i(t))}$$

similarly representing firm $i$'s beliefs conditional on having observed a High and Low signal, respectively.[12]

Note that in stark contrast to Poisson bandit models, each firm's posterior beliefs about

[12] Lemma A.1 shows that these beliefs are always uniquely pinned down by Bayes' rule in any PBE strategy profile. Note that if $\tau^{-i}(\emptyset) = t$, then firm $i$ is not able to observe this fact until after making his own initial investment decision, so its beliefs at time $t$ cannot condition on this fact. Hence the appropriate conditioning for "no investment by firm $-i$ up to time $t$" is the event $\tau^{-i}(\emptyset) \geq t$.

the state are independent of their own history of prospecting. Lack of signal acquisition in our model does not signal anything, positive or negative, about the true project state; no news truly is no news until a signal arrives. Nonetheless, if the other firm is prospecting and investing, then each firm's beliefs $\mu^i(t)$ *do* deteriorate over time due to the negative inference from continued lack of investment by the other firm. More precisely, continued inaction by firm $-i$ leads firm $i$ to infer that $-i$ is likely dormant due to receipt of a Low signal, rather than due to a long string of bad luck leading to no signal. (See Appendix B for an explicit characterization of this belief updating via Bayes' rule.) This effect is illustrated in Figure 1.

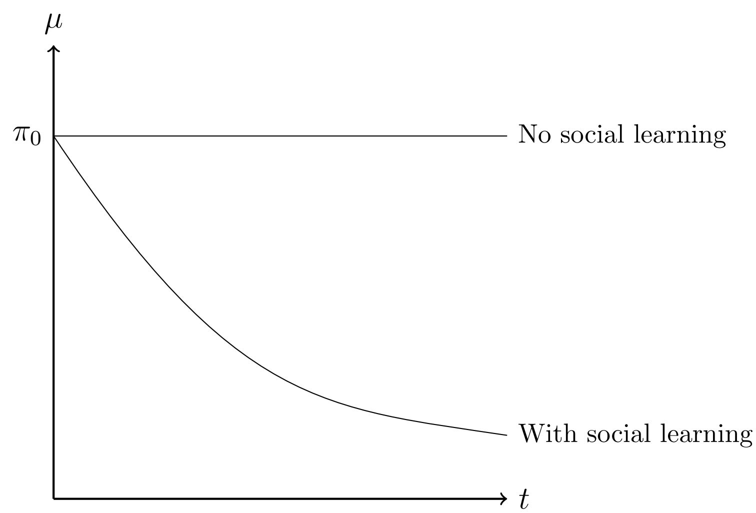


Figure 1: Evolution of beliefs prior to obtaining a signal

Of course, the rate at which beliefs deteriorate is an endogenous property of a particular equilibrium, and in particular will dependent on whether each firm expects the other to be prospecting and investing, or shirking and waiting. This linkage of beliefs about the state and the (unobserved) strategy of one's opponent will play a crucial role in the construction of equilibria in this setting.

## 2.3 Single-player benchmark

Consider a single firm prospecting and investing on its own, or equivalently with the social learning channel of our model shut down. We will refer to this benchmark setting as the *autarky case*. The firm's initial beliefs that the project is good will be taken to be $\mu < 1/R$.

So long as the firm has acquired no signal, it learns nothing about the project and its beliefs remain fixed at $\mu$. An optimal prospecting strategy is therefore stationary. (This behavior is very different from the cutoff strategies which are optimal when learning from Poisson bandits, and is driven by the very different learning dynamics as discussed in Section 2.2.) Once the firm has acquired a signal, no further information is available. It then faces a simple choice of whether to invest or abandon the project once and for all, by comparing its posterior beliefs to the investment threshold $1/R$.

The optimal prospecting strategy depends on whether the firm's initial beliefs $\mu$ lie above a critical threshold, which we will denote $\pi_A$ and refer to as the *autarky threshold*. If $\mu < \pi_A$, the firm abandons the project immediately and performs no prospecting. On the other hand, if $\mu > \pi_A$, then the firm prospects at the maximum rate $\overline{\lambda}$ until a signal is acquired.

The autarky threshold $\pi_A$ is computed by finding the smallest belief for which the value of obtaining a signal exceeds the flow cost of prospecting. It is formally characterized in the following lemma.

**Lemma 1.** *$\pi_A$ is the unique solution to*

$$h(\mu)(\mu_+ R - 1) = c,$$

*and has the explicit representation*

$$\pi_A = \frac{c + (1 - q^L)}{q^H(R-1) + (1 - q^L)}.$$

If firms' initial beliefs $\pi_0$ lie below $\pi_A$, then even in a model with social learning no prospecting or investing will ever take place.[13] Going forward we will assume that this is not the case. In particular, since $\pi_A$ is increasing in the cost parameter $c$, we will assume that prospecting costs are low enough that firms would want to prospect in the single-player benchmark.

**Assumption 3.** *$c$ is sufficiently small that $\pi_0 > \pi_A$.*

Note that as $c$ approaches 0, $\pi_A$ approaches the prior belief $\underline{\mu}$ for which $\underline{\mu}_+ R - 1 = 0$. Since by assumption $\pi_+ R - 1 > 0$, it must be that $\pi_0 > \pi_A$ for sufficiently small $c$.

## 2.4 Free-riding and investment delay

Our paper is motivated by environments in which incentives to delay both information acquisition and investment are present. We begin by formally defining these notions in the

[13]In particular, there can be no encouragement effect inducing prospecting below the single-player cutoff in this model. This is because in equilibrium some firm must invest first after obtaining a signal, and that firm does not benefit from inducing the other firm to act after it.

context of our model:

**Definition 2.** *Given a firm $i$ and strategy $\sigma^i$, firm $i$* free-rides *or* shirks *at time $t$ if $\mu^i(t) > \pi_A$ and $\lambda^i(t) < \overline{\lambda}$. The firm* delays investment (on the equilibrium path) *if* $\Pr(\tau^i(\emptyset) > \nu^i \mid s^i_{\nu^i} = H, \mu^i_+(\nu^i) > 1/R) > 0$.

We consider a firm to be free-riding (or, equivalently, shirking) if it exerts effort strictly below what would be optimal in a one-player setting at current beliefs. Any such effort reduction must be because the firm expects to learn from the other player's future actions, a motive typically labeled shirking in the literature on free-riding in teams. (See, for instance, Bonatti and Hörner (2011).)

Similarly, we consider a firm to be delaying investment on the equilibrium path if it sometimes obtains a High signal which 1) boosts its beliefs above the threshold for profitable investment, and 2) the firm does not immediately act on. This definition of investment delay is consistent with the convention used in the investment timing literature. (See, e.g., Chamley and Gale (1994).) We will occasionally omit the qualifier "on the equilibrium path" when we expect no confusion. Note that as we will characterize off-path play in our equilibrium characterizations, we will encounter information sets which a firm reaches only reach by deviating (in particular, acquiring a signal which its equilibrium strategy did not tell it to acquire), following which the firm optimally waits to invest after obtaining a High signal. We will not consider such behavior to constitute investment delay, as it does not arise on the path of play.

Because information acquisition is costly, the incentive to delay effort to acquire a signal is always present in our model. However, if signal acquisition costs are too large, investment delay will not play a role in equilibrium because no firm will ever find it profitable to acquire a signal it does not immediately act on. The following lemma establishes a bound on costs above which the substitution between shirking and investment delay is always resolved in favor of shirking in any equilibrium.

**Lemma 2.** *Suppose $c > \overline{c} \equiv h(\pi_+)(\pi_{++}R - 1) - (\pi_+ R - 1)$. Then invest delay does not arise on the equilibrium path in any perfect Bayesian equilibrium. Further, at least one firm must shirk at some time in every perfect Bayesian equilibrium.*

The cost threshold $\overline{c}$ represents the spread in expected profits, conditional on having observed a single High signal, between investing based solely on that information (yielding expected returns $\pi_+ R - 1$) and observing a second signal, then investing only if that additional signal is also High (yielding expected returns $h(\pi_+)(\pi_{++}R - 1)$). Note that $\overline{c} > 0$, as by

Assumption 2 the second signal provides pivotal information about whether investment is profitable.

Lemma 2 indicates that when costs are too high, investment delay is never observed in any equilibrium. Since our main goal is to understand the equilibrium interplay of shirking and investment delay when both are possible, we maintain the bound $c \leq \bar{c}$ as a standing assumption going forward:

**Assumption 4.** $c \leq \bar{c}$.

## 2.5 The value of multiple signals

One key determinant of equilibrium behavior is the optimal continuation strategy of a firm without a signal after observing the other firm invest. It can be shown that in any equilibrium, the first firm to invest is always in possession of a High signal. The remaining firm therefore finds itself in a stationary single-player environment with fixed beliefs $\pi_+$, analogous to the autarky case. It therefore either acts immediately or prospects at rate $\overline{\lambda}$ until it obtains another signal, depending on the cost and delay involved in prospecting.

The following definition characterizes the parameter values under which one or the other of these continuation outcomes prevails.

**Definition 3.** *Signals are* complements *if*

$$\frac{\overline{\lambda}}{\overline{\lambda}+r}(h(\pi_+)(\pi_{++}R-1)-c) \geq \pi_+ R - 1.$$

*Otherwise they are* substitutes.

Note that whenever signals are complements, Assumption 4 holds automatically. By contrast, when signals are substitutes, Assumption 4 imposes an additional restriction on the costs of signal acquisition.

When signals are complements, the firm optimally prospects for an additional signal in the post-investment continuation.[14] Conversely, when signals are substitutes, the firm invests immediately after seeing the other firm invest. The following lemma states this result formally:

**Lemma 3.** *In any perfect Bayesian equilibrium, for each firm $i \in \{1, 2\}$, $T \in \mathbb{R}_+$, $t \geq T$:*

[14] Technically, this is the unique optimal continuation strategy only when the complements inequality holds strictly. Otherwise a multiplicity of optimal continuation strategies exist. However, all such strategies are payoff-equivalent, and the selection does not affect any other aspects of equilibrium construction. So we will ignore the multiplicity arising in this edge case.

- *If signals are complements,* $\lambda^i(T)_t = \overline{\lambda}$ *and* $\iota^i(T)_t = \mathbf{1}\{s_t^i = H\}$,
- *If signals are substitutes,* $\iota^i(T)_t = \mathbf{1}\{s_t^i \in \{H, \emptyset\}\}$.

(Note that in the substitutes case, the firm's choice of prospecting strategy after observing investment is not payoff-relevant and need not be specified.) As we will see, several of our equilibrium predictions will depend on whether signals are complements or substitutes. We study equilibrium under complements in Section 3, and under substitutes in Section 4.

# 3 Complementary signals

In this section we characterize equilibrium behavior and welfare when signals are complements (as specified in Definition 3). We show that our model has exactly three perfect Bayesian equilibria. One equilibrium is symmetric and exhibits no free-riding or investment delay, but does suffer from eventual inefficient abandonment of the project. The remaining "leader-follower" equilibria feature distinct roles for the two firms, with one firm who takes the lead in prospecting and investing while the other firm plays a passive follower role. We further show that the leader-follower equilibria Pareto-dominate the symmetric equilibrium.

The section is structured as follows. In Section 3.1, we describe a class of threshold strategies for prospecting and investing which arise in every equilibrium. In Sections 3.2 and 3.3 we characterize the symmetric and leader-follower equilibria and provide intuition for their properties. In Section 3.4 we prove that no other equilibria exist. Finally, in Section 3.5 we perform welfare comparisons.

## 3.1 Threshold strategies

In principle, there are diverse possibilities for equilibrium prospecting and investing prior to observing investment by the other firm. In this subsection we introduce a class of strategies which will characterize the structure of firm behavior in any equilibrium.

**Definition 4.** *A strategy* $\sigma^i$ *for firm* $i \in \{1, 2\}$ *is a* threshold strategy *if:*

- $\lambda^i(\emptyset)_t = \overline{\lambda}\mathbf{1}\{t < \overline{T}_i\}$ *for some* $\overline{T}_i \in \mathbb{R}_+ \cup \{\infty\}$,
- $\iota^i(\emptyset)_t = \mathbf{1}\{s_t^i = H$ *and* $t < T_i^*\}$ *for some* $T_i^* \in \mathbb{R}_+ \cup \{\infty\}$.

Under a threshold strategy, prospecting and investment have a simple bang-bang structure prior to observing investment by the other firm. Until the threshold time $\overline{T}_i$, the firm prospects at the maximum possible rate $\overline{\lambda}$. At such times we will say that firm $i$ is *working*. Meanwhile after $\overline{T}_i$ the firm quits prospecting, at which point we will say that firm $i$ is *shirking*. And prior to the threshold $T_i^*$, the firm invests immediately whenever it is in possession of a High signal (on or off the equilibrium path), while after $T_i^*$ it never invests prior to the other firm investing. We will refer to the period prior to $T_i^*$ as the *investment* phase, and the period subsequent to $T_i^*$ as the *waiting* phase.

As we will see, in every equilibrium both firms follow threshold strategies. This fact, along with Lemma 3, reduces the description of an equilibrium to the characterization of the threshold times $\overline{T}_i$ and $T_i^*$ for each firm.

## 3.2 The symmetric equilibrium

In this subsection we characterize a symmetric equilibrium of the model. (In fact, this will turn out to be the unique symmetric equilibrium.) This equilibrium exhibits no free-riding or investment delay, but does involve eventual inefficient abandonment of the project.

To state the equilibrium, we define a time threshold at which a firm's posterior beliefs hit $\pi_A$ assuming the other firm acts as quickly as possible. Suppose that some firm $i$ prospects at rate $\overline{\lambda}$ forever and invests immediately whenever it obtains a High signal. Let $\mu^{\overline{\lambda}}$ denote the path of firm $-i$'s beliefs conditional on observing no investment by firm $i$. (See Appendix B for an explicit calculation of these beliefs.) These beliefs decline over time, asymptotically approaching $\pi_-$, and cross the autarky threshold $\pi_A$ at some finite time which we will denote $T^A \equiv (\mu^{\overline{\lambda}})^{-1}(\pi_A)$.

**Proposition 1** (The symmetric equilibrium)**.** *Suppose signals are complements. There exists a symmetric perfect Bayesian equilibrium in threshold strategies characterized by* $\overline{T}_1 = \overline{T}_2 = T^A$ *and* $T_1^* = T_2^* = \infty$.

This equilibrium unfolds as follows. Absent observing investment by the other firm, each firm works, i.e. prospects at rate $\overline{\lambda}$, until time $T^A$ (at which point the firm's beliefs reach the autarky threshold $\pi_A$). Afterward each firm abandons prospecting forever. If at any time a firm observes investment, it prospects at rate $\overline{\lambda}$ forever afterward until obtaining a signal. If at any time a firm is in possession of a High signal (on or off the equilibrium path), it invests immediately. In particular, prior to observing investment each firm is in the investment phase forever. Finally, no firm invests while in possession of no signal or a Low signal. The structure of prospecting and investing prior to observing investment by the other firm is represented diagrammatically in Figure 2. Note that the equilibrium is symmetric -

given the structure of threshold strategies and the optimal continuation play characterized by Lemma 3, equal prospecting and investment thresholds imply symmetric strategies.

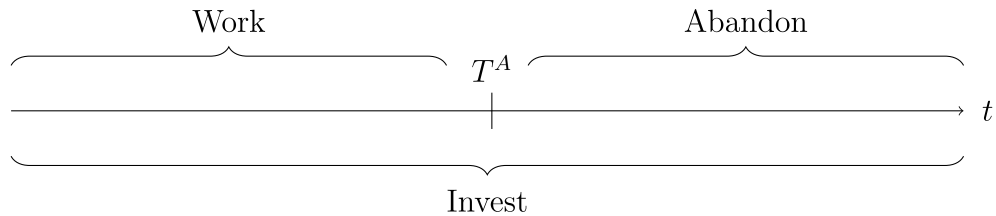


Figure 2: A timeline of prospecting and investment in the symmetric equilibrium

Why is investing immediately at all times optimal in this equilibrium? Because both firms quit prospecting at time $T^A$, each firm's beliefs are always at least $\pi_A$ for all times, even absent a signal or any observed investment. This means that after obtaining a High signal, each firm's beliefs lie above $1/R$ forever. So by waiting to invest, no firm ever obtains enough negative information to change their optimal investment decision, meaning any delay in investing is suboptimal. In other words, immediate investment upon receipt of a High signal is always optimal. Note that each firm's investment rule takes into account only the firm's own signal and *not* any information provided by the other firm's investment. This fact will be important to understanding the optimality of the equilibrium prospecting rule.

The key to understanding the equilibrium prospecting rule is that each firm never learns anything from the other firm's actions which is pivotal to their decision to prospect or invest. In particular, consider continuations in which the other firm has or has not invested. In the first case, under the complements assumption it continues to be optimal to acquire another signal. And in the second case, the firm's beliefs deteriorate over time, but critically not past the threshold level $\pi_A$ given the cutoff time $T^A$. Thus no information ever arrives which makes the firm so pessimistic about the project that signal acquisition isn't optimal. The result is that in all continuations, whether the other firm has invested or not, each firm's optimal prospecting decision remains the same as in the autarky case. It is then certainly optimal to prospect at rate $\overline{\lambda}$ prior to time $T^A$. Subsequent to $T^A$ the firm is made indifferent between prospecting or not, so it is (weakly) optimal to cease prospecting at that point.

One key feature of this equilibrium is the eventual abandonment of the project - if by time $T^A$ no firm has invested, both firms cease efforts to acquire a signal forever afterward. As just noted, this is only one of many best responses in this continuation. However, it is the unique best response which can be sustained as part of an equilibrium. For it is precisely the lack of information arriving after beliefs hit $\pi_A$ which makes it optimal for firms to prospect at all times prior to $T^A$. If some firm were to continue prospecting, they would drive the other firm's beliefs below $\pi_A$ in finite time, and that firm would then no longer optimally prospect until time $T^A$. The dynamics of belief updating in this equilibrium prior to obtaining a signal are illustrated in Figure 3.

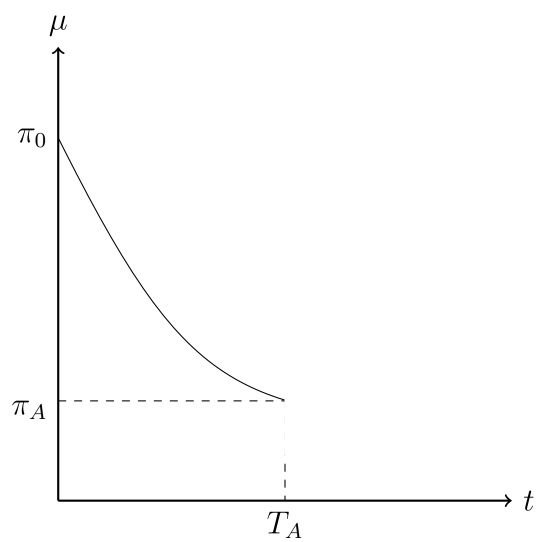


Figure 3: Evolution of beliefs in the symmetric equilibrium

Another important feature of this equilibrium is that it exhibits neither free-riding nor investment delay. In particular, at no point in time does a firm shirk from acquiring a signal while the other firm is working, nor does any firm in possession of a High signal ever delay investment. This is in contrast to models of strategic experimentation and investment timing, in which these effects typically arise. Despite this fact, eventual abandonment of the project generates an inefficiency which drives down aggregate welfare. In particular, a (non-equilibrium) strategy profile in which one firm continued prospecting after time $T^A$ would constitute a Pareto improvement over this equilibrium. As we will see, this inefficiency is so severe that welfare in this equilibrium is lower than in all other equilibria of the model.

## 3.3 The leader-follower equilibrium

In this subsection we characterize a pair of asymmetric equilibria in which one firm takes the lead to prospect and invest. (We will show later that these constitute all remaining equilibria of the model aside from the symmetric equilibrium characterized in Section 3.2.) Unlike the symmetric equilibrium, these equilibria do not feature abandonment of the project - with probability 1 at least one firm eventually acquires a signal about the project. However, they feature a different sort of inefficiency, namely effort reduction and delayed investment by the non-lead firm.

**Proposition 2** (The leader-follower equilibrium)**.** *Suppose signals are complements. For each $i \in \{1, 2\}$, there exists a unique perfect Bayesian equilibrium in which firm $i$ follows the threshold strategy $\overline{T}_i = T_i^* = \infty$. In this equilibrium, firm $-i$ follows the threshold strategy $T_{-i}^* = 0$ and $\overline{T}_{-i} = (\mu^{\overline{\lambda}})^{-1}(\overline{\mu}) < \infty$ for a belief threshold $\overline{\mu} \in (\pi_-, \pi_0]$ which is independent of $i$.*

(Recall that $\mu^{\overline{\lambda}}$ is the posterior belief process of a firm who believes its opponent prospects at rate $\overline{\lambda}$ and invests immediately.)

This equilibrium unfolds as follows. One firm, say firm $i$, takes the role of the *leader*. The leader works (prospects at rate $\overline{\lambda}$) until it obtains a signal, regardless of whether it has seen the other firm invest or not. If at any time the leader is in possession of a High signal (on or off the equilibrium path), it invests immediately. It is therefore in the investment phase forever. Meanwhile the other firm $-i$ takes the role of the *follower*. Absent observing investment, the follower works only up until the threshold time $\overline{T}_i < \infty$. After this time it stops prospecting until it observes investment. This constitutes shirking prior to time $T^A$, and abandonment of an unprofitable project afterward.

If at any time the follower is in possession of a High signal (on or off the equilibrium path), it invests if and only if it has previously observed the leader invest. It is therefore in the waiting phase at all times. Finally, if at any time the leader invests and the follower has not yet obtained a signal, the follower prospects at rate $\overline{\lambda}$ until obtaining a signal. The structure of prospecting and investing prior to observing investment by the other firm is represented diagrammatically in Figure 4. The evolution of each player's prior to obtaining a signal is illustrated in Figure 5.

The intuition for optimality of the leader's strategy is straightforward. At no point on the equilibrium path does the leader ever expect to see the follower invest. Therefore the leader is effectively in autarky, with prior beliefs $\pi_0 > \pi_A$. The leader therefore has a unique best response of working until a signal is obtained and then immediately acting on that signal, just as in the single-player benchmark.

The follower's optimal investment strategy is driven by the complementary signals assumption. Recall that under complementary signals, when a firm finds itself in the post-investment continuation (that is, after having seen the other firm invest), it optimally works to obtain a second signal before investing. This is similar to the post-signal continuation that the follower finds itself in after obtaining a High signal, since under the leader's strategy

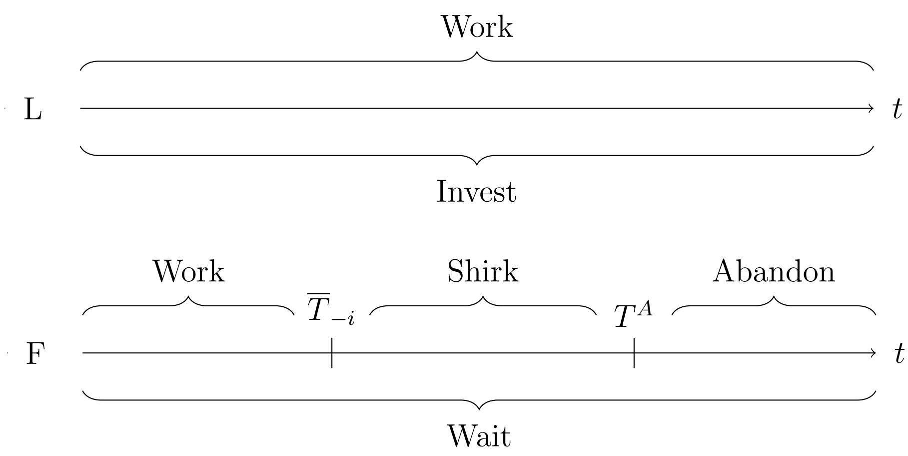


Figure 4: A timeline of prospecting and investing in the leader-follower equilibrium

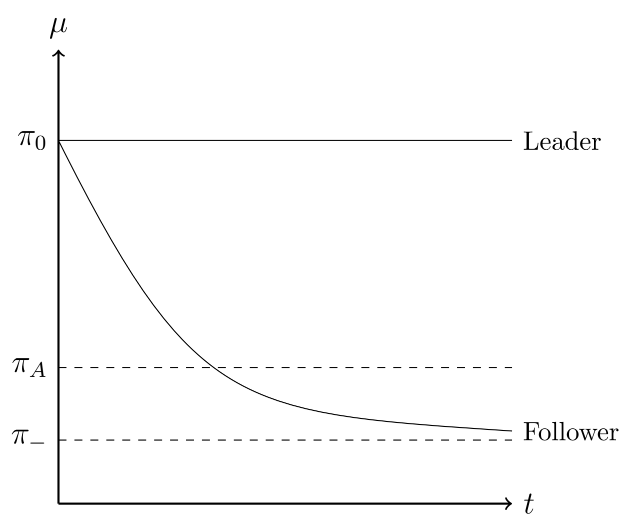


Figure 5: Evolution of beliefs in the leader-follower equilibrium under complements

a second signal arrives[15] at rate $\overline{\lambda}$. These two continuations do differ in two important ways, but both differences add to the incentive for the follower to wait post-signal. First, in the post-signal continuation the second signal requires no prospecting costs to obtain, unlike the post-investment continuation. Second, in the post-investment continuation the firm's beliefs are initially at level $\pi_+$. By contrast, in the post-signal continuation its beliefs are generally below this level due to the negative inference of not having seen investment. However, this effect only makes an additional signal *more* attractive, since the value of investing

[15] Of course, if this signal is Low the follower does not directly observe it. However, it can implement the same optimal investment strategy, investing if and only if the signal is High, as if it could directly observe the signal.

immediately is lower and the revision to beliefs from a second High signal is larger.

The follower's optimal prospecting strategy prior to observing investment balances two factors. On the one hand, if it observes the leader invest, by Lemma 3 it will then work to obtain a second signal before investing itself. It thus anticipates some delay costs in the event it is caught without a signal when the leader invests. On the other hand, it may be that the leader never invests, in which case the follower doesn't either and the prospecting costs of obtaining a signal are wasted. As time passes the first force becomes weaker while the second becomes stronger, because lack of investment makes the follower more pessimistic that the leader will ever invest. Eventually the second force must always dominate, because the follower becomes almost certain that the leader has obtained a Low signal. So $\overline{T}_{-i} < \infty$. As we establish below, if prospecting costs are sufficiently low, then the first force dominates early on - the delay costs in the event of being caught without a signal are more important than the wasted prospecting costs when a signal is obtained and never acted on, in which case $\overline{T}_{-i} > 0$.

We conclude our discussion of the leader-follower equilibrium by establishing several important properties of the follower's prospecting threshold, which characterize when shirking, investment delay, or both arise on the equilibrium path.

**Lemma 4.** *In the leader-follower equilibrium with firm $i$ as the leader, firm $-i$'s prospecting threshold $\overline{T}_{-i}$ satisfies the following properties:*

- *$\overline{T}_{-i}$ is decreasing in $c$ and strictly decreasing whenever $\overline{T}_{-i} > 0$,*
- *If $\overline{\lambda}/r$ is sufficiently large, then $\overline{T}_{-i} = 0$ for $c$ sufficiently large,*[16]
- *For $c$ sufficiently small, $\overline{T}_{-i} > T^A$.*

This lemma establishes that the follower shirks, delays investment, or both along the equilibrium path depending on the cost of prospecting. When costs are high (and firms are patient enough), $\overline{T}_{-i} = 0$ and the follower shirks until time $T^A$, when its beliefs fall below $\pi_A$. (Recall that we have defined shirking to mean effort which is strictly below a level optimal given current beliefs in a one-player setting.) However, because the follower never acquires a signal on the equilibrium path, no investment delay arises. When costs are intermediate, $0 < \overline{T}_{-i} < T_A$ and the follower both shirks and delays investment - investment delay arises first, over the time interval $[0, \overline{T}_{-i}]$, while shirking occurs subsequently, over the time interval

[16]More precisely, there exists a range of costs over which signals are complements and $\overline{T}_{-i} = 0$. For costs above the maximum of the range, signals are no longer complements and the characterization of Proposition 2 does not apply. See Proposition 6 for a similar result when signals are substitutes.

$[\overline{T}_{-i}, T_A]$. Finally, when costs are low the firm delays investment over the interval $[0, \overline{T}_{-i}]$ but never shirks. In fact, in this case the firm actually exerts effort even at times when it would be strictly suboptimal under autarky because beliefs are below $\pi_A$.

When $\overline{T}_{-i} > 0$, the leader-follower equilibrium exhibits a form of frontrunning: the follower works for a period of time to obtain a signal, even though it anticipates that it won't act on the signal until the leader acts. As mentioned above, this work is motivated by the anticipation that the leader will likely act soon, driving the follower to become informed first so that it can immediately react to the leader. This front-running incentive can be quite strong when prospecting costs are low - in particular, when $\overline{T}_{-i} > T^A$ the desire to front-run drives the follower to obtain a signal even when its beliefs are so low that this action would not be justified under autarky.

Unlike the symmetric equilibrium of Proposition 1, the leader-follower equilibrium exhibits shirking, investment delay, or a combination of the two. These behaviors are often viewed as sources of inefficiency in models of strategic experimentation and investment timing. However, as we will see, in our setting the leader-follower equilibria actually Pareto-dominate the symmetric equilibrium.

## 3.4 Characterization of the equilibrium set

So far we have demonstrated the existence of three equilibria: a symmetric equilibrium and two leader-follower equilibria (which are identical up to permutation of firms). We now establish that these equilibria constitute the entire equilibrium set.[17]

**Proposition 3.** *Suppose signals are complements. There exist no perfect Bayesian equilibria, in pure or mixed strategies, beyond those characterized in Propositions 1 and 2.*

The bulk of the proof involves showing that, up to some technicalities, all equilibria must be in threshold strategies. The optimality of a threshold investing rule relies on an argument ruling out waiting for a (possibly random) period and then investing. Such a strategy would merely delay investment without conditioning it on the arrival of information in any useful way. So once it becomes optimal to wait at all, any optimal strategy must

[17] More precisely, the proposition characterizes the equilibrium set up to payoff-irrelevant degeneracies. Because of the continuous-time setting, changing a prospecting or investment rule on a set of times of measure zero does not change payoffs or incentives, on or off the equilibrium path. Further, a strategy profile technically speaking specifies a prospecting rule even at points in time when the profile also directs the firm to invest. At such points in time the prospecting rule chosen does not affect payoffs, inducing a further degeneracy. Our uniqueness result is up to equivalence classes of equilibria which identify all such strategy profiles.

involve waiting until the other firm has invested. The optimality of a threshold prospecting rule is more technical, and requires studying the dynamics of the HJB equation. Essentially, once shirking becomes momentarily even weakly optimal, the HJB equation evolves in such a way that shirking remains strictly optimal forever afterward.

Within the class of equilibria in threshold strategies, the equilibrium set can be narrowed down by a straightforward classification argument. The symmetric equilibrium can be characterized as the unique equilibrium in which both firms stop investing on-path at the same time. Within this class, the only way that both firms can become passive at the same time in equilibrium is if both firms' beliefs reach $\pi_A$ at this time. If some firm's terminal beliefs were any higher that firm would prefer to continue prospecting and investing afterward, and if its beliefs were any lower it would prefer to become passive sooner. Backward induction then pins down the symmetric equilibrium as the unique behavior consistent with this outcome.

The leader-follower equilibria can be characterized as the unique equilibria in the remaining case that some firm stops investing on-path ahead of the other. Call this halting time $\widehat{T}$ and the firm who halts at this time the follower. In this case the other firm, the leader, is effectively in autarky after time $\widehat{T}$ and works and invests immediately at all future times. To sustain an equilibrium, it must then be a best response to the leader's continuation strategy for the follower to stop investing on-path at time $\widehat{T}$. This constraint turns out to pin down $\widehat{T} = 0$ as the only possible equilibrium outcome. Further, this behavior is driven by the follower's incentive to wait rather than invest after obtaining a High signal, implying $T^* = 0$. This reasoning characterizes all aspects of equilibrium except for the follower's optimal prospecting strategy, which is determined by a simple option value calculation, yielding the leader-follower equilibrium.

Our characterization of the equilibrium set stands in stark contrast to existing models of strategic experimentation and investment timing. In those models there typically exist a large multiplicity of equilibria, leading papers to focus on a particular class. (See the literature review for further discussion.) The simplicity of the equilibrium set of our model allows us to make sharp predictions about equilibrium behavior and welfare.

## 3.5 Welfare

We have seen that our model has exactly two equilibria (up to permutation of firms). One involves no shirking and no delay in investment by either firm, but does involve eventual inefficient abandonment of the project. The other involves free-riding and (if costs are low) investment delay on the equilibrium path by one firm. How do these equilibria compare in terms of individual and aggregate firm welfare, and to what extent do they improve on autarky?

Let $V^A$ be each firm's payoff under autarky, $V^S$ be the expected payoff of each firm in the symmetric equilibrium, and $V^L$ and $V^F$ be the expected payoffs the leader and follower, respectively, in the leader-follower equilibrium. The following proposition ranks these payoffs.

**Proposition 4.** *Suppose signals are complements. Then* $V^F > V^L = V^S = V^A$.

Proposition 4 implies the following welfare comparisons. First, the leader-follower equilibrium Pareto-dominates the symmetric equilibrium. Second, the symmetric equilibrium does not improve at all on the autarky outcome. Third, all of the gains from social learning in the leader-follower equilibrium are collected by the follower - the leader does not benefit at all from the presence of the follower.

This proposition establishes the surprising result that social learning provides no gains over autarky in the symmetric equilibrium. At first glance, each firm should be able to improve on their autarky payoff by learning from whether the other firm invests. However, as we discussed following Proposition 1, not enough learning occurs to impact the firm's optimal prospecting or investment decisions. So each firm gains nothing in equilibrium from social learning.

Meanwhile, in the leader-follower equilibrium exactly one firm gains from social learning. It's easy to see that the leader does not benefit from social learning, because the follower never invests ahead of the leader on the equilibrium path. So the leader is effectively in autarky. On the other hand, because the leader prospects until acquiring a signal, the follower's beliefs eventually drop below $\pi_A$. Thus the follower *does* benefit by conditioning its actions on the actions of the leader, boosting its payoffs over autarky.

These results indicate that groups of firms investigating a common investment may benefit from coordinating on a "market leader" who investigates and invests first, even when firms have homogeneous cost structures and knowledge. Despite the incentive this structure gives other firms to free ride and delay their own activity, it prevents an even more costly miscoordination when firms misinterpret inaction by their peers. In our model with symmetric firms, the choice of market leader is not pinned down by welfare considerations. However, in reality firms are likely at least somewhat heterogeneous at their skill in evaluating particular investments; in that case, efficiency considerations naturally select a leader.

# 4 Substitutable signals

In this section we characterize equilibrium behavior and welfare when signals are substitutes. The results closely parallel the complements case, with a few notable changes. First, in the leader-follower equilibrium the follower does not necessarily play a totally passive role. In particular, there may be an initial time period in which the follower both works and invests, prior to eventually assuming a passive stance. Second, the leader-follower equilibrium does not unambiguously dominate the symmetric equilibrium in payoff terms. Depending on parameter values, aggregate welfare may actually be higher in the symmetric equilibrium, and in general the symmetric equilibrium improves on autarky for each firm.

## 4.1 The equilibrium set

In this subsection we present a series of propositions paralleling the characterization of the equilibrium set in Section 3 for the complements case. These propositions characterize one symmetric and two leader-follower equilibria, and establish that there are no other equilibria of the model. As much of equilibrium behavior is unchanged whether signals are complements or substitutes, our discussion of results focuses on highlighting new behaviors which emerge in the substitutes case.

**Proposition 5** (The symmetric equilibrium)**.** *Suppose signals are substitutes. There exists a symmetric perfect Bayesian equilibrium in threshold strategies characterized by* $\overline{T}_1 = \overline{T}_2 = T^A$ *and* $T_1^* = T_2^* = \infty$.

This equilibrium is the analog of the symmetric equilibrium characterized in Proposition 1 for complementary signals. Equilibrium behavior prior to observing investment is identical to the complements case. The sole difference lies in how firms without a signal react to observing investment - recall from Lemma 3 that under substitutable signals they immediately follow the other firm and invest, in contrast to the complements case in which they worked to obtain their own signal.

The intuition for the existence of this equilibrium is more subtle than in the complements case. In particular, social learning *does* have value for each firm when signals are substitutes, as observing investment allows them to save on further prospecting costs by investing immediately afterward. There is then a potential incentive for firms to shirk in order to free-ride on the prospecting of the other firm. It turns out that when Assumption 4 holds, this incentive is never strong enough to induce shirking.

**Proposition 6** (The leader-follower equilibrium)**.** *Suppose signals are substitutes. For each $i \in \{1,2\}$, there exists a unique perfect Bayesian equilibrium in which firm $i$ follows the threshold strategy $\overline{T}_i = T_i^* = \infty$. In this equilibrium, firm $-i$ follows the threshold strategy $T_{-i}^* = (\mu^{\overline{\lambda}})^{-1}(\mu^*) < \infty$ and $\overline{T}_{-i} < (\mu^{\overline{\lambda}})^{-1}(\overline{\mu}) < \infty$, where $\mu^* \in (\pi_-, \pi_0]$ and $\overline{\mu} \in (\pi_-, \pi_0]$ are independent of $i$.*

This equilibrium is the analog of the leader-follower equilibrium characterized in Proposition 5 for complementary signals. The intuition for existence of this equilibrium is very similar to the complements case. The one complication is that when the follower is active, there is some incentive for the leader to free-ride off the follower. However, this incentive is always weaker than the incentive for the follower to free-ride off the leader. Thus whenever the follower is optimally active, so is the leader.

The following lemma establishes important properties of the follower's strategy, including key comparative statics in costs and the discount rate:

**Lemma 5.** *In the leader-follower equilibrium with firm $i$ as the leader, the follower's investment and prospecting thresholds $T_{-i}^*$ and $\overline{T}_{-i}$ satisfy the following properties:*

- $\min\{T_{-i}^*, \overline{T}_{-i}\} > 0$ *for $\overline{\lambda}/r$ sufficiently small.*
- $\min\{T_{-i}^*, \overline{T}_{-i}\} < T^A$,
- $T_{-i}^*$ *is independent of* $c$,
- $\overline{T}_{-i}$ *is decreasing in $c$, and strictly decreasing whenever $\overline{T}_{-i} > 0$,*
- *If $\overline{\lambda}/r$ is sufficiently large, then $\overline{T}_{-i} = 0$ for $c$ sufficiently large,*[18]
- $\overline{T}_{-i} > T^A > T_{-i}^*$ *for $c$ sufficiently small.*

The first result of Lemma 5 illustrates a qualitative feature of equilibrium under substitutes which does not arise in the complements case - the follower does not necessarily wait upon acquiring a High signal early on in the game. In particular, $T_{-i}^* > 0$ (recall that $T_{-i}^*$ is the time at which the follower being waiting after acquiring a High signal) when firms are impatient. The structure of prospecting and investing prior to observing investment by the other firm under such parameters is represented diagrammatically in Figure 6.

The possibility that $T_{-i}^* > 0$ means that the follower may transition from an active to a passive role as the equilibrium folds. This transition arises whenever $\min\{\overline{T}_{-i}, T_{-i}^*\} > 0$,

[18] More precisely, there a range of costs whose upper limit is $\overline{c}$ on which $\overline{T}_{-i} = 0$.

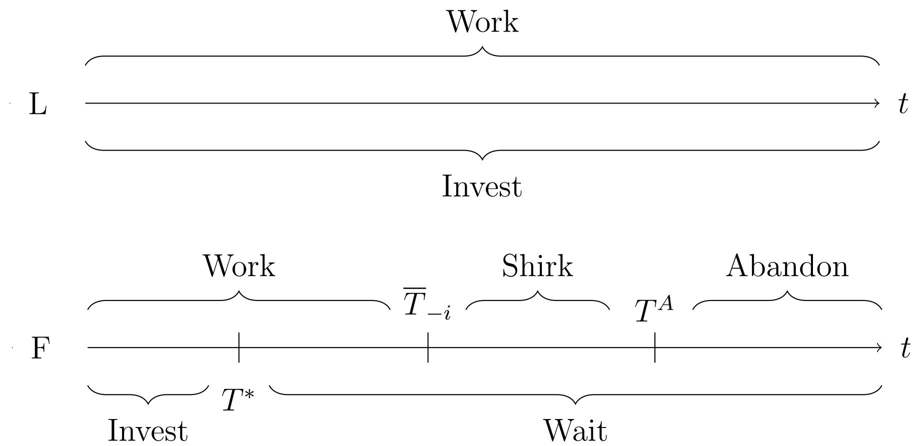


Figure 6: A timeline of prospecting and investing in the leader-follower equilibrium

an inequality holding if firms are sufficiently impatient, i.e. $\overline{\lambda}/r$ is sufficiently small. In the "active" phase, when $t < \min\{\overline{T}_{-i}, T^*_{-i}\}$, the follower works to obtain a signal and invests immediately just as the leader does. Afterward the follower becomes "passive", never investing ahead of the leader on the equilibrium path. Note that just as in the complements case, frontrunning still arises whenever costs are sufficiently low - even in the passive phase the follower may acquire a signal which it then waits to act on. The evolution of beliefs in the leader-follower equilibrium under substitutes is illustrated in figure 7.

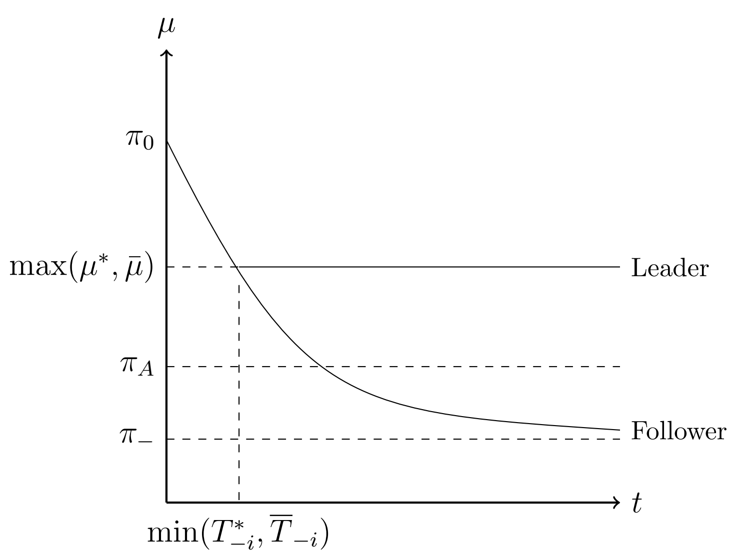


Figure 7: Evolution of beliefs in the leader-follower equilibrium under substitutes

The remainder of Lemma 5 establishes that as in the complements case, the follower either shirks, delays investment, or both at some point along the equilibrium path, with at least one form of delay guaranteed by the inequality $\min\{T^*_{-i}, \overline{T}_{-i}\} < T^A$. And as in the complements case, the mix of shirking and investment delay depends on the cost of prospecting. When costs are high (and firms are sufficiently patient) the follower shirks but does not delay investment on the equilibrium path. When costs are intermediate the follower initially delays investment on the interval $[T^*_{-i}, \overline{T}_{-i}]$, and then shirks on the interval $[\overline{T}_{-i}, T^A]$. And when costs are sufficiently low the follower delays investment on the interval $[T^*_{-i}, \overline{T}_{-i}]$ but never shirks, and in fact exerts effort at beliefs for which effort would be strictly optimal in a one-player setting.

Just as in the complements case, the equilibria characterized by Propositions 5 and 6 are the only perfect Bayesian equilibria of the model. This result is stated formally in the following proposition.

**Proposition 7.** *Suppose signals are substitutes. There exist no perfect Bayesian equilibria, in pure or mixed strategies, beyond those characterized in Propositions 5 and 6.*

The proof of this result is very similar to the complements case. The only difference stems from the optimality condition characterizing when the follower stops investing on-path in equilibrium. In the complements case this time was always 0. In the substitutes case, the optimal halting time is still uniquely determined, but need not be zero.

## 4.2 Welfare

We saw in Section 3.5 that when signals are complements, the leader-follower equilibrium Pareto dominates the symmetric equilibrium, and that the symmetric equilibrium in turn does not improve welfare over the autarky outcome. We now show that when signals are substitutes, the welfare comparison between the two equilibria is less clear-cut.

**Proposition 8.** *Suppose signals are substitutes. Then $V^F > V^S > V^L \geq V^A$, and $V^L > V^A$ iff $\min\{\overline{T}, T^*\} > 0$. When $r$ is sufficiently small, $V^L + V^F > 2V^S$. There exists a $\underline{c} < \overline{c}$ (independent of $r$ and $c$) such that if $c > \underline{c}$, then $2V^S > V^L + V^F$ for $r$ sufficiently large.*

One important distinction between welfare under complements and substitutes is that under substitutable signals, the symmetric equilibrium strictly improves on autarky. This is because observing investment at *any* time improves a firm's welfare, due to the unique optimality of immediate follow-on investment. In other words, in the substitutes case observing investment is pivotal in changing behavior from prospecting (or abandonment) to investment. In the complements case, by contrast, observing investment is pivotal only in changing behavior from abandonment to prospecting, which never occurs in the symmetric equilibrium. The essential point is that when the firm cares more about prospecting and delay costs than reduced signal precision, it has more scope for changing behavior to reap gains from social learning.

This expanded scope for positive information externalities also explains why in the substitutes case the leader is strictly worse off than it would have been had the symmetric equilibrium been played. This is because the follower becomes passive at a time $\min\{\overline{T}, T^*\}$ which is strictly earlier than $T^A$ by Proposition 6. Hence in the leader-follower equilibrium the leader enjoys a strictly shorter window for profitable information externalities than it would have in the symmetric equilibrium, meaning $V^S > V^L$. In the extreme case where $\min\{\overline{T}, T^*\} = 0$, e.g. if $\overline{\lambda}/r$ is sufficiently large, then the leader is effectively in autarky and receives the autarky payoff, in which case $V^L = V^A$.

Given that the leader is strictly worse off than in the symmetric equilibrium, the two equilibria are no longer Pareto-ranked. Continuity of equilibrium payoffs in model parameters implies that if signals are not *too* substitutable, then it will continue to be true that $V^L + V^F > 2V^S$. In particular, if the discount rate $r$ isn't too high, then the payoff loss to the leader from not observing investment in the time interval $[\min\{\overline{T}, T^*\}, T^A]$ isn't very large and aggregate welfare is higher under the leader-follower equilibrium. However, when costs are sufficiently high, for very impatient firms the aggregate welfare comparison reverses and firms do better overall in the symmetric equilibrium.

To understand this result, recall that in the symmetric equilibrium, the key inefficiency is eventual shutdown when neither firm obtains a signal early enough, generating expected flow losses versus an efficient benchmark beginning at time $T^A$. On the other hand, in the substitutes case the follower generates an inefficiency beginning at time $\min\{\overline{T}, T^*\} < T^A$ by failing to internalize the lost benefit to the leader when it chooses to remain passive.[19] This latter inefficiency, which arises at earlier times than the shutdown inefficiency, can be strong enough to dominate when prospecting costs are high[20] and the discount rate is high enough.

[19] In particular, unlike in the complements case, investment by one firm provides information which *always* boosts the other firm's payoffs in case they hadn't yet obtained a signal, no matter their current beliefs. Thus even a small increase in $\min\{\overline{T}, T^*\}$ would have a first-order impact on the leader's payoffs in the substitutes case. Such a change would boost overall efficiency given the second-order loss to the follower's payoffs when $\overline{T}$ and $T^*$ are initially at their equilibrium, hence follower-optimal, levels.

[20] This cost constraint need not be very stringent. In the proof of the proposition, we show that there are model parameters under which $\underline{c} < 0$.

# 5 Conclusion

We study a model of strategic investment timing with endogenous information acquisition, with the aim of understanding the interplay of incentives for free-riding and investment delay present in settings such as the market for entrepreneurial fundraising. We find that neither free-riding nor investment delay arise in the unique symmetric equilibrium of a two-firm model, but a novel discouragement effect arises which reduces welfare by inducing occasional inefficient abandonment of the project. By contrast, free-riding, investment delay, or both always arise in the remaining leader-follower equilibria, with the mix determined by the costs of prospecting, tilting toward free-riding as costs rise.

We find that when firms are patient, the discouragement effect arising in the symmetric equilibrium is sufficiently costly that the leader-follower equilibrium yields higher aggregate welfare than or even Pareto dominates the symmetric equilibrium, regardless of prospecting costs. Thus in situations of investing with social learning, firms may benefit from coordinating on a market leader who investigates and invests first, despite the incentive this gives other firms to free-ride and delay investment.

One natural extension of our model would be to consider asymmetric firms, with one firm possessing an advantage either in prospecting costs or signal precision. Our conjecture is that if heterogeneity is not too large, the equilibrium set would look qualitatively similar to the symmetric case, with two leader-follower equilibria and one "symmetric" equilibrium in which both firms eventually become discouraged and abandon the project simultaneously. The efficiency gains of the leader-follower equilibrium with the efficient firm as the leader over the symmetric equilibrium would be naturally enhanced by heterogeneity. When heterogeneity is sufficiently large, we conjecture that the leader-follower equilibrium with the efficient firm as the leader would emerge as the unique equilibrium.[21]

It would also be interesting to extend our model to capture potential payoff externalities often present in investment environments. For instance, moving first may yield some advantage relative to following, or innovations may pay off more when they are more widely adopted. Another extension would be to incorporate more firms, in particular to investigate whether efficient equilibria in large markets might involve only a few active investigators and a large fringe of passive firms who don't prospect at all.

[21] See also Awaya and Krishna (2019) for a related model of patent races in which the more-informed firm always takes an irreversible public action first in equilibrium when the gap in information is large.

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# Appendices

## A Regular strategies

One crucial element of any equilibrium is the inference made by a firm based on investment, or lack of investment, by the other firm. Intuitively, it should be the case that lack of investment is (weakly) bad news about the state, while investment indicates that the other firm has received a High signal. In this appendix we establish that such behavior is exhibited by any perfect Bayesian equilibrium.

**Definition A.1.** *A firm's strategy is* regular *if:*

- *Investment never occurs after receipt of a Low signal,*
- *Investment without a signal occurs only in histories in which the other firm has invested.*

Regular strategies treat receipt of a Low signal as a terminal node of the game, with the firm abandoning the project at that point. They also ignore the opportunity to invest until some information about the project's value has been received, either by receiving a signal or observing investment.

**Lemma A.1.** *In any perfect Bayesian equilibrium, each firm's strategy is regular.*

*Proof.* First consider a firm who has received a Low signal. Then regardless of his beliefs about the content of any signal received by the other firm, his posterior belief that the state is Good cannot be higher than $\pi_0$. As $\pi_0 R - 1 < 0$ by assumption, investing is never optimal at any point in the future.

To establish the second property of regularity, we first show that in equilibrium firms are always able to use Bayes' rule to update their beliefs about the state no matter the history of the game. Fix an equilibrium $\sigma$, and suppose by way of contradiction that at some time $t^*$ and following some history, 1) neither firm has invested by time $t^*$; 2) Bayes' rule applies for both firms at all $t < t^*$, but at $t^*$ firm $i$ cannot use Bayes' rule to form a posterior probability about $\theta$. These two conditions imply that according to $\sigma$, firm $-i$ should have invested with probability 1 prior to $t^*$ absent any investment by firm $i$. (Otherwise Bayes' rule would still be applicable at time $t^*$.) We already know that in any PBE, firm $-i$ will never invest if in possession of a Low signal. Therefore if firm $-i$ had a Low signal with some probability by time $t^*$, Bayes' rule would still apply for firm $i$. Hence firm $-i$ cannot have obtained a signal, hence cannot have prospected prior to $t^*$. In this case any updating to firm $-i$'s beliefs prior to $t^*$ must come solely from social learning due to the absence of firm $i$'s investment.

Bayes' rule applies for firm $-i$ at all $t < t^*$, meaning that with some probability under $\sigma^i$, firm $i$ did not invest prior to $t^*$. And we already know that in any PBE, firm $i$ never invests when in possession of a Low signal. Thus no matter how frequently it would invest when in possession of no signal or a High signal, the inference $-i$ must make about the state from lack of investment is weakly negative. Hence firm $-i$'s beliefs at all times prior to $t^*$ must be no higher than $\pi_0$, meaning the payoff from investing is no higher than $R\pi_0 - 1 < 0$. This contradicts the assumption that $\sigma$ is an equilibrium.

So under any equilibrium, if neither firm has invested by time $t^*$, and Bayes' rule applies for both firms at all prior times, then each firm must be able to use Bayes' rule to form beliefs at time $t^*$. Since this reasoning applies for every $t^*$ and every history, it must be that Bayes' rule applies for both firms at all times following all histories in which neither firm has invested. It then follows from the argument of the previous two paragraphs that a firm in possession of no signal and seeing no investment must at all times have beliefs no higher than $\pi_0$, and hence must never find immediate investment profitable. □

The intuition for this lemma is very simple. Once a Low signal has been received, a firm's posterior beliefs about the state can never rise above $\pi_{+-}$, no matter its inference about the other firm's information. Then as $\pi_{+-} < 1/R$ by assumption, it must never invest once in possession of a Low signal. Given this, observing lack of investment by the other firm must be weakly bad news about the state, so a firm's beliefs absent a signal or observation of investment cannot rise above $\pi_0$ in equilibrium. As $\pi_0 < 1/R$, investment can't be optimal at this point in the game either.

In light of Lemma A.1, in any perfect Bayesian equilibrium investment by the other firm indicates that it must have been in possession of a High signal. Thus lack of investment cannot be good news about the state, because under any regular strategy firms with no signal, or a Low signal, are at least as likely not to have invested yet as a firm with a High signal. We will freely invoke these properties of equilibrium in what follows, and will not explicitly specify investment behavior when a firm has no signal or a Low signal.

# B Belief updating identities

In this appendix we derive several useful identities involving posterior beliefs about the state in the event no investment by the other firm has been observed. These identities will be used in proofs elsewhere in the paper.

Fix a firm $i$ and any regular strategy for firm $-i$, and suppose that firm $-i$ does not randomize over its prospecting intensity and invests immediately upon receiving a High signal at all times. Let $\Omega^{-i}(t)$ be the (ex ante) probability that $-i$ has received no signal by time $t$. Then

$$\Omega^{-i}(t) = \exp\left(-\int_0^t \lambda^{-i}(s)\,ds\right).$$

**Lemma B.1.** *Suppose some firm $-i$ uses a regular strategy involving non-random prospecting and immediate investment upon receipt of a High signal. Then for almost every time $t$,*

$$\frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)} = -\lambda^{-i}(t)\frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-}.$$

*Proof.* Differentiating the definition of $\Omega^{-i}(t)$ yields

$$\dot{\Omega}^{-i}(t) = -\lambda^{-i}(t)\Omega^{-i}(t).$$

Meanwhile, by Bayes' rule

$$\mu^i(t) = \frac{(\Omega^{-i}(t) + (1-\Omega^{-i}(t))(1-q^H))\pi_0}{(\Omega^{-i}(t) + (1-\Omega^{-i}(t))(1-q^H))\pi_0 + (\Omega^{-i}(t) + (1-\Omega^{-i}(t))q^L)(1-\pi_0)}.$$

Using the identities $\pi_- = (1-q^H)\pi_0/l(\pi_0)$ in the numerator and $l(\pi_0) = (1-q^H)\pi_+ + q^L(1-\pi_0)$ in the denominator, this expression may be rewritten

$$\mu^i(t) = \frac{\Omega^{-i}(t)\pi_0 + (1-\Omega^{-i}(t))l(\pi_0)\pi_-}{\Omega^{-i}(t) + (1-\Omega^{-i}(t))l(\pi_0)}.$$

Solving this identity for $\Omega^{-i}(t)$ yields

$$\Omega^{-i}(t) = \frac{l(\pi_0)}{h(\pi_0)}\frac{\mu^i(t) - \pi_-}{\pi_+ - \mu^i(t)}.$$

Differentiating this expression and eliminating $\dot{\Omega}^{-i}(t)$ using the identity derived above yields the desired relationship. □

(Note that $\mu^i$ is not necessarily differentiable everywhere, as $\lambda^{-i}$ need not be continuous. However, it is certainly absolutely continuous, and lack of differentiability on a set of times of measure zero will not impact our analysis at any point.)

**Lemma B.2.** *For every* $\mu \in [\pi_-, \pi_+]$,

$$h(\pi_+)\frac{\mu - \pi_-}{\pi_+ - \pi_-} = h(\mu)\frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}}.$$

*Proof.* Note that both the lhs and rhs of the identity in the lemma statement are affine functions of $\mu$. (The lhs is immediate, while the numerator of the rhs may be rewritten $q^H\mu - \pi_{+-}h(\mu)$, which is affine in $\mu$ given that $h(\mu)$ is.) It is therefore enough to show that they coincide at two distinct values of $\mu$. Note that when $\mu = \pi_-$, both sides vanish, while when $\mu = \pi_+$, both sides reduce to $h(\pi_+)$, as desired. □

**Lemma B.3.**

$$h(\pi_+) = \frac{\pi_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}}.$$

*Proof.* Using the Bayes' rule identities $\pi_{++} = q^H\pi_+/h(\pi_+)$ and $\pi_{+-} = (1 - q^H)\pi_+/l(\pi_+)$, some algebra yields

$$\begin{aligned}
\frac{\pi_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}} &= 1 - \frac{\pi_{++} - \pi_+}{\pi_{++} - \pi_{+-}} \\
&= 1 - \frac{\frac{q^H\pi_+}{h(\pi_+)} - \pi_+}{\frac{q^H\pi_+}{h(\pi_+)} - \frac{(1-q^H)\pi_+}{l(\pi_+)}} \\
&= 1 - l(\pi_+)\frac{q^H - h(\pi_+)}{q^H l(\pi_+) - (1 - q^H)h(\pi_+)} \\
&= 1 - l(\pi_+) = h(\pi_+).
\end{aligned}$$

□

**Lemma B.4.** *Suppose some firm* $-i$ *uses a regular strategy involving immediate investment upon receipt of a High signal. Then for any time* $t$,

$$\Pr(S^{-i} = H \mid S^i = H, \tau^{-i}(\emptyset) \geq t) = \frac{\mu^i_+(t) - \pi_{+-}}{\pi_{++} - \pi_{+-}}.$$

*Proof.* By Bayes' rule,

$$\Pr(S^{-i} = H \mid S^i = H, \tau^{-i}(\emptyset) \geq t) = \frac{\Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H, S^{-i} = H)\Pr(S^{-i} = H \mid S^i = H)}{\Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H)}.$$

Now, $\Pr(S^{-i} = H \mid S^i = H) = h(\pi_+)$ by definition of $h$. And if $-i$'s signal is High, the only event in which he has not invested by time $t$ is when he has not yet received a signal:

$$\Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H, S^{-i} = H) = \Omega^{-i}(t).$$

Meanwhile, by the law of total probability,

$$\begin{aligned} & \Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H) \\ = \; & \Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H, S^{-i} = H) \Pr(S^{-i} = H \mid S^i = H) \\ & + \Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H, S^{-i} = L) \Pr(S^{-i} = L \mid S^i = H), \end{aligned}$$

and since a firm whose signal is low never invests, $\Pr(\tau^{-i}(\emptyset) \geq t \mid S^i = H, S^{-i} = L) = 1$ while $\Pr(S^{-i} = L \mid S^i = H) = l(\pi_+)$. Thus

$$\Pr(S^{-i} = H \mid S^i = H, \tau^{-i}(\emptyset) \geq t) = \frac{\Omega^{-i}(t) h(\pi_+)}{\Omega^{-i}(t) h(\pi_+) + l(\pi_+)}.$$

Meanwhile, another application of Bayes' rule yields

$$\mu^i_+(t) = \frac{(\Omega^{-i}(t) + (1 - \Omega^{-i}(t))(1 - q^H))\pi_+}{(\Omega^{-i}(t) + (1 - \Omega^{-i}(t))(1 - q^H))\pi_+ + (\Omega^{-i}(t) + (1 - \Omega^{-i}(t))q^L)(1 - \pi_+)}.$$

Using the identities $\pi_{+-} = (1 - q^H)\pi_+ / l(\pi_+)$ in the numerator and $l(\pi_+) = (1 - q^H)\pi_+ + q^L(1 - \pi_+)$ in the denominator, this expression may be rewritten

$$\mu^i_+(t) = \frac{\Omega^{-i}(t)\pi_+ + (1 - \Omega^{-i}(t)) l(\pi_+)\pi_{+-}}{\Omega^{-i}(t) + (1 - \Omega^{-i}(t)) l(\pi_+)}.$$

This representation allows us to write

$$\frac{\mu^i_+(t) - \pi_{+-}}{\pi_{++} - \pi_{+-}} = \frac{\pi_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}} \frac{\Omega^{-i}(t)}{\Omega^{-i}(t) + (1 - \Omega^{-i}(t)) l(\pi_+)}.$$

Now use the identity

$$h(\pi_+) = \frac{\pi_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}}$$

derived in Lemma B.3 to obtain

$$\frac{\mu^i_+(t) - \pi_{+-}}{\pi_{++} - \pi_{+-}} = \frac{\Omega^{-i}(t) h(\pi_+)}{\Omega^{-i}(t) h(\pi_+) + l(\pi_+)}.$$

This is exactly the representation of $\Pr(S^{-i} = H \mid S^i = H, \tau^{-i}(\emptyset) \geq t)$ derived above, completing the proof. ☐

**Lemma B.5.** *Suppose some firm $-i$ uses a regular strategy involving non-random prospecting and immediate investment upon receipt of a High signal. Then for almost every time $t$,*

$$\Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t)\lambda^{-i}(t) = -\frac{1}{h(\pi_0)} \frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}.$$

*Proof.* By Bayes' rule,

$$\Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t) = \frac{\Omega^{-i}(t)}{\Omega^{-i}(t) + (1 - \Omega^{-i}(t))l(\pi_0)}.$$

In the proof of Lemma B.1, we established the identity

$$\Omega^{-i}(t) = \frac{l(\pi_0)}{h(\pi_0)} \frac{\mu^i(t) - \pi_-}{\pi_+ - \mu^i(t)}.$$

Substituting into the previous expression to eliminate $\Omega^{-i}(t)$ in favor of $\mu^i(t)$ yields

$$\Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t) = \frac{1}{h(\pi_0)} \frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-}.$$

Multiplying through by $\lambda^{-i}(t)$ and using the identity derived in Lemma B.1 yields the expression in the lemma statement. ☐

# C The HJB equation

In this appendix we characterize the HJB equation determining each firm's optimal prospecting rule given a regular strategy by the other firm involving non-random prospecting and a threshold investment rule.[22] Note that Lemma A.1 establishes that each firm uses a regular strategy in equilibrium. Meanwhile, Lemma D.12 establishes that in equilibrium each firm uses an investment strategy outcome-equivalent to a threshold investment strategy. So this case is the relevant one for equilibrium analysis.

Let $\overline{V}$ be $i$'s continuation value upon seeing firm $-i$ invest. Given that $-i$ uses a regular strategy, $i$'s posterior beliefs from this point onward are fixed at $\pi_+$ so long as $i$ has not

[22]By threshold investment rule, we mean that there is a cutoff time before which the firm invests immediately upon obtaining a High signal, and after which the firm never invests first. This is a generalization of the threshold strategy concept, requiring only investment but not prospecting to exhibit a bang-bang structure.

acquired a signal. As in the single-player problem, $i$'s optimal continuation strategy in this contingency is either to invest immediately, or to acquire an additional signal and invest iff that signal is High. Therefore

$$\overline{V} = \max\left\{\pi_+ R - 1, \frac{\overline{\lambda}}{\overline{\lambda}+r}(h(\pi_+)(\pi_{++}R - 1) - c)\right\}$$

Let $V^i(t)$ be firm $i$'s equilibrium continuation value function conditional on receiving no signal and seeing no investment by firm $-i$ up to time $t$. Let $T^*_{-i}$ be the time at which firm $-i$ begins waiting upon receiving a High signal. For all times $t \geq T^*_{-i}$, the firm's beliefs are fixed at $\mu^i(T^*_{-i})$, and $V^i(t)$ is the value of the corresponding single-player problem. So consider times $t < T^*_{-i}$. By standard arguments, $V^i$ is an absolutely continuous function satisfying the HJB equation

$$\begin{aligned} rV^i(t) = & \max_{\lambda \in [0,\overline{\lambda}]} \left\{\lambda\left(\widetilde{V}^i(t) - c - V^i(t)\right)\right\} \\ & + \Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t)\lambda^{-i}(t)h(\pi_0)(\overline{V} - V^i(t)) + \dot{V}^i(t). \end{aligned}$$

for almost all times $t < T^*_{-i}$, where $\widetilde{V}^i(t)$ is firm $i$'s continuation value after having received a signal at time $t$. ($V^i$ need not be differentiable whenever $\mu^i$ is not, but this occurs at most on a set of times of measure 0 and does not affect the equilibrium analysis in any way.)

The first term on the rhs is the contribution to the firm's value from obtaining a signal, while the second is the contribution from the observation of investment by the other firm. Note that in writing down the HJB equation, we have used the fact that firm $i$ does not invest absent either obtaining a signal or observing the other firm invest, as required by any regular strategy and thus any best response.

Using Lemma B.5 to eliminate $\Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t)$, the HJB equation may be written

$$rV^i(t) = \max_{\lambda \in [0,\overline{\lambda}]} \left\{\lambda\left(\widetilde{V}^i(t) - c - V^i(t)\right)\right\} - \frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}(\overline{V} - V^i(t)) + \dot{V}^i(t).$$

Further, the maximization on the rhs is over a linear function of $\lambda$, so the HJB equation may be more compactly written

$$rV^i(t) = \overline{\lambda}\left(\widetilde{V}^i(t) - c - V^i(t)\right) - \frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}(\overline{V} - V^i(t)) + \dot{V}^i(t),$$

with the associated maximizer correspondence

$$\lambda^*(t) = \begin{cases} 0, & \widetilde{V}^i(t) - c < V^i(t) \\ [0, \overline{\lambda}], & \widetilde{V}^i(t) - c = V^i(t) \\ \overline{\lambda}, & \widetilde{V}^i(t) - c > V^i(t). \end{cases}$$

Suppose that following receipt of a High signal at some time $t$, it is optimal for firm $i$ to invest immediately. Then

$$\widetilde{V}^i(t) - c = h(\mu^i(t))(\mu^i_+(t)R - 1) - c,$$

where the calculation of $\widetilde{V}^i$ takes into account the ex post probability that the signal is High.

**Lemma C.1.** *For any* $\mu$,

$$h(\mu)(\mu_+ R - 1) - c = K(\mu - \pi_A),$$

*where* $\mu_+ = q^H \mu / h(\mu)$ *and*

$$K \equiv q^H(R-1) + (1 - q^L) > 0.$$

*Proof.* Some algebra shows that

$$h(\mu)(\mu_+ R - 1) - c = (q^H(R-1) + (1-q^L))\left(\mu - \frac{(1-q^L)+c}{q^H(R-1)+(1-q^L)}\right)$$

Recall from Lemma 1 that $\pi_A = ((1-q^L)+c)/(q^H(R-1)+(1-q^L))$, establishing the identity. □

The following lemma establishes an important upper bound on $\overline{V}$ which is used in our characterization of the equilibrium set.

**Lemma C.2.** $\overline{V} \leq K(\pi_+ - \pi_-)$.

*Proof.* If signals are complements, then $\overline{V} = \frac{\overline{\lambda}}{\overline{\lambda}+r} K(\pi_+ - \pi_A)$, in which case $\overline{V} < K(\pi_+ - \pi_A)$. If signals are substitutes, then $\overline{V} = \pi_+ R - 1$, and so by Assumption 4, $\overline{V} \leq K(\pi_+ - \pi_A)$. □

# D Proofs

## D.1 Proof of Lemma 1

The firm's expected profit from investing after obtaining a High signal is $\mu_+ R - 1$, where $\mu_+ = q^H \mu / h(\mu)$. The expected continuation value of obtaining a signal is therefore $h(\mu)(\mu_+ R - 1)$, where the rewards of a High signal are discounted by the probability $h(\mu)$ that the signal is actually High. (In the event of a Low signal, the firm receives nothing.) Prospecting for a signal is then profitable if and only if the value of obtaining a signal exceeds its flow cost $c$. The threshold $\pi_A$ is therefore the unique solution to the equation

$$h(\mu)(\mu_+ R - 1) = c,$$

which after distributing the leading term $h(\mu)$ reduces to a linear equation in $\mu$. Some algebra yields the explicit solution reported in the lemma statement.

## D.2 Proof of Lemma 2

We first establish that investment delay is not possible in equilibrium when $c > \bar{c}$ Fix a perfect Bayesian equilibrium, and consider any time $t$ and information set in which some firm $i$ has not seem firm $-i$ invest. As established in Appendix C (whose results, aside from Lemma C.2, hold whether or not Assumption 4 is imposed), prospecting on the equilibrium path is optimal for $i$ iff $\widetilde{V}^i(t) - V^i(t) \geq c$, where $V^i(t)$ is firm $i$'s time-$t$ continuation value prior to acquiring a signal, while $\widetilde{V}^i(t)$ is its continuation value afterward.

Suppose that at time $t$, firm $i$ optimally defers investment upon obtaining a High signal. Note that Lemmas A.1 and D.12 hold regardless of whether Assumption 4 is imposed. So it must be that firm $i$ optimally defers investment indefinitely upon obtaining a High signal. (This may not be the unique optimal strategy, but it suffices that this be one optimal strategy.) In this case $\widetilde{V}^i(t)$ is the probability that $i$'s signal is High while firm $-i$ has not yet obtained a signal, times the expected discounted value of investment $\pi_{++}R - 1$ following investment by firm $-i$ supposing that it eventually obtains a High signal as well:

$$\widetilde{V}^i(t) = \Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t) h(\pi_0) h(\pi_+)(\pi_{++}R - 1)\, \mathbb{E}\left[e^{-r(\tau^{-i}(\emptyset) - t)} \mid s_t^{-i} = \emptyset, s_{\nu^{-i}}^{-i} = H\right] \Bigg($$

(Note that the expectation need not be conditioned on the event $s_t^i = H$, as conditional on $s_{\nu^{-i}}^{-i} = H$ the distribution of $\tau^{-i}(\emptyset)$ is independent of $s^i$.)

Meanwhile, a lower bound on $V^i(t)$ can be obtained by noting that one feasible continuation strategy for firm $i$ is to perform no prospecting going forward, and to invest whenever $-i$ does. Thus

$$V^i(t) \geq \Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t) h(\pi_0)(\pi_+ R - 1) \left[ e^{-r(\tau^{-i}(\emptyset)-t)} \mid s_t^{-i} = \emptyset, s_{\nu^{-i}}^{-i} = H \right] \Big($$

Combining the characterization of $\widetilde{V}^i(t)$ with the lower bound on $V^i(t)$ yields

$$\begin{aligned} \widetilde{V}^i(t) - V^i(t) \leq \ & \Pr(s_t^{-i} = \emptyset \mid \tau^{-i}(\emptyset) \geq t) h(\pi_0) \left[ e^{-r(\tau^{-i}(\emptyset)-t)} \mid s_t^{-i} = \emptyset, s_{\nu^{-i}}^{-i} = H \right] \Big( \\ \Big( & \times (h(\pi_+)(\pi_{++} R - 1) - (\pi_+ R - 1)). \end{aligned}$$

By hypothesis, the final term is bounded strictly above by $c$. Thus $\widetilde{V}^i(t) - V^i(t) < c$, implying that prospecting at time $t$ is strictly suboptimal at time $t$. So at no time when deferring investment is optimal does firm $i$ prospect on the equilibrium path.

We now establish that at least one firm must shirk at some point in equilibrium. Suppose that some firm, say firm 1, never shirks in equilibrium. It is enough to show that firm 2 must shirk over some time interval. Recall by the result just proven that firm 1 also never delays investment. Thus firm 2's beliefs must evolve as $\mu^{\overline{\lambda}}$ until time $T^A$, where these objects are as defined prior to Proposition 1. As firm 2 also never delays investment, its HJB equation characterizing its prospecting decision for times prior to $T^A$ is then

$$rV^2(t) = \max_{\lambda \in [0,\overline{\lambda}]} \left\{ \lambda \left( \Big( K(\mu^{\overline{\lambda}}(t) - \pi_A) - V^2(t) \right) \right\} \Big( - \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)} (\overline{V} - V^2(t)) + \dot{V}^2(t).$$

(See Appendix C for a derivation.) Note that when $c > \overline{c}$ it must be that

$$\frac{\overline{\lambda}}{\overline{\lambda} + r}(h(\pi_+)(\pi_{++} R - 1) - c) < \frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_+ R - 1) < \pi_+ R - 1,$$

and therefore $\overline{V} = \pi_+ R - 1$. We will establish that for $t < T^A$ sufficiently close to $T^A$, $V^2(t) > K(\mu^{\overline{\lambda}}(t) - \pi_A)$, implying that firm 2 must shirk for an interval of time in any best response to any firm 1 strategy involving no shirking.

Toward this end, define the functional

$$F(w, t) \equiv rw - \overline{\lambda}(K(\mu^{\overline{\lambda}}(t) - \pi_A) - w(t))_+ + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}(\pi_+ R - 1 - w(t)) - \dot{w}(t).$$

Let $V^\dagger(t) \equiv K(\mu^{\overline{\lambda}}(t) - \pi_A)$. We will show that for $t < T^A$ sufficiently close to $T^A$, $F(V^\dagger, t) < 0$.

Evaluating the functional at $V^\dagger$ yields

$$F(V^\dagger, t) = K(\mu^{\bar{\lambda}}(t) - \pi_A) + \frac{\dot{\mu}^{\bar{\lambda}}(t)}{\pi_+ - \mu^{\bar{\lambda}}(t)}(\pi_+ R - 1 - K(\pi_+ - \pi_A)).$$

Now, $K(\pi_+ - \pi_A) = h(\pi_+)(\pi_{++}R - 1) - c$, which given $c > \bar{c}$ is strictly less than $\pi_+ R - 1$. Then as $\dot{\mu}^{\bar{\lambda}}(t) < 0$ for all $t$ and $\mu^{\bar{\lambda}}(T^A) = \pi_A$ by definition of $T^A$, it must be that

$$F(V^\dagger, T^A) = \frac{\dot{\mu}^{\bar{\lambda}}(T^A)}{\pi_+ - \mu^{\bar{\lambda}}(T^A)}(\pi_+ R - 1 - K(\pi_+ - \pi_A)) < 0.$$

By continuity, $F(V^\dagger, t)$ is also strictly negative for $t$ sufficiently close to $T^A$.

The previous result establishes that $V^\dagger$ is a strict subsolution to the HJB equation for times close to $T^A$. Further, $V^\dagger(T^A) = 0 \leq V^2(T^A)$, as firm 2's value function cannot be negative. It follows that for times $t$ close to $T^A$, $V^2(t) > V^\dagger(t)$, as desired.

## D.3 Proof of Lemma 3

Upon observing firm $-i$ invest, Lemma A.1 implies that firm $i$'s beliefs that the state is Good are $\pi_+$ if $s^i_t = \emptyset$. The continuation payoff from investing immediately is $\pi_+ R - 1$. Meanwhile receiving a signal has expected continuation payoff $h(\pi_+)(\pi_{++}R - 1)$, as the firm invests only if the signal is High. Let $V$ be firm $i$'s continuation value from an optimal policy. Then $V$ satisfies the HJB equation

$$rV = \max\left\{ r(\pi_+ R - 1), \max_{\lambda \in [0, \bar{\lambda}]} \{\lambda(h(\pi_+)(\pi_{++}R - 1) - c - V)\} \right\}$$

Suppose first that $\pi_+ R - 1 \geq \frac{\bar{\lambda}}{\bar{\lambda}+r}(h(\pi_+)(\pi_{++}R - 1) - c)$. Then as $V \geq \pi_+ R - 1$, it must be that

$$h(\pi_+)(\pi_{++}R - 1) - c - V \leq \frac{r}{\bar{\lambda}}(\pi_+ R - 1),$$

so the second term on the rhs of the HJB equation is weakly smaller than $r(\pi_+ R - 1)$, and is strictly smaller except in the knife-edge case. Hence $V = \pi_+ R - 1$ is the unique solution to the HJB equation, with immediate investment a corresponding optimal policy. Except in the knife-edge case this is the unique optimal policy; in the knife-edge case the two arguments of the max are equal, hence any stopping rule for investing prior to receiving a signal is optimal, so long as prospecting is undertaken at rate $\bar{\lambda}$.

On the other hand, suppose that $\pi_+ R - 1 < \frac{\bar{\lambda}}{\bar{\lambda}+r}(h(\pi_+)(\pi_{++}R - 1) - c)$. If $V = \pi_+ R - 1$

then

$$h(\pi_+)(\pi_{++}R-1)-c-V>\frac{r}{\overline{\lambda}}(\pi_+R-1),$$

so that the second term on the rhs of the HJB equation is strictly greater than $r(\pi_+R-1)$, a contradiction. Hence

$$rV=\max_{\lambda\in[0,\overline{\lambda}]}\{\lambda(h(\pi_+)(\pi_{++}R-1)-c-V)\}$$

and the rhs is strictly greater than $r(\pi_+R-1)$, meaning $\lambda=\overline{\lambda}$. Solving for $V$ yields

$$V=\frac{\overline{\lambda}}{\overline{\lambda}+r}(h(\pi_+)(\pi_{++}R-1)-c),$$

which by assumption is strictly greater than $r(\pi_+R-1)$. Thus this value function is the unique solution to the HJB equation, and the associated optimal policy is to prospect at rate $\overline{\lambda}$ until receiving a signal.

## D.4 Proof of Propositions 1 and 5

Fix a firm $i$, and consider any continuation game in which it has already obtained a High signal. Because $\mu^i\geq\pi_A$, therefore $\mu^i_+(t)>1/R$ for all time. So investment is always profitable at each future time, regardless of whether the other firm has invested or not. Therefore the payoff of any investment strategy which occasionally never invests is dominated by the payoff of an investment strategy which always eventually invests, and due to time discounting all strategies involving delay in investment yield a strictly lower payoff than a strategy which invests immediately. So investing immediately is an optimal continuation strategy in all such continuation games, implying optimality of $T^*_i=\infty$ for each firm.

Now consider firm $i$'s optimal prospecting problem prior to obtaining a signal. Subsequent to the cutoff time $T^A$ its beliefs are exactly $\pi_A$, so no prospecting is trivially an optimal strategy at this point. So consider times prior to $T^A$. We first show that $V^\dagger(t)=K(\mu^i(t)-\pi_A)$ is a supersolution to firm $i$'s HJB equation on $[0,T^A]$. Recall from Appendix C that the HJB equation for firm $i$ in this regime is

$$rV^i(t)=\overline{\lambda}\left(K(\mu^{\overline{\lambda}}(t)-\pi_A)-V^i(t)\right)\left(-\frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+-\mu^{\overline{\lambda}}(t)}\left(\overline{V}-V^i(t)\right)\right(+\dot{V}^i(t),$$

where we have used the representation for firm $i$'s continuation payoff following receipt of a signal derived in Lemma C.1. So define the functional

$$F(w,t) \equiv rw(t) - \overline{\lambda}\left(K(\mu^{\overline{\lambda}}(t) - \pi_A) - w(t)\right) + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}\left(\overline{V} - w(t)\right) - \dot{w}(t).$$

We will show that $F(V^\dagger, t) \geq 0$ for all $t \leq T^A$. Inserting $V^\dagger$ into the definition of $F$ and combining terms shows that

$$F(V^\dagger, t) = rV^\dagger(t) + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

Now that for $t \leq T^A$, $V^\dagger(t) \geq 0$ and $\frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)} < 0$. Meanwhile by Lemma C.2, $\overline{V} \leq K(\pi_+ - \pi_A)$. So $F(V^\dagger, t) \geq 0$, i.e. $V^\dagger$ is a supersolution to the HJB equation on $[0, T^A]$.

Now, note that $V^\dagger(T^A) = 0$ by definition of $T^A$, while also $V^i(T^A) = 0$ given that firm $i$ is in autarky with beliefs $\pi_A$ subsequent to $T^A$. Therefore $V^\dagger(T^A) = V^i(T^A)$, and since $V^i$ satisfies the HJB equation while $V^\dagger$ is a supersolution on $[0, T^A]$, it must be that $V^\dagger(t) \geq V^i(t)$ for all $t \in [0, T^A]$. The HJB equation then implies that prospecting at the maximum rate prior to $T^A$ is an optimal strategy.

As both the prospecting and investment strategy of each firm under the specified strategy profile are best responses to the other firm's strategy, the strategy profile constitutes a perfect Bayesian equilibrium.

## D.5 Proof of Propositions 2 and 6

We first characterize the follower's best response to the leader. To this end, we first define a pair of belief thresholds, which will turn out to pin down the times at which the follower stops prospecting and stops investing. Suppose that a firm $i$ has current posterior beliefs $\mu \in [\pi_-, \pi_0]$ about the state, following a history in which it has no signal and has not seen firm $-i$ invest. Further suppose firm $-i$ employs the leader strategy. If firm $i$ then receives a High signal, let $\Delta(\mu)$ be the difference in continuation payoffs between waiting for firm $-i$ to invest, then investing immediately afterward, versus investing immediately.

**Lemma D.1.** *For every* $\mu \in [\pi_-, \pi_0]$,

$$\Delta(\mu) = \frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}} \frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_{++}R - 1) - (\mu_+ R - 1).$$

*Proof.* Suppose firm $i$ receives a High signal at time $t$ when its posterior beliefs are $\mu^i(t) = \mu$.

The continuation value of investing immediately is exactly $\mu_+ R - 1$. Meanwhile, given that firm $-i$ employs the leader strategy, the value of waiting for firm $-i$ to invest is

$$\Pr(S^{-i} = H \mid S^i = H, \tau^{-i}(\emptyset) \geq t)\frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_{++}R - 1),$$

where the expression following the probability is the discounted value of eventual investment conditional on firm $-i$ eventually receiving a High signal. In Lemma B.4 we show that

$$\Pr(S^{-i} = H \mid S^i = H, \tau^{-i}(\emptyset) \geq t) = \frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}},$$

establishing the expression for $\Delta$ in the lemma statement. □

In the following lemma, we establish that the value of waiting rises relative to the value of investing immediately as beliefs drop, and that eventually waiting dominates immediate investing.

**Lemma D.2.** $\Delta$ *is a strictly decreasing function of* $\mu$, *and* $\Delta(\pi_-) > 0$. *Also,*

$$\Delta(\pi_0) = \frac{\overline{\lambda}}{\overline{\lambda} + r}h(\pi_+)(\pi_{++}R - 1) - (\pi_+ R - 1).$$

*In particular,* $\Delta(\pi_0) > 0$ *whenever signals are complements.*

*Proof.* Differentiating $\Delta$ yields

$$\Delta'(\mu) = \left(\frac{1}{\pi_{++} - \pi_{+-}}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_{++}R - 1) - R\right)\frac{d\mu_+}{d\mu}.$$

By assumption $\pi_{+-} < 1/R < \pi_{++}$, so

$$\Delta'(\mu) < -\frac{r}{\overline{\lambda} + r}R\frac{d\mu_+}{d\mu} < 0.$$

Further, $\Delta(\pi_-) = -(\pi_{+-}R - 1) > 0$. Finally,

$$\Delta(\pi_0) = \frac{\pi_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_{++}R - 1) - (\pi_+ R - 1),$$

and in Lemma B.3 we establish that

$$h(\pi_+) = \frac{\pi_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}}.$$

So

$$\Delta(\pi_0) = \frac{\overline{\lambda}}{\overline{\lambda} + r} h(\pi_+)(\pi_{++}R - 1) - (\pi_+ R - 1).$$

When signals are complements

$$\frac{\overline{\lambda}}{\overline{\lambda} + r} h(\pi_+)(\pi_{++}R - 1) \geq \pi_+ R - 1 + \frac{\overline{\lambda}}{\overline{\lambda} + r} c > \pi_+ R - 1,$$

so $\Delta(\pi_0) > 0$. □

Now, define a belief threshold $\mu^* \in (\pi_-, \pi_0]$ to be the belief at which the continuation payoffs of investing and waiting are equalized:

$$\mu^* \equiv \begin{cases} \pi_0, & \Delta(\pi_0) \geq 0, \\ \Delta^{-1}(0), & \Delta(\pi_0) < 0. \end{cases}$$

(If waiting is always superior to investing, then by convention we set $\mu^*$ equal to time-zero beliefs.) When mapped onto a corresponding time at which these beliefs are reached, $\mu^*$ will pin down the time at which the follower stops investing in equilibrium. The following corollary is an immediate consequence of the definition of $\mu^*$ and Lemma D.2.

**Corollary D.1.** $\mu^* = \pi_0$ *when signals are complements.*

Next, suppose firm $i$ has current posterior beliefs $\mu \in [\pi_-, \pi_0]$ about the state, following a history in which it has no signal and has not seen firm ${}_i$ invest. Further suppose firm $-i$ employs the leader strategy. If firm $i$ receives a signal, define $\check{V}(\mu)$ to be its expected continuation value, averaging over uncertainty in the realized signal. By definition of $\Delta$,

$$\check{V}(\mu) = h(\mu)(\mu_+ R - 1 + \max\{\Delta(\mu), 0\}).$$

The term $h(\mu)$ reflects the probability that firm $i$'s signal is High given its current posterior, and the remaining terms reflect the choice between investing immediately or waiting for the other firm to invest. (By Lemma D.12, the firm's optimal continuation strategy must be one of these two choices.)

**Lemma D.3.** *For all* $\mu \in [\pi_-, \pi_0]$,

$$\check{V}(\mu) = \max\left\{h(\mu)(\mu_+ R - 1), \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} h(\pi_+)(\pi_{++}R - 1)\right\} \Bigg($$

*Proof.* When $\Delta(\mu) \leq 0$, the definition of $\check{V}$ implies that $\check{V}(\mu) = h(\mu)(\mu_+ R - 1)$. Meanwhile when $\Delta(\mu) > 0$, the definition implies that

$$\check{V}(\mu) = h(\mu) \frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}} \frac{\overline{\lambda}}{\overline{\lambda} + r} (\pi_{++} R - 1).$$

In Lemma B.2, we prove the identity

$$h(\pi_+) \frac{\mu - \pi_-}{\pi_+ - \pi_-} = h(\mu) \frac{\mu_+ - \pi_{+-}}{\pi_{++} - \pi_{+-}},$$

establishing the expression for $\check{V}$ in the lemma statement. □

Note that the first term in the max dominates when $\mu \geq \mu^*$, and otherwise the second term does.

Now define a function $\widetilde{\Delta}$ for $\mu \in [\pi_-, \pi_0]$ by

$$\widetilde{\Delta}(\mu) \equiv \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{V} - \check{V}(\mu) + c.$$

This function represents the marginal change in firm $i$'s continuation value from a decrease in prospecting intensity.

**Lemma D.4.** *$\widetilde{\Delta}$ is a strictly decreasing function and $\widetilde{\Delta}(\pi_-) > 0$.*

*Proof.* Let

$$\hat{\Delta}(\mu) \equiv \frac{\mu - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} (\overline{V} - h(\pi_+)(\pi_{++} R - 1)) + c.$$

Differentiate $\hat{\Delta}$ to obtain

$$\hat{\Delta}'(\mu) = \frac{1}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} (\overline{V} - h(\pi_+)(\pi_{++} R - 1)).$$

By Lemma C.2, $\overline{V} \leq K(\pi_+ - \pi_A)$, i.e. $\overline{V} - h(\pi_+)(\pi_{++} R - 1) \leq -c$ and so $\hat{\Delta}'(\mu) < 0$ for all $\mu$.

Now, $\widetilde{\Delta}(\mu) = \hat{\Delta}(\mu)$ for $\mu \leq \mu^*$, while $\widetilde{\Delta}(\mu) \leq \hat{\Delta}(\mu)$ for $\mu > \mu^*$. Clearly $\widetilde{\Delta}'(\mu) < 0$ for $\mu < \mu^*$. Meanwhile as $\widetilde{\Delta}$ is continuous at $\mu^*$ and an affine function of $\mu$ on $[\mu^*, \pi_0]$, to ensure $\widetilde{\Delta} \leq \hat{\Delta}$ it must be that $\widetilde{\Delta}'(\mu) = \widetilde{\Delta}'(\mu^* +) \leq \hat{\Delta}'(\mu^*) < 0$ for $\mu \in (\mu^*, \pi_0]$. Hence $\widetilde{\Delta}$ is a strictly decreasing function. Finally, note that $\widetilde{\Delta}(\pi_-) = c > 0$. □

In light of the previous lemma, define a belief threshold $\overline{\mu} \in (\pi_-, \pi_0]$ by

$$\overline{\mu} \equiv \begin{cases} \pi_0, & \widetilde{\Delta}(\pi_0) \geq 0, \\ \widetilde{\Delta}^{-1}(0), & \widetilde{\Delta}(\pi_0) < 0. \end{cases}$$

This threshold will pin down the time at which the follower stops prospecting.

The following lemma characterizes the follower's unique best response to the leader's strategy, by showing that the follower prospects until its beliefs hit $\overline{\mu}$, then shirks; and invests immediately until its beliefs hit $\mu^*$, then waits. Note that as $\overline{\mu}, \mu^* > \pi_-$, both thresholds are reached in finite time.

**Lemma D.5.** *Suppose some firm $-i$ chooses the threshold strategy $T^*_{-i} = \overline{T}_{-i} = \infty$. Then firm $i$'s unique best response is the threshold strategy characterized by $T^*_i = (\mu^{\overline{\lambda}})^{-1}(\mu^*)$ and $\overline{T}_i = (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$.*

*Proof.* Assume firm $-i$ employs the strategy of the lemma statement. As firm $-i$'s strategy is regular, by Lemma D.12 firm $i$'s optimal investment strategy involves a cutoff time $T^*_i$ before which it invests immediately and after which it waits for firm $-i$ to invest. Further, given firm $-i$'s strategy, $\mu^i = \mu^{\overline{\lambda}}$. Thus by the definition of $\mu^*$, the time $(\mu^{\overline{\lambda}})^{-1}(\mu^*)$ is exactly when the value to firm $i$ of investing immediately falls below the value of waiting for firm $-i$ to invest first. So the firm's unique optimal investment strategy sets $T^*_i = (\mu^{\overline{\lambda}})^{-1}(\mu^*)$.

Next we derive firm $i$'s optimal prospecting strategy. Appendix C establishes that firm $i$'s continuation value function $V^i$ satisfies

$$rV^i(t) = \overline{\lambda} \left(\check{V}(\mu^{\overline{\lambda}}(t)) - c - V^i(t)\right)^+ - \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}(\overline{V} - V^i(t)) + \dot{V}^i(t)$$

for all time. Define the functional

$$F(w,t) \equiv rw - \overline{\lambda}\left(\check{V}(\mu^{\overline{\lambda}}(t)) - c - w(t)\right)^+ + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}\left(\overline{V} - w(t)\right) - \dot{w}(t).$$

We now show that $V^\dagger(t) \equiv \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\overline{V}$ satisfies the HJB equation for $t \geq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$. In other words, $F(V^\dagger, t) = 0$ for such times. By definition of $\overline{\mu}$, $\check{V}(\mu^{\overline{\lambda}}(t)) - c - V^\dagger(t) = -\widetilde{\Delta}(\mu^{\overline{\lambda}}(t)) \leq 0$ for $t \geq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$. So

$$F(V^\dagger, t) = r\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\overline{V} + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}\left(1 - \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\right)\overline{V} - \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\overline{V}.$$

Now use the identity

$$-\frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)} = \overline{\lambda}\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-},$$

derived in Lemma B.1, to eliminate $\dot{\mu}^{\overline{\lambda}}(t)$ from the rhs. The result, after simplifying, is $F(V^\dagger, t) = 0$, as desired.

As $V^\dagger$ is a bounded absolutely continuous function, it follows by a standard verification argument that $V^i(t) = V^\dagger(t)$ for $t \geq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$. Further, $\check{V}(\mu^{\overline{\lambda}}(t)) - c - V^i(t) = -\widetilde{\Delta}(\mu^{\overline{\lambda}}(t)) < 0$ for $t > (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$, hence firm $i$'s unique optimal prospecting strategy for $t \geq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$ is $\lambda^i(t) = 0$.

Now consider times $t < (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$. If $\overline{\mu} = \pi_0$ then this time interval is empty, so assume $\overline{\mu} < \pi_0$. We will show that $V^\ddagger(t) \equiv \check{V}(\mu^{\overline{\lambda}}(t)) - c$ is a strict supersolution to the HJB equation for $t < (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$. That is, $F(V^\ddagger, t) > 0$ for all such times. This is sufficient to establish the unique optimality of the prospecting policy $\lambda^i(t) = \overline{\lambda}$ a.e. on $[0, \overline{T}]$, by the following argument. Recall that $\widetilde{\Delta}(\overline{\mu}) = 0$ by definition, and therefore by the definition of $\widetilde{\Delta}$

$$V^\ddagger((\mu^{\overline{\lambda}})^{-1}(\overline{\mu})) = \check{V}(\overline{\mu}) - c = \frac{\overline{\mu} - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}\overline{V} = V^i((\mu^{\overline{\lambda}})^{-1}(\overline{\mu})).$$

Then if $V^\ddagger$ is a strict supersolution of the HJB equation for $t \leq \overline{T}$, it must be that $V^\ddagger(t) > V^i(t)$ for all $t < \overline{T}$. The HJB equation then implies that firm $i$'s unique optimal prospecting policy is $\lambda^i(t) = \overline{\lambda}$ for $t \leq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$.

As a first step toward establishing the supersolution result, note that $t \leq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$ implies $\widetilde{\Delta}(\mu^{\overline{\lambda}}(t)) \geq 0$ and therefore

$$V^\ddagger(t) = \check{V}(\mu^{\overline{\lambda}}(t)) - c \geq \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}\overline{V}.$$

As $\mu^{\overline{\lambda}} > \pi_-$, this inequality implies $V^\ddagger(t) > 0$ for every $t \leq (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$.

First consider times $t \leq \min\{(\mu^{\overline{\lambda}})^{-1}(\overline{\mu}), (\mu^{\overline{\lambda}})^{-1}(\mu^*)\}$. On this time range $\check{V}(\mu^{\overline{\lambda}}(t)) - c = K(\mu^{\overline{\lambda}}(t) - \pi_A)$, and so $F(V^\ddagger, t)$ evaluates to

$$F(V^\ddagger, t) = rV^\ddagger(t) + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

By Lemma C.2, $\overline{V} \leq K(\pi_+ - \pi_A)$, meaning $F(V^\ddagger, t) > 0$ given $\dot{\mu}^{\overline{\lambda}}(t) < 0$ and $V^\ddagger(t) > 0$.

If $\overline{\mu} \geq \mu^*$ then we're done, so suppose instead that $\overline{\mu} < \mu^*$. Recall that for $t \in$

$((\mu^{\overline{\lambda}})^{-1}(\mu^*), (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})]$,

$$\check{V}(\mu^{\overline{\lambda}}(t)) = \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r} h(\pi_+)(\pi_{++}R - 1) = \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(K(\pi_+ - \pi_A) + c).$$

The function $F(V^\ddagger, t)$ then evaluates to

$$\begin{aligned} F(V^\ddagger, t) = r & \left( \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(K(\pi_+ - \pi_A) + c) - c \right) \\ & + \frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)} \left( \overline{V} - \frac{\overline{\lambda}}{\overline{\lambda} + r}(K(\pi_+ - \pi_A) + c) + c \right) \end{aligned}$$

Now use the identity

$$-\frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)} = \overline{\lambda}\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}$$

to eliminate $\dot{\mu}^{\overline{\lambda}}(t)$ from $F(V^\ddagger, t)$, yielding

$$F(V^\ddagger, t) = -\overline{\lambda}\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}(\overline{V} - K(\pi_+ - \pi_A)) - rc.$$

Now, $\mu^{\overline{\lambda}}(t) \in [\overline{\mu}, \mu^*)$ for $t \in ((\mu^{\overline{\lambda}})^{-1}(\mu^*), (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})]$, and therefore

$$\widetilde{\Delta}(\mu^{\overline{\lambda}}(t)) = \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\overline{V} - K(\pi_+ - \pi_A) - c) + c \leq 0,$$

or equivalently

$$-\overline{\lambda}\frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}(\overline{V} - K(\pi_+ - \pi_A)) \geq (\overline{\lambda} + r)c - \frac{\mu^{\overline{\lambda}}(t) - \pi_-}{\pi_+ - \pi_-}\overline{\lambda}c > rc.$$

This bound establishes that $F(V^\ddagger, t) > 0$ on $[(\mu^{\overline{\lambda}})^{-1}(\mu^*), (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})]$, as desired. $\square$

We next establish that when $c$ is sufficiently small, the follower prospects for a period of time after it begins waiting to invest.

**Lemma D.6.** $\overline{\mu} < \min\{\mu^*, \pi_A\}$ *when $c$ is sufficiently small.*

*Proof.* Note that $c$ appears nowhere in the definition of $\Delta$, and so $\mu^*$ is independent of $c$.

First suppose that $\mu^* = \pi_0$. Note that $\widetilde{\Delta}(\pi_0)$ may be written

$$\widetilde{\Delta}(\pi_0) = h(\pi_0)\left(\frac{\overline{\lambda}}{\overline{\lambda}+r}\overline{V} - \max\left\{\pi_+R-1, \frac{\overline{\lambda}}{\overline{\lambda}+r}h(\pi_+)(\pi_{++}R-1)\right\}\right) + c.$$

When $c \downarrow 0$, $\overline{V} \to \max\left\{\pi_+R-1, \frac{\overline{\lambda}}{\overline{\lambda}+r}h(\pi_+)(\pi_{++}R-1)\right\}$. Thus the first term approaches a strictly negative value in this limit, while the second term approaches zero. This means $\widetilde{\Delta}(\pi_0) < 0$ for small $c$, i.e. $\overline{\mu} < \pi_0 = \mu^*$.

Next suppose $\mu^* < \pi_0$. In this case $\Delta(\mu^*) = 0$ and hence $\check{V}(\mu^*) = \frac{\mu^*-\pi_-}{\pi_+-\pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}h(\pi_+)(\pi_{++}R-1)$. So $\widetilde{\Delta}(\mu^*)$ may be written

$$\widetilde{\Delta}(\mu^*) = \frac{\mu^*-\pi_-}{\pi_+-\pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\left(\overline{V} - h(\pi_+)(\pi_{++}R-1)\right) + c.$$

$\overline{V}$ is decreasing in $c$, but due to time discounting $\overline{V} < h(\pi_+)(\pi_{++}R-1)$ even in the limit as $c \downarrow 0$. So the first term is negative and bounded away from 0 for all $c$, meaning that for sufficiently small $c$, it must be that $\widetilde{\Delta}(\mu^*) < 0$. Hence $\overline{\mu} < \mu^*$ in this case as well.

Finally, note that when $c = 0$, $\pi_A$ satisfies $h(\pi_A)(\pi_{A+}R-1) = 0$, i.e. $\pi_{A+} = 1/R$. Hence $\pi_{A+} > \pi_-$ given that $\pi_{+-} < 1/R$. Also, when $c = 0$,

$$\widetilde{\Delta}(\pi_A) = \frac{\pi_A-\pi_-}{\pi_+-\pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}\overline{V} - \check{V}(\pi_A) \leq \frac{\pi_A-\pi_-}{\pi_+-\pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}(\overline{V} - h(\pi_+)(\pi_{++}R-1)).$$

Note that

$$h(\pi_+)(\pi_{++}R-1) > \max\left\{\pi_+R-1, \frac{\overline{\lambda}}{\overline{\lambda}+r}h(\pi_+)(\pi_{++}R-1)\right\} = \overline{V},$$

hence $\widetilde{\Delta}(\pi_A) < 0$ when $c = 0$. Thus $\overline{\mu} < \pi_A$ when $c = 0$, and so by continuity also for $c$ sufficiently small. □

In the remainder of the proof, we establish that the leader's strategy is a best reply to the follower strategy characterized in Lemma D.5. We first show that the leader strategy is a unique best reply to any threshold strategy satisfying an auxiliary condition on beliefs.

**Lemma D.7.** *Suppose that some firm $i$ employs a threshold strategy satisfying $\mu^{\overline{\lambda}}(\min\{T_i^*, \overline{T}_i\}) > \pi^A$. Then firm $-i$'s unique best reply is the threshold strategy $T_{-i}^* = \overline{T}_{-i} = \infty$.*

*Proof.* Subsequent to time $\min\{T_i^*, \overline{T}_i\}$, firm $-i$ is in autarky with beliefs

$$\mu^{-i}(t) = \mu^{\overline{\lambda}}(\min\{T_i^*, \overline{T}_i\}) > \pi^A.$$

Thus its unique best reply at all such times is to prospect at rate $\overline{\lambda}$ and invest immediately. By Lemma D.12, it follows that firm $-i$'s unique optimal investment strategy is the cutoff rule $T^*_{-i} = \infty$. It remains only to characterize $-i$'s optimal prospecting behavior prior to time $\min\{T^*_i, \overline{T}_i\}$.

We proceed by establishing that $V^\dagger(t) = K(\mu^{-i}(t) - \pi_A)$ is a strict supersolution to firm $-i$'s HJB equation on $[0, \min\{T^*_i, \overline{T}_i\}]$. Recall that the HJB equation for firm $-i$ in this regime is

$$rV^{-i}(t) = \overline{\lambda}\left(K(\mu^{-i}(t) - \pi_A) - V^{-i}(t)\right) - \frac{\dot{\mu}^{-i}(t)}{\pi_+ - \mu^{-i}(t)}\left(\overline{V} - V^{-i}(t)\right) + \dot{V}^{-i}(t).$$

So define the functional

$$F(w, t) \equiv rw(t) - \overline{\lambda}\left(K(\mu^{-i}(t) - \pi_A) - w(t)\right) + \frac{\dot{\mu}^{-i}(t)}{\pi_+ - \mu^{-i}(t)}\left(\overline{V} - w(t)\right) - \dot{w}(t).$$

The claim that $V^\dagger$ is a strict supersolution is equivalent to $F(V^\dagger, t) > 0$ for $t < \min\{T^*_i, \overline{T}_i\}$. Evaluating the functional at $V^\dagger$ yields

$$F(V^\dagger, t) = rV^\dagger(t) + \frac{\dot{\mu}^{-i}(t)}{\pi_+ - \mu^{-i}(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

Note that $\overline{V} \leq K(\pi_+ - \pi_A)$ by Lemma C.2, so the second term on the rhs is non-negative. Meanwhile $\mu^{-i}(t) > \pi_A$ for $t \leq \min\{T^*_i, \overline{T}_i\}$, so $F(V^\dagger, t) > 0$ as claimed.

Now note that as firm -i is in autarky at time $\min\{T^*_i, \overline{T}_i\}$, its value function at this point is

$$V^{-i}(\min\{T^*_i, \overline{T}_i\}) = \frac{\overline{\lambda}}{\overline{\lambda} + r} K(\mu^{-i}(\min\{T^*_i, \overline{T}_i\}) - \pi_A) < V^\dagger(\min\{T^*_i, \overline{T}_i\}).$$

This boundary condition combined with the fact that $V^\dagger$ is a strict supersolution implies $V^\dagger(t) > V^{-i}(t)$ for all $t \in [0, \min\{T^*_i, \overline{T}_i\}]$. The HJB equation then implies that $\lambda^{-i}(t) = \overline{\lambda}$ is firm $-i$'s unique best reply for all times. □

The following lemma completes the equilibrium verification by showing that the follower's strategy satisfies the auxiliary belief condition of Lemma D.7.

**Lemma D.8.** $\max\{\overline{\mu}, \mu^*\} > \pi_A$.

*Proof.* If $\overline{\mu} = \pi_0$ then the result is automatic. So assume $\overline{\mu} < \pi_0$, in which case $\overline{\mu}$ is pinned down by the condition $\widetilde{\Delta}(\overline{\mu}) = 0$. If $\overline{\mu} \geq \mu^*$, then $\widetilde{\Delta}(\overline{\mu}) = 0$ may be written

$$\frac{\overline{\mu} - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{V} - K(\overline{\mu} - \pi_A) = 0.$$

As $\overline{\mu} > \pi_-$, the first term on the rhs is strictly positive, meaning so must be the second. Hence $\overline{\mu} > \pi_A$.

Suppose instead that $\mu^* > \overline{\mu}$. In this case $\widetilde{\Delta}(\mu^*) < 0$, which is equivalently

$$\frac{\mu^* - \pi_-}{\pi_+ - \pi_-} \frac{\overline{\lambda}}{\overline{\lambda} + r} \overline{V} - K(\mu^* - \pi_A) < 0.$$

As $\mu^* > \pi_-$, the first term is strictly positive. Therefore so must the second, implying $\mu^* > \pi_A$. □

Finally, the following lemmas establish properties of $\mu^*$ and $\overline{\mu}$ useful for proving important properties $T^*_{-i}$ and $\overline{T}_{-i}$, including comparative statics in $c$ and $\overline{\lambda}/r$.

**Lemma D.9.** *When signals are substitutes,* $\max\{\mu^*, \overline{\mu}\} < \pi_0$ *for* $\overline{\lambda}/r$ *sufficiently small.*

*Proof.* Note that when $\overline{\lambda}/r = 0$, $\Delta(\pi_0) = -(\pi_+ R - 1) < 0$ while $\widetilde{\Delta}(\pi_0) = -(h(\pi_0)(\pi_+ R - 1) - c) < 0$, with the second equality following from the assumption that $\pi_0 > \pi_A$. Thus $\max\{\mu^*, \overline{\mu}\} < \pi_0$ for $\overline{\lambda}/r = 0$, and so by continuity also for $\overline{\lambda}/r$ positive but small. In particular, $\overline{\lambda}/r$ may be taken small enough that

$$\frac{\overline{\lambda}}{\overline{\lambda} + r}(h(\pi_+)(\pi_{++} R - 1) - c) < \pi_+ R - 1,$$

in which case signals are substitutes. □

**Lemma D.10.** *When signals are complements and* $\overline{\lambda}/r$ *is sufficiently high,* $\overline{\mu} = \pi_0$ *for an interval of costs whose upper boundary is the largest cost at which signals are complements. When signals are substitutes and* $\overline{\lambda}/r$ *is sufficiently high,* $\overline{\mu} = \pi_0$ *for costs sufficiently close to* $\overline{c}$.

*Proof.* Suppose first that signals are substitutes. Then

$$\begin{aligned}
\widetilde{\Delta}(\pi_0) &= h(\pi_0) \frac{\overline{\lambda}}{\overline{\lambda} + r} (\overline{V} - h(\pi_+)(\pi_{++} R - 1)) + c \\
&= -h(\pi_0) \frac{\overline{\lambda}}{\overline{\lambda} + r} \left(1 - \frac{\overline{\lambda}}{\overline{\lambda} + r}\right) h(\pi_+)(\pi_{++} R - 1) + \left(1 - h(\pi_0) \left(\frac{\overline{\lambda}}{\overline{\lambda} + r}\right)^2\right) c.
\end{aligned}$$

Define $\widehat{c}$ to be the unique $c < \bar{c}$ such that

$$\frac{\bar{\lambda}}{\bar{\lambda}+r}(h(\pi_+)(\pi_{++}R-1)-c) = \pi_+R-1.$$

This is the maximum cost at which signals are complements. (By assumption that signals are complements, it must be the case that $\widehat{c} > 0$.) Note that when $c = \widehat{c}$, taking the limit $\bar{\lambda}/r \to \infty$ yields $\widetilde{\Delta}(\pi_0) \to l(\pi_0)\widehat{c} > 0$. Hence for sufficiently large $\bar{\lambda}/r$, it must be the case that $\widetilde{\Delta}(\pi_0) > 0$ when $c = \widehat{c}$, thus also when $c$ is sufficiently close to $\widehat{c}$ by continuity. So there exist a range of costs sufficiently high but bounded above by $\widehat{c}$ such that signals are complements and $\widetilde{\Delta}(\pi_0) > 0$, i.e. $\bar{\mu} = \pi_0$.

Now suppose signals are substitutes. Then

$$\widetilde{\Delta}(\pi_0) = h(\pi_0)\left(\frac{\bar{\lambda}}{\bar{\lambda}+r}(\pi_+R-1) - \max\left\{\pi_+R-1, \frac{\bar{\lambda}}{\bar{\lambda}+r}h(\pi_+)(\pi_{++}R-1)\right\}\right) + c.$$

Setting $c = \bar{c}$ and taking $\bar{\lambda}/r \to \infty$ yields $\widetilde{\Delta}(\pi_0) \to l(\pi_0)\bar{c} > 0$. So it must be that for $\bar{\lambda}/r$ sufficiently large, $\widetilde{\Delta}(\pi_0) > 0$ when $c = \bar{c}$, thus also when $c$ is sufficiently close to $\bar{c}$ by continuity. Note also that for any finite $\bar{\lambda}/r$, signals are substitutes when $c$ is sufficiently close to $\bar{c}$. Thus fixing $\bar{\lambda}/r$ and then taking $c$ close to $\bar{c}$ does not violate the assumption that signals are substitutes. □

**Lemma D.11.** *$\bar{\mu}$ is increasing in $c$, and strictly increasing whenever $\bar{\mu} < \pi_0$.*

*Proof.* Recall that

$$\widetilde{\Delta}(\mu) = \frac{\mu-\pi_-}{\pi_+-\pi_-}\frac{\bar{\lambda}}{\bar{\lambda}+r} - \check{V}(\mu) + c,$$

where $\check{V}(\mu)$ is not a function of $c$. When signals are substitutes, $\bar{V}$ doesn't depend on $c$ either, so that $\partial\widetilde{\Delta}/\partial c = 1 > 0$. When signals are complements,

$$\bar{V} = \frac{\bar{\lambda}}{\bar{\lambda}+r}\left(h(\pi_+)(\pi_{++}R-1)-c\right)$$

and so

$$\frac{\partial\widetilde{\Delta}}{\partial c} = -\frac{\mu-\pi_-}{\pi_+-\pi_-}\left(\frac{\bar{\lambda}}{\bar{\lambda}+r}\right)^2 + 1 > 0.$$

so $\widetilde{\Delta}$ is strictly increasing in $c$ for every $\mu$ in all cases. Thus $\bar{\mu}$ is increasing in $c$, and strictly increasing whenever $\bar{\mu} < \pi_0$. □

## D.6 Proof of Lemma 4

These properties are immediate corollaries of lemmas established in the proof of Proposition 2. Monotonicity of $\overline{T}_{-i}$ in $c$ follows from Lemma D.11. The result that $\overline{T}_{-i} = 0$ when $c$ is large and $\overline{\lambda}/r$ is sufficiently large follows from Lemma D.10. And the fact that $\overline{T}_{-i} > T^A$ for $c$ sufficiently small follows from Lemma D.6.

## D.7 Proof of Lemma 5

These properties are immediate corollaries of lemmas established in the proof of Proposition 2. The fact that $\min\{T^*_{-i}, \overline{T}_{-i}\} > 0$ for $\overline{\lambda}/r$ small follows from Lemma D.9. The inequality $\min\{T^*_{-i}, \overline{T}_{-i}\} < T^A$ follows from Lemma D.8. The independence of $T^*_{-i}$ from $c$ follows from the fact that the function $\Delta$ pinning down $\mu^*$ does not involve $c$. Monotonicity of $\overline{T}_{-i}$ in $c$ follows from Lemma D.11. The result that $\overline{T}_{-i} = 0$ when $c$ is large and $\overline{\lambda}/r$ is sufficiently large follows from Lemma D.10. And the fact that $\overline{T}_{-i} > T^A > T^*_{-i}$ for $c$ sufficiently small is a consequence of Lemma D.6, which establishes that $\overline{T}_{-i} > T^A$, in combination with the earlier result that at least one of $T^*_{-i}$ and $\overline{T}_{-i}$ must be strictly less than $T^A$.

## D.8 Proof of Propositions 3 and 7

Propositions 1, 2, 5, and 6 establish that the symmetric and leader-follower strategy profiles are sufficient for equilibrium. We now prove that that these profiles are also necessary for equilibrium.

We begin by establishing that essentially any best response to a regular strategy features a threshold rule for investment.

**Lemma D.12.** *Suppose firm $-i$ uses a regular strategy. Let $T^0_i \equiv \inf\{t \ : \ \mu^i_+(t) \leq 1/R\}$ and $\underline{T}_i \equiv \inf\{t \ : \ \mu^i_+(t) < 1/R\}$. Then:*

- *If $\underline{T}_i < \infty$, there exists a cutoff time $T^*_i \leq T^0_i$ such that every best reply for firm $i$ after obtaining a High signal at time $t$ entails investing immediately if $t < T^*_i$ and waiting forever if $t > T^*_i$,*
- *If $\underline{T}_i = \infty$, every best reply for firm $i$ after obtaining a High signal at time $t$ entails investing immediately if $t < T^0_i$. Upon receiving a signal at any $t \geq T^0_i$, any continuation strategy by firm $i$ is a best reply.*

*Proof.* We show first that, for any time $t$ such that $\mu^i_+(t) > 1/R$, any best reply for $i$ either invests immediately or waits until $-i$ invests. Suppose by way of contradiction that firm $i$

had a best reply such that upon receiving a High signal at time $t$, $i$ invests at the random time $\widetilde{\tau}^i \in [t,\infty) \cup \{\infty\}$ conditional on no investment by firm $-i$, with $\Pr(\widetilde{\tau}^i \in (t,\infty)) > 0$. Then there must exist another best reply such that firm $i$ waits until some deterministic time $t' \in (t,\infty)$ and then invests w.p. 1 at time $t'$ conditional on no investment by firm $-i$. In particular, it must be that $\mu^i_+(t')R - 1 \geq 0$.

Let $\tau^{-i}(\emptyset)$ be the (possibly random) time at which firm $-i$ invests, absent investment by firm i. Whenever $\tau^{-i}(\emptyset) \geq t'$, $i$'s ex post continuation payoff as of time $t$ is $e^{-rt'}(\mu^i_+(t')R-1) \leq \mu^i_+(t')R - 1$. In particular, if $\Pr(\tau^{-i}(\emptyset) \geq t' \mid \tau^{-i}(\emptyset) \geq t, S^i = H) = 1$, then $\mu^i_+(t') = \mu^i_+(t)$ and the previous inequality is strict. And whenever $\tau^{-i}(\emptyset) \in [t,t']$, $i$'s continuation payoff is $e^{-r\tau^{-i}}(\pi_{++}R - 1) < \pi_{++}R - 1$. Then $i$'s continuation payoff from this best reply is strictly less than

$$\begin{aligned} U' = \; & \Pr(\tau^{-i}(\emptyset) < t' \mid \tau^{-i}(\emptyset) \geq t, S^i = H)(\pi_{++}R - 1) \\ & + \Pr(\tau^{-i}(\emptyset) \geq t' \mid \tau^{-i}(\emptyset) \geq t, S^i = H)(\mu^i_+(t')R - 1). \end{aligned}$$

As

$$\mu^i_+(t) = \Pr(\tau^{-i}(\emptyset) < t' \mid \tau^{-i}(\emptyset) \geq t, S^i = H)\pi_{++} + \Pr(\tau^{-i}(\emptyset) \geq t' \mid \tau^{-i}(\emptyset) \geq t, S^i = H)\mu^i_+(t'),$$

$U' = \mu^i_+(t)R - 1$, which is exactly $i$'s payoff from investing immediately at time $t$. Thus waiting until $t'$ and then investing cannot be cannot be a best reply, yielding the desired contradiction.

Suppose first that $\underline{T}_i < \infty$. Consider times $t$ such that $\mu^i_+(t) \leq 1/R$. For any $t$ such that $\mu^i_+(t) < 1/R$, trivially the unique best response is for $i$ to wait forever, since investing at any time after $t$ yields a strictly negative payoff. It is also true that whenever $\mu^i_+(t) = 1/R$, waiting forever is $i$'s unique best response. For investing at any time before $-i$ invests yields a non-positive continuation payoff, whereas given $\underline{T}_i < \infty$ there is a positive probability that $-i$ invests in the future, so that waiting for $-i$ to invest yields a strictly positive payoff.

Next fix any time $t$ for which investing immediately is a best response for $i$. In this case $\mu^i_+(t) > 1/R$. We claim that for every $t' < t$, investing immediately is $i$'s unique best response. Suppose by way of contradiction that for some $t' < t$, there existed a best response which does not invest immediately. Then by the discussion above, waiting until $-i$ invests must be a best response. But by assumption upon reaching time $t$ with no investment by $-i$, investing immediately is a best response. Hence waiting until time $t$ and then investing must also be a best response. This is a contradiction of earlier results, as desired.

Hence when $\underline{T}_i < \infty$, the set of best responses by $i$ must have a simple structure - there exists a cutoff time $T_i^* \leq T_i^0$ such that any best response by $i$ invests immediately for $t < T_i^*$ and waits for any $t > T_i^*$.

Now consider the case $\underline{T}_i = \infty$. Suppose first that $T_i^0 < \infty$. For times $t \geq T_i^0$, the firm's beliefs upon obtaining a High signal are fixed at $1/R$, and so any investment strategy is optimal. In particular, investing immediately is optimal. So suppose by way of contradiction that for some $t < T_i^0$, firm $i$ optimally waits forever to invest. Then another best reply is to wait until time $T_i^0$ and then invest, a contradiction of earlier results. So it must be that for every $t < T_i^0$, immediate investment is optimal. Finally, suppose $T_i^0 = \infty$. By a variant of the arguments above, it can be shown that waiting to invest delays a profitable investment when $-i$ does eventually invest, without avoiding any losses when $-i$ does not eventually invest, given that $\mu_+^i(\infty) \geq 1/R$. So waiting forever must yield strictly lower profits than investing immediately at all times. □

This lemma does not entirely rule out existence of best replies which do not take a threshold form. The one case in other best replies exist is if $\mu_+^i(t)$ eventually drops to $1/R$ and then stays fixed there forever some firm $i$. In this case any investment policy by firm $i$ is optimal once beliefs have dropped to $1/R$. However, no best reply by firm $i$ ever leaves it in possession of a High signal in such a history. This is because either the firm obtained a High signal earlier and invested; or else the firm has not obtained a signal and $\mu_+^i(t) = 1/R$, in which case a signal has no value and the firm would never optimally acquire one at that point. Therefore any other best replies differ only off-path, regardless of the strategy chosen by firm $-i$.

Our proof will proceed by restricting attention to equilibria in threshold investment investment strategies. This restriction will not rule out any equilibrium paths, and therefore we will be able to determine whether there exist any equilibria in which $\mu_+^i(t)$ is eventually fixed at $1/R$ for some $i$. It will turn out there are not, and thus we do not exclude any equilibria by restricting attention to threshold investment strategies. We will further restrict attention to equilibria in pure prospecting strategies. At the end of the proof we verify that no equilibria with mixed prospecting strategies can exist.

Given any belief process $\mu^i$, define

$$t_i^A \equiv \inf\{t \ : \ \mu^i(t) \leq \pi_A\}$$

to be the time at which firm $i$'s beliefs reach the autarky threshold. We first establish an important technical result about the dynamics of the value of effort prior to time $t_i^A$. This result will be critical to establishing that firms follower a threshold rule in effort in any equilibrium.

**Lemma D.13.** *Fix any firm $i$ and any regular threshold investment strategy by firm $-i$. Let $V^i$ be firm $i$'s value function given firm $-i$'s strategy, and define $f_i(t) \equiv V^i(t) - K(\mu^i(t) - \pi_A)$. Then for almost every $t \in [0, \min\{T_i^*, t_i^A\}]$, either $f_i(t) < 0$ or $f_i'(t) > 0$.*

*Proof.* Fix a firm $i$. Suppose first that $T_{-i}^* \leq t < t_i^A$. Then at time $t$ firm $i$ is in autarky with beliefs $\mu^i(t) > \pi_A$, meaning its continuation value is $V^i(t) = \frac{\overline{\lambda}}{\overline{\lambda}+r} K(\mu^i(t) - \pi_A) < K(\mu^i(t) - \pi_A)$. Thus $f_i(t) < 0$ for all such times. So it is sufficient to establish the result for $t < \hat{t}_i \equiv \min\{T_i^*, t_i^A, T_{-i}^*\}$.

First note that as $V^i$ and $\mu^i$ are both absolutely continuous, $f_i$ is absolutely continuous as well and $f_i'$ is defined a.e. Let

$$T^* = \{t \in [0, \hat{t}_i) \ : \ f_i(t) \geq 0\}.$$

For almost every $t \in T^*$, the HJB equation

$$rV^i(t) = -\frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}(\overline{V} - V^i(t)) + \dot{V}^i(t)$$

must hold. This may be rewritten in terms of $f$ and $f'$ as

$$f_i'(t) = rf_i(t) + rK(\mu^i(t) - \pi_A) + \frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}(\overline{V} - K(\pi_+ - \pi_A) - f_i(t)).$$

As $f_i(t) \geq 0$ on $T^*$, the inequality

$$f_i'(t) \geq rK(\mu^i(t) - \pi_A) + \frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)}(\overline{V} - K(\pi_+ - \pi_A))$$

must hold almost everywhere on $T^*$. Now use the identity

$$\frac{\dot{\mu}^i(t)}{\pi_+ - \mu^i(t)} = -\lambda^{-i}(t)\frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-},$$

which holds for every $t < T_{-i}^*$, to rewrite this inequality as

$$f_i'(t) \geq \frac{\mu^i(t) - \pi_-}{\pi_+ - \pi_-}\left(-\lambda^{-i}(t)(\overline{V} - K(\pi_+ - \pi_A)) + rK(\mu^i(t) - \pi_A)\frac{\pi_+ - \pi_-}{\mu^i(t) - \pi_-}\right)\Bigg($$

Regardless of firm $-i$'s strategy, $\mu^i(t) > \pi_-$ for all time. So the lemma is proven if we can show that the final term on the rhs is strictly positive for $t < \hat{t}_i$. By Lemma C.2, $\overline{V} \leq K(\pi_+ - \pi_A)$. So the first term on the rhs is non-negative. Meanwhile $t_i^A$ implies that the second term on the rhs is strictly positive. So $f_i'(t) > 0$. □

We proceed by splitting the analysis into two cases: either $T_i^* < \infty$ for some firm $i$, or else $T_1^* = T_2^* = \infty$. We will show that in the first case, the only permissible equilibrium behavior is the leader-follower strategy profile, while in the second case, the only permissible behavior is the symmetric equilibrium profile. Consider first the $T_i^* < \infty$ case. The following lemma establishes that the remaining firm $-i$ must employ the leader strategy in any equilibrium.

**Lemma D.14.** *Fix any perfect Bayesian equilibrium in threshold investment strategies such that $T_i^* < \infty$ for some firm $i$. Then firm $-i$'s strategy must be the threshold strategy $\overline{T}_{-i} = T_{-i}^* = \infty$.*

We establish this result by first proving a series of auxiliary lemmas which restrict the permissible scope of equilibrium behavior and beliefs in response to a firm using a threshold investment rule with $T_i^* < \infty$.

**Lemma D.15.** *Fix any perfect Bayesian equilibrium in threshold investment strategies such that $T_i^* < \infty$ for some firm $i$. Then $T_{-i}^* = \infty$ and $\mu^{-i}(T_i^*) \geq \pi_A$.*

*Proof.* Without loss take $i = 1$. Suppose by way of contradiction that $\mu^2(T_1^*) < \pi_A$. In this case clearly $T_1^* > 0$ given that $\mu^2(0) = \pi_0 > \pi_A$. Further, as the belief process is continuous under a threshold investment strategy it must be that $t_2^A < T_1^*$. Then as $V^2 \geq 0$, for $t \in (t_2^A, T_1^*]$ it must be that $V^2(t) > K(\mu^2(t) - \pi_A)$, so that by the HJB equation 2's essentially unique optimal prospecting policy is $\lambda^2(t) = 0$ for $t \in (t_2^A, T_1^*]$. And after $T_1^*$ firm 2 is in autarky with beliefs below the autarky threshold, so it must also be that $\lambda^2(t) = 0$ for almost every $t \geq T_1^*$.

Then on the equilibrium path firm 2 never invests first after $t_2^A$, meaning firm 1 is in autarky with constant beliefs $\mu^1(t) = \mu^1(t_2^A)$ for all $t > t_2^A$. But then $\mu_+^1(t_2^A) = 1/R$, otherwise it could not be optimal for firm 1 to invest immediately prior to $T_1^*$ and wait forever after $T_1^*$. In this case $\mu^1(t_2^A) < \pi_A$, so $\lambda^1(t) = 0$ for all $t \geq t_2^A$. But then on the equilibrium path firm 1 does not invest first on $[t_2^A, T_1^*]$, implying $\mu^2(t_2^A) = \mu^2(T_1^*)$. This contradicts our assumption on $\mu^2(T_1^*)$ and the definition of $t_2^A$. So it must be that $\mu^2(T_1^*) \geq \pi_A$.

As firm 1 does not invest first on the equilibrium path after $T_1^*$, firm 2 is in autarky with constant beliefs $\mu^2(t) \geq \pi_A$ for all $t \geq T_1^*$. Hence $\mu_+^2(t) > 1/R$ and so immediate investing is strictly superior to waiting forever for every $t \geq T_1^*$. In other words, $T_2^* = \infty$. □

**Lemma D.16.** *Fix any perfect Bayesian equilibrium in threshold investment strategies in which $T_i^* < \infty$ for some firm $i$. Then $\mu^{-i}(T_i^*) > \pi_A$.*

*Proof.* Without loss take $i = 1$. By Lemma D.15 we already know that $\mu^2(T_1^*) \geq \pi_A$ and $T_2^* = \infty$. Assume by way of contradiction that $\mu^2(T_1^*) = \pi_A$. Clearly $T_1^* > 0$ in this case given $\mu^2(0) = \pi_0 > \pi_A$, and further $t_2^A \leq T_1^*$ given the definition of $t_2^A$.

Next observe that in equilibrium, firm 1 never invests first after $t_2^A$. This is automatically true for $t \geq T_1^*$, so the remaining thing to show is that $\lambda^1(t) = 0$ on $[t_2^A, T_1^*)$ in the case that this interval is non-empty. But $\mu^2(t)$ is constant on this interval, so given that firm 1 invests immediately upon obtaining a High signal its prospecting rate must be zero. Thus $V^2(t_2^A) = 0$, since at time $t_2^A$ firm 2 is in autarky with beliefs $\pi_A$. Also $V^2(t) > 0$ for all $t < t_2^A$. For at any such time $\mu^2(t) > \pi_A$, so firm 2's autarky payoff as of time $t$ is strictly positive, and this payoff is a lower bound on $V^2(t)$.

Define $f_2(t) \equiv V^2(t) - K(\mu^2(t) - \pi_A)$. By Lemma D.13, for almost every $t \in [0, t_2^A]$ either $f_2(t) < 0$ or $f_2'(t) > 0$. Note also that $V^2(t_2^A) = 0$ and $\mu^2(t_2^A) = \pi_A$ implies $f_2(t_2^A) = 0$. We next establish that $\lambda^2(t) = \overline{\lambda}$ a.e. on $[0, t_2^A]$.

Suppose first that $f_2(t') > 0$ for some $t' \in [0, t_2^A)$. Then as $f_2(t_2^A) = 0$ and $f_2$ is absolutely continuous, there must exist a positive-measure subset of $[t', t_2^A]$ on which $f_2(t) > 0$ and $f_2'(t) < 0$, a contradiction. So certainly $f_2(t) \leq 0$ on $[0, t_2^A]$. If $f_2(t) = 0$ on a positive-measure subset of $[0, t_2^A]$, then a.e. on this set $f_2'(t) > 0$. But whenever $f_2(t) = 0$ and $f_2'(t) > 0$, the definition of $f_2'(t)$ implies that $f_2(t+\varepsilon) > f_2(t) = 0$ for sufficiently small $\varepsilon > 0$, a contradiction. So $f_2(t) < 0$ for almost every $t \in [0, t_2^A]$. Hence from the HJB equation $\lambda^2(t) = \overline{\lambda}$ for almost every $t \in [0, t_2^A]$.

This means in particular that $\mu^1(t) \leq \mu^2(t)$ for $t \in [0, t_2^A]$ and therefore $t_1^A \leq t_2^A$, no matter the prospecting policy firm 1 follows. If $t_1^A < t_2^A$, then the fact that firm 2 prospects with positive intensity and invests immediately on $[t_1^A, t_2^A]$ means $V^1(t_1^A) > 0$. Then as $\mu^1(t) \leq \pi_A$ for $t \geq t_1^A$, the HJB equation requires that $\lambda^1(t) = 0$ for almost every $t \in [t_1^A, t_2^A]$. But then $\mu^2$ is constant on this interval, a contradiction of the fact that $t_2^A$ is the first time $\mu^2$ hits $\pi_A$. So $t_1^A = t_2^A = t^A$, which can only hold if $\lambda^1(t) = \overline{\lambda}$ for almost every $t \in [0, t^A]$.

If $V^1(t^A) > 0$, then given continuity of $V^1$ and $\mu^1$, for sufficiently large $t < t^A$ it would be the case that $V^1(t) > K(\mu^1(t) - \pi_A)$. But then $\lambda^1(t) = 0$ by the HJB equation, a contradiction. So $V^1(t^A) = 0$. But as $T_2^* = \infty$ by Lemma D.15, this can only be true if $\lambda^2(t) = 0$ for a.e. $t > t^A$. As $\mu^1(t_1^A) = \pi_A$ by definition of $t_1^A$, it must be that $\mu^1(t) = \pi_A$ for all $t > t^A$, so $\mu_+^1(t) > 1/R$ on this time range. As firm 2 never invests along the equilibrium path after $t^A$, this means that investing immediately upon receiving a High signal must yield a strictly higher continuation payoff for firm 1 than waiting forever, contradicting $T_1^* < \infty$.

This is the desired contradiction ruling out $\mu^2(T_1^*) = \pi_A$. $\square$

*Proof of Lemma D.14.* Without loss suppose $i = 1$. By Lemma D.15 $T_2^* = \infty$, and by Lemma D.16 $\mu^2(T_1^*) > \pi^A$. The latter inequality implies that for $t > T_1^*$ firm 2 is in autarky with constant beliefs $\mu^2(t) = \mu^2(T_1^*) > \pi_A$, meaning 2's unique optimal prospecting policy subsequent to $T_1^*$ is $\lambda^2(t) = \overline{\lambda}$. It remains only to pin down firm 2's optimal prospecting behavior prior to $T_1^*$.

We proceed by establishing that $V^\dagger(t) = K(\mu^2(t) - \pi_A)$ is a strict supersolution to firm 2's HJB equation on $[0, T_1^*]$. Recall that the HJB equation for firm 2 in this regime is

$$rV^2(t) = \overline{\lambda}\left(K(\mu^2(t) - \pi_A) - V^2(t)\right) - \frac{\dot{\mu}^2(t)}{\pi_+ - \mu^2(t)}\left(\overline{V} - V^2(t)\right) + \dot{V}^2(t).$$

So define the functional

$$F(w, t) \equiv rw(t) - \overline{\lambda}\left(K(\mu^2(t) - \pi_A) - w(t)\right) + \frac{\dot{\mu}^2(t)}{\pi_+ - \mu^2(t)}\left(\overline{V} - w(t)\right) - \dot{w}(t).$$

The claim that $V^\dagger$ is a strict supersolution is equivalent to $F(V^\dagger, t) > 0$ for $t < T_1^*$. Evaluating the functional at $V^\dagger$ yields

$$F(V^\dagger, t) = rV^\dagger(t) + \frac{\dot{\mu}^2(t)}{\pi_+ - \mu^2(t)}(\overline{V} - K(\pi_+ - \pi_A)).$$

Note that $\overline{V} \leq K(\pi_+ - \pi_A)$ by Lemma C.2, so the second term on the rhs is non-negative. Meanwhile $\mu^2(t) > \pi_A$ for $t \leq T_1^*$, so $F(V^\dagger, t) > 0$ as claimed.

Now note that as firm 2 is in autarky at time $T_1^*$, its value function at this point is $V^2(T_1^*) = \frac{\overline{\lambda}}{\overline{\lambda}+r}K(\mu^2(T_1^*) - \pi_A) < V^\dagger(T_1^*)$. This boundary condition combined with the fact that $V^\dagger$ is a strict supersolution implies $V^\dagger(t) > V^2(t)$ for all $t \in [0, T_1^*]$. The HJB equation then implies that $\lambda^2(t) = \overline{\lambda}$ is the unique best reply for firm 2. In other words, in equilibrium firm 2 must employ a threshold prospecting strategy with $\overline{T}_2 = \infty$. $\square$

Lemma D.14 establishes that in any equilibrium in threshold investment strategies in which some $T_i^* < \infty$, then the other firm must follower the leader's strategy. Meanwhile Lemma D.5 establishes that the follower's strategy is a unique best reply to the leader's strategy. So there exists a unique equilibrium in threshold investment strategies with some $T_i^* < \infty$, namely the leader-follower equilibrium.

The following lemma treats the remaining case, in which $T_1^* = T_2^* = \infty$. It establishes that the symmetric equilibrium strategies are the only ones consistent with equilibrium in this case.

**Lemma D.17.** *Fix any perfect Bayesian equilibrium in threshold investment strategies in which* $T_1^* = T_2^* = \infty$. *Then* $\lambda^1(t) = \lambda^2(t) = \overline{\lambda}$ *for every* $t < T^A$, *while* $\lambda^1(t) = \lambda^2(t) = 0$ *for every* $t > T^A$.

*Proof.* Let $f_i(t) \equiv V^i(t) - K(\mu^i(t) - \pi_A)$ be the term in firm $i$'s HJB equation whose sign determines $i$'s optimal prospecting rate. We show first that each $t_i^A < \infty$. To establish this, maintain for the time being the opposite assumption, that some $t_i^A = \infty$, say $i = 1$.

By Lemma D.13, for almost every $t$ either $f_1(t) < 0$ or $f_1'(t) > 0$. If $f_1(t) < 0$ a.e., then the HJB equation would imply that $\lambda^1(t) = \overline{\lambda}$ a.e. But in this case eventually $\mu_+^2(t) < 1/R$, meaning $t_2^* < \infty$, a contradiction of $t_1^* = \infty$. So on some positive-measure set of times $T^\dagger$, $f_1(t) \geq 0$ and $f_1'(t) > 0$. But whenever $f_1(t) \geq 0$ and $f_1'(t) > 0$, the definition of $f_1'(t)$ implies that $f_1(t + \varepsilon) > 0$ for sufficiently small $\varepsilon > 0$. Therefore $t^0 \equiv \inf\{t \ : \ f_1(t) > 0\} < \infty$.

Suppose $f_1(t') \leq 0$ for some $t' > t^0$. Then by definition of $t^0$ there exists a $t'' < t'$ such that $f_1(t'') > 0$, and so absolute continuity of $f_1$ implies that $f_1(t) > 0$ and $f_1'(t) < 0$ on some positive-measure subset of $[t'', t']$. This contradicts our earlier finding, so $f_1(t) > 0$ for all $t > t^0$. Hence the HJB equation implies that $\lambda^1(t) = 0$ for a.e. $t > t^0$.

Further, by definition of $t^0$ it must be that $f_1(t) \leq 0$ for all $t < t^0$. If $f_1(t) = 0$ on some positive-measure subset of $[0, t^0)$, then $f_1'(t) > 0$ a.e. on this set and so there would exist a $t' < t^0$ such that $f_1(t') > 0$ a contradiction. Hence $f_1(t) < 0$ for almost every $t \in [0, t^0]$, implying by the HJB equation that $\lambda^1(t) = \overline{\lambda}$ almost everywhere on $[0, t^0]$.

The fact that firm 1 does not prospect subsequent to $t^0$ means that firm 2 is in autarky after $t^0$. If $\mu^2(t^0) < \pi_A$, then its optimal prospecting rate is 0. But the assumption of $t_1^A = \infty$ means $\mu^1(t^0) > \pi_A$, so firm 1 would also also be autarky after $t^0$ with beliefs above the autarky threshold, contradicting the optimality of $\lambda^1(t) = 0$. On the other hand, if $\mu^2(t^0) > \pi_A$, then firm 2's optimal prospecting rate after $t^0$ is $\overline{\lambda}$ forever. But in this case eventually $\mu_+^1(t) < 1/R$, meaning $t_1^* < \infty$, a contradiction of our assumption. So it must be that $\mu^2(t^0) = \pi_A$. This implies in particular that $V^2(t^0) = 0$ and $f_2(t^0) = 0$ given that firm 2 is in autarky after $t^0$.

Meanwhile $\mu^2(t) > \pi_A$ for all $t < t^0$ given that firm 1 prospects at a strictly positive rate and invests immediately until $t^0$. So $t^0 = t_2^A$, and Lemma D.13 tells us that for almost every $t \in [0, t^0]$, either $f_2(t) < 0$ or $f_2'(t) > 0$. If ever $f_2(t') > 0$ for some $t' \in [0, t^0)$, then $f_2(t^0) = 0$ and absolute continuity of $f_2$ would imply $f_2(t) > 0$ and $f_2'(t) < 0$ on a positive-measure subset of $[t', t^0]$, a contradiction. So $f_2(t) \leq 0$ on $[0, t^0]$, and further $f_2(t) < 0$ for almost every $t \in [0, t^0]$. Thus by the HJB equation $\lambda^2(t) = \overline{\lambda}$ a.e. on $[0, t^0]$, implying $\mu^1(t^0) = \pi_A$

given that $\mu^2(t^0) = \pi_A$ and both firms employ the same prospecting and investing strategy prior to $t^0$. This yields the desired contradiction of our hypothesis that $t_1^A = \infty$.

So it must be that each $t_i^A < \infty$. Wlog we assume that $t_1^A \leq t_2^A$ going forward. We next prove that $V^1(t_1^A) = 0$. For the time being, suppose instead that $V^1(t_1^A) > 0$.

Given the hypothesis on $V^1(t_1^A)$, the definition of $t_1^A$, and the continuity of $V^1$ and $\mu^1$, for sufficiently large $t < t_1^A$ it must be that $f_1(t) > 0$. Hence $\lambda^1(t) = 0$ from the HJB equation, meaning $\mu^2(t)$ is constant and therefore $t_2^A > t_1^A$. If there existed a $t' \in (t_1^A, t_2^A)$ such that $\mu^1(t) < \pi_A$, then $\lambda^1(t) = 0$ a.e. on $[t', t_2^A]$, meaning $\mu^2(t)$ would be constant on that interval, a contradiction of the definition of $t_2^A$. So $\mu^1(t) = \pi_A$ on $[t_1^A, t_2^A]$, reducing the HJB equation for $V^1$ to $rV^1(t) = \dot{V}^1(t)$, with solution $V^1(t) = e^{r(t-t_1^A)}V^1(t_1^A)$. So $V^1(t) > 0$ on $[t_1^A, t_2^A]$, meaning that $\lambda^1(t) = 0$ a.e. on the interval, yielding a constant $\mu^2$ and another contradiction. So it must be that $V^1(t_1^A) = 0$ and hence also $f_1(t_1^A) = 0$.

Given $f_1(t_1^A) = 0$, a nearly identical argument to that applied to $f_2$ prior to $t^0$ implies that $\lambda^1(t) = \overline{\lambda}$ a.e. on $[0, t_1^A]$. Then surely $t_2^A \leq t_1^A$ no matter what prospecting policy firm 2 chooses, meaning $t_1^A = t_2^A$ and $\lambda^2(t) = \overline{\lambda}$ for $t \leq t_1^A = t_2^A$. Given these prospecting strategies, it must be that $\hat{t}$ it must be that $t_1^A = t_2^A = T^A$.

Finally, suppose that for some $i$ and some positive-measure subset of $[T^A, \infty)$, that $\lambda^i(t) > 0$. Then given $t_i^* = \infty$, firm $-i$'s value at $T^A$ would be strictly positive. But then for sufficiently large $t < T^A$ we would have $f_{-i}(t) > 0$, contradicting $\lambda^{-i} = \overline{\lambda}$. So it must be that $\lambda^1(t) = \lambda^2(t) = 0$ a.e. on $[T^A, \infty)$. □

We now return briefly to the restriction to equilibria in threshold investment strategies employed throughout this proof. As mentioned earlier, this does not rule out any equilibrium paths, and any excluded equilibria, if any exist, must differ only in their investment behavior off-path. Further, any non-threshold investment behavior can occur only in case for some firm $i$ eventually $\mu_+^i(t) = 1/R$ forever. But in the symmetric equilibrium neither firm's beliefs drop below $1/R$, meaning $\mu_+^i > 1/R$ for each $i$; and in the leader-follower equilibrium the leader's beliefs lie always strictly above $\pi^A$ while the follower's beliefs have limit $\pi_-$, meaning that eventually beliefs following a High signal lie strictly below $1/R$ given $\pi_{+-} < 1/R$. So no equilibria in non-threshold investment strategies exist.

We complete the proof by ruling out mixed prospecting rules in equilibrium. This is accomplished by the following lemma, which establishes that any equilibrium involving randomization over prospecting co-exists with a pure-strategy equilibrium involving interior prospecting. As none of the equilibria characterized above exhibit such behavior, it was without loss to ignore the possibility of mixed prospecting strategies.

**Lemma D.18.** *Fix any perfect Bayesian in threshold investment strategies. Then there exists a payoff-equivalent perfect Bayesian equilibrium in pure strategies, exhibiting interior prospecting for any firm and time at which the firm randomized over prospecting in the original equilibrium.*

*Proof.* Fix a perfect Bayesian equilibrium in threshold investment strategies. As threshold investment strategies are automatically pure strategies, we need only consider randomization over prospecting. Suppose that some firm $-i$ mixes over prospecting, with prospecting rule $\lambda^{-i}$ conditioning on firm $-i$'s randomization device. Let $T^*_{-i}$ be firm $-i$'s cutoff time for investment. After this time, firm $-i$'s prospecting rule does not affect firm $i$'s payoffs or incentives at all; thus $\lambda^{-i}$ may be replaced with any pure strategy maximizing $-i$'s payoffs subsequent to time $T^*_{-i}$ without disturbing the equilibrium. So consider times $t < T^*_{-i}$.

Generalize the definition of $\Omega^{-i}$ from Appendix B by letting

$$\Omega^{-i}(t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda^{-i}(s)\,ds\right)\right].$$

Define a new pure-strategy prospecting rule $\widetilde{\lambda}^{-i}$ by letting

$$\widetilde{\lambda}^{-i}(t) = -\frac{d}{dt}\log\Omega^{-i}(t)$$

for all times (with the prospecting rule arbitrary at any point of non-differentiability of $\Omega^{-i}$.) The two strategies $\lambda^{i}$ and $\widetilde{\lambda}^{i}$ induce the same sequence of posterior beliefs for firm $i$ about the state conditional on observing no investment, by construction. Further, both prospecting rules induce the same distribution of investment times by $-i$. Thus firm $i$'s incentives are unchanged by replacing $\lambda^{-i}$ with $\widetilde{\lambda}^{-i}$.

It remains to check that $\widetilde{\lambda}^{-i}$ is both feasible and optimal for firm $-i$. Note that

$$\widetilde{\lambda}^{-i}(t) = \frac{1}{\Omega^{-i}(t)}\mathbb{E}\left[\lambda^{-i}(t)\exp\left(-\int_0^t \lambda^{-i}(s)\,ds\right)\right].$$

The second term is bounded above by $\overline{\lambda}\Omega^{-i}(t)$ and below by zero, hence $\widetilde{\lambda}^{-i}(t) \in [0, \overline{\lambda}]$, ensuring feasibility. As for optimality, suppose first that time $t$, the action $\lambda^{-i}(t)$ is strictly optimal for firm $-i$. Then it must be non-random, in which case

$$\widetilde{\lambda}^{-i}(t) = \frac{1}{\Omega^{-i}(t)}\lambda^{-i}(t)\mathbb{E}\left[\exp\left(-\int_0^t \lambda^{-i}(s)\,ds\right)\right] = \lambda^{-i}(t).$$

So at any times for which randomization is not optimal for firm $-i$, the modified prospecting rule specifies the same prospecting intensity as the original rule. And at all other times, any prospecting intensity is optimal, thus in particular the intensity specified by $\widetilde{\lambda}^{-i}$ is optimal. So $\widetilde{\lambda}^{-i}$ is an optimal prospecting rule.

This argument shows that firm $-i$'s randomized prospecting rule may be replaced by a non-random one without disturbing payoffs, the optimality of $-i$'s strategy, or firm $i$'s incentives. This procedure may be performed for both firms, yielding a pure strategy PBE.

Finally, for any time $t$ at which $\lambda^{-i}(t)$ is not deterministic, it must be that $\Pr(\lambda^{-i}(t) \in (0, \overline{\lambda})) > 0$, meaning $\widetilde{\lambda}^{-i}(t) \in (0, \overline{\lambda})$. So randomization by some firm at some time in the original equilibrium yields an interior prospecting rate by that firm at the same time in the new equilibrium. □

## D.9 Proof of Proposition 4

Consider first the symmetric equilibrium. In this equilibrium, one best response for each firm is the equilibrium strategy of prospecting at the maximum rate until time $T^A$ and then abandoning the project. However, for $t \geq T^A$ each firm is in autarky with beliefs fixed at $\pi_A$. Hence another best response subsequent to time $T^A$ is to continue prospecting at the maximum rate forever and to invest if a High signal is received. Hence there exists a best response of the following sort - as long as there is no investment by the other firm, prospect forever until receiving a signal; if the other firm invests, continue prospecting forever until receiving a signal; and when a signal is received, invest immediately iff it is High, otherwise never invest. But this strategy is exactly the optimal strategy under autarky. And as the presence of the other firm brings informational externalities but no payoff externalities, this strategy must yield the autarky payoff. Hence each firm's equilibrium payoff must be the same as the autarky payoff: $V^S = V^A$.

In the leader-follower equilibrium, the follower never invests before the leader. Thus the leader is effectively in autarky and receives his autarky payoff: $V^L = V^A$. Meanwhile, suppose the leader plays its equilibrium strategy. By Lemma D.5 and the fact that $T^* = 0$ under complementary signals, it is a unique optimal continuation strategy for the follower to wait forever upon receiving a High signal at any time. Note that this is true regardless of the follower's prospecting strategy. So consider the payoff to the follower of employing a modified autarky strategy which preserves the autarky prospecting rule but waits forever rather than investing immediately after obtaining a High signal. This modification must strictly improve on the payoff of playing the autarky strategy, given the positive probability of obtaining a signal under this strategy and the unique optimality of waiting upon receiving a signal at any point in time. And the follower's equilibrium payoff must be at least as high as this modified autarky strategy, so in equilibrium the follower earns strictly more than by playing the autarky strategy. Meanwhile, playing the autarky strategy yields $V^A$ regardless of the leader's strategy, so $V^F > V^A$.

## D.10 Proof of Proposition 8

Suppose some firm, say firm 1, plays the follower's equilibrium strategy, and consider firm 2's payoff from playing a modification of the leader's equilibrium strategy in which it prospects for another signal following investment by the follower. (In what follows, we will use $T^*$ and $\overline{T}$, without subscripts, to denote the follower's investment and prospecting cutoffs, as there will be no ambiguity by doing so.) This is a strictly suboptimal continuation strategy, and so delivers a payoff lower than the payoff of the leader's strategy if firm 1 invests on the equilibrium path. This occurs iff $\min\{\overline{T}, T^*\} > 0$. If not, the modified strategy delivers the same payoff as the leader's strategy. Following the same argument as in the proof of the complements case, the modified strategy yields the same payoff to firm 1 as its autarky strategy. Thus $V^L \geq V^A$, with the inequality strict iff $\min\{\overline{T}, T^*\} > 0$.

Next, suppose firm 1 plays the symmetric equilibrium strategy. Let firm 2 play a modification of the symmetric equilibrium strategy, in which it continues prospecting forever after time $T^A$. Since it is indifferent between prospecting at all times after $T^A$, this yields the same payoff $V^S$ as its equilibrium strategy. Now modify the strategy further, so that if firm 1 invests subsequent to time $\min\{\overline{T}, T^*\}$, firm 2 prospects for another signal rather than investing immediately. This is a strictly suboptimal continuation strategy, and it arises on the equilibrium path given that $\min\{\overline{T}, T^*\} < T^A$. So this modification yields firm 2 a strictly lower payoff than $V^S$. Further, this final strategy yields firm 2 the same payoff $V^L$ as its equilibrium payoff as the leader in the leader-follower equilibrium. This is because subsequent to time $\min\{\overline{T}, T^*\}$, under the modified strategy firm 2 ignores firm 1's actions and plays its autarky strategy, just as when it is the leader in the leader-follower equilibrium; and prior to that time, both firm's behaviors are the same in the two environments. So $V^S > V^L$.

Now, suppose both firms play the symmetric equilibrium strategy, and modify each firm's strategy as follows. First, modify firm 2's strategy so that if it observes investment subsequent to time $T^A$, it does nothing forever after. Given firm 1's strategy, which involves no investment after time $T^A$ on the equilibrium path, this modification does not change firm 2's payoff, which continues to be $V^S$. Next, modify firm 1's strategy so that firm 1 prospects at all times after $T^A$. Under this modification, firm 2's payoff continues to be $V^S$, as under its current strategy it does not react to any firm 1 actions after time $T^A$. Now revert firm 2's strategy back to the symmetric equilibrium strategy. Note that its payoff strictly rises from doing so, as firm 1 invests on the equilibrium path after $T^A$ and investing following an investment by firm 1 is a unique optimal continuation strategy for firm 2. So there exists a strategy for firm 2 yielding profits strictly higher than $V^S$ in response to firm 1's current strategy. But note further that firm 1 is currently playing the leader's equilibrium strategy. Thus the payoff to firm 2 from playing the follower's equilibrium strategy must be at least as high as any other strategy, meaning $V^F > V^S$.

The remainder of the proof is devoted to comparing aggregate welfare across the two equilibria. First, note that the symmetric and leader's equilibrium strategy are independent of $r$, and thus expected discounted profits $V^L$ and $V^S$ are immediately continuous in $r$. Meanwhile the follower's equilibrium strategy maximizes expected discounted profits over two scalar time thresholds at which prospecting and investing cease. Then as the expected profit function is continuous in $r$ for any choice of thresholds, by the theorem of the maximum $V^S$ is continuous in $r$ as well. Now note that $V^L + V^F > 2V^S$ for the threshold discount rate $r^*$ at which

$$\frac{\overline{\lambda}}{\overline{\lambda} + r^*}(h(\pi_+)(\pi_{++}R - 1) - c) = \pi_+ R - 1,$$

i.e. the largest $r$ under which signals are complementary. Continuity therefore implies that $V^L + V^F > 2V^S$ for $r > r^*$ sufficiently small.

We now consider the limit of large $r$. We proceed by explicitly calculating each value as a sum of expected discounted flow profits over each moment of time in which a firm has not received a signal or seen the other firm invest. The flow of profits accounts for both any effort expended in that instant, as well as the probability of arrival of a signal or another investment, which each yield an additional flow of expected profits. In the symmetric equilibrium, these profits are just

$$rV^S = \int_0^{T^A} dt\, r e^{-rt} e^{-\overline{\lambda} t} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) (\pi_1(t) + \pi_2(t)),$$

where

$$\pi_1(t) \equiv \overline{\lambda} \left(h(\mu^{\overline{\lambda}}(t)) \left(\mu_+^{\overline{\lambda}}(t) R - 1\right) - c\right)$$

and

$$\pi_2(t) \equiv -\frac{\dot{\mu}^{\overline{\lambda}}(t)}{\pi_+ - \mu^{\overline{\lambda}}(t)} (\pi_+ R - 1).$$

The terms $e^{-rt}e^{-\overline{\lambda}t}\left(1-\left(1-e^{-\overline{\lambda}t}\right)h(\pi_0)\right)$ discount for time and the probability that the firm has not yet received a signal or seen investment by the other firm. Meanwhile the term $\pi_1(t)+\pi_2(t)$ capture the firm's expected flow profits in this information set; $\pi_1(t)$ accounts for effort costs and the arrival of a signal, while $\pi_2(t)$ accounts for arrival of an investment by the other firm. The derivation of these flow profit representations is as in the HJB equation of Appendix C.

To calculate leader and follower flow profits, we first show that when $r$ is large, $\overline{T} < T^*$. To see this, note that $\overline{T} = (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$ and $T^* = (\mu^{\overline{\lambda}})^{-1}(\mu^*)$, where for large $r$ $\overline{\mu}$ solves

$$\frac{\mu-\pi_-}{\pi_+-\pi_-}\frac{\overline{\lambda}}{\overline{\lambda}+r}(\pi_+R-1) = h(\mu)(\mu_+R-1)-c,$$

while $\mu^*$ solves

$$\frac{\mu_+-\pi_{+-}}{\pi_{++}-\pi_{+-}}\frac{\overline{\lambda}}{\overline{\lambda}+r}(\pi_{++}R-1) = \mu_+R-1.$$

Thus for large $r$ $\overline{\mu}$ is close to the solution to $h(\mu)(\mu_+R-1)-c=0$, i.e. $\pi_A$, while $\mu^*$ is close to the solution to $\mu_+R-1=0$. This implies in particular that $\overline{\mu} > \mu^*$, i.e. $\overline{T} < T^*$. Note also that $\overline{\mu} > \pi_A$, so $\overline{T} < T^A$ for large $r$.

Using this result, for large $r$ expected profits for the leader may be written

$$rV^L = \int_0^\infty dt\, re^{-rt}e^{-\overline{\lambda}t}\left(1-\left(1-e^{-\overline{\lambda}\min\{t,\overline{T}\}}\right)h(\pi_0)\right)\left(\pi_1(\min\{t,\overline{T}\})+\mathbf{1}\{t<\overline{T}\}\pi_2(t)\right).$$

An equivalent, more convenient representation of this expression is obtained by explicitly evaluating the integral for times $t \geq \overline{T}$, yielding

$$\begin{aligned} rV^L = &\int_0^{\overline{T}} dt\, re^{-rt}e^{-\overline{\lambda}t}\left(1-\left(1-e^{-\overline{\lambda}t}\right)h(\pi_0)\right)(\pi_1(t)+\pi_2(t)) \\ &+\frac{r}{\overline{\lambda}+r}e^{-r\overline{T}}e^{-\overline{\lambda}\overline{T}}\left(1-\left(1-e^{-\overline{\lambda}\overline{T}}\right)h(\pi_0)\right)\pi_1(\overline{T}). \end{aligned}$$

Finally, in the same large-$r$ regime follower profits are

$$rV^F = \int_0^\infty dt\, re^{-rt}e^{-\overline{\lambda}\min\{t,\overline{T}\}}\left(1-\left(1-e^{-\overline{\lambda}t}\right)h(\pi_0)\right)\left(\mathbf{1}\{t<\overline{T}\}\pi_1(t)+\pi_2(t)\right).$$

We now compute the difference $r(2V^S-V^L-V^F)$. Note that each of $V^S, V^L, V^F$ contains an identical integral over the range $[0,\overline{T}]$, so these terms cancel out in the sum. Each remaining term contains a common overall discount of $e^{-r\overline{T}}$, which does not affect the sign of the sum and may be factored out. What remains is

$$
\begin{aligned}
re^{r\overline{T}}(2V^S - V^L - V^F) = 2 &\int_{\overline{T}}^{T^A} dt\, re^{-r(t-\overline{T})} e^{-\overline{\lambda} t} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) (\pi_1(t) + \pi_2(t)) \\
&- \frac{r}{\overline{\lambda} + r} e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda}\overline{T}}\right) h(\pi_0)\right) \pi_1(\overline{T}) \\
&- \int_{\overline{T}}^{\infty} dt\, re^{-r(t-\overline{T})} e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) \pi_2(t).
\end{aligned}
$$

We now evaluate the limit of this expression as $r \to \infty$. First, note that $\overline{T} \to T^A$ as $r \to \infty$, and $\pi_1(T^A) = 0$ by definition of $T^A$. Thus the second term vanishes in the limit. To evaluate the third integral, make the substitution $t' = r(t - \overline{T})$ and write

$$
\begin{aligned}
&\int_{\overline{T}}^{\infty} dt\, re^{-r(t-\overline{T})} e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) \pi_2(t) \\
&= \int_0^{\infty} dt'\, e^{-t'} e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda}(t'/r + \overline{T})}\right) h(\pi_0)\right) \pi_2(t'/r + \overline{T}).
\end{aligned}
$$

As $\pi_2$ is bounded, the entire integrand is uniformly bounded for all $t$ and $r$. Then by the bounded convergence theorem,

$$
\begin{aligned}
&\lim_{r\to\infty} \int_{\overline{T}}^{\infty} dt\, re^{-r(t-\overline{T})} e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) \pi_2(t) \\
&= \int_0^{\infty} dt'\, e^{-t'} e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda} T^A}\right) h(\pi_0)\right) \pi_2(T^A) \\
&= e^{-\overline{\lambda}\overline{T}} \left(1 - \left(1 - e^{-\overline{\lambda} T^A}\right) h(\pi_0)\right) \pi_2(T^A).
\end{aligned}
$$

Finally, to evaluate the remaining integral, make the substitution $t' = r(t - \overline{T})$ and write

$$
\begin{aligned}
&\int_{\overline{T}}^{T^A} dt\, re^{-r(t-\overline{T})} e^{-\overline{\lambda} t} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) (\pi_1(t) + \pi_2(t)) \\
&= \int_0^{\infty} dt'\, e^{-t'} e^{-\overline{\lambda}(t'/r + \overline{T})} \left(1 - \left(1 - e^{-\overline{\lambda}(t'/r + \overline{T})}\right) h(\pi_0)\right) \\
&\qquad \times (\pi_1(t'/r + \overline{T}) + \pi_2(t'/r + \overline{T})) \mathbf{1}\{t' \le r(T^A - \overline{T})\}.
\end{aligned}
$$

**Lemma D.19.** $\lim_{r\to\infty} r(T^A - \overline{T}) = (\pi_+ R - 1)/(h(\pi_+)(\pi_{++} R - 1) - c)$.

*Proof.* Recall that $T^A = (\mu^{\overline{\lambda}})^{-1}(\pi_A)$ while $\overline{T} = (\mu^{\overline{\lambda}})^{-1}(\overline{\mu})$, so to first order in $r^{-1}$,

$$
T^A - \overline{T} = -\frac{1}{\dot{\mu}^{\overline{\lambda}}(T^A)} (\overline{\mu} - \pi_A) + O(r^{-2}).
$$

Next, for large $r$ $\overline{\mu}$ solves

$$\frac{\mu - \pi_-}{\pi_+ - \pi_-}\frac{\overline{\lambda}}{\overline{\lambda} + r}(\pi_+ R - 1) = h(\mu)(\mu_+ R - 1) - c.$$

Recall the representation $h(\mu)(\mu_+ R - 1) - c = K(\mu - \pi_A)$ derived in Lemma C.1. This is therefore a linear equation in $\mu$, with solution

$$\overline{\mu} = \frac{K\pi_A - \frac{J\pi_-}{\overline{\lambda}+r}}{K - \frac{J}{\overline{\lambda}+r}},$$

where $J \equiv \overline{\lambda}(\pi_+ R - 1)/(\pi_+ - \pi_-)$. To first order in $r^{-1}$,

$$\overline{\mu} = \pi_A + \frac{\frac{J}{K}(\pi_A - \pi_-)}{\overline{\lambda} + r} + O(r^{-2}).$$

Thus

$$T^A - \overline{T} = -\frac{\overline{\lambda}(\pi_A - \pi_-)}{(\pi_+ - \pi_-)\dot{\mu}^{\overline{\lambda}}(T^A)}\frac{\pi_+ R - 1}{K(\overline{\lambda} + r)} + O(r^{-2}).$$

Now, Lemma B.1 implies the identity

$$\frac{\dot{\mu}^{\overline{\lambda}}(T^A)}{\pi_+ - \pi_A} = -\overline{\lambda}\frac{\pi_A - \pi_-}{\pi_+ - \pi_-}.$$

Hence

$$\begin{aligned} T^A - \overline{T} &= \frac{\pi_+ R - 1}{K(\pi_+ - \pi_A)(\overline{\lambda} + r)} + O(r^{-2}) \\ &= \frac{\pi_+ R - 1}{(h(\pi_+)(\pi_{++} R - 1) - c)(\overline{\lambda} + r)} + O(r^{-2}). \end{aligned}$$

So

$$\lim_{r\to\infty} r(T^A - \overline{T}) = \lim_{r\to\infty} \frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++} R - 1) - c}\frac{r}{\overline{\lambda} + r} = \frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++} R - 1) - c}.$$

□

Now, note that the $\pi_1$ and $\pi_2$ are both bounded functions, so the bounded convergence theorem implies

$$
\begin{aligned}
&\lim_{r\to\infty} \int_{\overline{T}}^{T^A} dt\, r e^{-r(t-\overline{T})} e^{-\overline{\lambda} t} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) (\pi_1(t) + \pi_2(t)) \\
&= \int_0^{\infty} dt'\, e^{-t'} e^{-\overline{\lambda} T^A} \left(1 - \left(1 - e^{-\overline{\lambda} T^A}\right) h(\pi_0)\right) \\
&\qquad \times (\pi_1(T^A) + \pi_2(T^A)) \mathbf{1}\{t' \le C\} \\
&= \left(1 - \exp\left(-\frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++} R - 1) - c}\right)\right) e^{-\overline{\lambda} T^A} \left(1 - \left(1 - e^{-\overline{\lambda} T^A}\right) h(\pi_0)\right) (\pi_1(T^A) + \pi_2(T^A)).
\end{aligned}
$$

Finally, note that $\pi_1(T^A) = 0$ by definition of $T^A$ to obtain the result

$$
\begin{aligned}
&\lim_{r\to\infty} \int_{\overline{T}}^{T^A} dt\, r e^{-r(t-\overline{T})} e^{-\overline{\lambda} t} \left(1 - \left(1 - e^{-\overline{\lambda} t}\right) h(\pi_0)\right) (\pi_1(t) + \pi_2(t)) \\
&= \left(1 - \exp\left(-\frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++} R - 1) - c}\right)\right) e^{-\overline{\lambda} T^A} \left(1 - \left(1 - e^{-\overline{\lambda} T^A}\right) h(\pi_0)\right) \pi_2(T^A).
\end{aligned}
$$

Combining these calculations yields

$$
\begin{aligned}
&\lim_{r\to\infty} r(2V^S - V^L - V^F) \\
&= \left(1 - 2\exp\left(-\frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++} R - 1) - c}\right)\right) e^{-\overline{\lambda} T^A} \left(1 - \left(1 - e^{-\overline{\lambda} T^A}\right) h(\pi_0)\right) \pi_2(T^A).
\end{aligned}
$$

Since $\pi_2(T^A) > 0$, the sign of $2V^S - V^L - V^F$ for large $r$ is therefore the same as the sign of $1 - 2\exp(-(\pi_+ R - 1)/(h(\pi_+)(\pi_{++} R - 1) - c))$. In particular, $2V^S > V^L + V^F$ for large $r$ whenever

$$\frac{\pi_+ R - 1}{h(\pi_+)(\pi_{++} R - 1) - c} > \log 2.$$

Letting

$$\underline{c} \equiv h(\pi_+)(\pi_{++} R - 1) - \frac{1}{\log 2}(\pi_+ R - 1),$$

this condition is equivalent to $c > \underline{c}$, given the additional constraint that $c \le \overline{c}$ by Assumption 4. Note that $\underline{c}$ is independent of $r$ and $\underline{c} < \overline{c}$ given that $\log 2 < 1$, as claimed in the proposition statement. Finally, note that as $q^H$ approaches 1, $h(\pi_+)(\pi_{++} R - 1)$ approaches $\pi_+ R - 1$, in which case $\underline{c} < 0$. This limit can be taken consistent with the other assumptions on model parameters by taking $q^L$ sufficiently close to $q^H$.