Single-Crossing Differences on Distributions*

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Abstract

We characterize when choices among lotteries over arbitrary allocations are monotonic in an expected-utility agent's type. Our necessary and sufficient condition is on the von Neumann-Morgenstern utility function; we identify an order over lotteries that generates the choice monotonicity when the condition holds. We discuss applications to cheap-talk games, costly signaling games, and collective choice problems. Our characterization requires some new results on monotone comparative statics and aggregating single-crossing functions, a by-product of which is a characterization of the monotone likelihood ratio property.

1. Introduction

There are many economic and game-theoretic problems in which agents are faced with a choice among lotteries. The outcomes of these lotteries could be monetary prizes, political policies, or any number of other variables. Suppose the agent is an expected-utility maximizer whose utility over outcomes depends on some parameter or type. When choosing among lotteries, which von Neumann-Morgenstern utility functions ensure monotone comparative statics (MCS): choice monotonicity in the agent's type, no matter the choice set?

MCS are tied to single-crossing properties (Milgrom and Shannon, 1994, Theorem 4). More specifically, then, this paper's central question is: given a von Neumann-Morgenstern

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utility function $v(a, \theta)$, where $a \in A$ is an outcome and $\theta \in \Theta$ the agent's type, is the difference in expected utility between every pair of probability distributions on A single crossing (either from below or above) in θ?[1] We call this property *single-crossing expectational differences*, SCED.

When there are only two outcomes, $A = \{a_1, a_2\}$, it is straightforward to check that SCED holds if and only if $v(a, \theta)$ has single-crossing differences, i.e., if and only if $v(a_1, \theta) - v(a_2, \theta)$ is single crossing. When $|A| > 2$, single-crossing differences of $v(a, \theta)$ is not enough, as shown by Example 1.

Our main results are in Section 2. Theorem 1 provides a full characterization of utility functions $v(a, \theta)$ that have SCED. The characterization owes to the structure of expected utility. Roughly, it requires a pair of type-*independent* utility functions, say $v_1(a)$ and $v_2(a)$, such that there is a representation of each type's preferences as a convex combination of v_1 and v_2, with the relative weight shifting monotonically in type. We confirm that SCED is satisfied by canonical functional forms used in mechanism designing and screening, such as $v((q, t), \theta) = q\theta - t$ (where $q \in \mathbb{R}$ is quantity, $t \in \mathbb{R}$ is a transfer, and $\theta \in \mathbb{R}$ is the agent's marginal rate of substitution; Corollary 2), and those used in communication and voting, such as $v(a, \theta) = -(a - \theta)^2$ (where $a \in \mathbb{R}$ is a policy and $\theta \in \mathbb{R}$ is the agent's bliss point; Corollary 1). On the other hand, our characterization also makes clear that SCED is quite restrictive. For example, within a familiar class of loss functions, the quadratic loss function is the only one that satisfies SCED (Corollary 1).

We show that SCED is necessary and sufficient for a form of MCS in our setting (Corollary 4). More broadly, we establish a novel result—independent of expected utility or choice among lotteries—that single-crossing differences characterizes this form of MCS (Theorem 3).[2]

We believe the characterization of SCED will be useful in applications. Section 3 elaborates, but to illustrate now, consider a sender-receiver cheap-talk problem à la Crawford and Sobel (1982). When the receiver's preferences are known—and satisfy some common assumptions, like concavity—each message of the sender will, in equilibrium, induce a particular action from the receiver. Consequently, a standard single-crossing property on the sender's preferences ensures monotonicity of communication in any equilibrium: the set of sender types that induce a particular action is connected. But what about when the sender does not know the receiver's preferences? Then, for typical receiver strategies, the

[1] Informally, a real-valued function is single crossing if crosses over 0 at most once; see Definition 1.

[2] Milgrom and Shannon (1994, Theorem 4) use a related but different notion of choice monotonicity; their notion is characterized by the combination of (a version of) single-crossing differences and quasi-supermodularity.

sender will view each message as a non-trivial lottery over the receiver's actions. SCED is the requisite property for "interval cheap talk".

A key step towards our results on SCED is a characterization of when a family of real-valued functions, $\{f_i(\theta)\}_{i=1}^n$, has the property that all linear combinations $\sum_i \alpha_i f_i(\theta)$ for $\alpha \in \mathbb{R}^n$ are single crossing (Proposition 1). This result relates to Quah and Strulovici (2012), as elaborated in Section 4; it implies, in particular, a characterization of the monotone likelihood-ratio property. Section 4 also compares our analysis of SCED to that of Kushnir and Liu (2017), which concerns monotonic expectational differences.[3]

Section 5 is the paper's conclusion. The Appendices contain various proofs and additional material.

2. Main Results

Let A be an arbitrary set endowed with some σ-algebra such that all elements of A are measurable and ΔA be the set of probability measures on A and its σ-algebra. Let $(\Theta, \leq)$ be a (partially) ordered set containing upper and lower bounds for its pairs.[4] We often refer to elements of Θ as *types*. Let $v : A \times \Theta \to \mathbb{R}$ be a utility function such that, for every θ, $v(\cdot, \theta) : A \to \mathbb{R}$ is integrable with respect to any probability measure. Define the expected utility $V : \Delta A \times \Theta \to \mathbb{R}$ as

$$V(P, \theta) \equiv \int_A v(a, \theta) \mathrm{d}P.$$

For any two probability measures, also referred to as distributions or lotteries, $P \in \Delta A$ and $Q \in \Delta A$,

$$D_{P,Q}(\theta) \equiv V(P, \theta) - V(Q, \theta)$$

[3] Athey (2002) considers the following problem. Let $u(a, s)$ be the payoff from action $a \in A$ in "state" $s \in S$; let $f(s, \theta)$ be the probability density over states given parameter $\theta \in \Theta$. What conditions on u and f ensure that for all choice sets $A' \subseteq A$, $\arg\max_{a \in A'} \int_s u(a, s) f(s, \theta) \mathrm{d}s$ is monotonic, in the sense of the strong set order, in θ? Athey establishes, among other things, that such monotonicity holds for all log-supermodular u if and only if (ignoring some details) f is log-supermodular. The problem we study in this paper is different: there is no "state" (i.e., no uncertainty about preferences); we consider comparative statics in the utility function rather than the distribution of states; and it is essential for us that the choice space is a lottery space. It may be helpful to summarize the difference as follows: Athey studies "monotone comparative statics in a parameter of uncertainty", while we study "monotone comparative statics in a preference parameter when choosing among lotteries". There is also recent work of Apesteguia et al. (2017) on single-crossing random utility models; they discuss some stochastic monotone comparative statics, but their paper and ours tackle rather different issues.

[4] A partial order—hereafter, also referred to as just an order—is a binary relation that is reflexive, anti-symmetric, and transitive (but not necessarily complete). An upper (resp., lower) bound of $\Theta_0 \subseteq \Theta$ is $\overline{\theta} \in \Theta$ (resp., $\underline{\theta} \in \Theta$) such that $\theta \leq \overline{\theta}$ (resp., $\underline{\theta} \leq \theta$) for all $\theta \in \Theta_0$. While none of our results require any assumptions on the cardinality of Θ, the results in Subsection 2.1 are trivial when $|\Theta| < 3$. Appendix H discusses how our results extend when $(\Theta, \leq)$ is only a pre-ordered set, i.e., when $\leq$ does not satisfy anti-symmetry.

is the *expectational difference*.

2.1. Single-Crossing Expectational Differences

2.1.1. The Characterization

Our goal is to characterize when the expectational difference between arbitrary probability measures is single crossing in the following sense.

Definition 1. A function $f : \Theta \to \mathbb{R}$ is

1. **single crossing from below** if

$$(\forall \theta < \theta') \quad f(\theta) \geq (>) 0 \implies f(\theta') \geq (>) 0;$$

2. **single crossing from above** if

$$(\forall \theta < \theta') \quad f(\theta) \leq (<) 0 \implies f(\theta') \leq (<) 0;$$

3. **single crossing** if it is single crossing either from below or from above.

Plainly, a function f is single crossing if and only if either f or $-f$ is single crossing from below, and it can be single crossing from both below and above (e.g., $(\forall \theta)\ f(\theta) = 1$).

Definition 2. Given any set X, a function $f : X \times \Theta \to \mathbb{R}$ has **Single-Crossing Differences (SCD)** if $\forall x, x' \in X$, the difference $f(x, \theta) - f(x', \theta)$ is single crossing in θ.

Definition 3. The utility function $v : A \times \Theta \to \mathbb{R}$ has **Single-Crossing Expectational Differences (SCED)** if the expected utility function $V : \Delta A \times \Theta \to \mathbb{R}$ has SCD.

Our definition of SCD is related to but different from Milgrom (2004), who stipulates that $f : X \times \Theta \to \mathbb{R}$ has single-crossing differences given a (partial) order $\succeq$ on X if for all $x' \succ x''$ (where $\succ$ is the strict component of $\succeq$), $f(x', \theta) - f(x'', \theta)$ is single crossing from below. We use a different notion because, in the context of choosing among lotteries, there is no obvious exogenous order on the lottery space ΔA. In Subsection 2.3 we define an order on ΔA and justify our definition of SCD (Theorem 3).

Remark 1. If $A = \{a_1, a_2\}$, then for any two distributions $P, Q \in \Delta A$ with probability mass functions p and q, $D_{P,Q}(\theta) = (p(a_1) - q(a_1))\,(v(a_1, \theta) - v(a_2, \theta))$. It follows that v has SCED if and only if $v(a_1, \theta) - v(a_2, \theta)$ is single crossing, i.e., if and only if v has SCD. $\square$

However, the following example shows that when $|A| > 2$, SCED is not implied by SCD.

Example 1. Let $\Theta = [-1, 1]$, $A = \{0, 1, 2\}$, and $v(a, \theta) = a$ for $a \neq 1$ while $v(1, \theta) = \theta^2 + 1/2$. The function v has SCD: for any $a' > a''$, $v(a', \theta) - v(a'', \theta) > 0$ for all θ. Now consider the probability measures $P, Q \in \Delta A$ with the probability mass functions $p(1) = 1$ and $q(0) = q(2) = 1/2$. $D_{P,Q}(\theta) = \theta^2 - 1/2$ is not single crossing, and so v does not have SCED. See Figure 1, in which the red dot-dashed curve is $\int_A v(a, \theta)\mathrm{d}Q$ while the others depict $v(a, \theta)$ for $a \in \{0, 1, 2\}$. $\square$

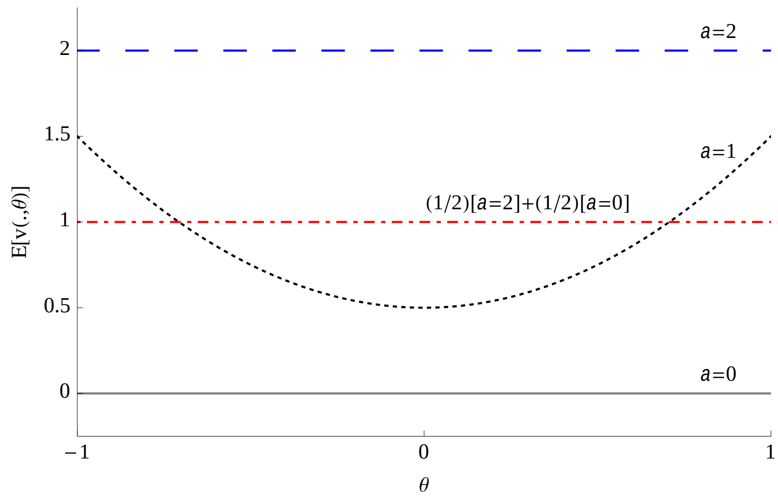


Figure 1: Single-crossing differences does not imply single-crossing expectational differences.

Our characterization of SCED (Theorem 1) uses the following definition.

Definition 4. Let $f_1, f_2 : \Theta \to \mathbb{R}$ each be single crossing.

1. f_1 **ratio dominates** f_2 if

$$(\forall \theta_l \leq \theta_h) \quad f_1(\theta_l)f_2(\theta_h) \leq f_1(\theta_h)f_2(\theta_l), \quad \text{and} \tag{1}$$

$$(\forall \theta_l \leq \theta_m \leq \theta_h) \quad f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l) \iff \begin{cases} f_1(\theta_l)f_2(\theta_m) = f_1(\theta_m)f_2(\theta_l), \\ f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m). \end{cases} \tag{2}$$

2. f_1 and f_2 are **ratio ordered** if either f_1 ratio dominates f_2 or f_2 ratio dominates f_1.

Since ratio dominance involves weak inequalities, f_1 can ratio dominate f_2 and vice-versa even when $f_1 \neq f_2$: consider $f_1 = -f_2$. We use the terminology "ratio dominance" because when f_2 is a strictly positive function, (1) is the requirement that the ratio $f_1(\theta)/f_2(\theta)$ must be (weakly) increasing in θ. Indeed, if both f_1 and f_2 are probability densities of random variables Y_1 and Y_2, then (1) says that Y_1 stochastically dominates Y_2 in the sense of likelihood ratios.[5]

Condition (1) is a natural generalization of the increasing ratio property to functions that may change sign. To get a geometric intuition, suppose f_1 "strictly" ratio dominates f_2 in the sense that (1) holds with strict inequality. For any θ, let $f(\theta) \equiv (f_1(\theta), f_2(\theta))$. For every $\theta_l < \theta_h$, $f_1(\theta_l)f_2(\theta_h) - f_1(\theta_h)f_2(\theta_l) < 0$ implies that the vector $f(\theta_l)$ moves to $f(\theta_h)$ through a rescaling of magnitude and a clockwise—rather than counterclockwise—rotation (throughout our paper, a "rotation" must be no more than 180 degrees); see Figure 2.

To confirm this point, recall that from the definition of cross product,

$$\begin{aligned}(f_1(\theta_l), f_2(\theta_l), 0) \times (f_1(\theta_h), f_2(\theta_h), 0) &= \|f(\theta_l)\| \|f(\theta_h)\| \sin(r) e_3 \\ &= (f_1(\theta_l)f_2(\theta_h) - f_1(\theta_h)f_2(\theta_l))\, e_3,\end{aligned}$$

where r is the counterclockwise angle from $f(\theta_l)$ to $f(\theta_h)$, $e_3 \equiv (0, 0, 1)$, $\times$ is the cross product, and $\|\cdot\|$ is the Euclidean norm. If $\sin(r) < 0$ (resp., $\sin(r) > 0$), then $f(\theta_l)$ moves to $f(\theta_h)$ through a clockwise (resp., counterclockwise) rotation.

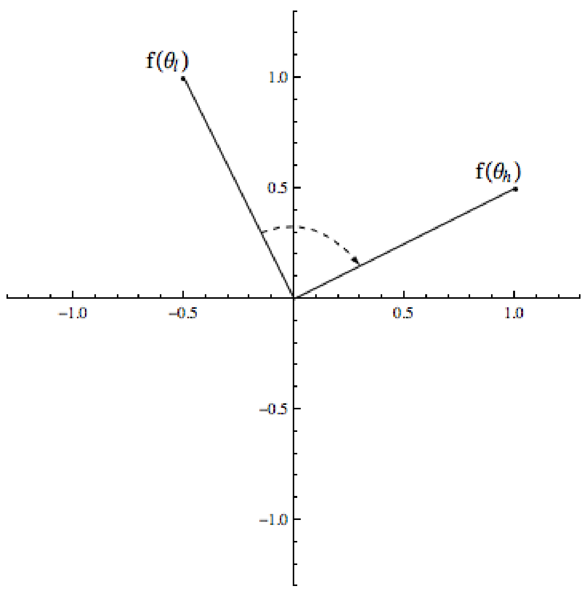


Figure 2: Geometric representation of Condition (1) for two points $\theta_l < \theta_h$.

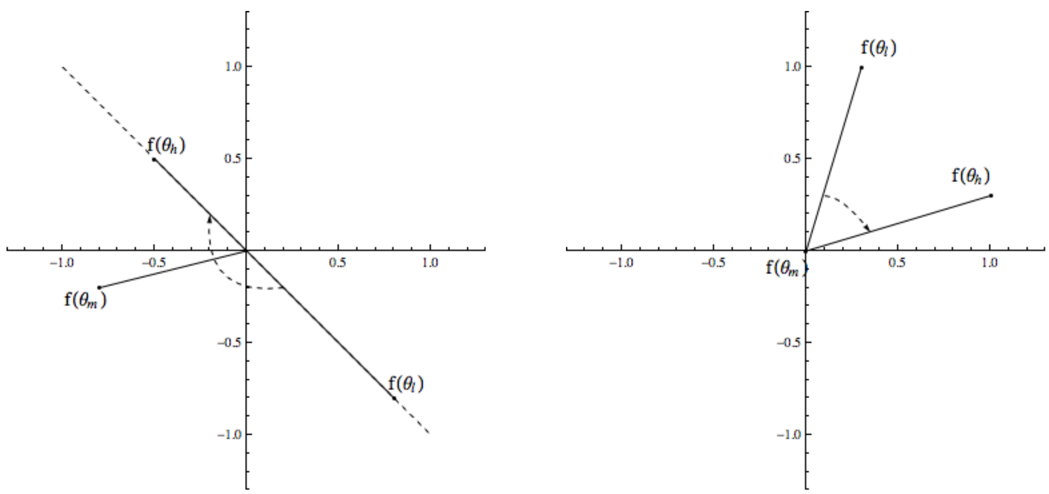
Hence, f_1 and f_2 are ratio ordered only if $f(\theta)$ *rotates monotonically* as θ increases, either always clockwise or always counterclockwise.[6] It follows that the set $\{f(\theta) : \theta \in \Theta\}$ must

[5] From the viewpoint of information economics, think of θ as a signal of a state $s \in \{1, 2\}$, drawn from the density $f(\theta|s) \equiv f_s(\theta)$. Condition (1) is Milgrom's (1981) monotone likelihood-ratio property for $f(\theta|s)$.

[6] The preceding discussion establishes this point under the presumption that Condition (1) holds strictly;

be contained in a closed half-space of $\mathbb{R}^2$: otherwise, there will be two pairs of vectors such that an increase in θ corresponds to a clockwise rotation in one pair and a counterclockwise rotation in the other. Plainly, when f_1 and f_2 are both strictly positive functions, monotonic rotation of $f(\theta)$ is equivalent to monotonicity of the ratio $f_1(\theta)/f_2(\theta)$.

Given Condition (1), Condition (2) holds automatically when f_1 and f_2 are both strictly positive functions. We explicitly impose it, however, to rule out cases in which, for some $\theta_l < \theta_m < \theta_h$, either (i) $f(\theta_l)$ and $f(\theta_h)$ are collinear in opposite directions while $f(\theta_m)$ is not, or (ii) $f(\theta_l)$ and $f(\theta_h)$ are non-zero vectors while $f(\theta_m)$ is not. See Figure 3, wherein panel (a) depicts case (i) and panel (b) depicts case (ii).



(a) Failure of $\implies$. (b) Failure of $\impliedby$.

Figure 3: f_1 and f_2 are not ratio ordered because Condition (2) fails for $\theta_l < \theta_m < \theta_h$.

Let (X, Σ) be a measurable space such that every $x \in X$ is measurable. We say that a function $X \mapsto \mathbb{R}$ is **finitely integrable** if it is integrable with respect to every finite signed measure on (X, Σ). Our main result is:

Theorem 1. *The function $v : A \times \Theta \to \mathbb{R}$ has SCED if and only if it takes the form*

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta), \tag{3}$$

with $g_1, g_2 : A \to \mathbb{R}$ each finitely integrable, $f_1, f_2 : \Theta \to \mathbb{R}$ each single crossing and ratio ordered, and $c : \Theta \to \mathbb{R}$.

however, because of the hypothesis in Definition 4 that f_1 and f_2 are single crossing and because of Condition (2), the conclusion holds without that presumption.

Example 1 can be understood using Theorem 1. Take $c(\theta) = 0$, $f_1(\theta) = \theta^2 + 1/2$, $f_2(\theta) = 2$, and for each $i \in \{1, 2\}$, $g_i(a) = \mathbb{1}_{\{a=i\}}$, where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. The Example then satisfies (3), with f_1 and f_2 each single crossing. But f_1 and f_2 are not ratio ordered, as ratio ordering reduces to monotonicity of f_1 when f_2 is constant.

An asymmetry between a and θ in the functional form (3) bears noting: the function $c : \Theta \to \mathbb{R}$ does not have a counterpart function $A \mapsto \mathbb{R}$. The reason is that whether the expectational difference between a pair of lotteries is single crossing or not could be altered by adding a function of a alone to the utility function $v(a, \theta)$. On the other hand, adding a function of θ alone to $v(a, \theta)$ has no effect on expectational differences. Indeed, SCED is an ordinal property of preferences that is invariant to positive affine transformations: if $v(a, \theta)$ has SCED, then so does $b(\theta)v(a, \theta) + c(\theta)$ for any $b : \Theta \to \mathbb{R}_{++}$ and $c : \theta \to \mathbb{R}$.

Theorem 1 implies that a necessary condition for SCED is that every lottery has two sufficient statistics that determine any type's preferences. Given the form (3), the relevant statistics for any lottery $P \in \Delta A$ are $\int_A g_1(a)\mathrm{d}P$ and $\int_A g_2(a)\mathrm{d}P$. Given any pair of lotteries, P and Q, the expected utility difference $D_{P,Q}(\theta)$ must be a linear combination of the functions f_1 and f_2 in (3). This observation underlies the following corollary.

Corollary 1. *Let $A = \mathbb{R}$ and $\Theta \subseteq \mathbb{R}$ with $|\Theta| \geq 3$, with the interpretation that a is a decision or policy and θ parameterizes the agent's bliss point. A loss function of the form $v(a, \theta) = -|a - \theta|^z$ with $z > 0$ has SCED if and only if $z = 2$.*

Beyond restricting preferences to only depend on two statistics of any lottery, SCED further constrains how preferences vary across types. This aspect is captured by Theorem 1's requirement that the functions f_1 and f_2 in (3) be ratio ordered. It underlies the following corollary.

Corollary 2. *Let $A \subseteq \mathbb{R}^2$ with $a \equiv (q, t)$ and $\Theta \subseteq \mathbb{R}$, with the interpretation that q is a quantity of consumption, t is a payment, and θ parameterizes the agent's marginal rate of substitution. A quasilinear utility function $v((q, t), \theta) = g(q)f(\theta) - t$, where g is finitely integrable and not constant, has SCED if and only if f is monotonic.*

Finally, notice that when $v(a, \theta)$ has the form (3) with strictly positive functions f_1 and f_2, any type's utility is, up to a positive affine transformation (viz., subtracting $c(\theta)$ and dividing by $f_1(\theta) + f_2(\theta)$), a convex combination of two type-independent utility functions over actions, g_1 and g_2. Theorem 1's ratio ordering requirement then simply says that the relative weight on g_1 and g_2 changes monotonically with the agent's type. This idea underlies the following corollary.

Corollary 3. *Let A and Θ be open subsets of $\mathbb{R}$, with the interpretation that a is money and θ parameterizes the agent's risk preferences. Assume that $v(a,\theta)$ has SCED; moreover, that (3) is satisfied with twice differentiable functions g_1 and g_2 such that $g_1' > 0$ and $g_2' > 0$, and differentiable functions f_1 and f_2 such that f_1 strictly ratio dominates f_2.*[7] *At any a, the Arrow-Pratt measure of absolute risk aversion, $-v_{aa}(a,\theta)/v_a(a,\theta)$, is decreasing (resp., increasing) in θ if and only if g_1 is less (resp., more) absolutely risk averse than g_2 at a.*[8]

The remainder of this subsection explains the logic behind Theorem 1. The central issue is when an arbitrary linear combination of single crossing functions is itself single crossing, which is of independent interest.

2.1.2. Aggregating Single-Crossing Functions

Suppose $v(a,\theta)$ has the form (3). Then, for any $P,Q \in \Delta A$, $D_{P,Q}(\theta) = \alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ for some $(\alpha_1,\alpha_2) \in \mathbb{R}^2$. What ensures that this function is single crossing?

Lemma 1. *Let $f_1, f_2 : \Theta \to \mathbb{R}$ each be single crossing. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is single crossing $\forall(\alpha_1,\alpha_2) \in \mathbb{R}^2$ if and only if f_1 and f_2 are ratio ordered.*

Lemma 1 implies a characterization of likelihood-ratio ordering for random variables with single-crossing densities, e.g., those with strictly positive densities.[9] While this likelihood-ratio ordering characterization is not well-known among economists (to our knowledge), it is a special case of Karlin's (1968) results on the variation diminishing property of totally positive functions. More generally, however, we believe the full force of Lemma 1 cannot be derived from the variation diminishing property.[10]

Lemma 1 sheds light on Example 1 and its discussion after Theorem 1. Both $f_1(\theta) = \theta^2 + 1/2$ and $f_2(\theta) = 2$ are single crossing, but $2f_1 - f_2$ is not single crossing because f_1

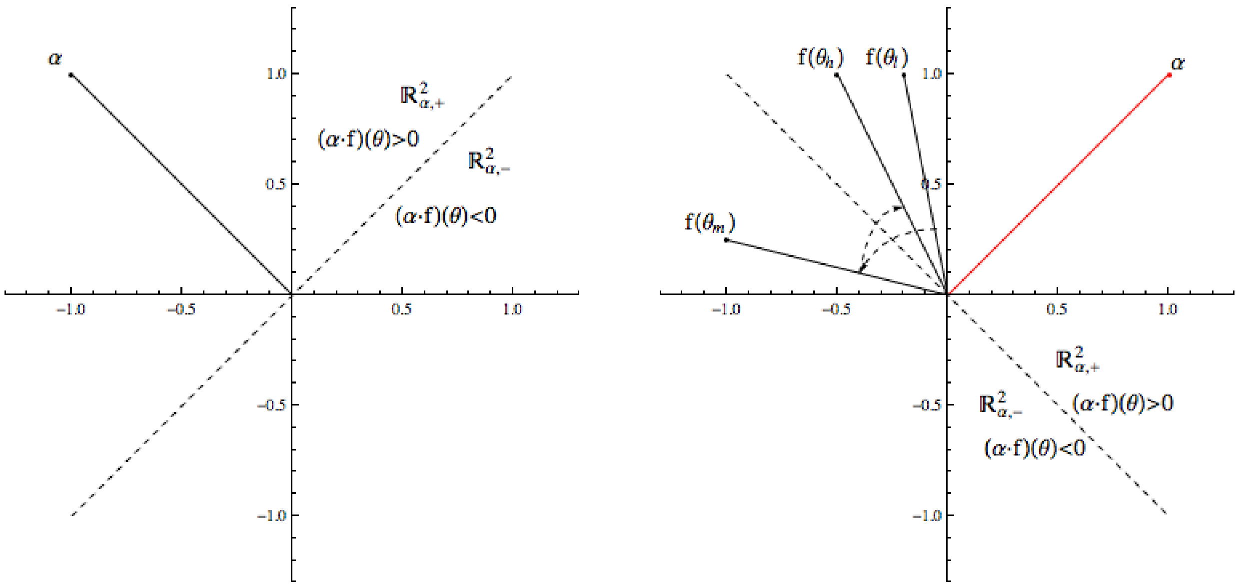
[7] That is, Condition (1) holds with strict inequality; see Definition 6 in Subsection 2.2. Primes on g_1 and g_2 denote derivatives.

[8] Subscripts on v denote partial derivatives. Our assumptions ensure that $-v_{aa}(a,\theta)/v_a(a,\theta)$ is well defined; in particular, $v_a(a,\theta) = g_1'(a)f_1(\theta) + g_2'(a)f_2(\theta) \neq 0$ because $g_1'(a) > 0$, $g_2'(a) > 0$, and either $f_1(\theta)$ or $f_2(\theta)$ is non-zero by strict ratio dominance of f_1 over f_2. The function g_1 is less absolutely risk averse than g_2 at a if it has a smaller Arrow-Pratt measure of absolute risk aversion at a, i.e., if $-g_1''(a)/g_1'(a) \leq -g_2''(a)/g_2'(a)$, and similarly for more absolutely risk averse.

[9] To see the role of Lemma 1's hypothesis that each function is single crossing, note that when $f_1(\theta) = 0$ for all θ, then f_1 and f_2 are ratio ordered even if f_2 is not single crossing. (A similar observation can be made for probability densities whose values are zero at some points.) Nonetheless, Lemma 2 in Subsection 2.2 provides a variant of the characterization for arbitrary f_1 and f_2.

[10] Karlin (1968) assumes a completely ordered domain, so consider a completely ordered Θ, with $|\Theta| > 2$ to avoid trivialities. Let $K(i,\theta) \equiv K_i(\theta)$ for $i = 1,2$ and some functions $K_i : \Theta \to \mathbb{R}$. The function K is said to be totally positive of order two, abbreviated TP_2, if K_1 and K_2 are both non-negative functions and $(\forall \theta_l \leq \theta_h)\ K_1(\theta_l)K_2(\theta_h) \leq K_1(\theta_h)K_2(\theta_l)$. The variation diminishing property of Karlin (1968, Theorem 3.1 in Chapter 5) implies that if—and, more or less, only if—K is TP_2, then any linear combination of K_1 and K_2 is

and f_2 are not ratio ordered. Note that in this example any linear combination of f_1 and f_2 with both positive (or both negative) coefficients is single crossing; the characterization in Lemma 1 relies on allowing coefficients of opposite signs.[11]



(a) Sufficiency of ratio ordering.

(b) Necessity of ratio ordering, with $\theta_l < \theta_m < \theta_h$.

Figure 4: Ratio ordering and single crossing of all linear combinations.

Here is Lemma 1's intuition. For sufficiency of Condition (1) of ratio ordering, consider any linear combination $\alpha_1 f_1 + \alpha_2 f_2$. Assume $\alpha \equiv (\alpha_1, \alpha_2) \in \mathbb{R}^2 \backslash \{0\}$, as otherwise the linear combination is trivially single crossing. The vector α defines two open half spaces $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$ and $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, where $\cdot$ is the dot product; see Figure 4(a). Ratio ordering of f_1 and f_2 implies that the vector $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ rotates monotonically as θ increases. If the rotation occurs from $\mathbb{R}^2_{\alpha,-}$ to $\mathbb{R}^2_{\alpha,+}$ (resp., from $\mathbb{R}^2_{\alpha,+}$ to $\mathbb{R}^2_{\alpha,-}$), then $\alpha \cdot f \equiv \alpha_1 f_1 + \alpha_2 f_2$ is single crossing only from below (resp., only from above). If $\bigcup_{\theta\in\Theta} f(\theta) \subseteq \mathbb{R}^2_{\alpha,-}$ or $\bigcup_{\theta\in\Theta} f(\theta) \subseteq \mathbb{R}^2_{\alpha,+}$, then $\alpha \cdot f$ is single crossing both from below and

single crossing. There are, however, ratio-ordered f_1 and f_2 such that there is no TP_2 function K with

$$\{\alpha_1 f_1 + \alpha_2 f_2 : \alpha \in \mathbb{R}^2\} = \{\alpha_1 K_1 + \alpha_2 K_2 : \alpha \in \mathbb{R}^2\}. \tag{4}$$

When f_1 and f_2 are ratio ordered and linearly independent (i.e., $(\forall \alpha \in \mathbb{R}^2\backslash\{0\})\ (\exists\theta)\ \alpha_1 f_1(\theta) + \alpha_2 f_2(\theta) \neq 0$), a TP_2 function K satisfying (4) exists if and only if the set $\{f(\theta) : \theta \in \Theta\}$ lies in an open half space of $\mathbb{R}^2$. (A proof is available from the authors on request.) Ratio ordering does imply that the set lies in a half space, as noted earlier, but the half space need not be open.

[11] Indeed, even if two functions are both increasing or both decreasing (in which case all positive and all negative linear combinations are monotonic, and hence single crossing), ratio ordering remains necessary for all their linear combinations to be single crossing. Lemma 5 in Subsection 4.1 characterizes when all linear combinations of a pair of monotonic functions are monotonic, rather than just single crossing.

above. Other cases are similar.

To see why Condition (1) of ratio ordering is necessary, suppose the vector $f(\theta)$ does not rotate monotonically. Figure 4(b) illustrates a case in which, for $\theta_l < \theta_m < \theta_h$, $f(\theta_l)$ rotates counterclockwise to $f(\theta_m)$, but $f(\theta_m)$ rotates clockwise to $f(\theta_h)$. As shown in the Figure, one can find $\alpha \in \mathbb{R}^2$ such that $f(\theta_m) \in \mathbb{R}^2_{\alpha,-}$ while both $f(\theta_l), f(\theta_h) \in \mathbb{R}^2_{\alpha,+}$, which implies that $\alpha \cdot f$ is not single crossing. The necessity of Condition (2) can be seen by returning to Figure 3. In panel (a), $(f_1 + f_2)(\theta_l) = (f_1 + f_2)(\theta_h) = 0$ while $(f_1 + f_2)(\theta_m) < 0$; in panel (b), $(f_1 + f_2)(\theta_l) > 0$ and $(f_1 + f_2)(\theta_h) > 0$ while $(f_1 + f_2)(\theta_m) = 0$.

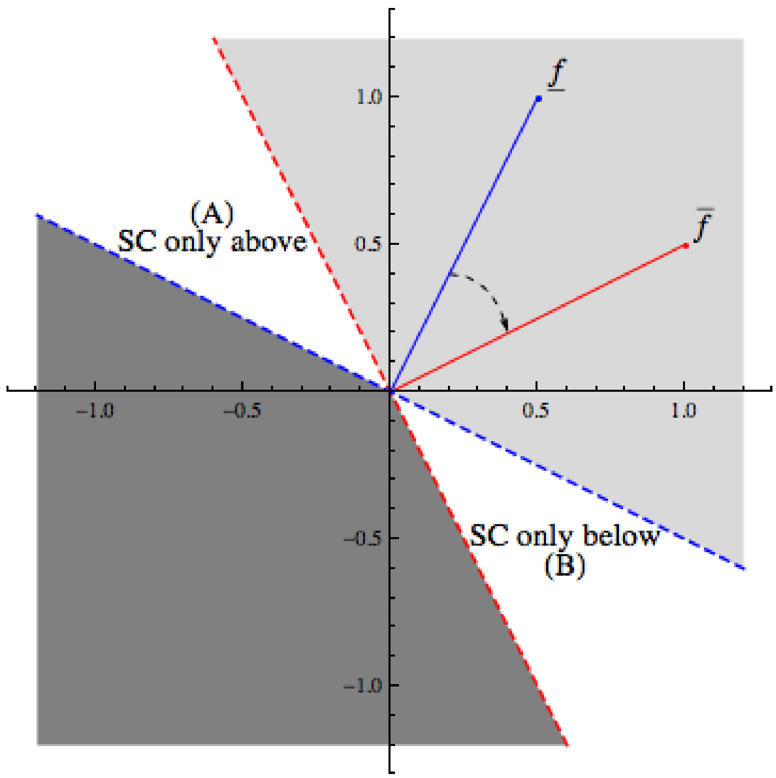


Figure 5: Identifying the direction of single crossing of $\alpha_1 f_1 + \alpha_2 f_2$.

For any two single-crossing and ratio-ordered functions f_1 and f_2, the direction in which $f(\theta)$ rotates as θ increases determines the direction of single crossing of any given linear combination. Suppose f_1 ratio dominates f_2, so that $f(\theta)$ rotates clockwise from $\underline{f}$ to $\overline{f}$ as θ increases, as illustrated in Figure 5. To be precise, $\underline{f}$ and $\overline{f}$ are extreme rays of the closed convex cone generated by $\{f(\theta) : \theta \in \Theta\}$. There are four relevant regions of $\mathbb{R}^2$ defined by the orthogonals of $\underline{f}$ (the blue dashed line in Figure 5) and $\overline{f}$ (the red dashed line). For any $\alpha \in \mathbb{R}^2$ in region (A), $\alpha \cdot \underline{f} > 0 > \alpha \cdot \overline{f}$, and so $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing only from above; for any α in (B), $\alpha \cdot \underline{f} < 0 < \alpha \cdot \overline{f}$, and so $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing only from below. If

α is in other regions, then $(\alpha \cdot f)(\theta)$ is strictly positive for all θ (light gray area) or strictly negative for all θ (dark gray area), and so $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing both from below and above.

Lemma 1 holds without change with only affine combinations, i.e., if we require $\alpha_1 + \alpha_2 = 1$. Nothing would be lost either if we strengthened the hypothesis of Lemma 1 to impose that both functions are single crossing from below (or both from above); the reason is that for any f_1 and f_2, $\{\alpha_1 f_1 + \alpha_2 f_2 : (\alpha_1, \alpha_2) \in \mathbb{R}^2\} = \{\alpha_1 f_1 + \alpha_2(-f_2) : (\alpha_1, \alpha_2) \in \mathbb{R}^2\}$ and $-f_2$ is single crossing from below if and only if f_2 is single crossing from above.

Theorem 1 requires an extension of Lemma 1 to more than two functions. Let (X, Σ) be a measurable space such that every $x \in X$ is measurable, and $f : X \times \Theta \to \mathbb{R}$. We say that f is **linear combinations SC-preserving** if $\int_X f(x, \theta) \mathrm{d}\mu$ is single crossing in θ for every finite signed measure μ on (X, Σ).

Proposition 1. *Let (X, Σ) be a measurable space and $f : X \times \Theta \to \mathbb{R}$ such that (i) $(\forall x)$ $f(x, \theta)$ is a single-crossing function of θ and (ii) $(\forall \theta)$ $f(x, \theta)$ is a finitely-integrable function of x. The function f is linear combinations SC-preserving if and only if there exist $x_1, x_2 \in X$ such that*

1. *$f(x_1, \cdot) : \Theta \to \mathbb{R}$ and $f(x_2, \cdot) : \Theta \to \mathbb{R}$ are ratio ordered, and*
2. *$(\forall \theta)$ $f(x, \theta) = \lambda_1(x) f(x_1, \theta) + \lambda_2(x) f(x_2, \theta)$ with $\lambda_1, \lambda_2 : X \to \mathbb{R}$ finitely integrable.*

The gist of Proposition 1 is that a family of single-crossing functions $\{f(x, \cdot)\}_{x \in X}$, where each $f(x, \cdot) : \Theta \to \mathbb{R}$, preserves single crossing of all linear combinations if and only if the family is "linearly generated" by two functions that are ratio ordered.[12] In particular, given any three single-crossing functions, f_1, f_2, and f_3, all their linear combinations will be single crossing if and only if there is a linear dependence in the triple, say $\lambda_1 f_1 + \lambda_2 f_2 = f_3$ for some $\lambda \in \mathbb{R}^2$, and f_1 and f_2 are ratio ordered.

The sufficiency direction of Proposition 1 follows from Lemma 1, as does necessity of the "generating functions" being ratio ordered. The intuition for the necessity of linear dependence is as follows. Assume Θ is completely ordered. For any θ, let $f(\theta) \equiv (f_1(\theta), f_2(\theta), f_3(\theta))$. If $\{f_1, f_2, f_3\}$ is linearly independent, then there exist $\theta_l < \theta_m < \theta_h$ such that $\{f(\theta_l), f(\theta_m), f(\theta_h)\}$ spans $\mathbb{R}^3$. Take any $\alpha \in \mathbb{R}^3 \setminus \{0\}$ that is orthogonal to the plane S_{θ_l, θ_h} that is spanned by $f(\theta_l)$ and $f(\theta_h)$, as illustrated in Figure 6. The linear combination $\alpha \cdot f$ is not single crossing because $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$ while $(\alpha \cdot f)(\theta_m) \neq 0$.

[12] The technical assumption (ii) in Proposition 1 is needed for the "only if" direction; it's role is to guarantee finite integrability of the λ_1 and λ_2 deduced in part 2. Conversely, finite integrability of λ_1 and λ_2 are needed for the "if" direction, so as to apply Lemma 1.

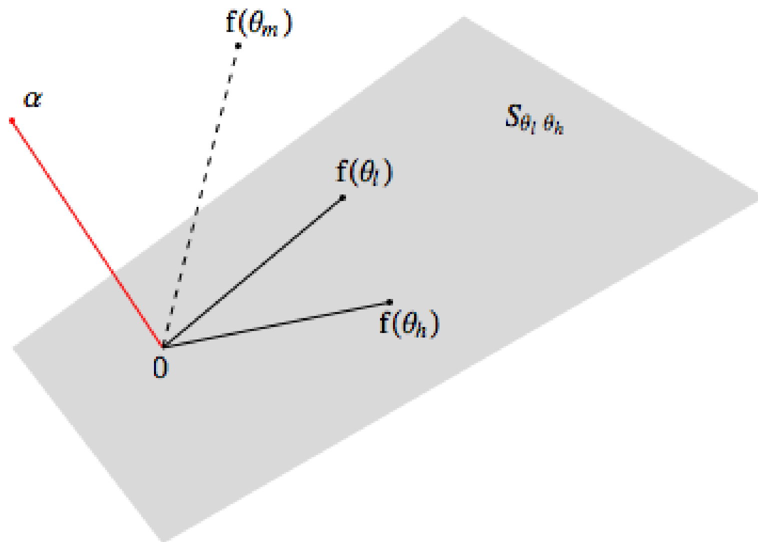


Figure 6: The necessity of linear dependence in Proposition 1.

While the necessity portion of Proposition 1 only asserts ratio ordering of the "generating functions", Lemma 1 implies that if $f : X \times \Theta \to \mathbb{R}$ is linear combinations SC-preserving, then for all $x, x' \in X$, the pair $f(x, \cdot) : \Theta \to \mathbb{R}$ and $f(x', \cdot) : \Theta \to \mathbb{R}$ must be ratio ordered. This observation can simplify the task of showing when f is *not* linear combinations SC-preserving.

2.1.3. Proof Sketch of Theorem 1

We can now sketch the argument for Theorem 1. That its characterization is sufficient for SCED is straightforward from Lemma 1. For necessity, suppose, as a simplification, $A = \{a_0, \ldots, a_n\}$ and $v : A \times \Theta \to \mathbb{R}$ is such that $(\forall \theta)\ v(a_0, \theta) = 0$.[13] If v has SCED, then for every a, $v(a, \theta) = v(a, \theta) - v(a_0, \theta)$ is single crossing: consider the expectational difference with distributions that put probability one on a and a_0 respectively. For any $(\lambda_0, \ldots, \lambda_n)$, we build on the Hahn-Jordan decomposition of $(\lambda_1, \ldots, \lambda_n)$ to write the linear combination $\sum_{i=0}^{n} \lambda_i v(a_i, \theta)$ as $M \sum_{i=0}^{n} (p(a_i) - q(a_i)) v(a_i, \theta)$, where p and q are probability mass functions on A, and M is a scalar.[14] (Unless $\sum_{i=1}^{n} \lambda_i = 0$, we have $\sum_{i=1}^{n} p(a_i) \neq \sum_{i=1}^{n} q(a_i)$; the assumption that $v(a_0, \cdot) = 0$ permits us to assign all the "excess difference" to a_0, as detailed in fn. 14.) Since v has SCED, every such linear combination is single crossing, and so Proposition 1 guarantees a' and a'' such that for all a, $v(a, \cdot) = g_1(a) v(a', \cdot) + g_2(a) v(a'', \cdot)$,

[13] The latter is a normalization, since $v(a, \theta)$ has SCED if and only if $\tilde{v}(a, \theta) \equiv v(a, \theta) - v(a_0, \theta)$ has SCED.

[14] Let $L \equiv \sum_{i=1}^{n} \lambda_i$. For $i > 0$, set $p'(a_i) \equiv \max\{\lambda_i, 0\}$ and $q'(a_i) \equiv -\min\{\lambda_i, 0\}$. If $L \geq 0$, set $p'(a_0) = 0$ and $q'(a_0) \equiv L$; if $L < 0$, set $p'(a_0) \equiv -L$ and $q'(a_0) \equiv 0$. Let $M \equiv \sum_{i=0}^{n} p'(a_i) = \sum_{i=0}^{n} q'(a_i)$. Finally, for all $a \in A$, set $p(a) \equiv p'(a)/M$ and $q(a) \equiv q'(a)/M$.

with $v(a', \cdot)$ and $v(a'', \cdot)$ each single crossing and ratio ordered.

2.2. Strict Single-Crossing Expectational Differences

For multiple reasons—e.g., to subsequently derive an analog of the standard monotone selection theorem (Milgrom and Shannon, 1994, Theorem 4′)— "strict variants" of the previous subsection's results are useful. This subsection provides them.

We now assume the existence of a strictly increasing real-valued function on $(\Theta, \leq)$. That is, we assume

$$\exists h : \Theta \to \mathbb{R} \text{ such that } \underline{\theta} < \overline{\theta} \implies h(\underline{\theta}) < h(\overline{\theta}). \tag{5}$$

Requirement (5) is related to utility representations for possibly incomplete preferences (Ok, 2007, Chapter B.4.3). Jaffray (1975, Corollary 1) implies that it is sufficient for (5) if Θ has a countable order dense subset, i.e., if there exists a countable set $\Theta_0 \subseteq \Theta$ such that

$$(\forall \underline{\theta}, \overline{\theta} \in \Theta \setminus \Theta_0) \quad \underline{\theta} < \overline{\theta} \implies \exists \theta_0 \in \Theta_0 \text{ s.t. } \underline{\theta} < \theta_0 < \overline{\theta}.$$

This is satisfied, for example, when $\Theta \subseteq \mathbb{R}^n$ is endowed with the usual order.

Definition 5. A function $f : \Theta \to \mathbb{R}$ is **strictly single crossing from below** if $(\forall \theta < \theta')$ $f(\theta) \geq 0 \implies f(\theta') > 0$; **strictly single crossing from above** if $(\forall \theta < \theta')$ $f(\theta) \leq 0 \implies f(\theta') < 0$; and **strictly single crossing** if it is either strictly single crossing from below or from above.

Equivalently, a function is strictly single crossing if it is single crossing and there are no $\theta' < \theta''$ such that $f(\theta') = f(\theta'') = 0$.

Definition 6. A function $f_1 : \Theta \to \mathbb{R}$ **strictly ratio dominates** $f_2 : \Theta \to \mathbb{R}$ if Condition (1) holds with strict inequality for every $\theta_l < \theta_h$; f_1 and f_2 are **strictly ratio ordered** if either f_1 strictly ratio dominates f_2 or vice-versa.

The definition of strict ratio dominance does not make reference to Condition (2) because that condition is vacuous when Condition (1) holds with strict inequality. Furthermore, unlike with Definition 4, Definition 6 is not restricted to single-crossing functions.

Lemma 2. *Let $f_1, f_2 : \Theta \to \mathbb{R}$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is strictly single crossing $\forall (\alpha_1, \alpha_2) \in \mathbb{R}^2 \setminus \{0\}$ if and only if f_1 and f_2 are strictly ratio ordered.*

Besides the change to strict single crossing and, correspondingly, strict ratio ordering, Lemma 2 has two other differences from Lemma 1. First, we rule out $(\alpha_1, \alpha_2) = 0$; this is unavoidable because a zero function is not strictly single crossing. Second, and more important, Lemma 2 does not need the hypothesis that f_1 and f_2 are each strictly single crossing. It turns out—as elaborated in the Lemma's proof—that when two functions are strictly ratio ordered, each of them must be strictly single crossing.

To extend Lemma 2 to more than two functions, we say that $f : X \times \Theta \to \mathbb{R}$ is **linear combinations SSC-preserving** if $\int_X f(x,\theta)\mathrm{d}\mu$ is either a zero function or strictly single crossing in θ for every finite signed measure μ on the measurable space (X, Σ). Two real-valued functions f_1 and f_2 are **linearly independent** if $(\forall \lambda \in \mathbb{R}^2 \backslash \{0\})$ $\lambda_1 f_1 + \lambda_2 f_2$ is not a zero function.

Proposition 2. *Let (X, Σ) be a measurable space and $f : X \times \Theta \to \mathbb{R}$ such that (i) $(\exists x_1, x_2)$ $f(x_1, \theta)$ and $f(x_2, \theta)$ are linearly independent functions of θ and (ii) $(\forall \theta)$ $f(x, \theta)$ is a finitely-integrable function of x. The function f is linear combinations SSC-preserving if and only if*

1. *$f(x_1, \cdot) : \Theta \to \mathbb{R}$ and $f(x_2, \cdot) : \Theta \to \mathbb{R}$ are strictly ratio ordered, and*
2. *$(\forall \theta)$ $f(x, \theta) = \lambda_1(x) f(x_1, \theta) + \lambda_2(x) f(x_2, \theta)$ with $\lambda_1, \lambda_2 : X \to \mathbb{R}$ finitely integrable.*

As with Lemma 2, Proposition 2 does not hypothesize that $(\forall x)$ $f(x, \theta)$ is strictly single crossing; this is instead an implication. The Proposition does assume that for some x_1, x_2, $f(x_1, \cdot)$ and $f(x_2, \cdot)$ are linearly independent, as any pair of linearly dependent functions is not strictly ratio ordered.[15]

Definition 7. Given any set X, a function $f : X \times \Theta \to \mathbb{R}$ has **Strict Single-Crossing Differences (SSCD)** if $\forall x, x' \in X$, the difference $f(x,\theta) - f(x', \theta)$ is either a zero function or strictly single crossing in θ. The utility function $v : A \times \Theta \to \mathbb{R}$ has **Strict Single-Crossing Expectational Differences (SSCED)** if the expected utility function $V : \Delta A \times \Theta \to \mathbb{R}$ has SSCD.

The zero-function possibility in the above definition cannot be avoided even if we restricted attention to distinct measures P and Q; for example, $D_{P,Q}$ is a zero function whenever P and Q have the same expectation and v is linear in a.

Theorem 2. *The function $v : A \times \Theta \to \mathbb{R}$ has SSCED if and only if it takes the form (3), with $g_1, g_2 : A \to \mathbb{R}$ each finitely integrable, $f_1, f_2 : \Theta \to \mathbb{R}$ strictly ratio ordered, and $c : \Theta \to \mathbb{R}$.*

[15] For the "if" direction of Proposition 2, the existence of a pair of linearly independent functions can be derived as a conclusion rather than a hypothesis. However, without that hypothesis, the "only if" direction would fail: given $X = \{x_1\}$, f is linear combinations SSC-preserving if and only if $f(x_1, \theta)$ is strictly single crossing in θ. Note also that linear combinations of more than two functions can result in zero functions even when not all coefficients are zero (unlike in Lemma 2).

2.3. Monotonicity of Choices

There is a sense in which SCED is necessary and sufficient for monotone comparative statics, while SSCED guarantees that any selection of choices is monotonic. A precise statement requires monotonicity theorems (Theorem 3 and Proposition 3 below) related to, but distinct from, the influential Theorem 4 and Theorem 4' of Milgrom and Shannon (1994).

Throughout this subsection, we consider an ordered set of alternatives, $(X, \succeq)$, and, as earlier, an ordered set of types, $(\Theta, \leq)$. We maintain the assumption that Θ contains upper and lower bounds for every pair of its elements. Neither $\succeq$ nor $\leq$ need be complete. We are interested in comparative statics for a function $f : X \times \Theta \to \mathbb{R}$. We assume the set X is **minimal** (with respect to f) in the sense that

$$(\forall x \neq x')(\exists \theta)\, f(x, \theta) \neq f(x', \theta).$$

Monotone Comparative Statics. For any $x, y \in X$, let $x \vee y$ and $x \wedge y$ denote the usual **join** and **meet** respectively.[16] In general, neither need exist. $(X, \succeq)$ is a **lattice** if each pair of elements has a join and meet in X. Given any $Y, Z \subseteq X$, we say that Y dominates Z in the **strong set order**, denoted $Y \succeq_{SSO} Z$, if for every $y \in Y$ and $z \in Z$, (i) $y \vee z$ and $y \wedge z$ exist, and (ii) $y \vee z \in Y$ and $y \wedge z \in Z$.

Remark 2. The strong set order is neither reflexive nor transitive: for any $S \subseteq X$, it holds that $S \succeq_{SSO} \emptyset$ and $\emptyset \succeq_{SSO} S$, whereas $S \succeq_{SSO} S$ if and only if $(S, \succeq)$ is a lattice. However, the strong set order is transitive on non-empty subsets of $(X, \succeq)$. While this transitivity is well-known when $(X, \succeq)$ is a lattice, it is a general property.[17]

Definition 8. $f : X \times \Theta \to \mathbb{R}$ has **monotone comparative statics (MCS)** on $(X, \succeq)$ if

$$(\forall S \subseteq X) \text{ and } (\forall \theta \leq \theta') : \arg\max_{s \in S} f(s, \theta') \succeq_{SSO} \arg\max_{s \in S} f(s, \theta).$$

Our definition of MCS is closely related to but not the same as Milgrom and Shannon

[16] $z \in X$ is the join (or supremum) of $\{x, y\}$ if (i) $z \succeq x$ and $z \succeq y$, and (ii) if $w \succeq x$ and $w \succeq y$, then $w \succeq z$. The meet (or infimum) of $\{x, y\}$ is defined analogously.

[17] A proof is as follows. The Dedekind-MacNeille (DM) completion $(X', \succeq')$ of $(X, \succeq)$ is a lattice (in fact, the smallest complete lattice) that order embeds $(X, \succeq)$, i.e., there exists an injection $h : X \to X'$ such that $x \succeq y$ if and only if $h(x) \succeq' h(y)$. It is known that the DM completion preserves all meets and joins that exist in $(X, \succeq)$. To show that $\succeq_{SSO}$ is transitive on non-empty subsets of $(X, \succeq)$, assume, without loss of generality, that $X \subseteq X'$ and h is the identity function, and take $W, Y, Z \in 2^X \setminus \{\emptyset\}$ such that $W \succeq_{SSO} Y$ and $Y \succeq_{SSO} Z$. As the DM completion preserves all existing meets and joins, it holds that $W \succeq'_{SSO} Y$ and $Y \succeq'_{SSO} Z$. By transitivity of $\succeq'_{SSO}$ on non-empty subsets of X', it follows that $W \succeq'_{SSO} Z$. That is, for every $w \in W$ and $z \in Z$, $w \vee' z \in W$ and $w \wedge' z \in Z$. Since $w \vee' z \in X$ and $w \wedge' z \in X$, they are the join and meet of w and z in $(X, \succeq)$. It follows that $W \succeq_{SSO} Z$.

(1994). We take $(X, \succeq)$ to be any ordered set while they require a lattice. We focus only on monotonicity of choice in θ but require the monotonicity to hold for every subset $S \subseteq X$; Milgrom and Shannon require monotonicity of choice jointly in the pair (θ, S), but this implicitly only requires choice monotonicity in θ to hold for every sub-lattice $S \subseteq X$.

Define binary relations $\succ_{SCD}$ and $\succeq_{SCD}$ on X as follows: $x \succ_{SCD} x'$ if $D_{x,x'}(\theta) \equiv f(x,\theta) - f(x',\theta)$ is single crossing from *only* below; $x \succeq_{SCD} x'$ if either $x \succ_{SCD} x'$ or $x = x'$. Recalling the definition of SCD (Definition 2), it is intuitive that:

Lemma 3. *If $f : X \times \Theta \to \mathbb{R}$ has SCD, then $(X, \succeq_{SCD})$ is an ordered set.*

Conversely, if $(X, \succeq_{SCD})$ is a completely ordered set, the function f has SCD.

Given two orders $\succeq$ and $\succeq'$ on X, the order $\succeq'$ is a **refinement** of $\succeq$ if

$$(\forall x, x' \in X) \quad x \succeq x' \implies x \succeq' x'.$$

Theorem 3. *$f : X \times \Theta \to \mathbb{R}$ has monotone comparative statics on $(X, \succeq)$ if and only if f has SCD and $\succeq$ is a refinement of $\succeq_{SCD}$.*

Our definition of SCD does not require an order on the set of alternatives, whereas MCS does. Theorem 3 says that MCS obtains if and only if the function has SCD and the set of alternatives is ordered by a refinement of the order induced by SCD. In this sense, SCD is necessary and sufficient for MCS. Moreover, the Theorem justifies viewing $\succeq_{SCD}$ as the prominent order for MCS: MCS do not obtain with any order that either coarsens $\succeq_{SCD}$ or reverses a ranking by $\succ_{SCD}$.

The argument for necessity in Theorem 3 uses binary choice sets: choices are not monotonic in type on some binary choice set if either SCD fails or the order $\succeq$ is not a refinement of $\succeq_{SCD}$. Note that the direction of ordering in $\succeq_{SCD}$ (and hence $\succeq$) owes to the direction of monotonicity in the definition of MCS. Regarding sufficiency, we remark only that MCS is assured by SCD and the order $\succeq_{SCD}$; moreover, the ranking of any pair of alternatives that are unranked by $\succeq_{SCD}$ has no bearing on MCS because, given SCD (and that X is minimal), one alternative from the pair will never be chosen by any type when the other is available.[18]

[18] Milgrom and Shannon (1994, Theorem 4) identify their version of single-crossing differences and quasisupermodularity as jointly necessary and sufficient for their monotone comparative statics. (On a lattice $(X, \succeq)$, $h : X \to \mathbb{R}$ is quasisupermodular if $h(x) \geq (>) h(x \wedge x') \implies h(x \vee x') \geq (>) h(x')$.) It bears explanation why quasisupermodularity plays no role in Theorem 3. Regarding necessity, the reason is that, as noted after Definition 8, we do not require monotonicity of choice as the choice set varies. Regarding sufficiency, the reason is that our construction of the order $\succeq_{SCD}$ is, to put it succinctly, "fine enough". In

Remark 3. If $f : X \times \Theta \to \mathbb{R}$ has SCD, then the Szpilrajn extension theorem implies the existence of a *complete* order, $\succeq^*$, on X that refines $\succeq_{SCD}$. f has MCS on $(X, \succeq^*)$.

We next apply Lemma 3 and Theorem 3 to our context of choice among lotteries.

Definition 9. For any $P, Q \in \Delta A$: (i) $P \succ_{SCED} Q$ if $D_{P,Q}$ is single crossing only from below; (ii) $P \sim_{SCED} Q$ if $(\forall\theta)\ D_{P,Q}(\theta) = 0$; and (iii) $P \succeq_{SCED} Q$, read P **dominates** Q **in SCED**, if $P \succ_{SCED} Q$ or $P \sim_{SCED} Q$.

Note that $P \sim_{SCED} Q$ if and only if all θ are indifferent between P and Q. Without any loss of generality then, we can focus on the quotient space of ΔA consisting of the set of equivalence classes defined by $\sim_{SCED}$; denote this quotient space by $\widetilde{\Delta}A$. For any $P, Q \in \widetilde{\Delta}A$, $P \sim_{SCED} Q$ if and only if $P = Q$, which implies $\widetilde{\Delta}A$ is minimal and the relation $\succeq_{SCED}$ on $\widetilde{\Delta}A$ is anti-symmetric (and reflexive).[19]

If v has SCED, the expected utility function V has SCD. By Lemma 3, $(\widetilde{\Delta}A, \succeq_{SCED})$ is an ordered set. According to Theorem 3, any order $\succeq$ on $\widetilde{\Delta}A$ that ensures the monotonicity of choices in θ for every choice set in $\widetilde{\Delta}A$ must be a refinement of $\succeq_{SCED}$. We summarize this MCS result as follows:

Corollary 4. *Let $\succeq$ be an arbitrary order on $\widetilde{\Delta}A$. V has monotone comparative statics on $(\widetilde{\Delta}A, \succeq)$ if and only if v has SCED and $\succeq$ is a refinement of $\succeq_{SCED}$. If v has SCED, then V has monotone comparative statics with respect to a complete order on $\widetilde{\Delta}A$ that refines $\succeq_{SCED}$.*

Monotone Selections. If v has SSCED, then we obtain a stronger notion of monotonicity of choices, given by the next definition.

Definition 10. $f : X \times \Theta \to \mathbb{R}$ has **Monotone Selection (MS)** on $(X, \succeq)$ if for any $S \subseteq X$, every selection $s^*(\theta)$ from $\arg\max_{s \in S} f(s, \theta)$ is (weakly) increasing in θ.

Define binary relations $\succ_{SSCD}$ and $\succeq_{SSCD}$ on X as follows: $x \succ_{SSCD} x'$ if $D_{x,x'}$ is strictly single crossing only from below; $x \succeq_{SSCD} x'$ if either $x \succ_{SSCD} x'$ or $x = x'$. From the definition of SSCD (Definition 7), it follows that:

more detail: let $(X, \succeq)$ be a lattice for comparability with Milgrom and Shannon (1994), and for any $S \subseteq X$, let $C(S) \equiv \{x' : (\exists\theta)\ x' \in \arg\max_{x \in S} f(x, \theta)\}$. If f has SCD, then for any $S \subseteq X$, the set $C(S)$ is completely ordered by $\succeq_{SCD}$ (as elaborated in the sufficiency proof of Theorem 3), and hence by any refinement of $\succeq_{SCD}$. One can check that quasisupermodularity is only needed in the sufficiency argument of Milgrom and Shannon (1994, Theorem 4) when, for a sub-lattice S, $C(S)$ is not completely ordered by $\succeq$.

[19] For readability, we abuse notation and treat elements of $\widetilde{\Delta}A$ as measures rather than equivalence classes of measures, and similarly use the notation $\succeq_{SCED}$ on $\widetilde{\Delta}A$ instead of the induced quotient relation. We adopt this convention throughout the paper.

Lemma 4. *If f has SSCD, then $(X, \succeq_{SSCD})$ is an ordered set.*

Proposition 3. *If $f : X \times \Theta \to \mathbb{R}$ has SSCD and $\succeq$ is a refinement of $\succeq_{SSCD}$, then f has monotone selection on $(X, \succeq)$.*

It remains only to apply Lemma 4 and Proposition 3 to our context of choice among lotteries.

Definition 11. For any $P, Q \in \Delta A$: (i) $P \succ_{SSCED} Q$ if $D_{P,Q}$ is strictly single crossing only from below; (ii) $P \sim_{SSCED} Q$ if $D_{P,Q}$ is a zero function; and (iii) $P \succeq_{SSCED} Q$, read P **dominates** Q **in SSCED**, if $P \succ_{SSCED} Q$ or $P \sim_{SSCED} Q$.

Analogous to the discussion preceding Corollary 4, we focus on the quotient space $\widetilde{\Delta} A$, which is minimal and on which $\succeq_{SSCED}$ is an order if v has SSCED.

Corollary 5. *Let $\succeq$ be an arbitrary order on $\widetilde{\Delta} A$. If v has SSCED and $\succeq$ is a refinement of $\succeq_{SSCED}$, then V has monotone selection on $(\widetilde{\Delta} A, \succeq)$. If v has SSCED, then V has monotone selection with respect to a complete order on $\widetilde{\Delta} A$ that refines $\succeq_{SSCED}$.*

3. Applications

This section illustrates how our results can be applied in four applications.

3.1. Cheap Talk with a Stochastic Receiver

There are two players, a sender (S) and a receiver (R). The sender's type is $\theta \in \Theta$, where Θ is partially ordered by $\leq$. After learning his type, S chooses a payoff-irrelevant message $m \in M$, where $|M| > 1$. After observing m but not θ, R takes an action $a \in A$. The sender's von Neumann-Morgenstern utility function is $v(a, \theta)$; the receiver's is $u(a, \theta, \psi)$, where $\psi \in \Psi$ is a preference parameter that is unknown to S when choosing m, and known to R when choosing a.[20] Note that ψ does not affect the sender's preferences. The variables θ and ψ are independently drawn from commonly-known probability measures.[21]

An example is $\Theta = [0, 1]$, $A = \mathbb{R}$, $\psi = (\psi_1, \psi_2) \in \Psi \subseteq \mathbb{R}^2$, $v(a, \theta) = -(a - \theta)^2$ and $u(a, \theta, \psi) = -(a - \psi_1 - \psi_2\theta)^2$. Here the variable ψ_2 captures the receiver's relative "sensitivity" to the sender's type. If ψ were commonly known and θ uniformly distributed, this would be the model of Melumad and Shibano (1991), which itself generalizes the canonical example from Crawford and Sobel (1982) that obtains when $\psi_1 \neq 0$ and $\psi_2 = 1$.

[20] Ψ is a measurable space endowed with some σ-algebra such that all singletons are measurable.

[21] Similar results could also be obtained in an alternative formulation in which the receiver's preferences are commonly known but there is exogenous communication noise, as in Blume et al. (2007).

We focus on (weak Perfect Bayesian) equilibria in which S uses a pure strategy, $\mu : \Theta \to M$, and R plays a possibly-mixed strategy, $\alpha : M \times \Psi \to \Delta A$.[22] Given any α, every message m induces some lottery over actions from the sender's viewpoint, $P_\alpha(m)$. An equilibrium (μ, α) is: (i) *connected* if every message is used by a connected set of sender types, i.e., if $\theta_l < \theta_m < \theta_h$ and $\mu(\theta_l) = \mu(\theta_h)$, then $\mu(\theta_m) = \mu(\theta_l)$; (ii) *sender minimal* if for all on-the-equilibrium-path $m \neq m'$, $\exists \theta : V(P_\alpha(m), \theta) \neq V(P_\alpha(m'), \theta)$; and (iii) *sender strict* if all sender types have uniquely optimal messages among the set of on-path messages. In words, connected equilibria are those with "interval cheap talk"; sender minimality rules out equilibria in which there is a distinct pair of on-path messages over which all sender types are indifferent;[23] and sender strictness is the usual notion of strictness applied to only the sender and adapted to the current cheap-talk setting. We say that v *strictly violates SCED* if there are $P, Q \in \Delta A$ and $\theta_l < \theta_m < \theta_h$ such that $\text{sign}[D_{P,Q}(\theta_l)]\,\text{sign}[D_{P,Q}(\theta_m)] = \text{sign}[D_{P,Q}(\theta_m)]\,\text{sign}[D_{P,Q}(\theta_h)] = -1$. (For $x \in \mathbb{R}$, $\text{sign}[x] = 1$ if $x > 0$, $\text{sign}[x] = 0$ if $x = 0$, and $\text{sign}[x] = -1$ if $x < 0$.) We say that v has *shared preference* if all types have the same preferences over ΔA; it is straightforward to check that this is equivalent to the form (3) with $f_1(\theta) > 0$ (or $f_1(\theta) < 0$) and $f_2(\theta) = 0$ for all θ.

Claim 1. *Focussing on sender-minimal equilibria:*

1. *If v has the form stated in Theorem 2, and hence has SSCED, then every equilibrium is connected; if v strictly violates SCED, then under some parameters, a non-connected sender-strict equilibrium exists.*

2. *If v has shared preference, then every equilibrium is uninformative; if v does not have shared preference, then under some parameters, an informative equilibrium exists.*

Proof. Part 1: First assume v has the form stated in Theorem 2 and take a sender-minimal equilibrium (μ^*, α^*). Suppose, to contradiction, there are $\theta_l < \theta_m < \theta_h$ such that $m' \equiv \mu^*(\theta_l) = \mu^*(\theta_h) \neq \mu^*(\theta_m) \equiv m''$. Let P' and P'' denote the equilibrium distributions of the receiver's actions induced by the messages m' and m'' respectively, from the viewpoint of the sender. Both P' and P'' are independent of θ because ψ and θ are independent. By Theorem 2, $D_{P',P''}$ is either (i) a zero function, or (ii) strictly single crossing only from below, or (iii) strictly single crossing only from above. Case (i) contradicts sender minimality of

[22] Our notion of equilibrium requires optimal play for every (not just almost every) type of sender. The restriction to pure strategies for the sender is for expositional simplicity; it is essentially without loss of generality for our purposes when the distribution of θ is non-atomic.

[23] In Crawford and Sobel (1982) and Melumad and Shibano (1991), all equilibria are outcome equivalent to sender-minimal equilibria. More generally, all equilibria are sender-minimal when there is a complete order over messages under which higher messages are infinitesimally more costly for all sender types.

the equilibrium; case (ii) contradicts m' being optimal for θ_l and m'' being optimal for θ_m; case (iii) contradicts m' being optimal for θ_h and m'' being optimal for θ_m.

Next assume v strictly violates SCED: without loss of generality, suppose there are $P, Q \in \Delta A$ and $\theta_l < \theta_m < \theta_h$ such that $\min\{D_{P,Q}(\theta_l), D_{P,Q}(\theta_h)\} > 0 > D_{P,Q}(\theta_m)$. Let $u(\cdot)$ be the zero function and the distribution of sender types have support $\{\theta_l, \theta_m, \theta_h\}$. Fix any $m', m'' \in M$ with $m' \neq m''$, and for any ψ, let $\alpha(m', \psi) = P$ while $\alpha(m, \psi) = Q$ for all $m \neq m'$. There is a sender-minimal non-connected and sender-strict equilibrium in which R plays α and S plays $\mu(\theta_l) = \mu(\theta_h) = m'$ and $\mu(\theta_m) = m''$.

Part 2: The first point is immediate given sender minimality; the second point follows from a construction in which the receiver is totally indifferent and chooses different lotteries following different messages such that two sender types do not share preferences over the two lotteries. *Q.E.D.*

Part 1 is the more important part of the above claim. In a sense, it establishes that (S)SCED is necessary and sufficient for interval cheap talk. The result relates to Seidmann (1990), who first considered an extension of Crawford and Sobel (1982) to sender uncertainty about the receiver's preferences. His goal was to illustrate how such uncertainty could facilitate informative communication. The first portion of part 2 of Claim 1 generalizes a point he makes on p. 454 of his article; indeed, with some additional structure, he identifies the functional form restrictions for "shared preference" as essentially sufficient for all equilibria to be uninformative. On the other hand, one of Seidmann's examples (Example 2 in his paper) constructs a non-connected informative equilibrium. Part 1 of Claim 1 clarifies that the key is a failure of (S)SCED.

3.2. Collective Choice

Collective choice over lotteries arises naturally in many contexts. For example, in elections there is uncertainty about what policies some politicians will implement if elected; when hiring a CEO, a board of directors may view each candidate as a probability distribution over earnings. Zeckhauser (1969) first pointed out that pairwise-majority comparisons in these settings can be cyclical, even when comparisons over deterministic outcomes are not. Our results shed light on when such difficulties do not arise.

Consider a finite group of individuals indexed by $i \in \{1, 2, \ldots, N\}$. The group must choose from a set of lotteries, $\mathcal{A} \subseteq \Delta A$, where A is the space of outcomes (political policies, earnings, etc.) with generic element a. Each individual i has von Neumann-Morgenstern utility function $v(a, \theta_i)$, where $\theta_i \in \Theta$ is a preference parameter or i's type. We assume Θ is completely ordered; without further loss of generality, let $\Theta \subset \mathbb{R}$ and $\theta_1 \leq \cdots \leq \theta_N$. The expected utility for an individual of type θ from lottery $P \in \mathcal{A}$ is $V(P,\theta) \equiv \int_A v(a,\theta)\mathrm{d}P$; integrability is assumed for all $\theta \in \{\theta_1,\dots,\theta_N\}$ and $P \in \mathcal{A}$.

Define the group's preference relation, $\succeq_{maj}$, over lotteries $P,Q \in \mathcal{A}$ by majority rule:

$$P \succeq_{maj} Q \text{ if } |\{i : V(P,\theta_i) \geq V(Q,\theta_i)\}| \geq N/2.$$

The relation $\succeq_{maj}$ is said to be *quasi-transitive* if the corresponding strict relation is transitive. Quasi-transitivity of $\succeq_{maj}$ is the key requirement for collective choice to be "rational": it ensures—given completeness, which $\succeq_{maj}$ obviously satisfies—that a preference-maximizing choice (equivalently, a Condorcet Winner) exists for the group under standard conditions on the choice set, e.g., if $\mathcal{A}$ is finite. We say there is a *unique median* if either (i) N is odd, in which case we define $M \equiv (N+1)/2$ or (ii) N is even and $\theta_{N/2} = \theta_{N/2+1}$, in which case $M \equiv N/2$.

Claim 2. *If v has the form stated in Theorem 1, and hence has SCED, then the group's preference relation is quasi-transitive. If v has the form stated in Theorem 2 and hence has SSCED, and there is a unique median, then the group's preference relation is transitive and is represented by $V(\cdot,\theta_M)$.*

Proof. Suppose v has SCED. Define the quotient space $\tilde{\mathcal{A}}$ from $\mathcal{A}$ using the equivalence relation $\sim_{SCED}$ (Definition 9). We work with $\tilde{\mathcal{A}}$ instead of $\mathcal{A}$ in order to invoke a result from Gans and Smart (1996) that assumes a completely ordered set. Corollary 4 implies there is a complete order $\succeq$ on $\tilde{\mathcal{A}}$, which refines $\succeq_{SCED}$, such that

$$(\forall P \succ Q,\ \forall \theta' > \theta)\ V(P,\theta) \geq V(Q,\theta) \implies V(P,\theta') \geq V(Q,\theta'), \tag{6}$$

$$(\forall P \succ Q,\ \forall \theta' > \theta)\ V(P,\theta) > V(Q,\theta) \implies V(P,\theta') > V(Q,\theta'). \tag{7}$$

Conditions (6) and (7) imply that Gans and Smart's (1996) "single-crossing condition" is satisfied on $\tilde{\mathcal{A}}$; their Corollary 1 implies the group's preference relation is quasi-transitive on $\tilde{\mathcal{A}}$. Since $\mathcal{A}$ differs from $\tilde{\mathcal{A}}$ only by distinguishing lotteries that all individuals are indifferent among, it follows that $\succeq_{maj}$ is quasi-transitive on $\mathcal{A}$.

The claim's second statement follows similarly from Corollary 5 and Gans and Smart's (1996) Corollary 2. *Q.E.D.*

Claim 2 can be applied to a well-known problem in political economy (Shepsle, 1972). The policy space is a finite set $A \subset \mathbb{R}$ (for simplicity) and there are an odd N number of voters ordered by their ideal points in $\mathbb{R}$, $\theta_1 \leq \cdots \leq \theta_N$ (i.e., for each voter i, $\{\theta_i\} = \arg\max_{a\in\mathbb{R}} v(a,\theta_i)$). Let $M \equiv (N+1)/2$. There are two office-motivated candidates two office-motivated candidates, L and R; each $j \in \{L, R\}$ can commit to any lottery from some given set $\mathcal{A}_j \subseteq \Delta A$. A restricted set $\mathcal{A}_j$ may capture various kinds of constraints; for example, Shepsle (1972) assumed the incumbent candidate could only choose degenerate lotteries. In our setting, what ensures the existence of an equilibrium, and which policy lotteries are offered in an equilibrium?[24]

Claim 2 implies that if voters' utility functions v have SSCED, and if voter M is indifferent between her most-preferred lottery in $\mathcal{A}_L$ and in $\mathcal{A}_R$ (e.g., if $\mathcal{A}_L = \mathcal{A}_R$, or if both sets contain the degenerate lottery on θ_M, denoted δ_{θ_M} hereafter), then there is a unique equilibrium: each candidate offers the best lottery for voter M; in particular, both candidates converge to δ_{θ_M} if that is feasible for both. A special case is when $v(a, \theta) = -(a - \theta)^2$ and $\delta_{\theta_M} \in \mathcal{A}_L \cap \mathcal{A}_R$. It bears emphasis, however, that there will be policy convergence at the median ideal point (so long as $\delta_{\theta_M} \in \mathcal{A}_L \cap \mathcal{A}_R$) given SSCED not because all voters need be globally "risk averse"; rather, it is because strict single-crossing differences over distributions ensures the existence of a decisive voter whose most-preferred lottery is degenerate.[25]

There is a sense in which (S)SCED is necessary to guarantee that each candidate j will offer the median ideal-point voter's most-preferred lottery from the feasible set $\mathcal{A}_j$. Suppose $v(a, \theta)$ *strictly* violates SCED, as defined in Subsection 3.1 before Claim 1.[26] For concreteness, suppose that for some $P, Q \in \Delta A$ and $\theta_l < \theta_m < \theta_h$, $\min\{D_{P,Q}(\theta_l), D_{P,Q}(\theta_h)\} > 0 > D_{P,Q}(\theta_m)$. Then, if the population of voters is just $\{l, m, h\}$ and $\mathcal{A}_L = \mathcal{A}_R = \{P, Q\}$, the unique equilibrium is for both candidates to offer lottery P, which is voter m's less preferred lottery.

3.3. Costly Signaling

Consider a version of Spence's (1973) signaling model. A worker is privately informed of his type θ that is drawn from some distribution with support $\Theta \subseteq \mathbb{R}$ and then chooses education $e \in \mathbb{R}_+$. There is a reduced-form market that observes e (but not θ) and allocates wage, or some other statistic of job characteristics, $w \in \mathbb{R}$ to the worker. The worker's von Neumann-Morgenstern payoff is given by $v(w, e, \theta)$. It is convenient to let $a \equiv (w, e)$, so that we can also write $v(a, \theta)$.

[24] More precisely: the two candidates simultaneously choose their lotteries, and each voter then votes for his preferred candidate (assuming, for concreteness, that a voter randomizes between the candidates with equal probability if indifferent). A candidate wins if he receives a majority of the votes. Candidates maximize the probability of winning. We seek a Nash equilibrium of the game between the two candidates.

[25] An example may be helpful. Let $A = [-1, 1]$, $\Theta = \{-1, 0, 1\}$, and $v(a, \theta) = a\theta + 1/(|a| + 1) + 1$. The corresponding functions $f_1(\theta) = \theta$ and $f_2(\theta) = 1$ are each strictly single crossing from below and strictly ratio ordered. For all θ, $v(\cdot, \theta) : A \to \mathbb{R}$ is maximized at $a = \theta$ but convex on a sub-interval of the policy space.

[26] A similar point could be made without a strict violation, but it would require endogenous tie-breaking by an indifferent voter.

In the standard model, (i) w is an exogenously-given strictly increasing function of the market's expectation $\mathbb{E}[\theta|e]$, (ii) $v(w,e,\theta)$ is strictly increasing in w and strictly decreasing in e, and (iii) $v(w,e,\theta)$ has strict single-crossing differences in θ.[27]

Our results allow us to generalize some central conclusions about education signaling to settings in which there is uncertainty about what wage the worker will receive, even conditional on the market belief about his type. Such uncertainty is, of course, economically plausible. Accordingly, in our specification, we allow for $w \sim F_\mu$, i.e., w is drawn from an exogenously-given cumulative distribution F that depends on $\mu \in \Delta\Theta$, the market belief about θ. Let $V(F,e,\theta) \equiv \int_w v(w,e,\theta)\mathrm{d}F$ and $\mathcal{F} \equiv \{F_\mu : \mu \in \Delta\Theta\}$ be the family of feasible wage distributions. We assume that for $\mu,\mu' \in \Delta\Theta$, if μ (strictly) *support-dominates* μ' (i.e., $\inf \mathrm{Supp}[\mu] \geq (>) \sup \mathrm{Supp}[\mu']$), then $(\forall e,\theta)\; V(F_\mu,e,\theta) \geq (>) V(F_{\mu'},e,\theta)$. This is a weak sense in which the worker wants to convince the market that his type is higher. We also assume that $v(w,e,\theta)$ is strictly decreasing in e.

A (weak Perfect Bayesian) equilibrium is described by a pair of functions $\sigma^*(\theta)$ and $\mu^*(e)$, where σ^* denotes the worker's mixed strategy (for each θ, $\sigma^*(\theta) \in \Delta\mathbb{R}_+$) and μ^* the market belief (for each e, $\mu^*(e) \in \Delta\Theta$). For notational and technical simplicity, we will restrict attention to equilibria in which for all θ, $\sigma^*(\theta)$ has countable support. When the equilibrium is pure we write $e^*(\theta)$ instead of $\sigma^*(\theta)$. A pure-strategy equilibrium exists: all types pool on $e = 0$ and off-path beliefs are the same as the prior.

A fundamental conclusion of the standard model is that in any equilibrium higher types acquire (weakly) more education. Our results deliver this conclusion in our specification; see Liu and Pei (2017) for related work.

We say that a strategy σ is *increasing* if for all $\underline{\theta} < \overline{\theta}$ and $\underline{e} < \overline{e}$, $\sigma(\overline{e}|\underline{\theta}) > 0 \implies \sigma(\underline{e}|\overline{\theta}) = 0$. In other words, a strategy is increasing if a higher type never acquires (with positive probability) strictly less education than a lower type.

Claim 3. *Assume $v(a,\theta) \equiv v(w,e,\theta)$ has the form stated in Theorem 2, and hence has SSCED. If*

$$F, G \in \mathcal{F}, \overline{e} \neq \underline{e} \implies (\exists\theta)\; V(F,\overline{e},\theta) \neq V(G,\underline{e},\theta), \tag{8}$$

then in any equilibrium $\sigma^*(\theta)$ is increasing.

Condition (8) is a mild richness condition; in particular, given SSCED, it is automatically satisfied if the worker's utility is separable in wage and education.[28]

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[27] Given point (ii), point (iii) is implied by the Spence-Mirlees single crossing condition: v_w/v_e is increasing in θ, where a subscript on v denotes a partial derivative (assuming differentiability).

[28] Suppose v has SSCED and is separable in w and e, so that it has the form $v(w,e,\theta) = g_1(w)f_1(\theta) +$

Proof of Claim 3. Suppose, to contradiction, that $\sigma^*(\theta)$ is not increasing. Then there exist $\underline{\theta} < \overline{\theta}$ and $\underline{e} < \overline{e}$ such that $\min\{\sigma^*(\overline{e}|\underline{\theta}), \sigma^*(\underline{e}|\overline{\theta})\} > 0$. Let $\underline{F}$ and $\overline{F}$ be the wage distributions resulting from $\underline{e}$ and $\overline{e}$ respectively. By Corollary 5, there is a complete order, say $\succeq^*_{SSCED}$, on the quotient space of wage distribution and education pairs with respect to which V has monotone selection. Condition (8) implies that either $(\overline{F}, \overline{e}) \succ^*_{SSCED} (\underline{F}, \underline{e})$ or $(\underline{F}, \underline{e}) \succ^*_{SSCED} (\overline{F}, \overline{e})$. As $(\overline{F}, \overline{e}) \succ^*_{SSCED} (\underline{F}, \underline{e})$ would contradict optimality of $\overline{e}$ for $\underline{\theta}$ and $\underline{e}$ for $\overline{\theta}$ (by monotone selection), it holds that $(\underline{F}, \underline{e}) \succ^*_{SSCED} (\overline{F}, \overline{e})$.

Monotone selection now implies that $\sigma^*(\underline{e}|\theta) > 0 \implies (\forall \theta' > \theta)\ \sigma^*(\overline{e}|\theta') = 0$. Consequently, $\mu^*(\underline{e})$ support-dominates $\mu^*(\overline{e})$. Given the support-dominance, type $\underline{\theta}$ can profitably deviate from switching mass from $\overline{e}$ to $\underline{e}$, because the reduction in education is strictly preferred and the change in market belief is weakly preferred, a contradiction. *Q.E.D.*

We can also study when there is a separating equilibrium. Let F_θ denote the wage distribution when the market puts probability one on θ.

Claim 4. *Let $\Theta \equiv \{\theta_1, \ldots, \theta_N\}$. If*

1. *$v(a, \theta) \equiv v(w, e, \theta)$ has the form stated in Theorem 1 and hence has SCED, and is continuous in e,*

2. *$\lim_{e \to \infty} V(F_{\theta_n}, e, \theta_{n-1}) < V(F_{\theta_{n-1}}, 0, \theta_{n-1})$ for all $n \in \{2, \ldots, N\}$, and*

3. *for all $n \in \{2, \ldots, N\}$ and $\overline{e} > \underline{e}$*

$$V(F_{\theta_n}, \overline{e}, \theta_{n-1}) = V(F_{\theta_{n-1}}, \underline{e}, \theta_{n-1}) \implies V(F_{\theta_n}, \overline{e}, \theta_n) \geq V(F_{\theta_{n-1}}, \underline{e}, \theta_n), \tag{9}$$

then there is a pure-strategy equilibrium in which $e^*(\theta)$ is strictly increasing.

The conditions in the result above are related to those in Cho and Sobel (1990). The inequalities in part 2 of the Claim merely require that no type θ_n ($n \in \{2, \ldots, N\}$) would be willing to pay the cost of acquiring arbitrarily high education to shift the market's belief from probability one on θ_{n-1} to probability one on θ_n. Despite a resemblance, Condition (9) is not by itself a single-crossing condition. Rather, loosely speaking, it ensures that the SCED order over wage distribution and education pairs goes in the "right direction"; the proof below clarifies.

$g_2(e)f_2(\theta)+c(\theta)$. Fix any $F, G \in \mathcal{F}$ and $\overline{e} \neq \underline{e}$. We compute the expectational difference $V(F, \overline{e}, \theta) - V(G, \underline{e}, \theta) = \left[\int g_1(w)\mathrm{d}F - \int g_1(w)\mathrm{d}G\right] f_1(\theta) + [g_2(\overline{e}) - g_2(\underline{e})]\, f_2(\theta)$. The maintained assumption that v is strictly decreasing in e implies $g_2(\overline{e}) - g_2(\underline{e}) \neq 0$. As strict ratio ordering of f_1 and f_2 implies they are linearly independent, it follows that the expectational difference is non-zero for some θ.

Proof of Claim 4. Set $e_1 = 0$. For $n > 1$, inductively construct e_n as the solution to

$$V(F_{\theta_n}, e_n, \theta_{n-1}) = V(F_{\theta_{n-1}}, e_{n-1}, \theta_{n-1}). \tag{10}$$

Our assumptions ensure there is a unique solution and that $e_n > e_{n-1}$ for all $n \in \{2, \dots, N\}$.

We claim $(\forall n)\ e^*(\theta_n) = e_n$ can be supported as an equilibrium. To see this, first note that by Corollary 4, there is a complete order, say $\succeq^*_{SCED}$, on the quotient space of wage distribution and education pairs with respect to which V has monotone comparative statics. It follows that for $\forall n \in \{2, \dots, N\}$, (9) and (10) imply $(F_{\theta_n}, e_n) \succeq^*_{SCED} (F_{\theta_{n-1}}, e_{n-1})$. Hence, no type can profitably deviate to any on-path $e \in \{e_1, \dots, e_N\}$; off-path deviations can be deterred by simply setting off-path beliefs to put probability one on θ_1. *Q.E.D.*

3.4. Altruism, Spite, and Utilitarianism

Finally, we illustrate how our results on aggregating single-crossing functions are useful even when the choice space is not a set of lotteries.

Consider two agents, denoted 1 and 2. Each agent i has a direct or material payoff function $v_i(a, \theta)$, where $a \in A$ is a single choice variable that affects both agents and $\theta \in \Theta$ is some parameter. Both A and Θ are ordered sets. Assume each v_i has SCD, i.e., for any a and a', $D^i_{a,a'}(\theta) \equiv v_i(a, \theta) - v_i(a', \theta)$ is single crossing. To allow for other-regarding preferences, consider the adjusted utility function

$$u(a, \theta) \equiv \sum_{i=1}^{2} \alpha_i v_i(a, \theta), \tag{11}$$

with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$.[29] A positive α_i represents altruism towards i, while a negative α_i represents spite (e.g., Levine, 1998).

By Theorem 3, u must have SCD for all α to yield monotone comparative statics no matter the other-regarding coefficient vector, α. Lemma 1 says that u has SCD for all α if and only if

$$(\forall a, a')\ D^1_{a,a'} \text{ and } D^2_{a,a'} \text{ are ratio ordered.} \tag{12}$$

While the above discussion focuses on two agents, Proposition 1 can be applied when there are more agents.

One may also interpret (11) as a weighted utilitarian criterion by restricting attention to $\alpha \in \mathbb{R}^2_+ \setminus \{0\}$. Under this restricted domain of α, (12) is not necessary for u to have SCD

[29] We have not written u to be agent specific purely to lighten notation.

given arbitrary v_1 and v_2. However, suppose there is always disagreement between the agents in their ordinal ranking of any pair of alternatives:

$$(\forall a, a')(\forall \theta)\ D^1_{a,a'}(\theta) D^2_{a,a'}(\theta) < 0. \tag{13}$$

Under (13), u has SCD for all $\alpha \in \mathbb{R}^2_+ \setminus \{0\}$ if and only if (12) holds. In particular, (13) implies that if (12) fails, then for some a and a', there is $\alpha \in \mathbb{R}^2_+ \setminus \{0\}$ such that $\alpha_1 D^1_{a,a} + \alpha_2 D^2_{a,a}$ is not single crossing; the essence of the argument can be seen in Figure 4(b).[30]

4. Discussion

4.1. Single Crossing vs. Monotonicity

We have characterized when $v : A \times \Theta \to \mathbb{R}$ has SCED. Viewing ΔA as a choice set and Θ as a parameter set, SCED is an *ordinal* property. Analogous to supermodularity on an ordered space, one might also be interested in a stronger *cardinal* property in our setting, which we call **Monotonic Expectational Differences** (MED):

$$(\forall P, Q \in \Delta A)\ \ D_{P,Q}(\theta) \text{ is monotonic in } \theta.$$

(A function $f : \Theta \to \mathbb{R}$ is **monotonic** if (i) $(\forall \theta \le \theta') f(\theta) \le f(\theta')$, or (ii) $(\forall \theta \le \theta')\ f(\theta) \ge f(\theta')$.)

To study which functions have MED, we begin with the following analog of Lemma 1.

Lemma 5. *Let $f_1, f_2 : \Theta \to \mathbb{R}$ be monotonic functions. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is monotonic $\forall(\alpha_1, \alpha_2) \in \mathbb{R}^2$ if and only if either f_1 or f_2 is an affine transformation of the other, i.e., there exists $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that either $f_2 = \lambda_1 f_1 + \lambda_2$ or $f_1 = \lambda_1 f_2 + \lambda_2$.*

We say that $f : X \times \Theta \to \mathbb{R}$ is **linear combinations monotonicity-preserving** if $\int_X f(x, \theta) d\mu$ is a monotonic function of θ for every finite signed measure μ on (X, Σ).

Proposition 4. *Let (X, Σ) be a measurable space and $f : X \times \Theta \to \mathbb{R}$ such that (i) $(\forall x)\ f(x, \theta)$ is a monotonic function of θ and (ii) $(\forall \theta)\ f(x, \theta)$ is a finitely-integrable function of x. The function f is linear combinations monotonicity-preserving if and only if there exists $x' \in X$ such that $(\forall \theta) f(x, \theta) = \lambda_1(x) f(x', \theta) + \lambda_2(x)$ with $\lambda_1, \lambda_2 : X \to \mathbb{R}$ finitely integrable.*

[30] To be precise, this logic only proves the analog of Condition (1) of ratio ordering. But (13) also ensures that the analog of Condition (2) is satisfied; cf. Figure 3.

Theorem 4. *The function $v : A \times \Theta \to \mathbb{R}$ has MED if and only if it takes the form*

$$v(a,\theta) = g_1(a)f_1(\theta) + g_2(a) + c(\theta), \tag{14}$$

where $g_1, g_2 : A \to \mathbb{R}$ are finitely integrable, $f_1 : \Theta \to \mathbb{R}$ is monotonic, and $c : \Theta \to \mathbb{R}$.

The proof of Theorem 4 in the Appendix uses Proposition 4, exhibiting a parallel with Proposition 1 and Theorem 1. There is, however, a simple intuition for Theorem 4 based on the von Neumann-Morgenstern expected utility theorem. Suppose $\Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$ and $v_\theta(a,\theta)$, the partial derivative of $v(a,\theta)$ with respect to θ, exists and is continuous. Consider the following strengthening of MED: $(\forall P, Q \in \Delta A)$ $D_{P,Q}(\theta)$ is either a zero function or strictly monotonic in θ. Then, for any P and Q, $\text{sign}\left[\int_A v_\theta(a,\theta)\mathrm{d}P - \int_A v_\theta(a,\theta)\mathrm{d}Q\right]$ is independent of θ. In other words, for all θ, $v_\theta(\cdot,\theta)$ is a von Neumann-Morgenstern representation of the same preferences over lotteries. The conclusion of Theorem 4 follows from the expected utility theorem's implication that for any $\theta', \theta'' \in \Theta$, $v_\theta(\cdot,\theta')$ must be a positive affine transformation of $v_\theta(\cdot,\theta'')$.[31] We are not aware of any related argument for the SCED characterization, Theorem 1.

Comparing Theorem 1 and Theorem 4, we see that a function v with MED is a special case of a function v with SCED, in which the function f_2 in (3) is identically equal to one. Note that when this is the case, f_1 and f_2 being ratio ordered is equivalent to f_1 being monotonic. SCED is more general than MED; for example, given a function $v(a,\theta)$ of the form (14) with $f_1(\cdot) > 0$ and domain $\Theta \subseteq \mathbb{R}_{++}$, the function $\tilde{v}(a,\theta) \equiv \theta v(a,\theta)$ will satisfy SCED but generally violate MED. The generality translates into preferences: the set of preferences with SCED representations is larger than that with MED representations; see Example 4 in Appendix G.4. Proposition 5 in Appendix G.4 characterizes exactly when preferences with an SCED representation have an MED representation. It is when (a) there is a pair of types that do not share the same strict preference over any pair of lotteries, or (b) there is a pair of lotteries over which all types share the same strict preference.

The MED characterization in Theorem 4 has largely been obtained by Kushnir and Liu (2017). They restrict attention to $\Theta \subseteq \mathbb{R}$, $A \subset \mathbb{R}^k$, and functions v that have some smoothness. Modulo minor differences, Kushnir and Liu establish that for their environment, a strict version of MED (in fact the strengthening discussed in the paragraph after Theorem 4) is equivalent to the characterization in Theorem 4 with f_1 strictly monotonic.[32]

[31] Pick any $\theta^* \in \Theta$. The expected utility theorem implies that for some $F_1 : \Theta \to \mathbb{R}_{++}$ and $C : \Theta \to \mathbb{R}$, $(\forall a, \theta)$ $v_\theta(a,\theta) = F_1(\theta)v_\theta(a,\theta^*) + C(\theta)$. Equation 14 follows from integrating up θ at each a, as $v(a,\theta) = \int_{\underline{\theta}}^{\theta} v_\theta(a,t)\mathrm{d}t + v(a,\underline{\theta})$.

[32] The statement in Proposition 3 of their paper is that f_1 is strictly increasing; monotonicity vs. increasing

Kushnir and Liu's focus is on the equivalence between Bayesian and dominant-strategy implementation; their methodology requires MED rather than SCED.

4.2. The Relationship with Quah and Strulovici (2012)

Lemma 1 is related to Quah and Strulovici (2012, Proposition 1). They establish that for any two functions f_1 and f_2 that are each single crossing from below, $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing from below for all $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ if and only if f_1 and f_2 satisfy **signed-ratio monotonicity below**:[33] for all $i, j \in \{1, 2\}$,

$$(\forall \theta_l \leq \theta_h) \quad f_j(\theta_l) < 0 < f_i(\theta_l) \implies f_i(\theta_h) f_j(\theta_l) \leq f_i(\theta_l) f_j(\theta_h). \tag{15}$$

The following example shows that even for a pair of functions that are single crossing from below, ratio ordering does not imply signed-ratio monotonicity below.

Example 2. Let $\Theta = [0, 1]$, $f_1(\theta) = 1$, and $f_2(\theta) = -\theta - 1$. Since f_1 is constant and f_2 is monotonic, they are ratio ordered; moreover, both functions are single crossing from below. However, they do not satisfy signed-ratio monotonicity below; $f_1 + f_2$, while single crossing, is not single crossing from below. □

Thus motivated, let us say that f_1 and f_2 satisfy **signed-ratio monotonicity** if either f_1 and f_2 satisfy signed-ratio monotonicity below or $-f_1$ and $-f_2$ satisfy signed-ratio monotonicity below; equivalently, f_1 and f_2 satisfy, for all $i, j \in \{1, 2\}$, either (15) or its reversed-inequality version:

$$(\forall \theta_l \leq \theta_h) \quad f_j(\theta_l) < 0 < f_i(\theta_l) \implies f_i(\theta_l) f_j(\theta_h) \leq f_i(\theta_h) f_j(\theta_l). \tag{16}$$

Ratio ordering is still not stronger (nor weaker) than signed-ratio monotonicity:

Example 3. Let $\Theta = [-1, 1]$, and for some $\varepsilon \in (0, 1/2)$,

$$f_1(\theta) = \begin{cases} 1 & \text{if } \theta \geq 0 \\ \varepsilon(\theta - 1) & \text{if } \theta < 0, \end{cases} \quad \text{and} \quad f_2(\theta) = \begin{cases} 1 & \text{if } \theta \leq 0 \\ -\varepsilon(1 + \theta) & \text{if } \theta > 0. \end{cases}$$

is immaterial, as the direction of monotonicity can be reversed by flipping the sign of the function g_1.

[33] Quah and Strulovici call this simply "signed-ratio monotonicity". We note that Quah and Strulovici also consider aggregating non-negative linear combinations of more than two functions that are each single-crossing from below (cf. Proposition 1).

Figure 7 depicts these functions (drawn for $\varepsilon = 1/4$). For all $\theta_l < \theta_h$, $f_1(\theta_l)f_2(\theta_h) < f_1(\theta_h)f_2(\theta_l)$, so f_1 ratio dominates f_2. As $f_1(\theta) > 0 > f_2(\theta)$ for all $\theta > 0$, (15) is violated; as $f_1(\theta) < 0 < f_2(\theta)$ for all $\theta < 0$, (16) is also violated. □

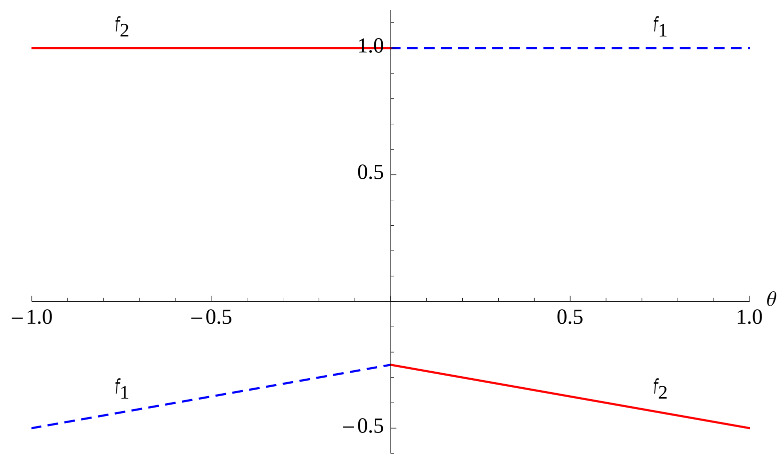

Figure 7: A violation of signed-ratio monotonicity that satisfies ratio ordering.

However:

Remark 4. For functions f_1 and f_2 that are both single crossing from below or both single crossing from above, ratio ordering implies signed-ratio monotonicity. To prove this, first note that if there is no θ at which $f_1(\theta)f_2(\theta) < 0$, signed-ratio monotonicity trivially holds. So assume there is a θ at which, without loss by re-labeling the functions, $f_1(\theta) > 0 > f_2(\theta)$. Since f_1 and f_2 are both single crossing from below or both from above, there is no θ' at which $f_1(\theta') < 0 < f_2(\theta')$. So we need only to show that fixing $i = 1$ and $j = 2$, either (15) or (16) holds. But this is implied by ratio ordering.

Moreover, it is straightforward that if for some i, j and all θ, $f_i(\theta) > 0 > f_j(\theta)$, then ratio ordering is equivalent to signed-ratio monotonicity. □

The reason why our Lemma 1 deduces the more demanding condition of ratio ordering even when f_1 and f_2 are single crossing in the same direction is because it considers aggregations $\alpha_1 f_1 + \alpha f_2$ with coefficients of arbitrary signs, while Quah and Strulovici (2012) require both coefficients to have the same sign. To illustrate, suppose $f_1 > 0$ and $f_2 > 0$. Signed-ratio monotonicity trivially holds and $\alpha f_1 + \alpha_2 f_2$ is single crossing from both below and above for all $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ and also for all $(\alpha_1, \alpha_2) \in \mathbb{R}^2_-$. However, ratio ordering can fail (f_1/f_2 need not be monotonic), in which case there exists $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $\alpha_1\alpha_2 < 0$ such that $\alpha_1 f_1 + \alpha_2 f_2$ is not single crossing, neither from below nor from above. Since a function f is single crossing from below if and only if $-f$ is single crossing from above, another perspective is that Lemma 1 accommodates cases in which the aggregating coefficients have the same sign but one of f_1 and f_2 is single crossing only from below while the other is single crossing only from above; see the utilitarianism application in Subsection 3.4. Quah and Strulovici's result ultimately requires joint restrictions on the signs of the aggregating coefficients and the directions in which the functions are single crossing.

5. Conclusion

The main result of this paper is a full characterization of which von Neumann-Morgenstern utility functions of outcome and type satisfy single-crossing expectational differences (SCED): the difference in expected utility between every pair of probability distributions on outcomes is single crossing in type (Theorem 1). We have established that this property is necessary and sufficient for a form of monotone comparative statics when an agent chooses among distributions over outcomes (Theorem 3).

We close by highlighting aspects of our analysis that suggest directions for future research.

Theorem 1's characterization that the utility function can be decomposed into a sum of *two* products (and a function independent of the outcome), $v(a, \theta) = g_1(a)f_1(\theta)+g_2(a)f_2(\theta)+c(\theta)$, owes to the expectational difference being *single*-crossing. If one were interested in (at most) n-crossings, for $n \in \mathbb{N}$,[34] then we believe that $v(a, \theta) = \sum_{i=1}^{n+1} g_i(a)f_i(\theta) + c(\theta)$ would be a necessary condition. It would be interesting to find the appropriate generalization of ratio ordering to make this form necessary and sufficient. Just as ratio ordering is related to total positivity of order two (recall the discussion after Lemma 1), we suspect the generalization would be related to total positivity of order $n + 1$.

Theorem 1's characterization leans on the requirement that the expectational difference must be single crossing for *all* pairs of distributions over outcomes. While a "sufficiently rich" set of pairs should suffice, there is a fundamental tradeoff. Requiring single crossing for only a subset of distributions would expand the set of utility functions satisfying the requirement, but any application must then have enough stochastic structure to validate the restriction on distributions.[35] Another possibility would be to weaken or alter the expected utility hypothesis.

[34] Choi and Smith (2016) generalize Quah and Strulovici (2012) in this vein.

[35] Recall that Claim 1 in Subsection 3.1 established a sense in which SCED (over all pairs of distributions) is necessary to guarantee that all cheap talk equilibria are connected in that application. Restricting the form of the receiver's preferences and/or the distributions of her private information could circumvent this necessity.

Our results have direct bearing on problems in which all types of an agent face the same choice set of distributions. Such situations arise naturally, as illustrated in Section 3. But consider a variation of the cheap-talk application (Subsection 3.1) in which the sender's type is correlated with the receiver's type. Even though the receiver's type does not affect the sender's payoff, different sender types will generally have different beliefs about the distribution of the receiver's action that any message induces in equilibrium. Effectively, different sender types will be choosing from different sets of distributions. An approach that synthesizes the current paper's with that of, for example, Athey's (2002) may be useful for such problems.

Appendices

A. Proofs of Corollaries in Subsection 2.1

A.1. Proof of Corollary 1

It is clear from Theorem 1 that $v(a,\theta) = -|a-\theta|^2 = -a^2+2a\theta-\theta^2$ has SCED, as $f_1(\theta) = -1$ and $f_2(\theta) = 2\theta$ are ratio ordered, $g_1(a) = a^2$ and $g_2(a) = a$ are each finitely integrable, and we take $c(\theta) = -\theta^2$.

For the converse, it is sufficient to prove the following claim.

Claim 5. *If there exist $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ and $f_1, f_2, c : \Theta \to \mathbb{R}$ such that*

$$v(a,\theta) \equiv -|a-\theta|^z = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta),$$

then $z = 2$.

Fix $a_0 \in \mathbb{R}$ and define $\tilde{v}(a,\theta) \equiv v(a,\theta) - v(a_0,\theta) = \tilde{g}_1(a)f_1(\theta) + \tilde{g}_2 f_2(\theta)$, where $\tilde{g}_1(a) \equiv g_1(a) - g_1(a_0)$ and $\tilde{g}_2 \equiv g_2(a) - g_2(a_0)$. Fix any $\theta_l < \theta_m < \theta_h$. There exists $(\lambda_l, \lambda_m, \lambda_h) \in \mathbb{R}^3 \backslash \{0\}$ such that

$$\begin{pmatrix} f_1(\theta_l) & f_1(\theta_m) & f_1(\theta_h) \\ f_2(\theta_l) & f_2(\theta_m) & f_2(\theta_h) \end{pmatrix} \begin{pmatrix} \lambda_l \\ \lambda_m \\ \lambda_h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, for every $a \in \mathbb{R}$,

$$
\begin{aligned}
h(a) &\equiv \lambda_l \tilde{v}(a, \theta_l) + \lambda_m \tilde{v}(a, \theta_m) + \lambda_h \tilde{v}(a, \theta_h) \\
&= \begin{pmatrix} \tilde{g}_1(a) & \tilde{g}_2(a) \end{pmatrix} \begin{pmatrix} f_1(\theta_l) & f_1(\theta_m) & f_1(\theta_h) \\ f_2(\theta_l) & f_2(\theta_m) & f_2(\theta_h) \end{pmatrix} \begin{pmatrix} \lambda_l \\ \lambda_m \\ \lambda_h \end{pmatrix} = 0.
\end{aligned}
$$

We hereafter consider $\lambda_l \neq 0$ (and omit the proof for the other two cases, $\lambda_m \neq 0$ and $\lambda_h \neq 0$, which are analogous). The previous equation implies that for any $a \in \mathbb{R}$,

$$
\tilde{v}(a, \theta_l) = -\frac{\lambda_m}{\lambda_l} \tilde{v}(a, \theta_m) - \frac{\lambda_h}{\lambda_l} \tilde{v}(a, \theta_h). \tag{17}
$$

At any $a < \theta$, $\tilde{v}(a, \theta) = -(\theta - a)^z - v(a_0, \theta)$ is differentiable in a, and hence (17) implies that the partial derivative $\tilde{v}_a(a, \theta_l)$ exists at $a = \theta_l$. Thus, the right partial derivative $\lim_{\varepsilon \downarrow 0} \frac{\tilde{v}(\theta_l + \varepsilon, \theta_l) - \tilde{v}(\theta_l, \theta_l)}{\varepsilon} = -\lim_{\varepsilon \downarrow 0} \varepsilon^{z-1}$ must equal the left partial derivative $\lim_{\varepsilon \downarrow 0} \frac{\tilde{v}(\theta_l - \varepsilon, \theta_l) - \tilde{v}(\theta_l, \theta_l)}{-\varepsilon} = \lim_{\varepsilon \downarrow 0} \varepsilon^{z-1}$, which implies $\lim_{\varepsilon \downarrow 0} \varepsilon^{z-1} = 0$, and thus $z > 1$.

Now suppose to contradiction that $z \neq 2$. At any $a > \theta_h$, (17) and $\tilde{v}(a, \theta) = -(a - \theta)^z - v(a_0, \theta)$ imply

$$
-\lambda_l (a - \theta_l)^z = \lambda_m (a - \theta_m)^z + \lambda_h (a - \theta_h)^z + (\lambda_m + \lambda_h - \lambda_l) v(a_0, \theta),
$$

and hence, differentiating with respect to a and simplifying using $z > 1$ and $z \neq 2$:

$$
-\lambda_l (a - \theta_l)^{z-1} = \lambda_m (a - \theta_m)^{z-1} + \lambda_h (a - \theta_h)^{z-1}, \tag{18}
$$

$$
-\lambda_l (a - \theta_l)^{z-2} = \lambda_m (a - \theta_m)^{z-2} + \lambda_h (a - \theta_h)^{z-2}, \tag{19}
$$

$$
-\lambda_l (a - \theta_l)^{z-3} = \lambda_m (a - \theta_m)^{z-3} + \lambda_h (a - \theta_h)^{z-3}. \tag{20}
$$

It follows that $\lambda_m \lambda_h \neq 0$ and $\lambda_h \neq 0$: if, for example, $\lambda_l = 0$, then (18) implies $\lambda_h \neq 0$ (as $\lambda_l \neq 0$), and then (18) and (19) imply $a - \theta_l = a - \theta_h$ for all $a > \theta_h$, contradicting $\theta_l < \theta_h$. Since $((a - \theta_l)^{z-2})^2 = (a - \theta_l)^{z-1}(a - \theta_l)^{z-3}$, we manipulate the right-hand sides of (18)–(20) to obtain

$$
2\lambda_m \lambda_h (a - \theta_m)^{z-2} (a - \theta_h)^{z-2} = \lambda_m \lambda_h \left((a - \theta_m)^{z-1} (a - \theta_h)^{z-3} + (a - \theta_m)^{z-3} (a - \theta_h)^{z-1} \right),
$$

which simplifies, using $\lambda_m \lambda_h \neq 0$, to

$$2 = \frac{a-\theta_h}{a-\theta_m} + \frac{a-\theta_m}{a-\theta_h}.$$

Therefore, $a-\theta_h = a-\theta_m$ for all $a > \theta_h$, contradicting $\theta_m < \theta_h$.

A.2. Proof of Corollary 2

If f is monotonic, it ratio dominates any positive constant function. It follows from Theorem 1 that $v((q,t),\theta)$ has SCED.

To prove the converse, suppose, toward contradiction, that f is not monotonic. Then there exist $\theta_l < \theta_m < \theta_h$ such that either $f(\theta_m) > \max\{f(\theta_l), f(\theta_h)\}$ or $f(\theta_m) < \min\{f(\theta_l), f(\theta_h)\}$. We proceed assuming the first case and omit the analogous argument for the other case.

Take any $z \in \mathbb{R}$ such that $f(\theta_m) > z > \max\{f(\theta_l), f(\theta_h)\}$, any $q_1, q_2 \in \mathbb{R}$ such that $g(q_1) - g(q_2) > 0$, and $t_1, t_2 \in \mathbb{R}$ such that $(g(q_1) - g(q_2))z - (t_1 - t_2) = 0$. The expectational difference between degenerate lotteries on $a_1 = (q_1, t_1)$ and $a_2 = (q_2, t_2)$,

$$D_{1,2}(\theta) \equiv v(a_1, \theta) - v(a_2, \theta) = (g(q_1) - g(q_2))f(\theta) - (t_1 - t_2),$$

is not single crossing as $D_{1,2}(\theta_m) > 0 > \max\{D_{1,2}(\theta_l), D_{1,2}(\theta_h)\}$.

A.3. Proof of Corollary 3

Fix any $a \in A$. Given the form (3),

$$\frac{v_{aa}(a,\theta)}{v_a(a,\theta)} = \frac{g_1''(a)f_1(\theta) + g_2''(a)f_2(\theta)}{g_1'(a)f_1(\theta) + g_2'(a)f_2(\theta)}.$$

Hence, $-\frac{v_{aa}(a,\theta)}{v_a(a,\theta)}$ is decreasing (resp., increasing) in θ if and only if the partial derivative of the right-hand side above with respect to θ is positive (resp., negative), or equivalently,

$$\begin{aligned}(\forall\theta)\quad & (g_1''(a)f_1'(\theta) + g_2''(a)f_2'(\theta))(g_1'(a)f_1(\theta) + g_2'(a)f_2(\theta)) \\ & \geq (\leq)\ (g_1'(a)f_1'(\theta) + g_2'(a)f_2'(\theta))(g_1''(a)f_1(\theta) + g_2''(a)f_2(\theta))\end{aligned}$$

which we can simplify, using the hypotheses that $g_1'(a) > 0$ and $g_2'(a) > 0$, as

$$(\forall\theta)\quad \frac{g_1''(a)}{g_1'(a)}(f_1'(\theta)f_2(\theta) - f_1(\theta)f_2'(\theta)) \geq (\leq)\ \frac{g_2''(a)}{g_2'(a)}(f_1'(\theta)f_2(\theta) - f_1(\theta)f_2'(\theta)).$$

The above inequality is equivalent to $-\frac{g_1''(a)}{g_1'(a)} \le (\ge) \; -\frac{g_2''(a)}{g_2'(a)}$ if

$$f_1'(\theta)f_2(\theta) \ge f_1(\theta)f_2'(\theta) \tag{21}$$

for all θ, with strict inequality for some θ. This latter property holds by our hypothesis that f_1 strictly ratio dominates f_2. Specifically, letting $k(\theta, \varepsilon) \equiv f_1(\theta+\varepsilon)f_2(\theta-\varepsilon)$, Condition (1) with a strict inequality implies that for any θ and (small) $\varepsilon > 0$, $k(\theta,\varepsilon) > k(\theta,-\varepsilon)$, and hence $k_\varepsilon(\theta,0) \ge 0$ (where k_ε is the partial derivative), which is equivalent to (21). Furthermore, Condition (1) with a strict inequality implies that $f_2(\theta^*) \ne 0$ for some θ^*, and hence, in a neighborhood of θ^*, f_1/f_2 is strictly increasing, which implies (21) with a strict inequality for some θ in that neighborhood.

B. Proof of Lemma 1

When $|\Theta| \le 2$, the proof is trivial as all functions are single crossing and every pair of f_1, f_2 are ratio ordered. Hereafter, we assume $|\Theta| \ge 3$.

B.1. ($\implies$)

To prove (1), we suppose towards contradiction that

$$\begin{aligned} &(\exists \theta_l < \theta_h) \quad f_1(\theta_l)f_2(\theta_h) < f_1(\theta_h)f_2(\theta_l) \text{ and} \\ &(\exists \theta' < \theta'') \quad f_1(\theta')f_2(\theta'') > f_1(\theta'')f_2(\theta'). \end{aligned} \tag{22}$$

Take any upper bound $\overline{\theta}$ of $\{\theta_l, \theta_h, \theta', \theta''\}$.

First, let $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$. Then, $(\alpha_l \cdot f)(\theta_l) = (f_2(\theta_l), -f_1(\theta_l)) \cdot (f_1(\theta_l), f_2(\theta_l)) = 0$, and $(\alpha_l \cdot f)(\theta_h) > 0$. Thus, $\alpha_l \cdot f$ is single crossing from below, and $(\alpha_l \cdot f)(\overline{\theta}) > 0$.

Second, let $\alpha' \equiv (f_2(\theta'), -f_1(\theta'))$. Then, $(\alpha' \cdot f)(\theta') = 0$ and $(\alpha' \cdot f)(\theta'') < 0$. Thus, $\alpha' \cdot f$ is single crossing from above, and $(\alpha' \cdot f)(\overline{\theta}) < 0$.

Let $\overline{\alpha} = (f_2(\overline{\theta}), -f_1(\overline{\theta}))$. Then,

$$\begin{aligned} &(\overline{\alpha} \cdot f)(\theta_l) = (f_2(\overline{\theta}), -f_1(\overline{\theta})) \cdot (f_1(\theta_l), f_2(\theta_l)) = -(\alpha_l \cdot f)(\overline{\theta}) < 0, \\ &(\overline{\alpha} \cdot f)(\theta') = -(\alpha' \cdot f)(\overline{\theta}) > 0, \text{ and} \\ &(\overline{\alpha} \cdot f)(\overline{\theta}) = 0. \end{aligned}$$

Therefore, $\overline{\alpha} \cdot f$ is not single crossing, a contradiction.

To prove (2), take any $\theta_l < \theta_m < \theta_h$.

First, we show that $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$ implies $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$ and $f_1(\theta_m)f_2(\theta_l) = f_1(\theta_l)f_2(\theta_m)$. Assume f_1 is not a zero function on $\{\theta_l, \theta_m, \theta_h\}$, as otherwise the proof is trivial. Since f_1 is single crossing, either $f_1(\theta_l) \neq 0$ or $f_1(\theta_h) \neq 0$. We consider the case of $f_1(\theta_h) \neq 0$ (and omit the proof for the other case). Let $\alpha_h = (f_2(\theta_h), -f_1(\theta_h))$. Since $\alpha_h \cdot f$ is single crossing and $(\alpha_h \cdot f)(\theta) = 0$ for $\theta = \theta_l, \theta_h$, it holds that $(\alpha_h \cdot f)(\theta_m) = f_2(\theta_h)f_1(\theta_m) - f_1(\theta_h)f_2(\theta_m) = 0$. It follows immediately that $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$. As $(f_1(\theta_m), f_2(\theta_m))$ and $(f_1(\theta_h), f_2(\theta_h))$ are linearly dependent and $(f_1(\theta_h), f_2(\theta_h))$ is a non-zero vector, there exists $\lambda \in \mathbb{R}$ such that $f_i(\theta_m) = \lambda f_i(\theta_h)$ for $i = 1, 2$. Thus,

$$f_1(\theta_l)f_2(\theta_m) = \lambda f_1(\theta_l)f_2(\theta_h) = \lambda f_2(\theta_l)f_1(\theta_h) = f_2(\theta_l)f_1(\theta_m).$$

Next, we show that if $f_1(\theta_l)f_2(\theta_m) = f_1(\theta_m)f_2(\theta_l)$ and $f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m)$, then $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$. Let $\alpha \equiv (f_2(\theta_l) - f_2(\theta_h), -f_1(\theta_l) + f_1(\theta_h))$. Then,

$$(\alpha \cdot f)(\theta_l) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_l) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_l) = f_1(\theta_h)f_2(\theta_l) - f_1(\theta_l)f_2(\theta_h),$$
$$(\alpha \cdot f)(\theta_h) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_h) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l) - f_1(\theta_l)f_2(\theta_h), \text{ and}$$
$$(\alpha \cdot f)(\theta_m) = (f_2(\theta_l) - f_2(\theta_h))\, f_1(\theta_m) - (f_1(\theta_l) - f_1(\theta_h))\, f_2(\theta_m) = 0.$$

As $\alpha \cdot f$ is single crossing, it follows that $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$, as we wanted to show.

B.2. ($\Longleftarrow$)

We provide a proof for the case in which f_1 ratio dominates f_2, and omit the other case's analogous proof. For any $\alpha \in \mathbb{R}^2$, we prove that $\alpha \cdot f$ is single crossing. We may assume that $\alpha \neq 0$, as the result is trivial otherwise.

Suppose, towards contradiction, that $\alpha \cdot f$ is not single crossing. We require the following:[36]

Claim 6. *There exist $\theta_l < \theta_m < \theta_h$ such that*

$$\operatorname{sign}[(\alpha \cdot f)(\theta_l)] < \operatorname{sign}[(\alpha \cdot f)(\theta_m)] \textit{ and } \operatorname{sign}[(\alpha \cdot f)(\theta_m)] > \operatorname{sign}[(\alpha \cdot f)(\theta_h)], \tag{23}$$

or

$$\operatorname{sign}[(\alpha \cdot f)(\theta_l)] > \operatorname{sign}[(\alpha \cdot f)(\theta_m)] \textit{ and } \operatorname{sign}[(\alpha \cdot f)(\theta_m)] < \operatorname{sign}[(\alpha \cdot f)(\theta_h)]. \tag{24}$$

Note that the claim is obvious when $(\Theta, \leq)$ is a completely ordered set.

[36] Recall that we defined $\operatorname{sign}[x] = 1$ if $x > 0$, $\operatorname{sign}[x] = 0$ if $x = 0$, and $\operatorname{sign}[x] = -1$ if $x < 0$.

Proof of Claim 6. Since $\alpha \cdot f$ is single crossing neither from below nor from above:

$$(\exists\theta_1 < \theta_2) \quad \text{sign}[(\alpha \cdot f)(\theta_1)] < \text{sign}[(\alpha \cdot f)(\theta_2)], \text{ and}$$
$$(\exists\theta_3 < \theta_4) \quad \text{sign}[(\alpha \cdot f)(\theta_3)] > \text{sign}[(\alpha \cdot f)(\theta_4)].$$

Let $\Theta_0 \equiv \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and $\overline{\theta}$ and $\underline{\theta}$ be an upper and lower bound of Θ_0. If $(\alpha \cdot f)(\underline{\theta}) = (\alpha \cdot f)(\overline{\theta}) = 0$, then $(\theta_l, \theta_m, \theta_h) = (\underline{\theta}, \theta_0, \overline{\theta})$ for some $\theta_0 \in \Theta_0$ with $(\alpha \cdot f)(\theta_0) \neq 0$ satisfies either (23) or (24). So assume $(\alpha \cdot f)(\overline{\theta}) \neq 0$, with a similar argument applying for $(\alpha \cdot f)(\underline{\theta}) \neq 0$. If $(\alpha \cdot f)(\overline{\theta}) < 0$, then $(\theta_l, \theta_m, \theta_h) = (\theta_1, \theta_2, \overline{\theta})$ satisfies (23). If $(\alpha \cdot f)(\overline{\theta}) > 0$, then $(\theta_l, \theta_m, \theta_h) = (\theta_3, \theta_4, \overline{\theta})$ satisfies (24). *Q.E.D.*

First, we consider the case in which $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ for all $\theta \in \{\theta_l, \theta_m, \theta_h\}$ are non-zero vectors. Take any $\theta_1, \theta_2 \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta_1 < \theta_2$. As f_1 ratio dominates f_2, by Condition (1), $f(\theta_1)$ moves to $f(\theta_2)$ in a clockwise rotation with an angle less than or equal to 180 degrees. Let r_{12} be the clockwise angle from $f(\theta_1)$ to $f(\theta_2)$. The vector $\alpha \neq 0$ defines a partition of $\mathbb{R}^2$ into $\mathbb{R}^2_{\alpha,+} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x > 0\}$, $\mathbb{R}^2_{\alpha,0} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x = 0\}$, and $\mathbb{R}^2_{\alpha,-} \equiv \{x \in \mathbb{R}^2 : \alpha \cdot x < 0\}$. In both cases (23) and (24), both $f(\theta_l)$ and $f(\theta_h)$ are not in the same part of the partition that $f(\theta_m)$ belongs to. Thus, $r_{lm} > 0$ and $r_{mh} > 0$. On the other hand, both $f(\theta_l)$ and $f(\theta_h)$ are in the same closed half-space, either $\mathbb{R}^2_{\alpha,+} \cup \mathbb{R}^2_{\alpha,0}$ or $\mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$, and $f(\theta_m)$ is in the other closed half-space, either $\mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$ or $\mathbb{R}^2_{\alpha,+} \cup \mathbb{R}^2_{\alpha,0}$, respectively. Thus, $r_{lh} \geq 180$. Since Condition (1) implies $r_{lh} \leq 180$, it follows that $r_{lh} = 180$. Hence, $f(\theta_l)$ and $f(\theta_m)$ are linearly independent ($0 < r_{lm} < 180$), and similarly for $f(\theta_m)$ and $f(\theta_h)$. However, $f(\theta_l)$ and $f(\theta_h)$ are linearly dependent ($r_{lh} = 180$). This contradicts (2).

Second, suppose either $f(\theta_l) = 0$ or $f(\theta_h) = 0$. We provide the argument assuming $f(\theta_l) = 0$; it is analogous if $f(\theta_h) = 0$. Under either (23) or (24), $f(\theta_m) \neq 0$. By Condition (2), $f(\theta_m)$ and $f(\theta_h)$ are linearly dependent. In particular, because $f(\theta_m) \neq 0$, there exists a unique $\lambda \in \mathbb{R}$ such that $f(\theta_h) = \lambda f(\theta_m)$. Under either (23) or (24), $\lambda \leq 0$, which contradicts the hypothesis that f_1 and f_2 are single crossing.

Last, suppose $f(\theta_l) \neq 0$, $f(\theta_m) = 0$, and $f(\theta_h) \neq 0$. By Condition (2), $f(\theta_l)$ and $f(\theta_h)$ are linearly dependent. Hence, there exists a unique $\lambda \in \mathbb{R}$ such that $f(\theta_l) = \lambda f(\theta_h)$. Under either (23) or (24), $\lambda > 0$, which contradicts the hypothesis that f_1 and f_2 are single crossing.

C. Proof of Proposition 1

Let (X, Σ) be a measurable space and $f : X \times \Theta \to \mathbb{R}$ be as described in the proposition. The result is trivial if $|X| = 1$ and it is equivalent to Lemma 1 if $|X| = 2$, so we may assume $|X| \geq 3$. The proof is also straightforward if all functions $f(x,\cdot)$ are multiples of one function $f(x_1,\cdot)$, i.e., if there is x_1 such that $(\exists\lambda : X \to \mathbb{R})(\forall x,\theta) f(x,\theta) = \lambda(x) f(x_1,\theta)$. Thus, we further assume there exist x', x'' such that $f(x',\cdot) : \Theta \to \mathbb{R}$ and $f(x'',\cdot) : \Theta \to \mathbb{R}$ are linearly independent, i.e., $(\forall\lambda \in \mathbb{R}^2\backslash\{0\})\ \lambda_1 f(x',\cdot) + \lambda_2 f(x'',\cdot)$ is not a zero function.

C.1. $(\Longleftarrow)$

Assume $f(x_1,\cdot)$ and $f(x_2,\cdot)$ are ratio ordered, and that there are finitely-integrable functions $\lambda_1, \lambda_2 : X \to \mathbb{R}$ such that for every x, $f(x,\cdot) = \lambda_1(x) f(x_1,\cdot) + \lambda_2(x) f(x_2,\cdot)$. Then, for every finite signed measure μ,

$$\int_X f(x,\theta)\mathrm{d}\mu = \int_X \lambda_1(x) f(x_1,\theta) + \lambda_2(x) f(x_2,\theta)\mathrm{d}\mu = \sum_{i=1,2}\left(\int_X \lambda_i(x)\mathrm{d}\mu\right) f(x_i,\theta),$$

which is single crossing in θ by Lemma 1, since for each $i = 1,2$, $\int_X \lambda_i(x)\mathrm{d}\mu \in \mathbb{R}$ exists.

C.2. $(\Longrightarrow)$

Take any $x_1, x_2 \in X$ such that $f_1(\cdot) \equiv f(x_1,\cdot)$ and $f_2(\cdot) \equiv f(x_2,\cdot)$ are linearly independent. Then, by Lemma 1, f_1 and f_2 are ratio ordered as their linear combinations are all single crossing.

For every θ', θ'', let

$$M_{\theta',\theta''} \equiv \begin{pmatrix} f_1(\theta') & f_2(\theta') \\ f_1(\theta'') & f_2(\theta'') \end{pmatrix}.$$

We first prove the following claim:

Claim 7. *There exists $\theta_l < \theta_h$ such that* $\mathrm{rank}[M_{\theta_l,\theta_h}] = 2$

Proof of Claim 7. As f_1 and f_2 are linearly independent, there exists θ_0 such that $f_2(\theta_0) \neq 0$. Let $\lambda \equiv -\frac{f_1(\theta_0)}{f_2(\theta_0)}$. Then, for some θ_λ, $f_1(\theta_\lambda) + \lambda f_2(\theta_\lambda) \neq 0$ and $\mathrm{rank}[M_{\theta_0,\theta_\lambda}] = 2$.

The proof is complete if $\theta_0 > \theta_\lambda$ or $\theta_0 < \theta_\lambda$. If not, take a lower and upper bound, $\underline{\theta}$ and $\overline{\theta}$, of $\{\theta_0, \theta_\lambda\}$. Then $\mathrm{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$. For otherwise, there exists $\alpha \in \mathbb{R}^2\backslash\{0\}$ such that $M_{\underline{\theta},\overline{\theta}}\alpha = 0$. As θ_0 and θ_λ are between $\underline{\theta}$ and $\overline{\theta}$, and $\alpha_1 f_1 + \alpha_2 f_2$ is single crossing, we have $M_{\theta_0,\theta_\lambda}\alpha = 0$, which contradicts $\mathrm{rank}[M_{\theta_0,\theta_\lambda}] = 2$. *Q.E.D.*

Now take any $x \in X$, the function $f_x(\cdot) \equiv f(x,\cdot)$, and θ_l, θ_h in Claim 7. As $\mathrm{rank}[M_{\theta_l,\theta_h}] = 2$, the system

$$\begin{pmatrix} f_x(\theta_l) \\ f_x(\theta_h) \end{pmatrix} = \begin{pmatrix} f_1(\theta_l) & f_2(\theta_l) \\ f_1(\theta_h) & f_2(\theta_h) \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{25}$$

has a unique solution $\lambda \in \mathbb{R}^2$. We will show that $f_x = \lambda_1 f_1 + \lambda_2 f_2$.

Suppose, towards contradiction, there exists θ_λ such that

$$f_x(\theta_\lambda) \neq \lambda_1 f_1(\theta_\lambda) + \lambda_2 f_2(\theta_\lambda). \tag{26}$$

Let $\underline{\theta}, \overline{\theta}$ be a lower and upper bound of $\{\theta_l, \theta_h, \theta_\lambda\}$. If $\operatorname{rank}[M_{\underline{\theta},\overline{\theta}}] < 2$, there is $\lambda' \in \mathbb{R}^2 \backslash \{0\}$ such that $\lambda'_1 f_1(\theta) + \lambda'_2 f_2(\theta) = 0$ for $\theta = \underline{\theta}, \overline{\theta}$. As θ_l and θ_h are between $\underline{\theta}$ and $\overline{\theta}$, and $\lambda'_1 f_1 + \lambda'_2 f_2$ is single crossing, we have $\lambda'_1 f_1(\theta) + \lambda'_2 f_2(\theta) = 0$ for $\theta = \theta_l, \theta_h$, which contradicts $\operatorname{rank}[M_{\theta_l,\theta_h}] = 2$.[37]

If, on the other hand, $\operatorname{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$, the system

$$\begin{pmatrix} f_x(\underline{\theta}) \\ f_x(\overline{\theta}) \end{pmatrix} = \begin{pmatrix} f_1(\underline{\theta}) & f_2(\underline{\theta}) \\ f_1(\overline{\theta}) & f_2(\overline{\theta}) \end{pmatrix} \begin{pmatrix} \lambda'_1 \\ \lambda'_2 \end{pmatrix}$$

has a unique solution $\lambda' \in \mathbb{R}^2$. As θ_l, θ_h, and θ_λ are between $\underline{\theta}$ and $\overline{\theta}$, and $f_x - \lambda'_1 f_1 - \lambda'_2 f_2$ is single crossing,

$$f_x(\theta_l) = \lambda'_1 f_1(\theta_l) + \lambda'_2 f_2(\theta_l) \quad \text{and} \quad f_x(\theta_h) = \lambda'_1 f_1(\theta_h) + \lambda'_2 f_2(\theta_h), \tag{27}$$

$$f_x(\theta_\lambda) = \lambda'_1 f_1(\theta_\lambda) + \lambda'_2 f_2(\theta_\lambda). \tag{28}$$

(27) implies that λ' solves (25). As the unique solution to (25) was λ, it follows that $\lambda' = \lambda$. But then (26) and (28) are in contradiction.

We have so far proved that

$$(\forall \theta) \quad f(x, \theta) = \lambda_1(x) f(x_1, \theta) + \lambda_2(x) f(x_2, \theta),$$

with $\lambda_1, \lambda_2 : X \to \mathbb{R}$. It remains to show that λ_1 and λ_2 are finitely integrable.

Take θ_l, θ_h in Claim 7. As $\operatorname{rank}[M_{\theta_l,\theta_h}] = 2$, we have

$$\det [M_{\theta_l,\theta_h}] \equiv f_1(\theta_l) f_2(\theta_h) - f_2(\theta_l) f_1(\theta_h) \neq 0. \tag{29}$$

In particular, either $f_1(\theta_l) f_2(\theta_h) \neq 0$ or $f_2(\theta_l) f_1(\theta_h) \neq 0$. We consider the case of $f_1(\theta_l) f_2(\theta_h) \neq 0$ (and omit the proof for the other case).

[37] The function $\lambda'_1 f_1 + \lambda'_2 f_2$ must be single crossing because we can apply the requirement to μ such that $\mu(\{x_1\}) = \lambda'_1$, $\mu(\{x_2\}) = \lambda'_2$, and $\mu(Y) = 0$ for any $Y \in \Sigma$ with $Y \cap \{x_1, x_2\} = \emptyset$. We use similar reasoning subsequently.

Let $\eta_l \equiv \frac{f_2(\theta_l)}{f_1(\theta_l)}$ and $\eta_h \equiv \frac{f_1(\theta_h)}{f_2(\theta_h)}$. Then

$$f(x,\theta_l) = \lambda_1(x)f_1(\theta_l) + \lambda_2(x)f_2(\theta_l) = (\lambda_1(x) + \eta_l\lambda_2(x))f_1(\theta_l) \quad \text{and}$$
$$f(x,\theta_h) = \lambda_1(x)f_1(\theta_h) + \lambda_2(x)f_2(\theta_h) = (\eta_h\lambda_1(x) + \lambda_2(x))f_2(\theta_h),$$

and, by hypothesis, both are finitely-integrable functions of x. As $f_1(\theta_l)$ and $f_2(\theta_h)$ are independent of x, it follows that $\lambda_1(x) + \eta_l\lambda_2(x)$ and $\eta_h\lambda_1(x) + \lambda_2(x)$ are finitely integrable, and so are their linear combinations

$$\eta_l\left(\eta_h\lambda_1(x) + \lambda_2(x)\right) - \left(\lambda_1(x) + \eta_l\lambda_2(x)\right) = (\eta_l\eta_h - 1)\lambda_1(x) \quad \text{and}$$
$$\eta_h\left(\lambda_1(x) + \eta_l\lambda_2(x)\right) - \left(\eta_h\lambda_1(x) + \lambda_2(x)\right) = (\eta_l\eta_h - 1)\lambda_2(x).$$

Since $\eta_l\eta_h \neq 1$ (as otherwise $\det[M_{\theta_l,\theta_h}] = f_1(\theta_l)f_2(\theta_h) - \eta_l\eta_h f_1(\theta_l)f_2(\theta_h) = 0$, contrary to (29)), λ_1 and λ_2 are also finitely integrable.

D. Proof of Theorem 1

D.1. ($\Longleftarrow$)

Suppose $v(a,\theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta)$, with $f_1 : \Theta \to \mathbb{R}$ and $f_2 : \Theta \to \mathbb{R}$ each single crossing and ratio ordered. Then, for any $P, Q \in \Delta A$,

$$D_{P,Q}(\theta) = \left[\int g_1(a)\mathrm{d}P - \int g_1(a)\mathrm{d}Q\right] f_1(\theta) + \left[\int g_2(a)\mathrm{d}P - \int g_2(a)\mathrm{d}Q\right] f_2(\theta).$$

Since g_1 and g_2 are finitely integrable by hypothesis, $D_{P,Q}$ is single crossing by Lemma 1.

D.2. ($\Longrightarrow$)

Assume, without loss of generality, that $|A| \geq 2$, and let Σ be a $\sigma-$algebra on A containing all singleton sets. Take any $a_0 \in A$, and define $A' \equiv A \setminus a_0$. Then (A', Σ') with $\Sigma' \equiv \{\tilde{A} \in \Sigma \,|\, \tilde{A} \subseteq A'\}$ is also a measurable space in which Σ' contains all singleton sets in A'. We will show that, in some sense, every finite signed measure μ' defined over Σ' can be represented as a multiple of the difference between two probability measures P, Q defined over Σ, and then apply Proposition 1.

Define $f : A \times \Theta \to \mathbb{R}$ as

$$f(a,\theta) \equiv v(a,\theta) - v(a_0,\theta).$$

It is clear that $(\forall a)\ f(a,\cdot)$ is single crossing: consider the expectational difference with probability measures that put probability one on a and a_0 respectively. Also, our σ-algebra contains all elements of A and any finite signed measure equals the difference between two rescaled probability measures.[38] It follows that $(\forall\theta)\ f(\cdot,\theta)$ is finitely-integrable as $v(a,\theta)$ is integrable with respect to any probability measure.

We will show that for every finite signed measure μ' over Σ', there exist $P, Q \in \Delta A$ such that $\int_{A'} f(a,\theta)\mathrm{d}\mu'$ is single crossing if and only if $D_{P,Q}$ is single crossing.

For any finite signed measure μ' on Σ', we define a finite signed measure μ over Σ as an extension of μ':

$$\mu(a_0) \equiv -\mu'(A') \text{ and } (\forall \tilde{A} \subseteq A)\ \mu(\tilde{A}) \equiv \mu'(\tilde{A}\backslash a_0) + \mathbb{1}_{\{a_0 \in \tilde{A}\}}\mu(a_0).$$

In a sense, we let a_0 absorb the signed measure of A'. In particular, note that

$$\mu(A) = \mu(A') + \mu(a_0) = \mu'(A') - \mu'(A') = 0.$$

Let (μ_+, μ_-) be the Hahn-Jordan decomposition of μ. That is, μ_+ and μ_- are two positive finite measures such that $\mu = \mu_+ - \mu_-$. Let $M = \mu_+(A) = \mu_-(A)$. If $M = 0$, pick an arbitrary $P \in \Delta A$ and let $Q = P$. If $M > 0$, define $P, Q \in \Delta A$ such that for any $\tilde{A} \subseteq A$,

$$P(\tilde{A}) = \frac{\mu_+(\tilde{A})}{M} \quad \text{and} \quad Q(\tilde{A}) = \frac{\mu_-(\tilde{A})}{M}.$$

Note that P and Q are well-defined probability measures defined over Σ: both are induced by finite positive measures μ_+ and μ_-, and $P(A) = Q(A) = 1$.

It follows that

$$\begin{aligned}
\int_{A'} f(a,\theta)\mathrm{d}\mu' &= \int_A f(a,\theta)\mathrm{d}\mu \quad \text{(because } f(a_0,\theta) = 0\text{)} \\
&= \int_A v(a,\theta)\mathrm{d}\mu - v(a_0,\theta)\mu(A) \\
&= \int_A v(a,\theta)\mathrm{d}\mu_+ - \int_A v(a,\theta)\mathrm{d}\mu_- \quad \text{(as } \mu(A) = 0\text{)} \\
&= M D_{P,Q}(\theta).
\end{aligned}$$

Thus, if v has SCED, then $f : A\backslash\{a_0\} \times \Theta \to \mathbb{R}$ is linear combinations SC-preserving. By Proposition 1, there exist $a_1, a_2 \in A\backslash\{a_0\}$ such that (i) $f(a_1,\theta)$ and $f(a_2,\theta)$ are ratio ordered, and (ii) $(\exists \lambda_1, \lambda_2 : A \setminus \{a_0\} \to \mathbb{R})\ (\forall\theta)\ f(a,\theta) = \lambda_1(a) f(a_1,\theta) + \lambda_2(a) f(a_2,\theta)$, with λ_1, λ_2

[38] Any finite signed measure μ has a unique Hahn-Jordan decomposition (μ_+, μ_-) of two positive finite measures such that $\mu = \mu_+ - \mu_-$.

finitely integrable. Hence, there exist functions $g_1, g_2 : A \to \mathbb{R}$, both finitely integrable and $g_1(a_0) = g_2(a_0) = 0$, such that $f(a,\theta) = g_1(a)f(a_1,\theta) + g_2(a)f(a_2,\theta)$, or equivalently,

$$v(a,\theta) = g_1(a)f(a_1,\theta) + g_2(a)f(a_2,\theta) + v(a_0,\theta).$$

E. Proofs for Strict Single Crossing (Subsection 2.2)

E.1. Proof of Lemma 2

When $|\Theta| \leq 2$.

If $|\Theta| = 1$, the proof is trivial as all functions are strictly single crossing and every pair of f_1, f_2 satisfy strict ratio ordering. So assume $|\Theta| = 2$ and denote $\Theta = \{\theta_l, \theta_h\}$; without loss, we may assume $\theta_h > \theta_l$ because of our maintained assumption that upper and lower bounds exist for all pairs.

$(\implies)$ Either $(f_1(\theta_l), f_2(\theta_l)) \neq 0$ or $(f_1(\theta_h), f_2(\theta_h)) \neq 0$: otherwise, for every $\alpha \in \mathbb{R}^2\backslash\{0\}$, $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$, and hence $\alpha \cdot f$ is a zero function, which is not strictly single crossing. Assume $(f_1(\theta_l), f_2(\theta_l)) \neq 0$; the proof for the other case is analogous. Let $\alpha_l \equiv (f_2(\theta_l), -f_1(\theta_l))$ and consider $(\alpha_l \cdot f)(\theta) = f_2(\theta_l)f_1(\theta) - f_1(\theta_l)f_2(\theta)$. We have $(\alpha_l \cdot f)(\theta_l) = 0$ and, by strict single crossing of $\alpha_l \cdot f$, $(\alpha_l \cdot f)(\theta_h) \neq 0$. That is, $f_2(\theta_l)f_1(\theta_h) \neq f_1(\theta_l)f_2(\theta_h)$, which means that f_1 and f_2 are strictly ratio ordered.

$(\impliedby)$ For any $\alpha \in \mathbb{R}^2\backslash\{0\}$, $\alpha \cdot f$ is not strictly single crossing if and only if $(\alpha \cdot f)(\theta_l) = (\alpha \cdot f)(\theta_h) = 0$. Then, $\alpha_1 f_1(\theta_l) = -\alpha_2 f_2(\theta_l)$ and $\alpha_1 f_1(\theta_h) = -\alpha_2 f_2(\theta_h)$, which result in

$$\begin{aligned}\alpha_1 f_1(\theta_l)f_2(\theta_h) &= -\alpha_2 f_2(\theta_l)f_2(\theta_h) = \alpha_1 f_1(\theta_h)f_2(\theta_l), \quad \text{and}\\ \alpha_2 f_1(\theta_l)f_2(\theta_h) &= -\alpha_1 f_1(\theta_l)f_1(\theta_h) = \alpha_2 f_1(\theta_h)f_2(\theta_l).\end{aligned}$$

As either α_1 or α_2 is not zero, $f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l)$, which violates strict ratio ordering of f_1 and f_2.

When $|\Theta| \geq 3$.

$(\implies)$ Suppose, towards contradiction, that

$$\begin{aligned}&(\exists \theta_l < \theta_h) \quad f_1(\theta_l)f_2(\theta_h) \leq f_1(\theta_h)f_2(\theta_l), \text{ and}\\ &(\exists \theta' < \theta'') \quad f_1(\theta')f_2(\theta'') \geq f_1(\theta'')f_2(\theta').\end{aligned} \tag{30}$$

Take any upper bound $\overline{\theta}$ of $\{\theta_l, \theta_h, \theta', \theta''\}$. First, let $\alpha_l = (f_2(\theta_l), -f_1(\theta_l))$. Then, $\alpha_l \cdot f$ is strictly single crossing only from below as $(\alpha_l \cdot f)(\theta_l) = (f_2(\theta_l), -f_1(\theta_l)) \cdot (f_1(\theta_l), f_2(\theta_l)) = 0$ and $(\alpha_l \cdot f)(\theta_h) \geq 0$. Second, let $\alpha' = (f_2(\theta'), -f_1(\theta'))$. Then, $\alpha' \cdot f$ is strictly single crossing only from above as $(\alpha' \cdot f)(\theta') = 0$ and $(\alpha' \cdot f)(\theta'') \leq 0$.

Let $\overline{\alpha} = (f_2(\overline{\theta}), -f_1(\overline{\theta}))$. Then,

$$
\begin{aligned}
&(\overline{\alpha} \cdot f)(\theta_l) = (f_2(\overline{\theta}), -f_1(\overline{\theta})) \cdot (f_1(\theta_l), f_2(\theta_l)) = -(\alpha_l \cdot f)(\overline{\theta}) \leq 0, \\
&(\overline{\alpha} \cdot f)(\theta') = -(\alpha' \cdot f)(\overline{\theta}) \geq 0, \text{ and} \\
&(\overline{\alpha} \cdot f)(\overline{\theta}) = 0.
\end{aligned}
$$

Therefore, $\overline{\alpha} \cdot f$ is not strictly single crossing from below nor from above.

($\Longleftarrow$) We provide a proof for the case in which f_1 strictly ratio dominates f_2, and omit the other case's analogous proof. For any $\alpha \in \mathbb{R}^2 \backslash \{0\}$, we prove that $\alpha \cdot f$ is single crossing. The argument is very similar to that used in proving Lemma 1, but note that here we do not assume either f_1 or f_2 are single crossing.

As f_1 strictly ratio dominates f_2,

$$
(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) < f_1(\theta_h) f_2(\theta_l). \tag{31}
$$

Suppose, towards contradiction, that $\alpha \cdot f$ is not strictly single crossing.

<u>Claim</u>: There exist $\theta_l, \theta_m, \theta_h$ with $\theta_l < \theta_m < \theta_h$ such that

$$
(\alpha \cdot f)(\theta_l) \leq 0, (\alpha \cdot f)(\theta_m) \geq 0, \text{ and } (\alpha \cdot f)(\theta_h) \leq 0, \tag{32}
$$

or

$$
(\alpha \cdot f)(\theta_l) \geq 0, (\alpha \cdot f)(\theta_m) \leq 0, \text{ and } (\alpha \cdot f)(\theta_h) \geq 0. \tag{33}
$$

<u>Proof of claim</u>: Since $\alpha \cdot f$ is not strictly single crossing either from below or from above,

$$
\begin{aligned}
&(\exists \theta_1 < \theta_2) \quad (\alpha \cdot f)(\theta_1) \geq 0 \geq (\alpha \cdot f)(\theta_2), \text{ and} \\
&(\exists \theta_3 < \theta_4) \quad (\alpha \cdot f)(\theta_3) \leq 0 \leq (\alpha \cdot f)(\theta_4).
\end{aligned}
$$

Let $\Theta_0 \equiv \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and let $\overline{\theta}$ and $\underline{\theta}$ be an upper and lower bound of Θ_0, respectively. Either $(\alpha \cdot f)(\underline{\theta}) \neq 0$ or $(\alpha \cdot f)(\overline{\theta}) \neq 0$, as otherwise $f_1(\underline{\theta}) f_2(\overline{\theta}) = f_2(\underline{\theta}) f_1(\overline{\theta})$, contradicting (31). Suppose $(\alpha \cdot f)(\overline{\theta}) \neq 0$. If $(\alpha \cdot f)(\overline{\theta}) < 0$, then we choose $(\theta_l, \theta_m, \theta_h) = (\theta_3, \theta_4, \overline{\theta})$, which satisfies (32). If $(\alpha \cdot f)(\overline{\theta}) > 0$, then we choose $(\theta_l, \theta_m, \theta_h) = (\theta_1, \theta_2, \overline{\theta})$, which satisfies (33). A similar argument applies when $(\alpha \cdot f)(\underline{\theta}) \neq 0$. $\|$

It is clear that $f(\theta) \equiv (f_1(\theta), f_2(\theta))$ for all $\theta \in \{\theta_l, \theta_m, \theta_h\}$ are non-zero vectors: otherwise, Condition (31) would not hold for some $\theta', \theta'' \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta' < \theta''$. Take any $\theta_1, \theta_2 \in \{\theta_l, \theta_m, \theta_h\}$ such that $\theta_1 < \theta_2$. By Condition (31), $f(\theta_1)$ moves to $f(\theta_2)$ in a clockwise rotation with an angle $r_{12} \in (0, 180)$. Suppose (32) holds; the argument is analogous if (33) holds. It follows from $0 < r_{lh} < 180$, $(\alpha \cdot f)(\theta_l) \leq 0$, and $(\alpha \cdot f)(\theta_h) \leq 0$ that $\{f(\theta_l), f(\theta_h)\} \subseteq \mathbb{R}^2_{\alpha,-} \cup \mathbb{R}^2_{\alpha,0}$ with $\{f(\theta_l), f(\theta_h)\} \not\subseteq \mathbb{R}^2_{\alpha,0}$. This, together with $r_{lm} > 0$ and $r_{mh} > 0$, implies $f(\theta_m) \in \mathbb{R}^2_{\alpha,-}$, which contradicts (32).

E.2. Proof of Proposition 2

Appendix C proved Proposition 1 assuming certain functions are linearly independent. Essentially the same proof can be used for Proposition 2; we need only to replace statements involving "single crossing" with "either a zero function or strictly single crossing".

E.3. Proof of Theorem 2

Most statements in the proof of Theorem 1 go through with strict single crossing when we replace "single crossing" with "either a zero function or strictly single crossing". We need only to rewrite the proof of the "only if" part in the following two special cases:

1. $(\forall a', a'')(\forall \theta)\ v(a', \theta) = v(a'', \theta)$, or
2. $(\exists a', a'')$ such that (i) $v(a'', \theta) - v(a', \theta)$ is not a zero function of θ, and (ii) $(\forall a)\ v(a, \theta) - v(a', \theta)$ and $v(a'', \theta) - v(a', \theta)$ are linearly dependent functions of θ.

In the first case, we can write $v(a, \theta)$ in form of (3) where g_1, g_2 are zero functions, $c(\theta) \equiv v(a_0, \theta)$ for any arbitrary a_0, $f_1(\theta) = 1$, and $f_2(\theta)$ is any strictly decreasing function of θ. Then,

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) = f_2(\theta_h) < f_2(\theta_l) = f_1(\theta_h) f_2(\theta_l).$$

In the second case, for every a, there exists $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \backslash \{0\}$ such that $\lambda_1 (v(a, \cdot) - v(a', \cdot)) + \lambda_2 (v(a'', \cdot) - v(a', \cdot))$ is a zero function. Note that $\lambda_1 \neq 0$, as otherwise $v(a'', \cdot) - v(a', \cdot)$ would be a zero function. It follows that there exists $\lambda : A \to \mathbb{R}$ such that

$$(\forall a) \quad v(a, \theta) - v(a', \theta) = \lambda(a) (v(a'', \theta) - v(a', \theta)),$$

or equivalently,

$$v(a, \theta) = \lambda(a) (v(a'', \theta) - v(a', \theta)) + v(a', \theta).$$

Note that $v(a'', \theta) - v(a', \theta)$ is a strictly single-crossing function of θ: consider the expectational difference with measures that put probability one on a'' and a' respectively. If the difference is strictly single crossing from below, we can write $v(a, \theta)$ in the form of (3) where $g_1(a) = \lambda(a)$, $g_2(a) = 0$, $f_1(\theta) = v(a'', \theta) - v(a', \theta)$, and $c(\theta) = v(a', \theta)$. If the difference is strictly single crossing only from above, we let $g_1(a) = -\lambda(a)$ and $f_1(\theta) = v(a', \theta) - v(a'', \theta)$.

Last, we take any strictly increasing function $h : \Theta \to \mathbb{R}$ and define

$$\hat{h}(\theta) \equiv \begin{cases} -e^{h(\theta)} & \text{if } f_1(\theta) \leq 0 \\ e^{-h(\theta)} & \text{otherwise,} \end{cases} \quad \text{and} \quad f_2(\theta) \equiv \begin{cases} \hat{h}(\theta) f_1(\theta) & \text{if } f_1(\theta) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

To verify that f_1 and f_2 are strictly ratio ordered, take any $\theta_l < \theta_h$. There are three possibilities to consider:

1. If $f_1(\theta_l) f_1(\theta_h) > 0$, then

$$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_h) < f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_l) = f_1(\theta_h) f_2(\theta_l),$$

   as $\hat{h}(\theta)$ is strictly decreasing over $\{\theta \mid f_1(\theta) < 0\}$ and $\{\theta \mid f_1(\theta) > 0\}$.

2. If $f_1(\theta_l) f_1(\theta_h) < 0$, as $f_1(\theta)$ is strictly single crossing from below, we have $f_1(\theta_l) < 0 < f_1(\theta_h)$. Hence,

$$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_h) < 0 < f_1(\theta_l) f_1(\theta_h) \hat{h}(\theta_l) = f_1(\theta_h) f_2(\theta_l).$$

3. If $f_1(\theta_l) f_1(\theta_h) = 0$, by strict single crossing from below of f_1, we have either (i) $f_1(\theta_l) < 0 = f_1(\theta_h)$, which results in $f_1(\theta_l) f_2(\theta_h) = f_1(\theta_l) < 0 = f_1(\theta_h) f_2(\theta_l)$, or (ii) $f_1(\theta_l) = 0 < f_1(\theta_h)$, which results in $f_1(\theta_l) f_2(\theta_h) = 0 < f_1(\theta_h) = f_1(\theta_h) f_2(\theta_l)$.

F. Proofs for Monotonicity of Choices (Subsection 2.3)

F.1. Proof of Lemma 3

We omit the proof that $\succeq_{SCD}$ is reflexive and anti-symmetric. To prove that $\succeq_{SCD}$ is transitive, take x, y, z such that $x \succeq_{SCD} y$ and $y \succeq_{SCD} z$. It is sufficient to show $x \succeq_{SCD} z$ assuming $x \succ_{SCD} y$ and $y \succ_{SCD} z$ (if $x = y$ or $y = z$, it trivially holds that $x \succeq_{SCD} z$). Since f has SCD, $D_{x,z}$ is single crossing. As both $D_{x,y}$ and $D_{y,z}$ are single crossing only from below, $D_{x,z} = D_{x,y} + D_{y,z}$ cannot be single crossing from above; since $D_{x,z}$ is single crossing by SCD, it is single crossing only from below. Thus, $x \succ_{SCD} z$.

F.2. Proof of Theorem 3

($\Longrightarrow$) We first prove the following claim.

Claim 8. *For every $x, x' \in X$, if $\exists \theta_l < \theta_h$ such that $\text{sign}[D_{x,x'}(\theta_l)] < \text{sign}[D_{x,x'}(\theta_h)]$, then $x \succ x'$.*

Proof. Consider $S = \{x, x'\}$. Since $\text{sign}[D_{x,x'}(\theta_l)] \neq \text{sign}[D_{x,x'}(\theta_h)]$, we have

$$\arg\max_{s\in S} f(x, \theta_l) \neq \arg\max_{s\in S} f(x, \theta_h).$$

Thus, either (i) $x \in \arg\max_{s\in S} f(x, \theta_l)$ and $x' \in \arg\max_{s\in S} f(x, \theta_h)$, or (ii) $x' \in \arg\max_{s\in S} f(x, \theta_l)$ and $x \in \arg\max_{s\in S} f(x, \theta_h)$. Since f has MCS on $(X, \succeq)$, we have $\arg\max_{s\in S} f(x, \theta_h) \succeq_{SSO} \arg\max_{s\in S} f(x, \theta_l)$. Therefore, $x \wedge x' \in \arg\max_{s\in S} f(s, \theta_l)$ and $x \vee x' \in \arg\max_{s\in S} f(s, \theta_h)$, which implies that either $x \succeq x'$ or $x' \succeq x$. Since $x' \neq x$, we have either $x \succ x'$ or $x' \succ x$. If $x' \succ x$, then $x' = x \vee x' \in \arg\max_{s\in S} f(s, \theta_h)$, contradicting $\text{sign}[D_{x,x'}(\theta_l)] < \text{sign}[D_{x,x'}(\theta_h)]$. Thus, $x \succ x'$. *Q.E.D.*

To show that f has SCD on X, suppose not, per contra. Then there exist $x, x' \in X$ and $\theta_l < \theta_m < \theta_h$ such that either,[39]

$$\text{sign}[D_{x,x'}(\theta_l)] < \text{sign}[D_{x,x'}(\theta_m)] \quad \text{and} \quad \text{sign}[D_{x,x'}(\theta_m)] > \text{sign}[D_{x,x'}(\theta_h)], \quad \text{or} \tag{34}$$

$$\text{sign}[D_{x,x'}(\theta_l)] > \text{sign}[D_{x,x'}(\theta_m)] \quad \text{and} \quad \text{sign}[D_{x,x'}(\theta_m)] < \text{sign}[D_{x,x'}(\theta_h)]. \tag{35}$$

Given either (34) or (35), Claim 8 implies $x \succ x'$ and $x' \succ x$, a contradiction.

To show that $\succeq$ is a refinement of $\succeq_{SCD}$, it suffices to show that

$$(\forall x, x' \in X) \quad x \succ_{SCD} x' \implies x \succ x', \tag{36}$$

because both $\succeq$ and $\succeq_{SCD}$ are anti-symmetric. Take any $x, x' \in X$ such that $x \succ_{SCD} x'$. As $D_{x,x'}$ is single crossing only from below, $\exists \theta_l < \theta_h$ such that $\text{sign}[D_{x,x'}(\theta_l)] < \text{sign}[D_{x,x'}(\theta_h)]$. Claim 8 implies $x \succ x'$, which proves (36).

($\Longleftarrow$) Take any $S \subseteq X$ and $\theta_l < \theta_h$. To establish that

$$\arg\max_{s\in S} f(s, \theta_h) \succeq_{SSO} \arg\max_{s\in S} f(s, \theta_l),$$

[39] The existence of such $\theta_l, \theta_m, \theta_h$ is immediate if $(\Theta, \leq)$ is a completely ordered set. More generally, the existence of $\theta_l, \theta_m, \theta_h$ follows from our maintained assumption that every pair in Θ has upper and lower bounds; see Appendix B.2.

we prove that

$$x \in \arg\max_{s \in S} f(s, \theta_l) \quad \text{and} \quad x' \in \arg\max_{s \in S} f(s, \theta_h) \tag{37}$$

implies

$$x \wedge x' \in \arg\max_{s \in S} f(s, \theta_l) \quad \text{and} \quad x \vee x' \in \arg\max_{s \in S} f(s, \theta_h). \tag{38}$$

Pick any x and x' satisfying (37) (if there is either no such x or no such x', we are done), which implies

$$D_{x,x'}(\theta_l) \geq 0 \geq D_{x,x'}(\theta_h). \tag{39}$$

Assume $x \neq x'$, as otherwise (38) holds trivially. $D_{x,x'}$ is single crossing in θ because f has SCD on X. As X is minimal and $x \neq x'$, (39) implies that $D_{x,x'}$ cannot be single crossing from both below and above.

Suppose first that $D_{x,x'}$ is single crossing only from below. Then $x \succeq_{SCD} x'$ (by definition of $\succeq_{SCD}$), and hence $x \succeq x'$ (since $\succeq$ is a refinement of $\succeq_{SCD}$), which in turn implies $x = x \vee x'$ and $x' = x \wedge x'$. Moreover, (39) implies that $D_{x,x'}(\theta_l) = 0 = D_{x,x'}(\theta_h)$. Hence, both x and x' are in $\arg\max_{s \in S} f(s, \theta_l)$ and in $\arg\max_{s \in S} f(s, \theta_h)$, which implies (38).

Suppose next that $D_{x,x'}$ is single crossing only from above, or equivalently, $D_{x',x}$ is single crossing only from below. Analogous to the previous paragraph, it follows that $x' \succeq_{SCD} x$, $x' \succeq x$, and hence $x' = x \vee x'$ and $x = x \wedge x'$. Plainly, (38) holds as it is equivalent to (37).

F.3. Proof of Lemma 4

The proof is omitted as it is analogous to the proof of Lemma 3 in Appendix F.1.

F.4. Proof of Proposition 3

Take any $S \subset X$. To show that any selection is monotonic, we assume that $\arg\max_{s \in S} f(s, \theta)$ is non-empty for all θ, as otherwise the result holds vacuously. Pick any $\theta_l < \theta_h$ and any $x \in \arg\max_{s \in S} f(s, \theta_l)$ and $x' \in \arg\max_{s \in S} f(s, \theta_h)$. We must show that $x' \succeq x$. As this is trivially true if $x = x'$, assume $x \neq x'$. $D_{x,x'}$ is strictly single crossing in θ because f has SSCD and X is minimal. Since $D_{x,x'}(\theta_l) \geq 0 \geq D_{x,x'}(\theta_h)$ from the hypotheses on x and x', it follows that $D_{x,x'}$ is strictly single crossing only from above, or equivalently, $D_{x',x}$ is strictly single crossing only from below. Thus, $x' \succeq_{SSCD} x$ (by definition of $\succeq_{SSCD}$) and $x' \succeq x$ (as $\succeq$ is a refinement of $\succeq_{SSCD}$).

G. Proofs for Monotonic Expectational Differences (Subsection 4.1)

G.1. Proof of Lemma 5

($\Longleftarrow$) Suppose there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $f_2 = \lambda_1 f_1 + \lambda_2$. Then, for any $\alpha \in \mathbb{R}^2$,

$$(\alpha \cdot f)(\theta) = \alpha_1 f_1(\theta) + \alpha_2(\lambda_1 f_1(\theta) + \lambda_2) = (\alpha_1 + \alpha_2\lambda_1) f_1(\theta) + \lambda_2,$$

which is monotonic.

($\Longrightarrow$) The proof is trivial if both f_1 and f_2 are constant functions. Thus, we suppose that one function, say f_1, is not constant:

$$(\exists \theta', \theta'') \quad f_1(\theta') \neq f_1(\theta''). \tag{40}$$

Then, $\operatorname{rank}[M_{\theta',\theta''}] = 2$, where

$$M_{\theta',\theta''} \equiv \begin{pmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{pmatrix}.$$

Hence, the system

$$\begin{pmatrix} f_2(\theta') \\ f_2(\theta'') \end{pmatrix} = \begin{pmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{41}$$

has a unique solution $\lambda \in \mathbb{R}^2$. We will show that $f_2 = \lambda_1 f_1 + \lambda_2$.

Suppose, towards contradiction, there exists θ_λ such that

$$f_2(\theta_\lambda) \neq \lambda_1 f_1(\theta_\lambda) + \lambda_2. \tag{42}$$

Let $\underline{\theta}$ and $\overline{\theta}$ be a lower and upper bound of $\{\theta', \theta'', \theta_\lambda\}$. If $\operatorname{rank}[M_{\underline{\theta},\overline{\theta}}] < 2$, $f_1(\overline{\theta}) = f_1(\underline{\theta})$. As θ' and θ'' are between $\underline{\theta}$ and $\overline{\theta}$, and f_1 is monotone, we have $f_1(\theta') = f_1(\theta'')$, which contradicts (40). If, on the other hand, $\operatorname{rank}[M_{\underline{\theta},\overline{\theta}}] = 2$, the system

$$\begin{pmatrix} f_2(\underline{\theta}) \\ f_2(\overline{\theta}) \end{pmatrix} = \begin{pmatrix} f_1(\underline{\theta}) & 1 \\ f_1(\overline{\theta}) & 1 \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix}$$

has a unique solution $\lambda' \in \mathbb{R}^2$. As θ', θ'', and θ_λ are in between $\underline{\theta}$ and $\overline{\theta}$ and $f_2 - \lambda_1' f_1$ is monotonic,

$$\begin{pmatrix} f_2(\theta') \\ f_2(\theta'') \end{pmatrix} = \begin{pmatrix} f_1(\theta') & 1 \\ f_1(\theta'') & 1 \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} \text{ and} \tag{43}$$

$$f_2(\theta_\lambda) = \lambda_1' f_1(\theta_\lambda) + \lambda_2'. \tag{44}$$

Equation 43 implies that λ' solves (41). As the unique solution to (41) was λ, it follows that $\lambda' = \lambda$. But then (42) and (44) are in contradiction.

G.2. Proof of Proposition 4

($\Longleftarrow$) We omit the proof as it is similar to the proof of Proposition 1 in Appendix C.1.

($\Longrightarrow$) For the proof of necessity, if $(\forall x)$ $f(x,\theta)$ is a constant function of θ, then we let $\lambda_1(x) = 0$ and $\lambda_2(x) = f(x,\theta)$. If there exists $x' \in X$ such that $f(x',\theta)$ is not a constant function of θ, then Lemma 5 implies $(\forall x,\theta)$ $f(x,\theta) = \lambda_1(x) f(x',\theta) + \lambda_2(x)$, with $\lambda_1, \lambda_2 : X \to \mathbb{R}$.

It remains to verify the λ_1 and λ_2 are finitely integrable. Take θ', θ'' such that $f(x',\theta') \neq f(x',\theta'')$, so that $f(x,\theta') = \lambda_1(x) f(x',\theta') + \lambda_2(x)$ and $f(x,\theta'') = \lambda_1(x) f(x',\theta'') + \lambda_2(x)$. By hypothesis, both $f(x,\theta')$ and $f(x,\theta'')$ are finitely-integrable functions of x. Since $f(x',\theta')$ and $f(x',\theta'')$ are independent of x, it follows that $f(x,\theta') - f(x,\theta'') = \lambda_1(x)\left(f(x',\theta') - f(x',\theta'')\right)$ is finitely integrable, and so is $\lambda_1(x)$. Moreover, $\lambda_2(x) = f(x,\theta') - \lambda_1(x) f(x',\theta')$ is finitely integrable.

G.3. Proof of Theorem 4

($\Longleftarrow$) We omit the proof as it is similar to the proof of Theorem 1 in Appendix D.1.

($\Longrightarrow$) The proof is trivial if $v(a,\theta) = 0$ for all $a \in A$ and $\theta \in \Theta$, so assume there exists a_0 such that $v(a_0,\cdot) : \Theta \to \mathbb{R}$ is not a zero function. Define $f : A \times \Theta \to \mathbb{R}$ by $f(a,\theta) \equiv v(a,\theta) - v(a_0,\theta)$. Note that (i) $(\forall a)$ $f(a,\cdot)$ is a monotonic function of θ (consider the expectational difference with measures that put probability one on a and a_0 respectively), and (ii) $(\forall \theta)$ $f(\cdot,\theta)$ is a finitely-integrable function of a.

Let Σ be a σ-algebra on A containing all singleton sets. Let $A' \equiv A \backslash a_0$. Then (A', Σ') with $\Sigma' \equiv \{\tilde{A} \in \Sigma \,|\, \tilde{A} \subseteq A'\}$ is also a measurable space, in which Σ' contains all singleton sets in A'. As in the proof of Theorem 1 in Appendix D.2, for every finite signed measure μ' over Σ', there exist $P, Q \in \Delta A$ such that $\int_{A'} f(a,\theta) d\mu'$ is monotonic if and only if $D_{P,Q}$ is monotonic. By Proposition 4, there exist $a' \in A \backslash a_0$ and $\lambda_1, \lambda_2 : A \backslash \{a_0\} \to \mathbb{R}$ such that

$(\forall a, \theta)\ f(a,\theta) = \lambda_1(a) f(a', \theta) + \lambda_2(a)$, with λ_1, λ_2 finitely integrable. Hence, there exist functions $g_1, g_2 : A \to \mathbb{R}$, both finitely integrable and $g_1(a_0) = g_2(a_0) = 0$, such that $f(a,\theta) = g_1(a) f(a', \theta) + g_2(a)$, or equivalently, $v(a,\theta) = g_1(a) f(a',\theta) + g_2(a) + v(a_0, \theta)$.

G.4. Further Results Comparing MED and SCED

Let $\succeq_\Theta \equiv \{\succeq_\theta :\ \theta \in \Theta\}$ be a family of type-dependent preferences (i.e., complete, reflexive, and transitive binary relations) over ΔA. We say that $v : A \times \Theta \to \mathbb{R}$ **represents** $\succeq_\Theta$ (in the expected utility form) if

$$(\forall\theta)(\forall P, Q \in \Delta A) \quad P \succeq_\theta Q \iff \int_A v(a,\theta) \mathrm{d}P \geq \int_A v(a,\theta) \mathrm{d}Q. \tag{45}$$

We say that $v' : A \times \Theta \to \mathbb{R}$ is a **type-dependent positive affine transformation** of v if there exist $b : \Theta \to \mathbb{R}_{++}$ and $d : \Theta \to \mathbb{R}$ such that $v'(a,\theta) = b(\theta) v(a,\theta) + d(\theta)$. For any $v : A \times \Theta \to \mathbb{R}$ and $\succeq_\Theta$ defined by (45), a function $v' : A \times \Theta \to \mathbb{R}$ is a type-dependent positive affine transformation of v if and only if v' represents $\succeq_\Theta$.

Proposition 5. *Let $v : A \times \Theta \to \mathbb{R}$ have SCED: i.e.,*

$$v(a,\theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta),$$

where $g_1, g_2 : A \to \mathbb{R}$ are each finitely integrable, $f_1, f_2 : \Theta \to \mathbb{R}$ are each single crossing and ratio ordered, and $c : \Theta \to \mathbb{R}$. Let $\succeq_\Theta$ be the family of type-dependent preferences over ΔA defined by (45). Then, $\succeq_\Theta$ can be represented by a function with MED if and only if (i) either g_1 is an affine transformation of g_2 or vice-versa, or (ii) f_1 and f_2 are linearly dependent, or (iii) there exists $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $(\forall\theta)\ (\lambda \cdot f)(\theta) > 0$.

We can interpret Proposition 5 as follows: given $\succeq_\Theta$ with an SECD representation, there is an MED representation if and only if either

(a) there is a pair of types that do not share the same strict preference over any pair of lotteries (i.e., $(\exists\theta', \theta'')\ (\forall P, Q \in \Delta A)\ D_{P,Q}(\theta') D_{P,Q}(\theta'') \leq 0$), or

(b) there is a pair of lotteries over which all types share the same strict preference (i.e., $(\exists P, Q \in \Delta A)\ (\forall\theta)\ D_{P,Q}(\theta) > 0$).

To see this interpretation, suppose Case (i) or (ii) holds in Proposition 5. Then, there are functions $\hat{g}_1$, $\hat{f}_1$, and $\hat{c}$ such that $v(a,\theta) = \hat{g}_1(a) \hat{f}_1(\theta) + \hat{c}(\theta)$, with $\hat{f}_1$ single crossing. If $\hat{g}_1$ is constant or $\hat{f}_1$ is a zero function, every type is indifferent across all lotteries, and (a) holds.

Suppose $\hat{g}_1$ is not constant. If $\hat{f}_1 > 0$ (or < 0), then (b) holds.[40] If $\hat{f}_1(\theta)$ is single crossing from only below or from only above, then for some θ' and θ'', $\hat{f}_1(\theta')\hat{f}_1(\theta'') \leq 0$; the pair θ' and θ'' does not share the same strict preference over any two lotteries, and so (a) holds. On the other hand, if Proposition 5's Case (iii) applies and Case (i) does not, then (b) holds, because $(\exists P, Q \in \Delta A, M \in \mathbb{R}_{++})\ (\forall \theta)\ MD_{P,Q}(\theta) = (\lambda \cdot f)(\theta) > 0$; see (46).[41]

Here is some geometric intuition for the "if" direction of Proposition 5. For Case (i) or (ii), $v(a, \theta) = \hat{g}_1(a)\hat{f}_1(\theta) + \hat{c}(\theta)$ with $\hat{f}_1$ single crossing, as already noted. We can rescale $\hat{f}_1(\theta)$ using a function $b : \Theta \to \mathbb{R}_{++}$ such that $b(\theta)\hat{f}_1(\theta)$ is monotonic. Thus, $v'(a, \theta) \equiv b(\theta)v(a, \theta)$ represents $\succeq_\Theta$ and has MED. For Case (iii), assume without loss of generality that $\|(\lambda_1, \lambda_2)\| = 1$. Let $b(\theta) \equiv \frac{1}{(\lambda \cdot f)(\theta)}$. It follows that $(\forall \theta)\ (\lambda \cdot (bf))(\theta) = 1$, i.e., the function b adjusts the lengths of vectors $\{f(\theta) \in \mathbb{R}^2 : \theta \in \Theta\}$ while maintaining their directions, as illustrated in Figure 8. The vector $(bf)(\theta)$ rotates monotonically as θ increases, while staying on the hyperplane $\{x \in \mathbb{R}^2 \mid \lambda \cdot x = 1\}$. Let $e_1 \equiv (1, 0)$ and $e_2 \equiv (0, 1)$. Suppose $\{e_1, \lambda\}$ is a basis for $\mathbb{R}^2$ (an analogous argument would hold if instead $\{e_2, \lambda\}$ were a basis). Then, for every θ, the vector $(bf)(\theta)$ is represented as $(b(\theta)f_1(\theta), 1)$ with respect to the new basis. We define $\tilde{f}_1(\theta) \equiv b(\theta)f_1(\theta)$ and

$$v'(a, \theta) \equiv b(\theta)v(a, \theta) = \tilde{g}_1(a)\tilde{f}_1(\theta) + \tilde{g}_2(a) + b(\theta)c(\theta),$$

with appropriately defined functions $\tilde{g}_1$ and $\tilde{g}_2$. Since $(bf)(\theta)$ rotates monotonically, $\tilde{f}_1(\theta)$ is monotonic. It follows that v' has MED.

Proof of Proposition 5. ($\Longleftarrow$) First we prove that if either (i) or (ii) holds, then we can write v as $v(a, \theta) = \hat{g}_1(a)\hat{f}_1(\theta) + \hat{c}(\theta)$, with $\hat{f}_1$ single crossing.

Suppose (i) holds; without loss, assume $(\exists d_1, d_2 \in \mathbb{R})\ (\forall a)\ g_2(a) = d_1 g_1(a) + d_2$. Then, $\hat{g}_1(a) = g_1(a)$, $\hat{f}_1(\theta) = f_1(\theta) + d_1 f_2(\theta)$, and $\hat{c}(\theta) = d_2 f_2(\theta) + c(\theta)$. Next suppose (ii) holds; without loss, assume $(\exists d \in \mathbb{R})\ (\forall \theta)\ f_2(\theta) = d f_1(\theta)$. Then, $\hat{g}_1(a) = g_1(a) + d g_2(a)$, $\hat{f}_1(\theta) = f_1(\theta)$, and $\hat{c}(\theta) = c(\theta)$. In either case, $\hat{f}_1$ is a linear combination of f_1 and f_2, so it is single crossing.

Define

$$b(\theta) \equiv \begin{cases} \frac{1}{|\hat{f}_1(\theta)|} & \text{if } \hat{f}_1(\theta) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

[40] Consider two degenerate lotteries over a' and a'' such that $\hat{g}_1(a') \neq \hat{g}_2(a'')$.

[41] Conversely, (a) implies either Case (i) or (ii) of Proposition 5. If g_1 and g_2 are affinely independent (i.e., a violation of (i)), then $(\forall \lambda \in \mathbb{R}^2 \backslash \{0\})\ (\exists P, Q \in \Delta A, M \in \mathbb{R}_{++})\ (\forall \theta)\ (\lambda \cdot f)(\theta) = MD_{P,Q}(\theta)$; see (46). Then, (a) implies that for some θ', θ'' and for some $\beta \leq 0$, $f(\theta') = \beta f(\theta'')$, which, together with Condition (2) of ratio ordering, implies that f_1 and f_2 are linearly dependent. Moreover, (b) implies Case (iii) of the Proposition: letting $\lambda_i = \int_A g_i(a)\mathrm{d}[P - Q]$ for $i = 1, 2$, it holds that $\lambda_1 f_1(\theta) + \lambda_2 f_2(\theta) = D_{P,Q}(\theta) > 0$ for all θ.

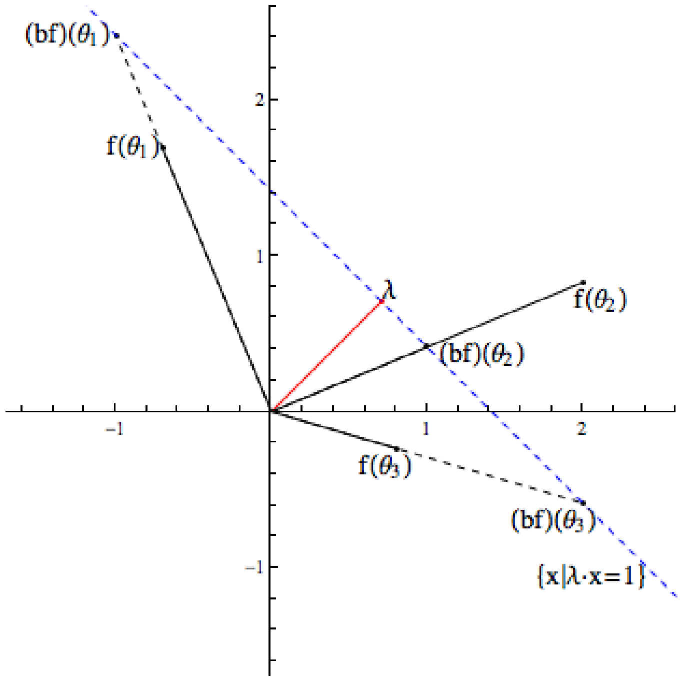

Figure 8: A geometric intuition for sufficiency of (iii) in Proposition 5.

Since $(\forall\theta)\ b(\theta) > 0$, $v'(a,\theta) \equiv b(\theta)v(a,\theta) = \hat{g}_1(a)b(\theta)\hat{f}_1(\theta) + b(\theta)\hat{c}(\theta)$ is a type-dependent positive affine transformation of v, and hence it also represents $\succeq_\Theta$. As $\hat{f}_1$ is single crossing, $b(\theta)\hat{f}_1(\theta) = \text{sign}[\hat{f}_1(\theta)]$ is monotonic, and v' has MED.

Now suppose (iii) holds: $(\exists\lambda \in \mathbb{R}^2\backslash\{0\})\ (\forall\theta)\ (\lambda \cdot f)(\theta) > 0$. Define $b(\theta) \equiv \frac{1}{(\lambda\cdot f)(\theta)}$. Then $(\lambda \cdot (bf))(\theta) = 1$, so that either $(bf_1)(\theta)$ is an affine transformation of $(bf_2)(\theta)$, or vice-versa: $(\exists\gamma,\omega \in \mathbb{R})$ such that either $b(\theta)f_1(\theta) = \gamma b(\theta)f_2(\theta) + \omega$ or $b(\theta)f_2(\theta) = \gamma b(\theta)f_1(\theta) + \omega$.

We consider the case in which $b(\theta)f_2(\theta) = \gamma b(\theta)f_1(\theta) + \omega$ and omit the other case's analogous proof. If $\lambda_2 \geq (\leq) 0$, then as f_1 ratio dominates f_2,

$$
\begin{aligned}
(\forall\theta' \leq \theta'') \qquad & f_1(\theta')f_2(\theta'') \leq f_1(\theta'')f_2(\theta') \\
\implies \quad & \lambda_1 f_1(\theta')f_1(\theta'') + \lambda_2 f_1(\theta')f_2(\theta'') \leq (\geq) \lambda_1 f_1(\theta')f_1(\theta'') + \lambda_2 f_1(\theta'')f_2(\theta') \\
\implies \quad & f_1(\theta')(\lambda \cdot f)(\theta'') \leq (\geq) f_1(\theta'')(\lambda \cdot f)(\theta') \\
\implies \quad & b(\theta')f_1(\theta') \leq (\geq) b(\theta'')f_1(\theta'').
\end{aligned}
$$

Thus, regardless of whether $\lambda_2 \geq 0$ or $\lambda_2 \leq 0$, $(bf_1)(\theta)$ is monotonic in θ. It follows that

$$\begin{aligned} v'(a,\theta) &\equiv b(\theta)v(a,\theta) = g_1(a)b(\theta)f_1(\theta) + g_2(a)b(\theta)f_2(\theta) + b(\theta)c(\theta) \\ &= g_1(a)b(\theta)f_1(\theta) + g_2(a)(\gamma b(\theta)f_1(\theta) + \omega) + b(\theta)c(\theta) \\ &= (g_1(a) + \gamma g_2(a))b(\theta)f_1(\theta) + \omega g_2(a) + b(\theta)c(\theta) \end{aligned}$$

has MED.

($\Longrightarrow$) We prove that if neither (i) or (ii) holds, then (iii) holds.

We first show that when (i) does not hold,

$$(\forall \lambda \in \mathbb{R}^2\backslash\{0\})\ (\exists P, Q \in \Delta A, M \in \mathbb{R}_{++}) \quad (\lambda \cdot f)(\theta) = MD_{P,Q}(\theta) \quad \text{for all } \theta. \tag{46}$$

As (i) does not hold, the three functions g_1, g_2, and 1 (where 1 represents the constant function whose value is 1) are linearly independent. For otherwise, either g_1 and g_2 are linearly dependent, or $(\exists \alpha \in \mathbb{R}^2\backslash\{0\})$ $(\forall a)$ $\alpha_1 g_1(a) + \alpha_2 g_2(a) = 1$; in either case, either g_1 would be an affine transformation of g_2 or vice-versa.

By the aforementioned linear independence, there exist $a_0, a_1, a_2 \in A$ such that

$$\operatorname{rank} \begin{pmatrix} g_1(a_0) & g_1(a_1) & g_1(a_2) \\ g_2(a_0) & g_2(a_1) & g_2(a_2) \\ 1 & 1 & 1 \end{pmatrix} = 3.$$

It follows that for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2\backslash\{0\}$ there exists $(d_0, d_1, d_2) \in \mathbb{R}^3$ such that

$$\begin{pmatrix} g_1(a_0) & g_1(a_1) & g_1(a_2) \\ g_2(a_0) & g_2(a_1) & g_2(a_2) \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix}.$$

Take a sufficiently large $M \in \mathbb{R}_{++}$ such that $|d_i| \leq M/3$ for $i = 0, 1, 2$, and define $P, Q \in \Delta A$ so that $P(\{a_i\}) = 1/3$ and $Q(\{a_i\}) = 1/3 - d_i/M$ for $i = 0, 1, 2$. Then, both P and Q are probability measures on $\{a_0, a_1, a_2\}$, and

$$\begin{aligned} (\lambda \cdot f)(\theta) &= (d_0 g_1(a_0) + d_1 g_1(a_1) + d_2 g_1(a_2))\, f_1(\theta) + (d_0 g_2(a_0) + d_1 g_2(a_1) + d_2 g_2(a_2))\, f_2(\theta) \\ &= \sum_{i=0,1,2} d_i \left(g_1(a_i) f_1(\theta) + g_2(a_i) f_2(\theta)\right) \\ &= MD_{P,Q}(\theta). \end{aligned}$$

Suppose $\succeq_\Theta$ is represented by a function $v' : A \times \Theta \to \mathbb{R}$ with MED. As both v and v' represent $\succeq_\Theta$ in the expected utility form, v' is a type-dependent positive affine transformation of v. It follows that there exist $b : \Theta \to \mathbb{R}_{++}$ and $d : \Theta \to \mathbb{R}$ such that

$$v'(a, \theta) = b(\theta)v(a, \theta) + d(\theta) = g_1(a)\hat{f}_1(\theta) + g_2(a)\hat{f}_2(\theta) + \hat{c}(\theta),$$

where $\hat{f}_1(\theta) = b(\theta)f_1(\theta)$, $\hat{f}_2(\theta) = b(\theta)f_2(\theta)$, and $\hat{c}(\theta) = b(\theta)c(\theta) + d(\theta)$.

Given (46), for any $\lambda \in \mathbb{R}^2 \backslash \{0\}$, there exist $P, Q \in \Delta A$ and $M \in \mathbb{R}_{++}$ such that

$$(\lambda \cdot \hat{f})(\theta) = b(\theta)(\lambda \cdot f)(\theta) = b(\theta)MD_{P,Q}(\theta) = M \int_A v'(a, \theta)d[P - Q],$$

which is monotonic.

We find $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $(\forall\theta)$ $(\lambda \cdot \hat{f})(\theta) = 1$. For any θ', θ'', let

$$M_{\theta'\theta''} \equiv \begin{pmatrix} f_1(\theta') & f_1(\theta'') \\ f_2(\theta') & f_2(\theta'') \end{pmatrix} \quad \text{and} \quad \hat{M}_{\theta'\theta''} \equiv \begin{pmatrix} \hat{f}_1(\theta') & \hat{f}_1(\theta'') \\ \hat{f}_2(\theta') & \hat{f}_2(\theta'') \end{pmatrix}.$$

As f_1 and f_2 are linearly independent, there exist θ_1, θ_2 such that $\text{rank}[M_{\theta_1,\theta_2}] = 2$, which implies that $\text{rank}[\hat{M}_{\theta_1,\theta_2}] = 2$. Let $\lambda^* \in \mathbb{R}^2 \backslash \{0\}$ be the unique solution of $\hat{M}_{\theta_1,\theta_2}\lambda = (1, 1)$. Take any θ_0, and let $\underline{\theta}$ and $\overline{\theta}$ be a lower and upper bound of $\{\theta_0, \theta_1, \theta_2\}$. It must be that $\text{rank}[\hat{M}_{\underline{\theta},\overline{\theta}}] = 2$. If otherwise, there exists $\lambda \in \mathbb{R}^2 \backslash \{0\}$ such that $(\lambda \cdot \hat{f})(\underline{\theta}) = (\lambda \cdot \hat{f})(\overline{\theta}) = 0$. By (46), $\lambda \cdot \hat{f}$ is monotonic, so $(\lambda \cdot \hat{f})(\theta_1) = (\lambda \cdot \hat{f})(\theta_2) = 0$, which contradicts $\text{rank}[\hat{M}_{\theta_1,\theta_2}] = 2$. Let λ^{**} be the unique solution of $\hat{M}_{\underline{\theta},\overline{\theta}}\lambda = (1, 1)$. By monotonicity of $\lambda^{**} \cdot \hat{f}$, $\hat{M}_{\theta_1,\theta_2}\lambda^{**} = (1, 1)$, which implies that $\lambda^{**} = \lambda^*$. It follows that $(\lambda^* \cdot \hat{f})(\theta_0) = 1$. As θ_0 is arbitrary, we have $(\forall\theta)$ $(\lambda^* \cdot \hat{f})(\theta) = 1$.

Finally, $(\forall\theta)$ $(\lambda^* \cdot f)(\theta) = \frac{(\lambda^* \cdot \hat{f})(\theta)}{b(\theta)} > 0$. *Q.E.D.*

We can use Proposition 5 to provide an example of type-dependent preferences representable by an SCED function that are not representable by any MED function:

Example 4. Let $\Theta \equiv (-1, 1] \subset \mathbb{R}$ and $A = \{a_0, a', a''\}$. Consider $v(a, \theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta)$, with

1. $g_1(a_0) = g_2(a_0) = 0$, $g_1(a') = g_2(a') = 1$, $g_1(a'') = 2$, $g_2(a'') = 3$, and
2. $f_1(\theta) = \theta$, $f_2(\theta) = 1 - \theta^2$.

Observe that f_1 ratio dominates f_2: if $\theta' < \theta''$, then $\theta'\theta'' < 1$, and hence $f_1(\theta')f_2(\theta'') < f_1(\theta'')f_2(\theta')$. It follows that v has SCED.

We claim the family of type-dependent preferences $\succeq_\Theta$ represented by v is not representable by any MED function. It is easy to verify that neither is g_1 is an affine transformation of g_2 nor vice-versa, and that f_1 and f_2 are linearly independent. By Proposition 5, it suffices to show that $\nexists \lambda \in \mathbb{R}^2$ such that $(\forall\theta)\ (\lambda \cdot f)(\theta) > 0$. Take any $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \backslash \{0\}$. If $\lambda_1 = 0$, then $(\lambda \cdot f)(1) = 0$. If, on the other hand, $\lambda_1 \neq 0$, then $\text{sign}[(\lambda \cdot f)(1)] = \text{sign}[\lambda_1]$ and $\lim_{\theta \to -1} \text{sign}[(\lambda \cdot f)(\theta)] = -\text{sign}[\lambda_1]$, and so $(\exists\theta)(\lambda \cdot f)(\theta) < 0$. □

H. Relaxing Anti-symmetry

This appendix shows how anti-symmetry of $\leq$ over Θ can be dropped by appropriately generalizing the definition of single crossing. This extension is useful, for example, because rankings over Θ based on norms (say, when $\Theta \subseteq \mathbb{R}^n$) generally violate anti-symmetry.

Assume $(\Theta, \leq)$ is a preordered set, i.e., $\leq$ is a binary relation that is reflexive and transitive, but not necessarily anti-symmetric. We write $\theta' \cong \theta''$ when $\theta' \geq \theta''$ and $\theta' \leq \theta''$.

Definition 12. When $(\Theta, \leq)$ is a preordered set, a function $f : \Theta \to \mathbb{R}$ is[42]

1. **single crossing from below** if

$$\begin{aligned}
&(\forall \theta < \theta') \quad f(\theta) \geq (>)0 \implies f(\theta') \geq (>)0, \text{ and} \\
&(\forall \theta' \cong \theta) \quad \text{sign}[f(\theta)] = \text{sign}[f(\theta')];
\end{aligned}$$

2. **single crossing from above** if

$$\begin{aligned}
&(\forall \theta < \theta') \quad f(\theta) \leq (<)0 \implies f(\theta') \leq (<)0, \text{ and} \\
&(\forall \theta' \cong \theta) \quad \text{sign}[f(\theta)] = \text{sign}[f(\theta')];
\end{aligned}$$

3. **single crossing** if it is single crossing either from below or from above.

Definition 12 reduces to Definition 1 when $(\Theta, \leq)$ is a partially-ordered set, because in that case $\theta' \cong \theta \iff \theta' = \theta$.

Lemma 6. *Let $f_1, f_2 : \Theta \to \mathbb{R}$ be single-crossing functions on a preordered set $(\Theta, \leq)$. The linear combination $\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is single crossing $\forall \alpha \in \mathbb{R}^2$ if and only if f_1 and f_2 are ratio ordered.*

The rest of our main results (Proposition 1, Theorem 1, and Theorem 3) and their proofs remain the same.

The proof of Lemma 6 consists of establishing Claim 9 and Claim 10 below.

[42] Recall that for $x \in \mathbb{R}$, $\text{sign}[x] = 1$ if $x > 0$, $\text{sign}[x] = 0$ if $x = 0$, and $\text{sign}[x] = -1$ if $x < 0$.

Claim 9. *Let* $f_1, f_2 : \Theta \to \mathbb{R}$ *each be single crossing. Then,* $(\forall \alpha \in \mathbb{R}^2)$

$$\begin{aligned} \textit{either } (\forall \theta < \theta') \quad (\alpha \cdot f)(\theta) \geq (>) 0 &\implies (\alpha \cdot f)(\theta') \geq (>) 0, \\ \textit{or } (\forall \theta < \theta') \quad (\alpha \cdot f)(\theta) \leq (<) 0 &\implies (\alpha \cdot f)(\theta') \leq (<) 0 \end{aligned}$$

if and only if f_i *and* f_j*, for either* $(i, j) = (1, 2)$ *or* $(i, j) = (2, 1)$*, satisfy*

$$(\forall \theta_l < \theta_h) \quad f_i(\theta_l) f_j(\theta_h) \leq f_i(\theta_h) f_j(\theta_l), \quad \textit{and}$$

$$(\forall \theta_l < \theta_m < \theta_h) \quad f_i(\theta_l) f_j(\theta_h) = f_i(\theta_h) f_j(\theta_l) \iff \begin{cases} f_i(\theta_l) f_j(\theta_m) = f_i(\theta_m) f_j(\theta_l), \\ f_i(\theta_m) f_j(\theta_h) = f_i(\theta_h) f_j(\theta_m). \end{cases}$$

Proof. The proof of Claim 9 is analogous to the proof of Lemma 1 given two observations: (i) the proof of Lemma 1 does not use anti-symmetry of $\leq$ over Θ; (ii) the weak inequalities $\theta_l \leq \theta_h$ in (1) and $\theta_l \leq \theta_m \leq \theta_h$ in (2) can be replaced with strict inequalities. *Q.E.D.*

Claim 10. *Let* $f_1, f_2 : \Theta \to \mathbb{R}$ *each be single crossing. Then,*

$$(\forall \alpha \in \mathbb{R}^2)(\forall \theta' \cong \theta) \quad \operatorname{sign}[(\alpha \cdot f)(\theta')] = \operatorname{sign}[(\alpha \cdot f)(\theta'')]$$

if and only if

$$(\forall \theta' \cong \theta'') \quad f_1(\theta') f_2(\theta'') = f_1(\theta'') f_2(\theta').$$

Proof. ($\implies$) Suppose, towards contradiction, that $(\exists \theta' \cong \theta'')$ $f_1(\theta') f_2(\theta'') \neq f_1(\theta'') f_2(\theta')$. Then, $\alpha' \equiv (-f_2(\theta'), f_1(\theta')) \neq 0$. It follows that $(\alpha' \cdot f)(\theta') = 0$ and $(\alpha' \cdot f)(\theta'') \neq 0$, which contradicts to single crossing of $\alpha' \cdot f$.

($\impliedby$) Take any $\alpha \in \mathbb{R}^2$ and $\theta' \cong \theta''$. If $f_1(\theta') f_2(\theta'') = f_1(\theta'') f_2(\theta') \neq 0$, then all four function values are non-zero. Thus,

$$\begin{aligned} \operatorname{sign}[(\alpha \cdot f)(\theta')] &= \operatorname{sign}\left[\frac{f_1(\theta'')}{f_1(\theta')} (\alpha_1 f_1(\theta') + \alpha_2 f_2(\theta'))\right] \quad \text{(because } \operatorname{sign}[f_1(\theta')] = \operatorname{sign}[f_1(\theta'')] \neq 0\text{)} \\ &= \operatorname{sign}[\alpha_1 f_1(\theta'') + \alpha_2 f_2(\theta'')] \quad \text{(using } f_2(\theta'') = f_2(\theta') f_1(\theta'')/f_1(\theta')\text{)} \\ &= \operatorname{sign}[(\alpha \cdot f)(\theta'')]. \end{aligned}$$

If, on the other hand, $f_1(\theta') f_2(\theta'') = f_1(\theta'') f_2(\theta') = 0$, then at least one function value, say $f_1(\theta')$, equals zero. As f_1 is single crossing, $f_1(\theta') = f_1(\theta'') = 0$. Thus,

$$\operatorname{sign}[(\alpha \cdot f)(\theta')] = \operatorname{sign}[\alpha_2 f_2(\theta')] = \operatorname{sign}[\alpha_2 f_2(\theta'')] = \operatorname{sign}[(\alpha \cdot f)(\theta'')].$$ *Q.E.D.*

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