On discrimination in procurement auctions*

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Abstract

With exogenous participation, strong bidders should be discriminated against weak bidders to maximize revenues (Myerson 1981). When participation is endogenous and the set of potential entrants is large, optimal discrimination if any takes a very different form. Without incumbents, there should be no discrimination even if entrants come from groups with different characteristics. With incumbents, those should be discriminated against entrants no matter how strong/weak they are even if some share of their surplus is internalized by the designer. The optimal reserve policy in standard auctions is also analyzed to shed light on situations in which discrimination is not permitted.

Keywords: auctions with endogenous entry, optimal auction design, Poisson games, asymmetric buyers, bid preference programs, incumbents, cartels, government procurement, favoritism.

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1 Introduction

In procurement auctions, governments are often tempted to use discrimination in favor of some bidders, for example according to whether a firm is domestic or not.[1] Yet, firms typically adjust their bidding strategies as a function of the chosen form of discrimination, and it is unclear how costly the discrimination is to the working of competition. A legitimate fear is that discrimination would distort competition in an inefficient way, and that it would lead to pay a higher price for the same service. This is an argument often put forward by the World Trade Organization or the European Commission to ban discrimination.[2]

However, if firms are ex ante asymmetric, it would seem some discrimination with the aim of inducing a more balanced competition would be desirable. The work of Myerson (1981) on optimal auctions can be interpreted as providing some support to this idea by giving a precise measure of how stronger bidders should be handicapped to generate more revenues (see McAfee and McMillan (1989) for proposing such an interpretation of Myerson's (1981) work).

The work of Myerson (1981) however assumes that the set of firms participating in the procurement auction is exogenously given. But, if effective participation is too costly to attract every possible firm, participation should be viewed as being endogenously determined. This adds another consideration. When participation is endogenous, how many firms show up typically depends on the auction format and thus on the form and magnitude of discrimination employed. It is then important to reassess the extent to which discrimination is desirable in the presence of asymmetries when participation is endogenous.

This paper considers the issue of discrimination in procurement auctions when the participation of some set of firms referred to as entrants is endogenously determined while allowing for the automatic presence of some other firms referred to as incumbents. Specifically, we consider a private value setup in which there may be several groups of potential entrants, each characterized by possibly different distributions of costs and possibly different participation costs, and assuming in each group that the set of potential entrants is arbitrarily large. We also assume that firms of the same group follow the same (possibly mixed) strategy. We ask whether and to what extent discrimination is desirable in such a setup. We refer to the outside option cost of the designer as the expected cost she would incur by postponing the project or by having the job done outside the set of participating firms.

Our first main result is that, when there are no incumbents, minimizing the expected cost

[1] Mougeot and Naegelen (1989) report that the Buy American Act (which starts in 1933) promotes bid subsidies ranging from 6 percent to 12 percent. Defense contracts have a special treatment and subsidies can be as high as 50%. Canadian and Australian legislations have similar dispositions. In other countries, e.g. European countries, favoritism with respect to domestic firms is not written in the law, but non-explicit discrimination rules lead to the same results.

[2] The WTO which struggles against barriers to trade, rules out discrimination in its Agreement on Government Procurement, the Buy American Act being a notable exception. The European Commission cares about helping SMEs winning public procurements but only through non-discriminatory approaches (see http://ec.europa.eu/enterprise/policies/sme/small-business-act/).

in the procurement auction requires that there be no discrimination and that the most efficient firm be the winner, as long as its cost is lower than the outside option cost. We also show that this cost minimizing outcome can be achieved using a standard first-price auction in which each participating firm is requested to submit a sealed bid and the firm with the lowest bid wins the contract if this bid is lower than the outside option cost and is paid the amount of its own bid.

Our second set of results shows that some form of discrimination is desirable in the presence of incumbents. More precisely, we characterize the optimal form of discrimination as a function of the extent to which the designer internalizes the profits of the incumbents and as a function of the distributions of valuations of the various bidders. We observe that incumbents should be discriminated against potential entrants no matter whether incumbents are ex ante more/less cost-efficient than entrants and no matter which (positive) share of the profit of the incumbents is internalized by the designer. Finally, assuming discrimination is not possible and that the sole instrument of the designer is the reserve cost above which the designer commits not to use any firm, we establish that the optimal reserve cost should be set below the designer's outside option cost in the presence of incumbents.

While we have phrased the above results for the procurement auction application, in the rest of the paper, we phrase the problem as a regular auction in which the designer is a seller and the firms are buyers. We now review our main results in light of the auction literature also to shed light on what is new in our results.

Our first non-discrimination result is somehow related to the result in Levin and Smith (1994) who consider an auction model with endogenous entry in which all potential entrants are ex ante symmetric and learn their valuation only after their entry decision. They show that in a private value setting, revenue maximization requires that the reserve price be set at the seller's valuation in second-price auctions. Compared to Levin-Smith (1994), our setup allows for potential ex ante asymmetries, and it allows for any mechanism in particular allowing for asymmetric treatments of participants coming from different groups, which are both essential to be able to speak of discrimination.

Another classic setup in the auction literature with endogenous participation is the one proposed by Samuelson (1985) in which buyers know their valuation prior to deciding whether or not to participate. Somehow in the vein of Samuelson (1985), McAfee (1993) considers a model of competing auctions in which buyers know their valuations from the start and a large number of sellers compete in mechanisms to attract bidders. He finds that one (competitive) equilibrium is such that sellers use second-price auctions with reserve prices set at the sellers' valuations assumed to be homogeneous, thereby resulting in an ex post efficient allocation. In the competitive equilibrium considered by McAfee (1993), the expected utility of buyers is assumed not to be affected by the choice of auction of a given seller so that his model of competing auctions can be viewed as a model of auction with endogenous entry in which the participation cost is just the utility this bidder would get elsewhere.[3] The result of McAfee (1993) is suggestive that distortions may not be revenue maximizing in the presence of endogenous participation, but it does not address the possibility that the seller observes some characteristics of the buyers and make the auction procedures depend on these, nor does it address the possibility that buyers refine their valuation after entry. It does not address either whether there could be alternative inefficient equilibria.

By contrast, our setup allows buyers to belong to different groups of potential entrants with different and arbitrary valuation distributions and participation costs and that the seller observes some (but not necessarily all) characteristics of the buyers. This means that we allow buyers to have some private information before they make their participation decision (as in McAfee 1993) and to acquire further extra information as they participate (as in Levin and Smith 1994). The possibility that entrants might belong to different groups and that the seller may have some information related to the groups bidders belong to is essential to be able to speak of (direct) discrimination, and our setup is we believe the first in the mechanism design literature on endogenous participation to be allowing this.[4,5] Besides, given that, in most applications, information would flow in both before and after the participation decision, we believe that the combined model that we study is consistent with a broader set of applications.

Our non-discrimination result can be decomposed into two steps. The first step consists in showing that when the seller seeks to maximize revenues, one equilibrium consists for her in choosing the second-price auction with a reserve price set at the seller's valuation -an auction format that we refer to as the pivotal mechanism in the rest of the paper- and for the entrants to choose (anonymous) participation strategies that maximize total welfare. The logic for this result is as follows. On the one hand, whatever the chosen mechanism, entrants exhaust their rents through entry so that the seller's expected revenue is equal to the total expected welfare (net of the entry costs). On the other hand, the pivotal mechanism is such that the welfare optimal participation strategies constitutes a participation equilibrium,[6] which obtains because the payoff of each entrant corresponds to his contribution to the welfare in such a mechanism. Combined together, these insights imply that one equilibrium of the auction game with endogenous entry

[3] An important difference between McAfee (1993) and Samuelson (1985) is that the opportunity cost of bidders depends on their valuation in McAfee (1993) but not in Samuelson (1985). We note that (unlike this paper) Samuelson's (1985) model cannot be viewed as a reduced form of competing auctions.

[4] Our analysis should be contrasted with that suggesting a potential benefit of discrimination in contexts with costly participation that arises in an attempt to better coordinate entry decisions (see, for example, Celik and Yilankaya (2009) who consider a model in the vein of Samuelson (1985) and obtain that discrimination may be desirable even in symmetric cases). By contrast, such coordination motives are absent from our model insofar as we assume the mechanism and bidders' strategies must respect the ex ante symmetry of bidders. Our approach fits better the idea that our model can be thought of as a reduced form model of competing auctions insofar that competition would make such discriminatory devices ineffective.

[5] Roberts and Sweeting (2012,2013) also consider a setup with ex ante asymmetric bidders who are privately informed. However, they do not develop a mechanism design analysis.

[6] When we use the terminology *optimal* we implicitly restrict ourselves to the set of participation strategies where buyers from the same group enter symmetrically. If buyers from the same group could coordinate their participation strategies, then the welfare could be raised.

requires that there be no discrimination.[7]

The second step consists in showing that all equilibria are equivalent to the one arising with the pivotal mechanism and optimal participation decisions. This step crucially depends on our assumptions that the set of potential entrants is arbitrarily large and that entrants of the same group follow the same participation strategy (we believe the latter symmetry assumption is natural in contexts in which entrants have no way to coordinate their participation decisions – it is an assumption also made in Levin and Smith 1994). Observe that with a small set of potential entrants, there is no guarantee that participation would be efficient in the pivotal mechanism: In such a case, there may be multiple equilibria including inefficient ones because of coordination problems that may arise even if we assume that entrants of the same group follow the same strategy (for example if due to the entry costs, there is room for only one entrant, there is no guarantee that the entrant from the more efficient group would be the one participating). Besides, if participation is inefficient when the pivotal mechanism is used, there is no guarantee that the seller would not be better off choosing another format (so as to get closer to the welfare optimal entry profile even if it is at the cost of inducing ex post allocative inefficiencies). As a result, it is unclear whether the non-discrimination result applies to all equilibria. By contrast, when the set of potential entrants is arbitrarily large (and buyers from the same group follow the same participation strategy), participation decisions must be optimal, therefore also implying that the seller cannot do better than choosing the non-discriminatory pivotal mechanism.

To the best of our knowledge, our paper is the first to establish such a strong form of non-discrimination result. While technically our result relies mainly on the concavity of the expected welfare function with respect to participation rates (where participation distributions are modeled as Poisson distributions), the intuition as to why an arbitrarily large set of potential participants helps is that in such a case (and in contrast to the case with sets of finite size) every potential participant expects to be facing the same distribution of competitors whatever his ex ante strength so that less efficient buyers have lower incentive to participate than more efficient buyers. The latter property -which is referred to as environmental equivalence in Myerson (1998) - is also the one that makes first- and second-price auctions equivalent in our context despite the fact that entrants are ex ante asymmetric.

When incumbents who participate for sure are present, we have the following results. When the seller fully internalizes the profits of incumbents, the non-discrimination result extends. This is a straightforward extension of our non-discrimination result without incumbents. When the seller cares only about her own revenue, the optimal auction takes the form of a modified pivotal mechanism in which the valuations of incumbents should be replaced by their virtual valuations as defined in the work of Myerson (1981). Such a format thus requires some discrimination against incumbents (given that virtual valuations are lower than valuations). We also characterize

[7]Somehow the works of McAfee (1993) and Peters (1997,2001) illustrate a similar efficiency insight in contexts in which buyers fully know their valuations ex ante before making their participation decisions.

the optimal auction when the seller internalizes partly the profit of incumbents in which case the modified valuation of each incumbent should be a convex combination between his virtual valuation and his valuation, thereby implying that the discrimination against incumbents applies no matter how the valuations of entrants and incumbents are distributed and no matter which share of the profit of incumbents is internalized by the seller, as long as it is not fully internalized.

Our setup with incumbents and entrants offers a mixture between models with exogenous participation à la Myerson (1981) and models with only endogenous participation à la Levin-Smith (1994), and it shows how the insights of the two approaches should be combined to derive the optimal auction and shed light on the issue of discrimination in procurement auctions. To the best of our knowledge, our paper is the first to offer this general perspective on discrimination. We also briefly discuss the cases in which the seller cannot employ discriminatory mechanisms. When the sole instrument is the reserve price, we establish in the context of second-price auctions that the seller should post a reserve price strictly above her valuation in a wide range of situations. We also establish that our main results carry over to the case in which the seller's valuation is privately known to her (so that the mechanism chosen by the seller may a priori convey some information about her valuation) and we also note that it extends to the case in which buyers enjoy an ex post quitting right assuming the entry cost is sunk (and cannot be recovered even if the buyer quits the auction ex post).

We note that our theoretical results showing the negative effects of discrimination on revenues in contexts without incumbents are somehow confirmed (and quantified) by the empirical findings reported in Marion (2007) who establishes that in California the five percent subsidy that accrues to small businesses in auctions for road construction projects using only state funds increases the procurement costs by 3.8 percent compared to projects using federal aid where there is no such bid preference program. It should also be mentioned that Marion (2007) shows that the main channel for this detrimental effect of discrimination comes from the reduced participation of lower cost large firms in those auctions where they are unfavored, which is precisely the theoretical channel that this paper identifies. In another context, Athey et al. (2013) find a positive effect of discrimination on revenues. Specifically, according to their structural estimates for timber auctions, the seller's revenue is increasing in the subsidy level on small firms, at least up to a 20% subsidy. This is consistent with our results insofar as for the largest part of their sample (the so-called large sales), Athey et al. (2013) consider that the participation of the large bidders (the mills) is inelastic. Those large firms correspond thus to incumbents according to our terminology, thereby explaining why some discrimination in favor of small firms (which is of course equivalent to discrimination against large firms) may improve revenues.

From a broader perspective, our work can be seen as belonging to the auction literature with pre-participation investments insofar as the participation decision can be viewed as affecting the distribution of valuations (the valuation is set at zero in case of no participation). Of course, the setting with endogenous participation offers additional specific properties that could not be obtained in more general settings (in particular, other forms of investments would typically lead to multiple equilibria).[8]

The rest of the paper is organized as follows: Section 2 introduces the endogenous entry model with the Poisson distribution structure and the corresponding equilibrium concept we use. Environments without incumbents are analyzed in Section 3 where our general non-discrimination result is established. In section 4, we characterize the optimal auction in the presence of incumbents and in Section 5 we discuss setups in which discrimination is prohibited. Section 6 concludes. Most of the proofs are relegated to the Appendix. The supplementary material (henceforth Supp. Mat.) is devoted to the exposition of some technical elements or extensions that are less central to the paper.

2 The model

One seller S has one object to sell. Her valuation X_S is assumed for now to be known to everybody. We will later on extend our results to the case in which the valuation of the seller is privately known to her. The seller chooses an auction mechanism m in some set $\mathcal{M}$. Buyers then decide whether or not to participate. Those who participate play the auction game resulting from the mechanism according to a Bayes-Nash equilibrium. The outcome of the mechanism (specifying who gets the good and monetary transfers) is implemented. The seller and the buyers are assumed to be risk-neutral.

Observe that the model is framed in an auction setup, i.e. where buyers are bidding in order to acquire an object. However, it could equally be phrased as a procurement in which the designer seeks to obtain a service from various potential providers. The latter better suits our motivations and policy implications as described in Introduction.

Coming back to our auction model, we assume that every mechanism allows bidders to get at least 0 (say by making an irrelevant bid), and we assume that there are several types of buyers: the incumbents who participate for sure irrespective of the mechanism and the potential entrants who adjust their participation decisions as a function of the chosen mechanism. There are $I \geq 0$ incumbents who are each characterized by a cumulative distribution $F_i^I(.|z)$, $i = 1, \dots, I$, from which their valuations are drawn conditional on the realization z of some underlying variable Z. The set of incumbents is denoted by $\mathcal{I} := \{1, \dots, I\}$. There are $K \geq 0$ groups of potential entrants, each group being composed of infinitely many potential buyers. A buyer from group $k \in \{1, \dots, K\}$ incurs the cost of entry $C_k > 0$ (which corresponds equivalently to his expected utility if he chooses an outside option and which could also include physical costs) while his

[8] Due to equilibrium multiplicity, Bag (1997) obtains in settings with pre-investment decisions that discrimination may be desirable even if buyers are ex ante symmetric. By contrast, Piccione and Tan (1996) provides some conditions on the investment technology that guarantees uniqueness so that the pivotal mechanism implements the first-best.

valuation is drawn from the cumulative distribution $F_k(.|z)$ conditional on the realization z of the underlying variable Z. Conditionally on z, the valuations of the various buyers are assumed to be drawn independently. Conditional independence is a general way to introduce some correlation between buyers' valuations. The set of the various groups of entrants is denoted by $\mathcal{K} := \{1, \ldots, K\}$.[9] When $K = 1$, we let $F \equiv F_1$ and $C = C_1$. We assume that the supports of $F_i^I(.|z)$ and $F_k(.|z)$ are bounded. Otherwise, we do not impose any specific restriction on the distributions $F_i^I(.|z)$ and $F_k(.|z)$, neither on C_k, nor on the distribution of z. From that perspective, our model is extremely general.[10] The special case in which $F_k(x|z) = \mathbf{1}[x > x_k]$ for any $k \in \mathcal{K}$ corresponds to the case in which potential entrants from group k know their valuation x_k before entry as in McAfee (1993).

Observe that we have not specified whether the seller observes the group from which an entrant comes. We have in mind that the seller is partially informed about the groups k effective participants belong to: This ensures that the designer may (a priori) use mechanisms that entail some form of (direct) discrimination while participating buyers still have some private information (both before and after entry).[11] While the various possibilities to discriminate between buyers from various groups obviously depend on the information held by the seller, those aspects can be formalized through the set of possible mechanisms $\mathcal{M}$ discussed below and for which the model leaves a lot of flexibility.

In a given mechanism m and for a given profile of entrants, the outcome is determined by bidders' strategies the set of which is denoted by $\Sigma(m)$. Under the terminology 'bidders', we consider both the participating buyers and the seller who is allowed to be active at the auction stage (if the seller decides so).[12] The outcome of an auction is defined as an assignment rule which specifies the probability of receiving the good for each agent (including the seller) and a set of monetary transfers. The feasibility constraints impose that the assignment probabilities sum to one and that the monetary transfers sum to zero. In order to avoid technical complications, we also assume that the monetary transfers of all agents (both the buyers and the seller) are

[9] $I = 0$ [resp. $K = 0$] is equivalent to $\mathcal{I} = \emptyset$ [resp. $\mathcal{K} = \emptyset$].

[10] A nice aspect of our model is also that we do not put any restriction on the number of groups while the structural empirical literature focus on the two-group case as e.g. in Athey et al. (2011,2013). The group structure can thus capture the idea of pre-entry signals about valuations as in Roberts and Sweeting (2012). Nevertheless, it does not capture models where the cost of entry is heterogenous among potential entrants as in Krasnokutskaya and Seim (2011) who consider a model à la Levin and Smith (1994) with two groups of buyers and heterogenous entry costs. Among a given group, buyers enter when their entry cost is below some threshold. We conjecture that there is no discontinuity when the entry cost becomes homogenous so that our model can be viewed as the limiting case of such a model when entry costs are homogenous.

[11] Observe that if the seller were perfectly informed of k, then she could adjust entrants' participation rates through appropriate choices of discriminatory entry fees and subsidies (see Bag (1997) in a setup with pre-participation investments). As a result, in the sole presence of entrants, the seller would choose an efficient mechanism (such as a second-price auction with the reserve price set at the seller's valuation) and she would adjust optimally the entry fees. Our non-discrimination result is of a different nature, since it applies no matter how much private information entrants possess prior to the entry decision. Besides, even assuming away the private information of participating buyers, efficient mechanisms with entry fees would lose their efficiency property if buyers were to enjoy ex post quitting rights. Imposing ex post participation constraints would lead to the same analysis as the one developed below.

[12] This matters essentially in the informed principal setup which is considered later in the analysis.

bounded by some fixed amount (we could alternatively assume that the participation constraints of all agents should be satisfied ex post). As already mentioned, for any mechanism m, we assume that there is a non-participation strategy in $\Sigma(m)$ which guarantees to any buyer who uses it that his monetary transfer is null while he does not receive the good. Finally, we impose that buyers from the same group of potential entrants are treated in a similar way which reflects an anonymity constraint. The set of all mechanisms with the above constraints is denoted by $\mathcal{M}^*$ and we assume in the sequel that $\mathcal{M} \subseteq \mathcal{M}^*$. We also assume that buyers from the same group follow the same strategy (both in terms of participation and at the bidding stage), which (together with our restriction on mechanisms) implies that all potential entrants from the same group derive the same expected utility from participating in the mechanism.

Assuming that the seller can pick any mechanism in $\mathcal{M}^*$, i.e. $\mathcal{M} = \mathcal{M}^*$, follows the tradition of the mechanism design literature in the vein of Myerson (1981). By contrast, toward the end of the paper we will consider the case in which the seller is restricted to choose a second-price auction and can only adjust the reserve price r. We refer to the latter set of mechanisms as $\mathcal{M}^r_{SP}$. Observe that if the seller picks a mechanism in $\mathcal{M}^r_{SP}$ when unrestricted, it means that she chooses not to discriminate among buyers (since all buyers are treated alike in a mechanism $m \in \mathcal{M}^r_{SP}$). When the seller is forced to pick a mechanism in $\mathcal{M}^r_{SP}$ (whereas she would have picked an alternative mechanism otherwise), we have in mind situations in which discrimination would be prohibited by law. In general, we use the notation $m \in \mathcal{M}$ for a given mechanism. When we consider a mechanism in the set $\mathcal{M}^r_{SP}$, then we use the notation $r \in R_+$ to denote a specific mechanism. We will also refer to the second-price auction with the reserve price set at the seller's valuation as the *pivotal mechanism*, denoted also by X_S.

A key aspect of our model is the modeling of buyers' participation as a Poisson game. Our assumption that there are infinitely many buyers in each group k of entrants together with our assumption that participation decisions are made symmetrically among buyers of the same group leads us to assume that the effective number of entrants from a given group $k \in \mathcal{K}$ of potential entrants is taken to be the realization of a random variable following a Poisson distribution with mean $\mu_k \geq 0$. That is, the probability that there are n_k entrants from group $k \in \mathcal{K}$ is equal to $e^{-\sum_{k=1}^{K} \mu_k} \cdot \prod_{k=1}^{K} \frac{[\mu_k]^{n_k}}{n_k!}$. The Poisson distribution corresponds to the limit distribution of the number of entrants of each group of a model with a finite number buyers per group and as the number of buyers in each group goes to infinity (and assuming every individual entrant of a given group follows the same participation strategy). By contrast with the voting literature with Poisson games initiated by Myerson (1998,2002) where the Poisson distributions are taken as exogenous, the parameters μ_k, $k \in \mathcal{K}$, will be endogenously determined in our competitive equilibrium.

Remark 2.1 The case in which $I = 0$, $K = 1$ and $\mathcal{M} = \mathcal{M}^r_{SP}$ corresponds to the limiting case of Levin and Smith's (1994) model where the total number of potential entrants goes to infinity so that the effective number of participants, which follows a binomial distribution in Levin and Smith (1994), follows then a Poisson distribution.

Remark 2.2 The case in which $I \geq 1$, $K = 0$ and $\mathcal{M} = \mathcal{M}^*$ corresponds to the private value environment à la Myerson (1981).

Remark 2.3 We are allowing any form of (a)symmetry between groups. While in general we have in mind that $F_k \neq F_{k'}$ and $C_k \neq C_{k'}$ for $k \neq k'$, our setup allows that $F_k = F_{k'}$ and/or $C_k = C_{k'}$ for $k \neq k'$.

Before we present the formal definition of equilibrium, some additional notation is required. We let

- $N = (n_1, \ldots, n_K) \in \mathbb{N}^K$ denote a realization of the profile of entrants. For a given vector N, we let $|N| = \sum_{k=1}^{K} n_k$, $N_{-k} = (n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1} \ldots, n_K)$ and $N_{+k} = (n_1, \ldots, n_{k-1}, n_k + 1, n_{k+1} \ldots, n_K)$.

- $F^{(j:N \cup I)}$ [resp. $F^{(j:N \cup I_{-i})}$] denote the CDF of the j^{th} order statistic among the set of entrants N and the I incumbents [resp. the I incumbents except incumbent i]. E.g., $F^{(1:N \cup I)}(x) = E_Z[\prod_{k=1}^{K} [F_k(x|z)]^{n_k} \cdot \prod_{i=1}^{I} F_i^I(x|z)]$. If $j > |N| + I$, then we adopt the convention $F^{(j:N \cup I)}(x) = 1$. If $I = 0$ [resp. $N = (0, \ldots, 0)$], we use the simplified notation $F^{(j:N)}$ [resp. $F^{(j:I)}$] for $F^{(j:N \cup I)}$.

- $\sigma(m) \in \Sigma(m)$ denote the bidding strategy profile used by the bidders in the mechanism m.

- $P(N|\mu) = e^{-\sum_{k=1}^{K} \mu_k} \cdot \prod_{k=1}^{K} \frac{[\mu_k]^{n_k}}{n_k!}$ denote the probability of the realization N when the entry rate vector is μ, namely when the Poisson distribution of group k buyer has mean μ_k for any $k \in \mathcal{K}$.

- $\Lambda_N(m, X_S; \sigma(m))$ denote the expected (interim) utility (or objective function) of the seller when the set of participants consists of the profile of potential entrants N and the set of incumbents, and when bidders follow the bidding profile $\sigma(m)$. We do not always adopt the view that the seller is a (pure) revenue-maximizer.

- $V_{k,N}(m; \sigma(m))$ [resp. $V_{i,N}^I(m; \sigma(m))$] denote the expected (interim) utility of a buyer from group k [resp. the incumbent i] participating in the mechanism m when the set of participants consists of the profile of potential entrants N with $n_k \geq 1$ [resp. N] and the set of incumbents, and when bidders follow the bidding profile $\sigma(m)$.

- $u_S(\mu_1, \ldots, \mu_K, m, X_S; \sigma(m)) = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot \Lambda_N(m, X_S; \sigma(m))$ denote the expected (ex ante) utility of a seller with valuation X_S in the mechanism m when the entry rate vector is μ and when bidders follow the bidding profile $\sigma(m)$.[13]

- $u_k(\mu_1, \dots, \mu_K, m; \sigma(m)) = \sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot V_{k,N_{+k}}(m;\sigma(m))$ denote the expected (ex ante) utility of a group k buyer in the mechanism m when the entry rate vector is μ and when bidders follow the bidding profile $\sigma(m)$. Note that from the perspective of any entrant no matter what his group k is, the probability that he faces the set of entrants N (excluding himself) is also equal to $P(N|\mu)$.

- $u_i^I(\mu_1, \dots, \mu_K, m; \sigma(m)) = \sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot V_{i,N}^I(m;\sigma(m))$ denote the expected (ex ante) utility of the incumbent i in the mechanism m when the entry rate vector is μ and when buyers follow the bidding profile $\sigma(m)$.

Finally, to present some of our results it is convenient to define

- $W_N(m, X_S; \sigma(m))$ as the expected (interim) gross welfare (i.e. the sum of all agents' utilities excluding the entry costs that are sunk) conditional on participation N and valuation X_S of the seller when the mechanism m is proposed and bidders follow the strategy $\sigma(m)$.

- $\Phi_N(m, X_S; \sigma(m))$ as the corresponding expected (interim) revenue of the seller, i.e.

$$\Phi_N(m, X_S; \sigma(m)) := W_N(m, X_S; \sigma(m)) - \sum_{k=1}^{K} n_k \cdot V_{k,N}(m;\sigma(m)) - \sum_{i=1}^{I} V_{i,N}^I(m;\sigma(m)). \tag{1}$$

In the sequel, we will assume that bidders use undominated strategies.[14] Thus, when the mechanism m is a second-price auction, i.e. $m \in \mathcal{M}_{SP}^r$, buyers bid their valuation. To alleviate notation, we then drop $\sigma(m)$ from the notation of the various expected utility functions as introduced above but also from the notation introduced later on. Specifically, $W_N(X_S, X_S)$ denotes the expected (gross) welfare in the pivotal mechanism with the set of participants N. Clearly, we have

$$W_N(m, X_S; \sigma(m)) \leq W_N(X_S, X_S) \tag{2}$$

for any profile of entrants N, any $m \in \mathcal{M}$ and any strategy profile $\sigma(m) \in \Sigma(m)$.

How many buyers of a given group enter a mechanism is determined by equilibrium conditions reflecting an arbitrage condition for every potential participant between entering the given

[13]Since transfers and valuations are uniformly bounded, then $\Lambda_N(m, X_S; \sigma(m))$ is also uniformly bounded and the previous sum is correctly defined. It also guarantees that the function $\mu \to u_S(\mu, m, X_S; \sigma(m))$ is differentiable on R_+^K. The same remark holds for the similar sums that appear in our analysis.

[14]In fact, for our non-discrimination result, we do not even use that bidding strategies form a Bayes-Nash equilibrium.

auction or using his outside option. To define the equilibrium formally, we introduce for each $k \in \mathcal{K}$ a Poisson parameter function $\mu_k^* : \mathcal{M} \to R_+$, where $\mu_k^*(m)$ characterizes the distribution of participation of buyers of type k in the mechanism m. An equilibrium is defined as:

Definition 1 *For a given set of possible mechanisms $\mathcal{M}$, an equilibrium with endogenous participation is defined as a strategy profile $(m^*, (\mu_k^*)_{k\in\mathcal{K}}, \sigma^*)$, where $m^* \in \mathcal{M}$ stands for the seller's chosen mechanism, $\mu_k^* : \mathcal{M} \to R_+$ describes the distribution of participation of group k buyers in the various possible mechanisms and $\sigma^*(m) \in \Sigma(m)$ describes the bidding profile of the bidders in the various possible mechanisms $m \in \mathcal{M}$ where*

(i) (Utility maximization for the seller)

$$m^* \in Arg \max_{m\in\mathcal{M}} \; u_S(\mu_1^*(m), \ldots, \mu_K^*(m), m, X_S; \sigma^*(m)). \tag{3}$$

(ii) (Utility maximization for potential entrants) for any $m \in \mathcal{M}$ and any $k \in \mathcal{K}$,

$$\mu_k^*(m) \underset{(resp.\ =)}{>} 0 \Longrightarrow u_k(\mu_1^*(m), \ldots, \mu_K^*(m), m; \sigma^*(m)) \underset{(resp.\ \leq)}{=} C_k. \tag{4}$$

(iii) (Equilibrium conditions) in any mechanism $m \in \mathcal{M}$, bidders are using the bidding profile $\sigma^*(m)$ which is a Bayes-Nash equilibrium (if an equilibrium exists) and are using undominated strategies.

Condition (3) implies that the seller is required to pick a mechanism which maximizes her objective given the entry rate vector $\mu^*(m) := (\mu_1^*(m), \ldots, \mu_K^*(m))$ and the equilibrium bidding profile $\sigma^*(m)$ attached to any mechanism m. Condition (4) implies that whatever the mechanism and for each group $k \in \mathcal{K}$, either the participation rate is positive and delivers an expected equilibrium utility of C_k to buyers of group k,[15] or the participation rate is zero and the corresponding expected payoff of a buyer is lower than C_k.

For a given mechanism $m \in \mathcal{M}$, we let

$$M(m) := \{\mu \in R_+^K | \text{ Condition } (4) \text{ holds for } m \text{ and all } k \in \mathcal{K}\}$$

denote the set of participation rates that are compatible with equilibrium behavior. Next, for a given equilibrium with endogenous participation, the entry rate vector $\mu^*(m^*)$ is referred to as the equilibrium participation rate. We show in the Supp. Mat. that $M(m) \neq \emptyset$ for any $m \in \mathcal{M}$.[16]

[15] Clearly, participation rates cannot be infinite as it would result in a zero payoff when participating (and entry costs C_k are assumed to be strictly positive). This also explains why if the participation rate of some group k is positive, the expected utility of k buyers obtained from participation should be set at C_k.

[16] This is the step where the technical assumption that the seller has a limited budget plays a role.

Observe that we have not specified the exact information structure of participants regarding the set N of effective participants (which could a priori be affected by the choice of mechanism) and their beliefs regarding the valuation of their opponents in the auction. This in fact plays no role for our non-discrimination result which relies only on players not using weakly dominated strategies (in second-price auctions). When we introduce incumbents in Section 4, we have to specify more precisely the information structure, in particular regarding the signals received by incumbents, and our characterization of the optimal mechanism there relies critically on the assumption that incumbents are using best-responses in the bidding profile $\sigma(m)$.

Comments: 1) The Poisson model for participation rates encompasses implicitly both a large market hypothesis and a symmetry assumption that both deserve some discussion. Formally, this specification corresponds to the limit of a (standard) Nash equilibrium concept with a finite number of potential entrants in each group that enter with the same probability when the number of potential entrants in each group goes to infinity (see the Supp. Mat. for details). Restricting attention to group-symmetric equilibria is a popular assumption in the empirical literature that deals with structural models. It is well-known when $K = 1$ that there exists many asymmetric equilibria: in particular equilibria where some agents enter the auction with probability one as it would occur in models with sequential entry (see Engelbrecht-Wiggans 1993). Nevertheless, those agents who enter for sure could be classified somehow as belonging to the set of incumbents, which would still fit with our general model.[17] Another reason why the group-symmetry assumption is not as restrictive as it may appear at first glance is that we are allowing groups to be symmetric as noted in Remark 2.3 so that we can split a given group into several identical groups if we wish to allow some asymmetric behavior among ex ante symmetric potential entrants. 2) Our model can be viewed as a reduced form model for richer models of competition between many (possibly heterogenous) sellers and many (possibly heterogenous) buyers in which the cost of entry of a group k buyer corresponds to the expected utility such a buyer could at best obtain by participating in another auction as in McAfee (1993). 3) Athey et al. (2011, 2013) estimate structural models where groups of entrants are of finite size. More precisely, those papers consider either $K = 2$ and $I = 0$ or $K = 1$ and $I > 1$. The model in Roberts and Sweeting (2012) is also closely related to ours: They consider Levin and Smith's (1994) model (with possibly several groups) with the additional feature that potential entrants receive a private signal about the future realization of their valuation. One can embed such a situation into our model by parameterizing groups by the signals privately observed by the entrants and assuming this characteristic of the group is not observed by the designer. From the

[17] To guarantee that it is properly an equilibrium, we should check that the expected payoff of such an "incumbent" from group k is larger than his entry cost C_k once $\mu_k^*(m) > 0$. It is true for second-price auctions that play a central role in our analysis. Consider a second-price auction with the reserve r and an incumbent i such that $F_i^I = F_k$ with $\mu_k^*(r) > 0$. Since $V_{i,N}^I(r) \geq V_{i,N_{+k}}^I(r) = V_{k,N_{+k}}(r)$, then we obtain that $u_i^I(\mu^*(r), r) \geq u_k(\mu^*(r), r) = C_k$. In words, the underlying argument is that compared to the incumbent i, an entrant from group k competes on average with the same set of bidders plus the given incumbent.

perspective of this empirical literature, one could object that our large market hypothesis is too strong if we have in mind that the number of participants is typically less than five. Nevertheless, it should be stressed that our large market hypothesis does not relate to the number of effective participants but to the number of potential participants, which is much larger as explained in Athey et al. (2011, 2013).

3 A general non-discrimination result

Throughout this section, we assume that the seller is a revenue-maximizer and there are no incumbents, i.e $\Lambda_N(m, X_S; \sigma(m)) = \Phi_N(m, X_S; \sigma(m))$ and $\mathcal{I} = \varnothing$.

For a given mechanism m proposed by a seller with reservation value X_S, for a given bidding profile $\sigma(m)$ and when the participation rate of group k buyers is μ_k for $k \in \mathcal{K}$, we define the total expected (ex ante) welfare (net of the expected opportunity cost of participation) by

$$TW(\mu_1, \ldots, \mu_K, m, X_S; \sigma(m)) := \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot W_N(m, X_S; \sigma(m)) - \sum_{k=1}^{K} \mu_k \cdot C_k. \quad (5)$$

Combining the expression for expected revenue (1) with the utility maximization conditions (4) for potential entrants, we obtain that the seller's revenue coincides with the total welfare for any mechanism m and any entry rate $\mu \in M(m)$, namely $u_S(\mu, m, X_S; \sigma^*(m)) = TW(\mu, m, X_S; \sigma^*(m))$ if $\mu \in M(m)$. From the utility maximization conditions (3), the mechanism chosen by the seller in any equilibrium solves thus the maximization program:

$$\max_{m \in \mathcal{M}} TW(\mu^*(m), m, X_S; \sigma^*(m)). \quad (6)$$

Consider then the relaxed maximization program:

$$\max_{\mu \in R_+^K, m \in \mathcal{M}, \sigma(m) \in \Sigma(m)} TW(\mu, m, X_S; \sigma(m)). \quad (7)$$

We say that an equilibrium implements the first-best if the mechanism chosen by the seller, the equilibrium participation rates and the equilibrium bidding profile followed by the bidders solve the maximization program (7).

From (2), we obtain that $TW(\mu, m, X_S; \sigma(m)) \leq TW(\mu, X_S, X_S)$ for any $m \in \mathcal{M}$ and any $\sigma(m) \in \Sigma(m)$ where $m = X_S$ denotes the pivotal mechanism.

Let

$$J(m, X_S; \sigma(m)) := \left\{ \mu \in R_+^K | \text{ for each } \ k \in \mathcal{K}, \frac{\partial TW}{\partial \mu_k}(\mu, m, X_S; \sigma(m)) \underset{(resp. \leq)}{=} 0 \ \ if \ \ \mu_k \underset{(resp. =)}{>} 0 \right\}$$

and $J^{MAX}(m, X_S; \sigma(m)) := \text{Arg}\max_{\mu \in R_+^K} TW(\mu, m, X_S; \sigma(m))$. Any local maximum of the function $\mu \to TW(\mu, m, X_S; \sigma(m))$ belongs to $J(m, X_S)$, and thus a fortiori global maxima of $\mu \to TW(\mu, m, X_S; \sigma(m))$ belong to $J(m, X_S)$ so that we have $J^{MAX}(m, X_S; \sigma(m)) \subseteq J(m, X_S; \sigma(m))$. It is straightforward to check that any pair $(\widetilde{\mu}, X_S)$ where $\widetilde{\mu} \in J^{MAX}(X_S, X_S)$ together with the truthful bidding strategy for the bidders is a solution of the maximization program (7).

A key property that plays a central role in our argument is that in the pivotal mechanism, the social contribution of an entrant to the expected welfare coincides with his expected payoff. This fundamental property of the pivotal mechanism can be stated formally as

$$W_{N_{+k}}(X_S, X_S) - W_N(X_S, X_S) = V_{k,N_{+k}}(X_S) \tag{8}$$

for all $N \in \mathbb{N}^K$ and $k \in \mathcal{K}$. As buyers obtain the incremental surplus they generate in the pivotal mechanism, we get (more details are given in the Appendix –Proof of Lemma 3.1) that

$$\frac{\partial TW(\mu, X_S, X_S)}{\partial \mu_k} = u_k(\mu, X_S) - C_k \tag{9}$$

which further implies that the set of participation rates that are compatible with equilibrium behavior in the pivotal mechanism corresponds to the set of participation rates μ such that the gradient of the total welfare is null, i.e. formally

$$J(X_S, X_S) = M(X_S). \tag{10}$$

Proposing the pivotal mechanism and having $\mu^*(X_S) \in J^{MAX}(X_S, X_S)$ forms then an equilibrium (independently of the various ways we could specify $\mu^*(m)$ for $m \neq X_S$). This is so because the pivotal mechanism is both ex post and ex ante optimal: On the one hand, it maximizes the total welfare for any given participation rate. On the other hand, it induces welfare-maximizing entry rates.

More generally, it is well-known that in private value setups when the pivotal mechanism is preceded by a stage in which agents are making private pre-participation investments (i.e. that influence only their own type), then any profile of investments that maximizes the welfare is an equilibrium (see Rogerson (1992) and more recently, Bergemann and Välimäki (2002) in a perspective with information acquisition, Arozamena and Cantillon (2004) when buyers can upgrade their valuation distribution and Stegeman (1996) in auctions with participation costs). The key point in those papers is that the maximization program faced by each agent corresponds to the maximization of the total welfare. From a broader perspective, the games that govern the choices of pre-investment strategies can be seen as potential games (Monderer and Shapley 1996) where the potential function is equal to the welfare. In our case, entry decisions can somehow be viewed as a specific form of pre-participation investments (not entering can be viewed equivalently as inducing a null valuation), thereby explaining why the efficient participation rates can arise in equilibrium in the pivotal mechanism. We note that in our setup there is a continuum of potential entrants, which the previous papers considering pre-participation investments did not have.

The aforementioned literature with pre-participation investments has also shown that in the pivotal mechanism there may exist other equilibrium investment profiles that are not a global optimum of the welfare but only a local optimum. For example, in second-price auctions with participation costs, Stegeman (1996) exhibits an example with symmetric bidders where the symmetric equilibrium (and so the most salient one) is inefficient.[18] In the present "auction with endogenous entry" setup, such inefficiencies could occur if there were to exist some $\widetilde{\mu} \in J(X_S, X_S)$ such that $\widetilde{\mu} \notin \text{Arg}\max_{\mu \in R_+^K} TW(\mu, X_S, X_S)$. The possible emergence of such a $\widetilde{\mu} \in J(X_S, X_S) \setminus J^{MAX}(X_S, X_S)$ would correspond to a miscoordination in the participation decisions in the pivotal mechanism, and it might well lead the seller to strictly benefit from proposing a different mechanism, in particular one that is ex post inefficient. The next lemma shows that this never occurs in our context because the function $\mu \to TW(\mu, X_S, X_S)$ is globally concave.

Lemma 3.1 *For any $X_S \in \mathbb{R}_+$, $\mu \to TW(\mu, X_S, X_S)$ is concave on R_+^K. As a corollary, we have $J^{MAX}(X_S, X_S) = M(X_S)$.*

Lemma 3.1 implies that if the seller proposes the pivotal mechanism, the vector of the equilibrium participating rates belongs to $J^{MAX}(X_S, X_S)$, which in turn allows us to establish that any equilibrium is "equivalent" to one arising with the pivotal mechanism where equivalence between two strategy profiles is formally defined as:

Definition 2 *We say that two strategy profiles $(m, (\mu_k)_{k=1,\dots,K}, \{\sigma(m)\}_{m \in \mathcal{M}})$ and $(\widetilde{m}, (\widetilde{\mu}_k)_{k=1,\dots,K}, \{\widetilde{\sigma}(m)\}_{m \in \mathcal{M}})$ are equivalent if the participation rates at the mechanism proposed by the seller are the same, namely $\mu(m) = \widetilde{\mu}(\widetilde{m})$, and if for any profile of entrants N that occurs with positive probability (i.e. $P(N|\mu(m)) > 0$), then the good is assigned in the same way with probability one (which implies in particular that $W_N(m, X_S; \sigma(m)) = W_N(\widetilde{m}, X_S; \widetilde{\sigma}(m))$).*

We can state our main non-discrimination result:

Proposition 3.2 *Assume that $X_S \in \mathcal{M}$. For any $\mu \in M(X_S)$, there exists an equilibrium in which the seller proposes the pivotal mechanism while the equilibrium participation rate is μ. Conversely, any equilibrium is equivalent to an equilibrium in which the seller proposes the pivotal mechanism. Any equilibrium implements the first-best.*

[18] Tan and Yilankaya (2006) provide some sufficient conditions on the curvature of the underlying valuation distributions that guarantee equilibrium uniqueness in auctions with participation costs.

The first part of Proposition 3.2 can be viewed as generalizing the insights obtained by Levin-Smith (1994) and McAfee (1993): There exists an equilibrium which results in an ex post efficient allocation. As we show, this insight extends to situations with ex ante asymmetries among entrants whatever the exact informational assumption regarding what the designer can observe ex ante about entrants' characteristics.[19] It should be noted that in the case in which the seller's reserve price lies below the lower bound of buyers' valuation distributions, then all reserve prices between the seller's valuation and this lower bound would achieve an ex post efficient allocation. Yet, only the reserve price set at the seller's valuation would correspond to an equilibrium as other reserve prices would fail to induce efficient participation rates (see the Supp. Mat.).[20]

The second part of Proposition 3.2 establishes a much stronger result by showing that in any equilibrium, the seller proposes a mechanism that implements -in equilibrium- an outcome that is equivalent to the one that arises with the pivotal mechanism. To the best of our knowledge, no such "uniqueness" result appears in the earlier auction literature with endogenous participation at least in a model of such generality.

As it turns out, the assumption that the set of potential entrants is large is key for the derivation of the second part of Proposition 3.2. To illustrate why it would not hold in general when the set of potential entrants is finite, consider the following simple scenario. There are two types of entrants $K = 2$ with $F_1(x) = \mathbf{1}[x \geq 1]$, $F_2(x) = \mathbf{1}[x \geq 1+\epsilon]$, $\epsilon > 0$, $C_1 = C_2 = C \in (\epsilon, 1)$ and $X_S = 0$, and assume that there is only one potential buyer per group. There are then three equilibria in the pivotal mechanism: the two pure strategy equilibria where one buyer participates for sure and not the other one, and a purely mixed equilibrium where buyer 1 [resp. 2] enters with probability $q_1 = 1+\epsilon-C$ [resp. $q_2 = 1-C$]. Only the equilibrium where buyer 1 enters for sure is efficient. Putting several potential buyers per group has a concavification effect on how total welfare depends on the participation rates, which in turn guarantees uniqueness.[21]

To illustrate why increasing the number of potential entrants alleviates the coordination problem, we develop further the previous toy example by considering multiple potential buyers per group. Specifically, suppose now that instead of one buyer in group 1, there are $\mathcal{N}$ buyers

[19] This insight is actually much more general and extends to general allocation problems with private values (see Jehiel and Lamy (2013) for an application of this idea to competition among jurisdictions) but also to environments with general private pre-participation investments where agents can invest prior to the bidding stage to modify the distribution of their type or to refine their knowledge about their type.

[20] This discussion points the finger on the discrepancy between our pivotal mechanism and the pivotal mechanism as defined by Krishna and Perry (1998). The latter leaves no rents to the buyers with the lowest type.

[21] When there is only one potential buyer per group, the entry profile is characterized by the vector $q = (q_1, \ldots, q_K)$ where q_k denotes the probability that buyer k enters the auction. Let $TW(q, r, X_S) := \sum_{\substack{n_k \in \{0,1\} \\ k=1,\ldots,K}} \prod_{k=1}^{K} \left[q_k^{n_k}(1-q_k)^{1-n_k}\right] W_N(r, X_S) - \sum_{k=1}^{K} q_k C_k$ denote the total welfare in the second-price auction with the reserve price r and when the seller's valuation is X_S. For each k, it is straightforward that $\frac{\partial^2 TW(q,r,X_S)}{\partial^2 q_k} = 0$ which prevents concavity (except in the degenerate case where $\frac{\partial^2 TW(q,r,X_S)}{\partial q_k \partial q_l} = 0$ also when $k \neq l$, i.e. when the Hessian matrix is always null.)

By contrast in our model with large numbers of potential entrants, $\mu \rightarrow TW(\mu, r, , r)$ is always concave with no further assumption on the distributions of valuations.

while we still assume for simplicity that there is just one potential buyer in group 2. There are three equilibrium candidates. First the one where only the buyers from group 1 are active ($q_1 > 0$ and $q_2 = 0$), second the one where only the buyer from group 2 is active ($q_1 = 1$ and $q_2 > 0$) and finally the one where buyers from both groups are active ($q_1, q_2 \neq 0$). The second candidate implements the efficient entry profile and is always an equilibrium. By contrast, the two other candidates involve inefficiencies and may not be equilibria. The first candidate is characterized by the equilibrium condition $(1 - q_1)^{\mathcal{N}-1} = C$ (for the buyers from group 1) and is an equilibrium if an only if the buyer from group 2 does not find profitable to enter, which requires that $\epsilon \leq C - C^{\frac{\mathcal{N}}{\mathcal{N}-1}}$. The third candidate is characterized by the equilibrium conditions $(1-q_1)^{\mathcal{N}-1} \cdot (1-q_2) = C$ and $(\epsilon + (1-q_1)^{\mathcal{N}}) = C$ and it can be checked that it is an equilibrium only if there exists an equilibrium where only the buyers from group 1 enter the auction. On the whole, inefficient equilibria arises only if $\epsilon \leq C - C^{\frac{\mathcal{N}}{\mathcal{N}-1}}$ a condition that cannot hold when $\mathcal{N}$ is large enough. For any finite $\mathcal{N}$ we can cook a setup such that inefficiencies occur by taking ϵ small enough. By contrast, in the limit with arbitrarily large number of potential entrants, participation is distributed according to a Poisson distribution, and as follows from Lemma 3.1, no inefficient equilibrium can then arise no matter how valuations are distributed and no matter how the entry costs are specified.[22]

Extensions:

1) **The seller is privately informed of her valuation**

In some applications, it makes sense to assume that only the seller knows X_S, where X_S is drawn from some arbitrary distribution. A priori, we move now in the territory of informed principal problems in which the choice of format may convey some information to the buyers. This signaling aspect is often a source of multiplicity in principal-agent settings. Yet, in our context, this is not so. Indeed, by choosing the pivotal mechanism (with respect to her private valuation), a given seller whatever her valuation generates the highest possible expected total welfare and her expected revenue is equal to this welfare (net of the participation costs of the entrants). Suppose now that some pool of sellers (with different valuations) were to pick the same auction format and that it generates some positive rate of participation. Given that first-best participation rates if positive cannot be the same with different reservation values, we obtain then that total welfare would be strictly lower than the one arising in the situation in which all these sellers would pick the pivotal mechanism. However, for this pool of sellers, the welfare and the revenue should coincide on average as entrants' expected payoffs should coincide with their entry cost. This further implies that at least one pooled seller would be strictly better off

[22]The equilibrium multiplicity issue is a well-known issue pointed in the structural literature. It calls for a selection rule which often consists in picking the welfare-maximizing equilibrium. Our work gives some theoretical support for this selection rule insofar as it is the equilibrium which survives when the number of bidders per group gets large. Interestingly, Roberts and Sweeting (2012) discuss informally that when there are more agents in each given type of potential entrants, then it "tends to favor the game having a unique equilibrium", which is consistent with Lemma 3.1.

choosing the pivotal mechanism, thereby leading to a contradiction. We conclude that:[23,24]

Proposition 3.3 *On the equilibrium path, a mechanism that attracts some entrants cannot be proposed by sellers with different valuations. For any realization of the type of the seller and any equilibrium, the strategy profile on the equilibrium path is equivalent to the one in which the seller proposes the pivotal mechanism. Any equilibrium implements the first-best.*

2) **Multi-object auctions**

Under some specific structures with multi-object assignment problems (in particular when a bundle is valued according to the sum of its individual values), the pivotal mechanism still induces efficient entry and our analysis extends as detailed in the Supp. Mat. We obtain thus as a corollary that bundling would be detrimental to the seller, an insight which contrasts with the multi-object literature with exogenous entry where the optimal mechanism involves a departure from full efficiency (Jehiel and Moldovanu 2001) and some form of bundling (Jehiel et al. 2007) even in the additive case.

A notable multi-object extension that we are able to cover is the sponsored search auction setup à la Edelman et al (2007) and Varian (2007). These authors consider the following model: there are L units of (possibly) different sizes taken from an homogenous good and bidders are allowed to win at most one unit. The size of the k^{th} unit is denoted by s_k and we label units so that $s_1 \geq \cdots \geq s_L$. Each buyer is characterized by a valuation x so that his valuation for the k^{th} unit is given by $s_k \cdot x$ for any k. In this environment, the pivotal mechanism consists in assigning the k^{th} unit to the buyer with the k^{th} highest valuation (or bid) which is denoted by p_k and making the latter pay $s_k \cdot p_{k+1} - \sum_{i=k+1}^{L} s_i \cdot (p_i - p_{i+1})$. Proposition 3.2 extends in this environment because the total welfare function is still concave with respect to the vector of participation rates. The efficiency of the optimal auction with endogenous participation contrasts with the optimal auction with exogenous participation characterized by Edelman and Schwarz (2010).

3) **Environments in which the pivotal mechanism is not available**

In some applications, the pivotal mechanism may not be available (it may not belong to $\mathcal{M}$) in which case Proposition 3.2 is not applicable. Nevertheless, Proposition 3.2 extends if there is a mechanism $\widetilde{m} \in \mathcal{M}$ that is payoff-equivalent to the pivotal mechanism in the sense that it allocates the good in an ex post efficient way and $u_k(\mu^*(X_S), \widetilde{m}; \sigma^*(\widetilde{m})) = u_k(\mu^*(X_S), X_S)$ for any $k \in \mathcal{K}$ and any bidding profile $\sigma^*(\widetilde{m})$ that is a Bayes-Nash equilibrium. If it is so then

[23] The argument as to why assuming the valuation X_S is private information to the seller makes no difference is somehow related to some insights appearing in the literature on informed principals. Here, we are in a private value setup (i.e., the private information of the seller does not directly affect buyers' preferences). Moreover, from an ex ante perspective the seller can do no better than in the situation in which her private information would be known to the buyers, which thereby is suggestive why the seller has no interest in not disclosing her information. Despite these general observations, we cannot rely on the existing results of the informed principal literature because here participation is endogenous unlike in this literature.

[24] In Appendix, we provide a formal definition of an equilibrium in this environment.

$\mu^*(X_S) \in M(\widetilde{m})$, $\widetilde{m}$ implements the first-best and it necessarily maximizes the seller's revenue among all possible mechanisms. In Section 5 we will apply this result to first-price auctions.

4 Optimal discrimination in the presence of incumbent buyers

We consider the presence of incumbents, $\mathcal{I} \neq \varnothing$, and we allow the seller to internalize any positive share of the rents of the incumbents. Specifically, we assume:

Assumption A 1 $\Lambda_N(m, X_S; \sigma(m)) := \Phi_N(m, X_S; \sigma(m)) + \sum_{i=1}^{I} \beta_i^I \cdot V_{i,N}^I(m; \sigma(m))$ *with* $0 \leq \beta_i^I \leq 1$ *for any* $i \in \mathcal{I}$ *and* $I > 0$ *.*

Our interest lies in understanding whether discrimination between incumbents and entrants is desirable in such a case and how the answer depends on whether incumbents are weaker or stronger than entrants (as measured by their respective CDFs).

Our main insight is that incumbents should be discriminated against entrants no matter whether they are stronger or weaker than entrants and no matter which share of their rent is internalized by the seller. Moreover, we characterize the exact form of optimal discrimination in the vein of Myerson's (1981) analysis.

To formalize that insight, we make the following simplifying assumption:

Assumption A 2 *1) The only information received by incumbents is their valuations. The information received by the entrants and the seller is not correlated with incumbents' valuations. For each* $i \in \mathcal{I}$*, the distribution* $F_i^I(.|z)$ *does not depend on* z *and is denoted by* $F_i^I(.)$*. Furthermore,* $F_i^I(.)$ *is continuously differentiable on its supports* $[\underline{x}_i, \overline{x}_i]$ *with density, denoted by* $f_i^I(.)$*,*[25] *that is strictly positive. 2) The CDFs* $F_i^I(.)$*,* $i \in \mathcal{I}$*, are regular, namely* $x \to \frac{1-F_i^I(x)}{f_i^I(x)}$ *is strictly decreasing on* $[\underline{x}_i, \overline{x}_i]$*.*

Assumption A2 has two parts. The first part rules out the possibility that the information held by each incumbent be somehow correlated with the information held by any other agents. This is somehow needed for our characterization result as otherwise the rents of incumbents could be eliminated via the use of mechanisms à la Crémer-McLean. Observe though that the information held by entrants may still be correlated between each other. The second part is a regularity assumption that allows us to simplify the exposition of the equilibrium (avoiding the need of ironing techniques). It is quite standard in the applied literature.

Some additional notation is required before we can state our main result. We let

- $V_i^I(x, m; \mu, \sigma(m))$ denote the expected utility of the incumbent i with valuation x in the mechanism m when the entry rate vector is μ and when buyers follow the bidding profile $\sigma(m)$. Note that $u_i^I(\mu, m; \sigma(m)) = \int_{\underline{x}_i}^{\overline{x}_i} V_i^I(x, m; \mu, \sigma(m)) dF_i^I(x)$.

[25] From now on, we use lowercase letters for corresponding densities.

- $x^{\mathcal{I}} = (x_1^{\mathcal{I}}, \ldots, x_I^{\mathcal{I}})$ [resp. $x_{-i}^{\mathcal{I}} = (x_1^{\mathcal{I}}, \ldots, x_{i-1}^{\mathcal{I}}, x_{i+1}^{\mathcal{I}}, \ldots, x_I^{\mathcal{I}})$] denote the realization of the vector of valuations of the incumbents [resp. the incumbents other than i].

- s_0 denote the realization of the signal of the seller (which incorporates her information about the groups of the various entrants). Note that we assume that the identity of incumbents is observed by the seller (and incumbents even if symmetric are not required to follow the same strategy even if the mechanism preserves the symmetry among incumbents).

- $x^N = (x_1^N, \ldots, x_{|N|}^N)$ [resp. $s^N = (s_1^N, \ldots, s_{|N|}^N)$] denote the realization of the vector of valuations [resp. signals] of the entrants given that the profile of entrants is N.

- $G_N(.)$ [resp. $G_{-i,N}(.)$] denote the distribution of $(x^{\mathcal{I}}, s^N, s_0)$ [resp. $(x_{-i}^{\mathcal{I}}, s^N, s_0)$] for a given profile of entrants N. From A2, we have $G_N(x^{\mathcal{I}}, s^N) = F_i^I(x_i^{\mathcal{I}}) \cdot G_{-i,N}(x_{-i}^{\mathcal{I}}, s^N, s_0)$.

- $Q_{i,N}^I(x^{\mathcal{I}}, s^N, s_0; \sigma(m))$ [resp. $Q_{j,N}(x^{\mathcal{I}}, s^N, s_0; \sigma(m))$] denote the corresponding probability that the incumbent $i \in \mathcal{I}$ [resp. the entrant $j \in \{1, \ldots, |N|\}$] receives the good.

- $Q_{0,N}(x^{\mathcal{I}}, s^N, s_0; \sigma(m)) = 1 - \sum_{i=1}^{I} Q_{i,N}^I(x^{\mathcal{I}}, s^N, s_0; \sigma(m)) - \sum_{j=1}^{|N|} Q_{j,N}(x^{\mathcal{I}}, s^N, s_0; \sigma(m))$, denote the corresponding probability that the seller keeps the good.

Standard calculations in the vein of Myerson (1981) reveal that the mechanism chosen by the seller in any equilibrium solves the maximization program (see the Appendix for details)

$$\max_{m \in \mathcal{M}} \sum_{N \in \mathbb{N}^K} P(N|\mu^*(m)) \cdot \overline{W}_N(m, X_S; \sigma^*(m)) - \sum_{k=1}^{K} \mu_k^*(m) \cdot C_k - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)) \tag{11}$$

where the term[26]

$$\overline{W}_N(m, X_S; \sigma^*(m)) := \int \Bigg\{ Q_{0,N}(x^{\mathcal{I}}, s^N, s_0; \sigma^*(m)) \cdot X_S + \sum_{j=1}^{N} Q_{j,N}(x^{\mathcal{I}}, s^N, s_0; \sigma^*(m)) \cdot x_j^N \\ + \sum_{i=1}^{I} Q_{i,N}^I(x^{\mathcal{I}}, s^N, s_0; \sigma^*(m)) \cdot \left[x_i^{\mathcal{I}} - (1 - \beta_i^I) \cdot \frac{1 - F_i^I(x_i^{\mathcal{I}})}{f_i^I(x_i^{\mathcal{I}})} \right] \Bigg\} d[G_N(x^{\mathcal{I}}, s^N, s_0)] \tag{12}$$

is referred to as the expected virtual welfare. It is the total expected welfare that would obtain if, while keeping the valuations of entrants unchanged, the valuations of incumbents were replaced by their virtual valuations where the mapping between true and virtual valuations of incumbent i is defined by[27]

[26] This is the unique step where the first part of A2 plays a role. What we need fundamentally here is that incumbents' rents can be expressed exactly as in Myerson (1981). This is the reason why our analysis would extend to an additive form of informational externalities coming from the incumbents as detailed in the Supp. Mat., a quite relevant extension if we have in mind that incumbents are informed on some common value features of the good.

[27] $\widetilde{x}_i^I(\cdot)$ is an increasing function thanks to Assumption A2. We let $(\widetilde{x}_i^I(\cdot))^{-1}$ denote its inverse function.

$$\widetilde{x}_i^I(x_i) := x_i - (1-\beta_i^I)\cdot\frac{1-F_i^I(x_i)}{f_i^I(x_i)} \tag{13}$$

for any $x_i \in [\underline{x}_i, \overline{x}_i]$. In the sequel, when we refer to the virtual valuation, we mean the true valuation for an entrant or the seller and the virtual valuation as just defined for incumbents. We also refer to the virtual pivotal mechanism, denoted by m_{β,X_S}^{VP}, as the auction that assigns the good to the agent with the highest virtual valuation (including the seller) and has the winner (if any) pay the valuation that would make him match the second-highest virtual valuation (including the seller) while losing bidders do not pay anything. The formal definition is as follows.

Definition 3 *The virtual pivotal mechanism is the direct mechanism such that for any $N \in R_+^K$ and any realization of $x^{\mathcal{I}}, s^N, s_0$ in the support of $G_N(.)$:*
1) The assignment rule is characterized by[28]

$$\left.\begin{array}{c} Q_{0,N}(x^{\mathcal{I}}, s^N, s_0) \\ Q_{j^*,N}(x^{\mathcal{I}}, s^N, s_0) \\ Q_{i^*,N}^I(x^{\mathcal{I}}, s^N, s_0) \end{array}\right\} = 1 \quad \text{if} \quad \left\{\begin{array}{c} X_S > \max\{\max_{i\in\mathcal{I}} \widetilde{x}_i^I(x_i^{\mathcal{I}}), \max_{j\in\{1,\dots,|N|\}} x_j^N\} \\ x_{j^*}^N > \max\{\max_{i\in\mathcal{I}} \widetilde{x}_i^I(x_i^{\mathcal{I}}), \max_{j\in\{1,\dots,|N|\}\setminus\{j^*\}} x_j^N, X_S\} \\ \widetilde{x}_{i^*}^I(x_{i^*}^{\mathcal{I}}) > \max\{\max_{i\in\mathcal{I}\setminus\{i^*\}} \widetilde{x}_i^I(x_i^{\mathcal{I}}), \max_{j\in\{1,\dots,|N|\}} x_j^N, X_S\} \end{array}\right. .$$

2) The payment rule is characterized by the fact that losing bidders do not pay anything and that if the winner is an incumbent [resp. an entrant], he pays $(\widetilde{x}_i^I)^{-1}(\max\{P, \widetilde{x}_i^I(\underline{x}_i)\})$ [resp. P] where P denotes the second-highest element in the set $\{\widetilde{x}_i^I(x_i^{\mathcal{I}}), x_j^N, X_S\}_{\substack{i=1,\dots,I \\ j=1,\dots,|N|}}$.

We show in the Appendix that the virtual pivotal mechanism belongs to the larger class of generalized second-price auctions which are such that bidders find it weakly optimal to bid truthfully. We also note that the participation constraints of the incumbents with lowest valuations are binding in the virtual pivotal mechanism, namely

$$V_i^I(\underline{x}_i, m_{\beta,X_S}^{DP}; \mu) = 0 \tag{14}$$

for any $\mu \in R_+^K$ and any $i \in \mathcal{I}$.[29]

Let $\overline{TW}(\mu, m, X_S; \sigma(m)) := \sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot \overline{W}_N(m, X_S; \sigma(m)) - \sum_{k=1}^K \mu_k \cdot C_k$ denote the total expected (ex ante) virtual welfare. We also use the notation $\overline{J}^{MAX}(m, X_S; \sigma(m))$ for the set of the maxima of the function $\mu \to \overline{TW}(\mu, m, X_S; \sigma(m))$ and similarly $\overline{J}(m, X_S; \sigma(m))$ for the analog of $J(m, X_S; \sigma(m))$ in Section 3. As before, for second-price auctions, when truth-

[28] The cases that are not covered correspond to ties. Any specification will work insofar as ties involving one incumbent occur with probability null whereas the way the remaining ties are broken do not matter in terms of bidders' payoffs.

[29] If the incumbent i wins the good then he has to pay $(\widetilde{x}_i^I)^{-1}(\max\{P, \widetilde{x}_i^I(\underline{x}_i)\}) \geq \underline{x}_i$. Consequently, the incumbent i with valuation $\underline{x}_i$ makes no profit. By contrast, entrants with lowest valuations may make some profit.

ful bidding is a (weakly) dominant strategy (in particular for $m = m^{VP}_{\beta,X_S}$) then we drop the dependence with respect to $\sigma(m)$ to alleviate notation.

From (11), the seller's maximization program can be written in a form that is very similar to (6):

$$\max_{m\in\mathcal{M}} \overline{TW}(\mu^*(m), m, X_S; \sigma^*(m)) - \sum_{i=1}^{I}(1-\beta_i^I)\cdot V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)). \tag{15}$$

Consider then the relaxed maximization program:

$$\max_{\substack{\mu\in R_+^K, m\in\mathcal{M}, \sigma(m)\in\Sigma(m)\\ V_i^I(\underline{x}_i, m;\mu,\sigma(m))\geq 0,\text{ for any } i\in\mathcal{I}}} \overline{TW}(\mu, m, X_S; \sigma(m)) - \sum_{i=1}^{I}(1-\beta_i^I)\cdot V_i^I(\underline{x}_i, m; \mu, \sigma(m)). \tag{16}$$

where the unique constraints left are the I participation constraints for the incumbents with lowest valuations.

Next we say that an equilibrium implements the virtual first-best if the mechanism chosen by the seller, the equilibrium participation rate and the equilibrium bidding profile followed by the bidders solve the maximization program (16).

By definition, the virtual pivotal mechanism implements the ex post efficient assignment when efficiency is defined according to virtual valuations. We have thus the analog of (2), namely

$$\overline{W}_N(m, X_S; \sigma(m)) \leq \overline{W}_N(m^{VP}_{\beta,X_S}, X_S) \tag{17}$$

for any profile of entrants N, any mechanism $m \in \mathcal{M}$ and any bidding profile $\sigma(m) \in \Sigma(m)$. As a corollary, we obtain that $\overline{TW}(\mu, m, X_S; \sigma(m)) \leq \overline{TW}(\mu, m^{VP}_{\beta,X_S}, X_S)$ for any $m \in \mathcal{M}$ and any bidding profile $\sigma(m) \in \Sigma(m)$. It follows then that any pair $(\widetilde{\mu}, m^{VP}_{\beta,X_S})$ where $\widetilde{\mu} \in \overline{J}^{MAX}(m^{VP}_{\beta,X_S}, X_S)$ joint with truthful bidding is a solution of the maximization program (16).

From the point of view of the potential entrants, the virtual pivotal mechanism in our setup with incumbents is equivalent to the pivotal mechanism in a setup without incumbents but in which the valuation of the seller would be stochastically determined according to $\widehat{X} := \max\{\max_{i\in\mathcal{I}}\{\widetilde{x}_i^{\mathcal{I}}(x_i^{\mathcal{I}})\}, X_S\}$ after the entry stage.[30] As shown by Lamy (2013) in Levin and Smith's (1994) model (and so with $K = 1$), the fact that the seller's valuation is determined after instead of before entry does not affect the fundamental property of the pivotal mechanism, namely that the payoff of a buyer coincides with his contribution to the total welfare. As a

[30] By pivotal mechanism, we mean here a second-price auction where the reserve price would be set ex post at the seller's valuation. This raises some implementation issues since the seller do not have the proper incentives to report truthfully her valuation. Lamy (2013) shows that when the seller's valuation is stochastic but all the private information of the seller comes from ex ante signals, the outcome of the pivotal mechanism can be implemented with a simple and commonly used auction format: the English auction with cancelation rights, no reserve price and the possibility to submit jump bids.

result, in our model with a continuum of potential entrants and with incumbents, this property translates thus into

$$\overline{J}(m^{VP}_{\beta,X_S}, r) = M(m^{VP}_{\beta,X_S}). \tag{18}$$

Since the total welfare function for the virtual pivotal mechanism, namely $\mu \to \overline{TW}(\mu, m^{VP}_{\beta,X_S}, X_S)$, can be viewed as a convex combination of welfare functions of the type considered in Lemma 3.1., it is necessarily concave and we can conclude that the virtual pivotal mechanism must induce participation rates that maximize the virtual welfare. Formally, we prove in the Appendix the following lemma.

Lemma 4.1 *For any $X_S \in R_+$ and $\beta = \{\beta^I_i\}_{i\in\mathcal{I}} \in [0,1]^I$, $\mu \to \overline{TW}(\mu, m^{VP}_{\beta,X_S}, X_S)$ is concave on R^K_+. As a corollary, we have $\overline{J}^{MAX}(m^{VP}_{\beta,X_S}, r) = M(m^{VP}_{\beta,X_S})$.*

As a corollary of Lemma 4.1, we obtain that the virtual pivotal mechanism solves the maximization program (16), which in turn implies our main characterization result:

Proposition 4.2 *Assume A1, A2 and $m^{VP}_{\beta,X_S} \in \mathcal{M}$. For any $\mu \in M(m^{VP}_{\beta,X_S})$, there exists an equilibrium where the seller proposes the virtual pivotal mechanism while the equilibrium participation rate is μ. Conversely, any equilibrium is equivalent to an equilibrium where the seller proposes the virtual pivotal mechanism. Any equilibrium implements the virtual first-best.*[31]

From the viewpoint of discrimination, Proposition 4.2 provides an exact characterization of the optimal shape of discrimination. Only incumbents should be discriminated. Moreover, they should always be discriminated against entrants irrespective of whether incumbents are stronger or weaker than entrants and irrespective of which share of incumbents' profits is internalized by the seller, given that virtual valuations are always below the valuations (namely $\widetilde{x}^I_i(x) \leq x$). Observe that in the special case in which the seller fully internalizes the rents of the incumbents ($\beta^I_i = 1$) there should be no discrimination against incumbents.[32] It should also be stressed that as in the case without incumbents there should be no discrimination (positive or negative) among entrants.

From the perspective of the literature on auctions, Proposition 4.2 can be viewed as providing a general setup in which the optimal design problem can accommodate both exogenous entry (and have thus Myerson (1981) as a special case) and endogenous entry (and have thus our previous non-discrimination result as a special case) while allowing for the mixed case (which

[31] It is straightforward from the aforementioned equivalence (from entrants' perspective) between incumbents and stochastic seller's valuations that the optimality property of the English auction considered by Lamy (2013) extends to our model with several groups of entrants $K > 1$.

[32] By contrast, in a setup with exogenous participation and when the seller fully internalizes the rents of the so-called domestic firms, McAfee and McMillan (1989) obtain the following result: The optimal mechanism involves favoritism towards domestic firms independently of the respective strengths of the different firms.

the previous literature did not consider and that we think is quite relevant for the study of discrimination).

Comments:

1) **The seller is privately informed of her valuation**

As in the setup without incumbents, the privacy of the principal's information is irrelevant here because the virtual pivotal mechanism, which is optimal once the seller's valuation is publicly known, implements the virtual first-best and is thus optimal from an ex-ante perspective.

2) **When incumbents receive information about entrants**

We have assumed that incumbents know only their valuation and so implicitly that they do not receive any information about the realization of the set of entrants (see Assumption 2). In some applications, it may be that the set of participants in an auction is common knowledge among bidders. Proposition 4.2 extends to this case. More generally, Proposition 4.2 extends if we assume that incumbents receive some extra information about the set of entrants (and also possibly about their valuations), as long as this information is common knowledge among them (so that Myerson's (1981) techniques can be applied).

3) **Dealing with non increasing virtual valuations**

If one of the virtual valuation functions $\widetilde{x}_i^{\mathcal{I}}(x)$ is not non-decreasing, then it is impossible to implement the virtual first-best insofar as it would violate the well-known monotonicity constraints, namely that incumbents with higher valuations should receive the good more often. The resolution of the maximization program (15) requires some so-called "ironing" techniques. We can apply Myerson's ironing technique to each of the functions $x - (1-\beta_i^I) \cdot \frac{1-F_i^I(x)}{f_i^I(x)}$, $i \in \mathcal{I}$, and construct a generalized second-price auction with "ironed virtual valuations". From Myerson (1981), the solution maximizes the seller's objective function for any realization of the set of effective participants. To conclude that this candidate solution is optimal, it is enough to realize that it still gives the right incentive in terms of participation rates (which follows from Lemma 6.2).

4) **Alternative objectives for the seller**

Alternative objectives could be considered for the seller. For example, when incumbent i with valuation x wins, the seller could enjoy an externality of the form $\alpha_i^I \cdot x + \gamma_i^I$ with the idea that the valuation of the incumbent may affect how many employees will be needed to do the job (in this respect, it may well be that $\alpha_i^I < 0$ if a higher valuation of the incumbent is due to a less labor intensive technology - in this case, one would need to assume that $\alpha_i^I > -1$ so as to avoid the need to use ironing techniques). Such a specification would lead to replace A1 by

$$\Lambda_N(m, X_S; \sigma(m)) := \Phi_N(m, X_S; \sigma(m)) + \int \sum_{i=1}^{I} Q_{i,N}^I(x^{\mathcal{I}}, s^N; \sigma(m)) \cdot [\alpha_i^I \cdot x_i^{\mathcal{I}} + \gamma_i^I] d[G_N(x^{\mathcal{I}}, s^N)]. \tag{19}$$

In this case, the definition of the virtual pivotal mechanism should be amended, and the virtual valuation of incumbent $i \in \mathcal{I}$ with valuation x should be defined as

$$\widetilde{x}_i^{\mathcal{I}}(x) := [1 + \alpha_i^I] \cdot x_i^{\mathcal{I}} + \gamma_i^I - \frac{1 - F_i^I(x)}{f_i^I(x)}.$$

Proposition 4.2 extends straightforwardly to this environment.

Up to now we have assumed that only the incumbents' rents are internalized by the seller. In some cases, it may make sense to assume that some positive share of the payoffs of entrants is internalized by the seller so as to cover applications in which some entrants might be local firms. This extension would lead to consider the following objective function for the seller

$$\Lambda_N(m, X_S; \sigma(m)) := \Phi_N(m, X_S; \sigma(m)) + \sum_{i=1}^{I} \beta_i^I \cdot V_{i,N}^I(m; \sigma(m)) + \sum_{k=1}^{K} \beta_k \cdot n_k \cdot V_{i,N}^I(m; \sigma(m)) \quad (20)$$

where the coefficient β_k, $k \in \mathcal{K}$, can be interpreted as reflecting the share of the payoff of the various groups of entrants that is internalized by the seller. The maximization program for the seller is now given by:

$$\max_{m \in \mathcal{M}} \overline{TW}(\mu^*(m), m, X_S; \sigma^*(m)) - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)) + \sum_{k=1}^{K} \beta_k \cdot \mu_k^*(m) \cdot C_k. \quad (21)$$

We wish to point out that we have no characterization of the optimal form of discrimination in this case, which is thus left for future research. The issue is that there will typically[33] be a conflict between ex post and ex ante efficiency: The mechanisms that assign the good efficiently according to the virtual welfare never induce an entry rate that maximizes the seller's objective. E.g., the virtual pivotal mechanism is no longer optimal because potential entrants do not internalize the new term $\sum_{k=1}^{K} \beta_k \cdot \mu_k^*(m) \cdot C_k$. When such a conflict arises, the characterization of the optimal mechanism is not covered by our analysis.

5) **What if discrimination takes the form of a linear distortion of bids?**

The optimal discrimination as arising from the shape of the virtual valuations need not be implementable using standard auctions, say second or first-price auctions, in which the submitted bids would be linearly transformed before the rule of the auction is applied. If one applies such additional constraints on the shape of discrimination, one may be interested in the shape of the optimal slopes that should be applied to the distortion of bids. In a working paper version of this

[33] A notable exception is the case where in each group k of potential entrants with $\beta_k > 0$, the bidders with the lowest type have strictly positive rents in the virtual pivotal mechanism while the seller observes that those bidders come from group k. In such a case, the seller has some flexibility insofar as she can induce the same assignment while giving different incentives for the potential entrants at the entry stage. The maximization program (21) may thus be solved with the virtual pivotal mechanism to which some extra fees have been added to the entrants.

paper, we have considered such forms of discrimination and shown in a number of cases and in line with Proposition 4.2 above that incumbents should be discriminated against entrants even in this restricted class of discrimination mechanisms.

5 When discrimination is prohibited

In this Section, we first establish that in the absence of incumbents, the outcome of the pivotal mechanism can also be implemented using a first-price auction in which the reserve price is set at the seller's valuation, which we believe should have some applied appeal given that, in most procurement auctions, the auction format is of that form. Then we review what happens when the reserve price is the sole auction design instrument in the context of second-price auctions (or equivalently ascending auctions).

Throughout this section, we also make the following additional assumption on the information structure:

Assumption A 3 *Buyers do not receive any information in addition to their private valuation. The distributions $F_i^I(.|z)$, $i \in \mathcal{I}$, and $F_k(.|z)$, $k \in \mathcal{K}$, do not depend on z and are continuously differentiable on their (common) support which is denoted by $[\underline{x}, \overline{x}]$.*

In particular, valuations are now assumed to be drawn independently. In the rest of the Section, we drop the dependence in z in the notation.

5.1 First-price auctions

Throughout this subsection, we consider that $\mathcal{M}$ contains the set of first-price auctions with possibly a reserve price which is denoted by $\mathcal{M}_{FP}^r$. We also assume that there are no incumbents, $\mathcal{I} = \varnothing$, and that $X_S \geq \underline{x}$.

From the so-called "environmental equivalence" property that arises with Poisson distributions, it must be the case that each potential entrant expects that the probability that the profile of entrants is N is given by $P(N|\mu)$, and this does not depend on which group of entrants this potential entrant belongs to. This implies that if all buyers are using the same strictly increasing bidding function $B : R_+ \to R_+$ with $B(r) = r$, then each entrant independently of the group he comes from expects to be facing the same distribution of bids of other participants. More precisely, the best-response of an entrant with valuation $x \geq r$ is to submit an active bid and he expects then to get[34]

$$u^{FP}(x;r) = \max_{x' \in [r,\overline{x}]} \left\{ (x - B(x')) \cdot \prod_{k=1}^{K} e^{-\mu_k(1-F_k(x'))} \right\} \tag{22}$$

[34]From A3, the CDFs $F_k(.)$ have no atoms so that ties would occur with a null probability and we can thus abstract from the possibility of ties.

no matter which group he comes from. Because this maximization program is independent of the group k, one can then ensure that the bidding strategy of participants is independent of the group (and depends solely on the valuation). More precisely, we obtain from standard arguments in auction theory (see Krishna 2002), that for the first-price auction with reserve price r and for any entry profile μ, there is a symmetric equilibrium at the bidding stage in which every bidder with valuation x bids according to $B(x) = \int_{\underline{x}}^{x} \max\{y, r\} \frac{d[\prod_{k=1}^{K} e^{-\mu_k(1-F_k(y))}]}{\prod_{k=1}^{K} e^{-\mu_k(1-F_k(x))}}$ if $x \geq r$ and $B(x) < r$ otherwise.

In equilibrium,[35] buyers are bidding the expectation of the highest valuation among his opponents (interpreting the reserve price as the valuation of the seller) conditional on having the highest valuation. If $r \geq \underline{x}$, this in turn gives rise to exactly the same expected payoff for bidders as the one arising in the second-price auction with reserve price r (this is an application of the celebrated revenue equivalence result given that in both mechanisms the allocation is the same and the expected payoff of the buyers with the lowest type is null and thus the same). As a result, the first-price auction with reserve price $r = X_S$ is payoff-equivalent to the pivotal mechanism and we can thus apply Comment 4) from Section 3 to conclude that Proposition 3.2 carries over, in particular that it is optimal for the seller to use a first-price auction with reserve price $r = X_S$.[36]

5.2 Optimal reserve price policy

In this Subsection, we consider second-price auctions in which the only instrument available to the seller is the reserve price. Starting from the pivotal mechanism, Bulow and Klemperer (1996) establish in the symmetric IPV model in which the seller has no intrinsic value for the object that the seller would be better off having a standard second-price auction with no reserve price but an extra bidder rather than a fine tuned reserve price and no extra bidder. Yet, it should be noted that Bulow and Klemperer (1996) do not formalize the channel through which this extra bidder would join the auction. By contrast, our model precisely models the impact of the auction format on the intensity of entrants' participation. Our interest in the rest of this section lies in understanding which reserve price the seller would optimally choose taking all effects into account. From another perspective, the analysis to be presented now can also be viewed as introducing incumbents in Levin and Smith's (1994) model with the aim of shedding light on what distortions on optimal reserve prices the presence of incumbents would induce.

[35] We assume now implicitly that the symmetric equilibrium exhibited above is always played. To the best of our knowledge, equilibrium uniqueness results have not been established in first-price auctions with a stochastic number of bidders. We conjecture that this is the unique (group-symmetric) equilibrium.

[36] This seems to be inconsistent with Athey et al.'s (2011) structural estimates which have a special focus on the non-equivalence between first- and second-price auctions. The difference comes from the fact that we assume implicitly in A3 that buyers do not know the set of participants at the bidding stage. By contrast, Athey et al. (2011) -as most of the empirical literature- assumes that the set of participants is common knowledge among bidders so that the bidding stage is the same as in models with exogenous entry where stronger buyers are bidding less aggressively (Maskin and Riley 2000). To the best of our knowledge, there is no empirical justification for one assumption or the other.

From our general description of the seller's objective, one purpose that the reserve price should serve is to reduce the rent of the incumbents. We note that the effect of the reserve price on the incumbents' rents is a priori ambiguous: On the one hand, for any given realization of the number of entrants, raising the reserve price reduces the incumbents' rents, which is beneficial to the seller. On the other hand, a higher reserve reduces the incentives to enter the auction for entrants, which benefits indirectly the incumbents because they face less tight competition from new entrants. Those two channels are reflected by the two popular guidelines when one suspects bid-rigging in a procurement auction: Firstly, "imposing an aggressive but credible reserve price [...] as it reduces the illegal gains". Secondly, "reducing barriers to entry and increasing bidders' participation" (OECD 2008, Policy Brief) which pleads for low reserves in order to make the auction more attractive to new entrants.[37]

From Bulow and Klemperer (1996), we might have conjectured that the optimal reserve is (or at least could be) below the seller's valuation, namely that the entry channel might dominate the informational rents channel with respect to the optimal reserve price policy. This turns out to be incorrect, as we now show.

Throughout this subsection, we assume

Assumption A 4 $\Lambda_N(m, X_S; \sigma(m)) := \Phi_N(m, X_S; \sigma(m)) + \sum_{i=1}^{I} \beta_i^I \cdot V_{i,N}^I(m; \sigma(m))$, *with* $\beta_i^I < 1$ *for any* $i \in \mathcal{I}$, $K = 1$ *and* $I > 0$.

When $\mathcal{M} = \mathcal{M}_{SP}^r$ the seller's maximization program in equilibrium is given by

$$\max_{r \in \mathbb{R}_+} TW(\mu^*(r), r, X_S) - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot u_i^I(\mu^*(r), r). \tag{23}$$

From the arguments detailed in Section 3, we have that $X_S \in \operatorname{Arg\,max}_{r \in R_+} TW(\mu^*(r), r, X_S)$. In order to show that the seller should propose a reserve price strictly above her valuation, it is thus sufficient to check that the incumbents' rents are decreasing in r. The monotonicity of the function $r \rightarrow u_i^I(\mu^*(r), r)$ is a priori ambiguous. On the one hand, a higher reserve makes the incumbent i worse-off ceteris paribus ($\frac{\partial u_i^I(\mu,r)}{\partial r} \leq 0$). On the other hand, however, a higher reserve price discourages the potential entrants from entering the auction ($\frac{d\mu^*(r)}{dr} \leq 0$), which is beneficial to the incumbent i ($\frac{\partial u_i^I(\mu,r)}{\partial \mu} \leq 0$). We show under mild additional (technical) assumptions that the first effect always dominates the second so that the optimal reserve price should always be strictly above X_S. We then ask how far above X_S it can be. As expected, we establish that it would be suboptimal to go beyond the optimal reserve in the case where the seller faces only the set of incumbents.

Assumption A 5 *The seller's reservation value* $X_S \in (\underline{x}, \overline{x})$ *and* $r \rightarrow u_S(0, r, X_S)$ *is strictly quasi-concave on* $[\underline{x}, \overline{x}]$ *with the mode denoted by* $r_I^M(X_S)$.

[37] See also Marshall and Meurer (2004).

Combined with $\beta_i^I < 1$, we show in the Appendix that A5 guarantees that $r_I^M(X_S) > X_S$. We also make the following assumption:

Assumption A 6 *For any $i \in \mathcal{I}$, $x \to \frac{(1-F_i^I(x))}{F_i^I(x)(1-F(x))}$ is strictly decreasing on $(\underline{x}, \overline{x})$.*

Note that A6 holds if $F_i^I = F$ for each $i \in \mathcal{I}$, i.e. in the case in which all buyers are symmetric in terms of valuation distribution. A6 a fortiori holds if F dominates F_i^I in terms of the hazard rate, i.e. if $\frac{f(x)}{1-F(x)} \leq \frac{f_i^I(x)}{1-F_i^I(x)}$ for any $x \in (\underline{x}, \overline{x})$ (which would hold if we have in mind that potential entrants are more efficient than incumbents). One may be interested for applications to cover situations in which the entrants may be less efficient than incumbents. We show in the Supp. Mat. that A6 still holds when $F_i^I(x) = [F(x)]^{s_i}$ for $s_i \geq 1$, for each $i \in \mathcal{I}$. In this case we can interpret i as being a cartel including s_i buyers where each individual buyer has a valuation drawn from the same CDF $F(\cdot)$ (see Graham et al. (1987) for such a modeling of cartels).

We can now state:

Proposition 5.1 *Assume A3, A4, A5, A6 and $\mathcal{M} = \mathcal{M}_{SP}^r$. Any equilibrium reserve price, denoted by r^*, is strictly above the seller's valuation, i.e. $r^* > X_S$, and below the optimal reserve price against the set of incumbents, i.e. $r^* \leq r_I^M(X_S)$.*

An important practical application of our analysis with incumbents is the case where the auctioneer suspects the presence of a cartel. It is well recognized that bid-rigging is a pervasive first-order issue in public procurements.[38] Concerning the reserve price policy, it is well-known that the buyer's optimal reserve is increasing in the size of the ring under exogenous entry.[39] In other words, in order to spot a cartel, the seller's best response consists somehow in raising the reserve price. Furthermore, this response lowers the rents of the cartel to such an extent that it can deter the ex ante incentives to collude as illustrated through examples in McAfee and McMillan (1992). To the best of our knowledge, our analysis is the first theoretical contribution that considers the use of reserve prices to fight a colluding ring in a context with endogenous participation. The take away insight that we get is that raising the reserve price above the seller's valuation is beneficial to the seller even after taking into account the negative effect it has on attracting new participants.

5.3 Practical anti-collusion policies

Sometimes, even if explicit discrimination is not allowed, it may take a disguised form. For example, it is common to observe special rules in the case where too few bidders show up at

[38] Porter and Zona (1993) report that more than half of the criminal cases filed between 1982 and 1988 by the Antitrust Division of the Department of Justice involved bid-rigging or price fixing in auction markets while Marshall and Meurer (2004) reports that 90% of the prison sentences for violation of antitrust law in US arise in bid-rigging cases.

[39] See Graham and Marshall (1987) for the seminal contribution with symmetric bidders and Krishna (2002) for a more general treatment.

the procurement auction. In China, the law on procurements states that "Where there are less than three bidders, the bid inviter shall, in accordance with this Law, invites bids anew".[40] In the State of Colorado, the law on procurements for transportation projects states that "In the event that there are less than three bidders on a highway project, no award shall be made if such award is more than ten percent over the estimate of the department of transportation on the project".[41] To summarize these practices and phrase them in the auction setup, one could say that when there are too few participants, the seller may be allowed not to sell the good under some circumstances.

From a theoretical perspective, we note that if the seller could use different reserve prices depending on whether the highest bidder is an entrant or an incumbent, then she would set the reserve price at Myerson's optimal level $r_I^M(X_S)$ in the case where she faces an incumbent. The argument is that this instrument does not alter the incentives of the potential entrants so that the reserve price should be set as under exogenous entry. Regarding entrants, it is a priori less clear how the reserve price should be set. Assuming there is only one group of entrants, it can be shown that in the presence of incumbents, the optimal reserve price for entrants should be set below the seller's valuation so as to better align entrants' interests with the virtual welfare as described in Section 5 and to better reduce the rent of the incumbents (as compared with the non distortionary reserve price policy).

Yet, reserve price policies that depend on the identity of the highest bidder are rarely observed (if at all) in practice. More commonly observed is a reserve price policy that depends simply on the total number of bidders showing up, which corresponds somehow to the aforementioned anti-collusive practices. Intuition suggests then to impose a higher reserve price when there are fewer bidders insofar as when there are fewer bidders it is more likely that the seller faces incumbents. In other words, the number of bidders can be used by the seller as an imperfect proxy regarding whether the bidder submitting the highest bid is an incumbent or an entrant so as to better adjust her reserve price policy.

5.4 Split-awards

Alternative non-discriminatory instruments sometimes used in practice that allows to discriminate indirectly between bidders are split awards: instead of assigning a contract entirely to a single firm, it consists of splitting the contract among the highest bidders. For example, the bidders submitting the two highest bids split the award in some proportion. One intuitive appeal of such a mechanism in contexts with a pre-auction investment stage is that guaranteing a share of the project to a bidder with non-maximal bid may give weaker bidders a stronger incentive to invest thereby inducing a more balanced competition and higher revenues (Anton-Yao 1989 and Gong, Li and McAfee 2011).

[40] http://english.cguardian.com/services/laws/2011-10-20/22_5.html
[41] http://www.state.co.us/gov_dir/leg_dir/olls/sl1999/sl_181.htm

In the context of our model, assuming that the good is divisible and that valuations are linear in quantity, we obtain as a by-product of Proposition 4.2 that split awards are necessary suboptimal once the set of possible mechanisms include the virtual pivotal mechanism.[42] Indeed in an optimal mechanism, the good should be put entirely into the hands of the buyer with the highest virtual valuation (which is generically unique) as in Myerson (1981).

Assuming that the designer cannot use the virtual pivotal mechanism, it would be of interest to analyze whether the use of split awards could increase revenues. Intuitively, split awards seem to be a way to reduce the incumbents' rents and seems thus desirable from a revenue viewpoint. Yet, split awards also have the drawback of reducing ex post welfare, which is not desirable. Trading-off these two effects would require further work.

6 Conclusion

Our main insight is that considering that participation is endogenously determined by the choice of the auction format deeply affects how one should think of discrimination in procurement auctions. There should be no discrimination among potential entrants even if entrants can come from different groups with different characteristics of valuation distributions and entry costs. When there are incumbents, those should be discriminated against entrants no matter whether they are ex ante stronger or weaker than entrants and no matter which share of their surplus is internalized by the designer.

The importance of taking into account the effect of auction formats on participation has been stressed by the recent and growing empirical econometrics literature on procurements that strongly suggested that the previous debates about bid preferences programs based on models with exogenous participation were missing a key ingredient. To name just a few, Marion (2007), Krasnokutskaya and Seim (2011) and Athey et al. (2013) all show that taking into account participation elasticities of the various groups of potential entrants significantly alters the assessment of discriminatory policies. In a model that unifies Levin and Smith (1994) and Samuelson (1985), Gentry and Li (2013) consider the problem of (nonparametric) identification of the joint distribution of bidders ex ante private signals and their valuations. Somehow our framework and results provide a theoretical benchmark to study the applications covered in this literature (once there is enough competition among potential entrants).

It should also be mentioned that the first papers in the empirical literature about endogenous participation (Li 2005, Li and Zheng 2009 and Marmer et al. 2011) were limited to symmetric IPV environments where a finite number of potential buyers decide whether to incur a sunk cost in order to enter the procurement and then be able to bid. In a way, these models were somehow not suited to study situations in which there are ex ante asymmetries between bidders (such as

[42] In other environments with pre-participation investments, Bag (1997) and Celik and and Yilankaya (2009) do not restrict the set of possible mechanisms and obtain also a no split awards result.

in timber auction). The aim of these early papers was instead to delineate what the appropriate model of entry is, namely whether bidders know their valuation before or after entry or whether entry decisions are simultaneous or sequential. It should be mentioned that for a highway mowing auction data from Texas, Li and Zheng (2009) found overwhelming evidence in favor of Levin and Smith's (1994) model against Samuelson (1985) and also against models where entry is coordinated as in Engelbrecht-Wiggans (1993). What our analysis establishes is that the details of the information structure does not play a major role in the analysis of discrimination, since our model somehow unifies the informational assumptions of Levin and Smith (1994) and McAfee (1993). Yet, the assumption that participation decisions are made simultaneously rather than sequentially is of great importance for our non-discriminatory result, and more empirical work should be devoted to whether participation should be modeled sequentially or simultaneously.[43]

When considering the issue of discrimination (against incumbents) from a practical viewpoint, there are a number of alternative instruments that can be used and that have not been discussed in this paper. Regarding explicit discrimination, these include the use of linear discriminatory distortions of bids in first-price auctions (Athey et al. 2013), the right of first refusal (Burguet and Perry 2006), reserving a share of the good/contract to a bidder belonging to a specific group such as an entrant as some governments did in the context of UMTS licenses (Jehiel and Moldovanu 2003), set asides (Athey et al. 2013). While set asides policies (which constitute the most widespread form of explicit discrimination in procurements) are typically suboptimal in models with exogenous entry, it is not clear how well they perform in models with endogenous entry.[44] Analyzing the effects of these instruments in a context with endogenous participation is left for future research.

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[43] We note that sequential entry may be a way to save the participation costs, and as such may be desirable from the viewpoint of the designer (Roberts and Sweeting 2013). We have not explored this theme here.

[44] In the European Union, set asides are typically forbidden. There is however an exception with cultural goods. E.g. the French state sometimes objects to exports which is somehow a set-asides with regards to foreign bidders. In art auctions, discrimination with regards to domestic versus foreign bidders arise also through export/import taxes. See Moulin (1997) for institutional details about protectionism for cultural goods.

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Appendix

Proof of Lemma 3.1

As a preliminary, we give some useful formulas for second-price auctions, i.e. for $m = r \in \mathcal{M}^r_{SP}$. Using standard results from auction theory, conditional on z, a buyer with valuation $u \geq r$ who participates in the seller's auction against the profile $N \in \mathbb{N}^K$ when the reserve price is r will receive the expected payoff of $\int_r^u \prod_{k=1}^K [F_k(x|z)]^{n_k} \cdot \prod_{i=1}^I F_i^I(x|z)dx$.[45] The corresponding (interim) payoff of a group k buyer from entering such an auction, i.e. before knowing what his valuation will be, is given (after simple calculations) by

[45] This is the integral of the (interim) probability that a bidder with valuation x wins the object as x varies from r to u conditional on z.

$$V_{k,N_{+k}}(r) = \int_r^{\infty} (F^{(1:N\cup I)}(x) - F^{(1:N_{+k}\cup I)}(x))dx. \tag{24}$$

Similarly we have $V_{i,N}^{I}(r) = \int_r^{\infty} (F^{(1:N\cup I_{-i})}(x) - F^{(1:N\cup I)}(x))dx$. The corresponding expected welfare is given by

$$W_N(r, X_S) = X_S \cdot F^{(1:N\cup I)}(r) + \int_r^{\infty} xd[F^{(1:N\cup I)}(x)] = X_S + \int_{X_S}^{\infty} (1 - F^{(1:N\cup I)}(x))dx. \tag{25}$$

where the last equality comes after an integration per part. From (25), we obtain

$$W_{N_{+k}}(r, X_S) - W_N(r, X_S) = -(r - X_S) \cdot \underbrace{\left[F^{1:N_{+k}\cup I}(r) - F^{1:N\cup I}(r)\right]}_{\leq 0} + V_{k,N_{+k}}(r), \tag{26}$$

In words, we have thus that when the reserve price is below [resp. above] the seller's valuation, then the social contribution of a new entrant is smaller [resp. larger] than his payoff. Eq. (8) corresponding to the special case where $r = X_S$.

We note then that $\frac{\partial P(N|\mu)}{\partial \mu_k} = -P(N|\mu)$ if $n_k = 0$ and $\frac{\partial P(N|\mu)}{\partial \mu_k} = -P(N|\mu) + P(N_{-k}|\mu)$ if $n_k \geq 1$. We first establish (10) formally. For any r, X_S and $k = 1, \ldots, K$, we have $\frac{\partial TW(\mu,r,X_S)}{\partial \mu_k} = \sum_{N\in\mathbb{N}^K} P(N|\mu) \cdot \left[W_{N_{+k}}(r, X_S) - W_N(r, X_S)\right] - C_k$. If $r = X_S$, from (8), we obtain also that $\frac{\partial TW(\mu,r,r)}{\partial \mu_k} = \sum_{N\in\mathbb{N}^K} P(N|\mu)V_{k,N_{+k}}(r) - C_k = u_k(\mu, r) - C_k$. As a corollary, we obtain then that having $\frac{\partial TW(\mu,r,r)}{\partial \mu_k} \underset{(resp.\ \leq)}{=} 0$ if $\mu_k \underset{(resp.\ =)}{>} 0$ for each $k = 1, \ldots, K$ is equivalent to $\mu \in M(r)$. In a nutshell, we have $J(r, r) = M(r)$. We are thus done with (10).

Carrying on our calculation and using (24), we obtain that

$$\begin{aligned}
\frac{\partial^2 TW(\mu, r, r)}{\partial \mu_k \partial \mu_l} &= \sum_{N\in\mathbb{N}^K} P(N|\mu) \cdot \left[V_{k,[N_{+k}]_{+l}}(r) - V_{k,N_{+k}}(r)\right] \\
&= \sum_{N\in\mathbb{N}^K} P(N|\mu) \cdot \Big[\int_r^{\infty} (F^{(1:N_{+l}\cup I)}(x) - F^{(1:[N_{+k}]_{+l}\cup I)}(x) + F^{(1:N_{+k}\cup I)}(x) - F^{(1:N\cup I)}(x))dx\Big] \leq 0 \\
&= -\sum_{N\in\mathbb{N}^K} P(N|\mu) \cdot E_Z\Big[\int_r^{\infty} \prod_{k=1}^{K} [F_k(x|Z)]^{n_k} \cdot \prod_{i=1}^{I} F_i^I(x|Z) \cdot (1 - F_l(x|Z))(1 - F_k(x|Z))dx\Big] \leq 0
\end{aligned} \tag{27}$$

for any $k, l \in \{1, \ldots, K\}$. Let $\mathbf{H}_r^{\mu}$ denote the Hessian matrix of the function $\mu \to TW(\mu, r, r)$ at the vector of participation rates μ. In order to show that $\mu \to TW(\mu, r, r)$ is concave on R_+^K, it is sufficient to show that $\mathbf{H}_r^{\mu}$ is negative semi-definite for any μ in R_+^K.

Let $Q(x, z) := [(1 - F_1(x|z)), \ldots, (1 - F_K(x|z))]$. For $X \in R^K$, let $X^{\top}$ its transpose. More generally, the notation $^\top$ is used for any matrix. We then have to show that $X^\top \cdot \mathbf{H}_r^\mu \cdot X \leq 0$ for any $X \in R^K$ and any $\mu \in R_+^K$. From (27), we have:

$$X^\top \mathbf{H}_r^\mu X = -\sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot \left[\int_r^\infty \prod_{k=1}^K [F_k(x|Z)]^{n_k} \cdot \prod_{i=1}^I F_i^I(x|Z) \cdot \underbrace{X^\top \cdot Q(x,Z)^\top Q(x,Z) \cdot X}_{=[Q(x,Z)X]^\top \cdot [Q(x,Z)X] \geq 0} dx \right] \leq 0. \tag{28}$$

In other words, (28) says that $\mathbf{H}_r^\mu$ can be viewed as a weighted sum (including integrals) with positive weights of the negative semi-definite matrices $-[Q(x,z)]^\top Q(x,z)$ and is thus also negative semi-definite.

Proof of Proposition 3.2

For a given $\widetilde{\mu} \in J^{MAX}(X_S, X_S)$, we let $m^* = X_S$ and $\mu^*(X_S) = \widetilde{\mu}$ and we pick any value in $M(m)$ for $m \in \mathcal{M} \setminus \{X_S\}$ (which is possible since $M(m) \neq \emptyset$). For any $m \in \mathcal{M}$, bidders' strategies $\sigma^*(m)$ are defined in order to guarantee the equilibrium conditions. In particular, $\sigma^*(m)$ corresponds to truthful bidding in second-price auctions. This is an equilibrium since $TW(\mu^*(m), m, X_S; \sigma^*(m)) \leq TW(\mu^*(m), X_S, X_S) \leq TW(\widetilde{\mu}, X_S, X_S)$ for any $m \in \mathcal{M}$, or equivalently $u_S(\mu^*(m), m, X_S; \sigma^*(m)) \leq u_S(\mu^*(m), X_S, X_S) \leq u_S(\widetilde{\mu}, X_S, X_S)$ for any $m \in \mathcal{M}$.

Consider a given equilibrium (m^*, μ^*, σ^*). From (3), we have $TW(\mu^*(m^*), m^*, X_S; \sigma^*(m^*)) \geq TW(\mu^*(X_S), X_S, X_S) = \max_{\mu \in R_+^K} TW(\mu, X_S, X_S)$ where the latter equality results from Lemma 3.1. Since $TW(\mu, m, X_S; \sigma^*(m)) \leq TW(\mu, X_S, X_S)$ for any μ and m, this further implies that $TW(\mu^*(m^*), m^*, X_S; \sigma^*(m^*)) = TW(\mu^*(m^*), X_S, X_S)$ and that $\mu^*(m) \in J^{MAX}(X_S, X_S)$. From (2), the equality $TW(\mu^*(m^*), m^*, X_S; \sigma^*(m^*)) = TW(\mu^*(m^*), X_S, X_S)$ implies that $W_N(m^*, X_S; \sigma^*(m^*)) = W_N(X_S, X_S)$ for any N such that $P(N|\mu^*(m^*)) > 0$ or equivalently that the good is assigned with probability one to the agents with the highest valuation for any N such that $P(N|\mu^*(m^*)) > 0$. Any assignment where the good is given to the agent with the highest valuation can be implemented with the pivotal mechanism provided that the breaking rule is well-specified (remember that we do not exclude that tie occur with a positive probability since valuation distributions may have some atoms). On the whole we have shown that the equilibrium (m^*, μ^*, σ^*) is equivalent to any pivotal equilibrium with the equilibrium participation rate $\mu^*(m^*)$.

Proof of Proposition 3.3

Once the seller is informed about her type, we have to extend our equilibrium concept in Definition 1. Now m^* should be replaced by a probability distribution over the set of possible mechanisms $\mathcal{M}$ for any possible realization X_S, denoted by $m^*(X_S)$ and (3) should be replaced by

$$\mathrm{Supp}(m^*(X_S)) \subset \mathrm{Arg}\max_{m\in\mathcal{M}} \ u_S(\mu_1^*(m), \ldots, \mu_K^*(m), m, X_S; \sigma^*(m)). \tag{29}$$

for any possible realization X_S. Concerning the buyers, they should be equipped with a belief for the types of the seller that announce a given mechanism. This corresponds equivalently to a distribution, denoted by H_m, over sellers' types. Once a mechanism is proposed by some sellers in equilibrium, then the beliefs should be consistent with the strategy of the seller. For any $m \in \mathcal{M}$ and $k \in \mathcal{K}$, (4) should be replaced by

$$\mu_k^*(m) \underset{(resp.\ =)}{>} 0 \Longrightarrow \int u_k(\mu_1^*(m), \ldots, \mu_K^*(m), m; \sigma^*(m, X_S))dH_m(X_S) \underset{(resp.\ \leq)}{=} C_k. \tag{30}$$

where the way buyers' beliefs matter in their computation of their expected payoff is through the bidding strategy $\sigma^*(m, X_S))$ which possibly depend on the realization of the seller's type. In a mechanism where the seller is inactive as in the pivotal mechanism, then H_m does not play any role in (30).

Consider now a given mechanism m such that $\mu^*(m) \neq (0, \ldots, 0)$ and that belongs to $\mathrm{Supp}(m^*(X_S))$ for at least two realizations X_S and $\widetilde{X}_S$. From (30), the expected revenue on the equilibrium path of the seller once she has chosen the mechanism m is given by
$\int TW(\mu^*(m), m, X_S; \sigma^*(m, X_S))dH_m(X_S) \leq \int \max_{\mu\in R_+^K, m\in\mathcal{M}, \sigma(m)\in\Sigma(m)} TW(\mu, m, X_S; \sigma(m))dH_m(X_S) =$ $\int TW(\mu^*(X_S), X_S, X_S)dH_m(X_S)$ where the last term corresponds to the expected revenue of the types of the seller choosing m if they were deviating to propose the pivotal mechanism. If all those sellers were deviating to propose the pivotal mechanism, then the equilibrium conditions (29) impose that they should raise a (weakly) lower revenue which implies that the previous inequality hold as an equality. Then we must have $TW(\mu^*(m), m, x; \sigma^*(m)) = TW(\mu^*(x), x, x)$ for $x = X_S, \widetilde{X}_S$ with say $X_S < \widetilde{X}_S$. This implies further that $\mu^*(m) \in J^{max}(X_S, X_S) \cap J^{max}(\widetilde{X}_S, \widetilde{X}_S)$ or equivalently, from Lemma 3.1, $\mu^*(m) \in M(X_S) \cap M(\widetilde{X}_S)$. Take $k \in \{1, \ldots K\}$ such that $\mu_k^*(m) \neq 0$. From (4), we have then $u_k(\mu^*(m), X_S) = u_k(\mu^*(m), \widetilde{X}_S) = C_k$. However, we have $V_{k,N_{+k}}(X_S) \geq V_{k,N_{+k}}(\widetilde{X}_S)$ for any $N \in \mathbb{N}^K$ with a strict inequality when $N = (0, \ldots, 0)$. Since $P((0, \ldots 0)|\mu^*(m)) > 0$, this further implies that $u_k(\mu^*(m), X_S) > u_k(\mu^*(m), \widetilde{X}_S)$ which raises thus a contradiction.

The result above implies that on the equilibrium path, if the seller proposes a mechanism which raises some entry, then the expected revenue of the seller coincides with the expected total welfare which then must coincides with the first-best (otherwise the seller would strictly benefit to deviate and propose the pivotal mechanism). On the equilibrium path, if the seller proposes a mechanism which raises no entry, then the seller's revenue corresponds to her valuation. Furthermore, the revenue in the pivotal mechanism should not be strictly larger than it (otherwise it would be profitable to deviate) which implies that there should be no entry. On the whole, for any possible realization of the valuation of the seller, we obtain the equivalence with the pivotal mechanism.

Generalized second-price auctions

Under exogenous participation, Myerson (1981) shows that the optimal auction can be implemented with a generalized second-price auction where bids are distorted in a very general (nonlinear) way. Similarly, bid distortions play a crucial role in presence of incumbents.

A generalized second-price auction with (general) non-linear distortion (or a bid preference program) is characterized by a reserve price $r \in \mathbb{R}_+$ and a set of right-continuous increasing functions, called next bid distortion functions, $A_k : \mathbb{R}_+ \to \mathbb{R}_+$ (for each group $k \in \mathcal{K}$) and $A_i^I : \mathbb{R}_+ \to \mathbb{R}_+$ (for each incumbent $i \in \mathcal{I}$). The rules of the generalized second-price auction are as follows:

1. The seller collects all the bids and computes a new (or distorted) bid $A_i^I(b)$ [resp. $A_k(b)$] for each bid b from an incumbent i [resp. from an entrant from group k].
2. The reserve price r is considered next as a bid from the seller.
3. One of the agents (including the seller) with the highest new bid is declared to be the winner and receives the good.[46]
4. Let p denote the maximum of the second highest new bid (if any) and the reserve price. If the winner is an incumbent i [resp. an entrant from group k], he has to pay $\min\{b \in \mathbb{R}_+ | A_i^I(b) \geq p\}$ [resp. $\min\{b \in \mathbb{R}_+ | A_k(b) \geq p\}$]. The monetary transfer of a buyer who does not receive the good is null.[47]

The price paid by the winner corresponds to the lowest bid he would have to submit in order to be still declared the winner (with some positive probability). Note that the price paid by the winner can never be strictly above his bid. Compared to truthful bidding, bidding below its valuation involves only the loss of some profitable opportunities. Compared to truthful bidding, bidding above its valuation changes the final outcome only in the case where p is above his valuation, i.e. in the events where the final price would have been greater than his valuation. On the whole, we obtain that

Lemma 6.1 *For any generalized second-price auction, truthful bidding is a (weakly) dominant strategy.*

[46] In case of multiple winning bids, we need also a tie-breaking rule to complete the description of a specific auction. Any rule would suit, e.g. the one consisting in picking the winner at random.

[47] When there are atoms the tie-breaking rule may matter in terms of the final assignment. Nevertheless, it does not matter in terms of final payoffs in equilibrium since the pricing rule under truthful bidding guarantees that the bidders involved in a tie obtain pay their valuation (this is because $\min\{b \in \mathbb{R}_+ | A_i^I(b) \geq p\} = x$ if $A(x) = p \geq r$).

This further implies that in equilibrium, bidders should bid truthfully in generalized second-price auctions (since we assume that bidders use undominated strategies).

Let $\mathcal{M}_{SP}^A \supset \mathcal{M}_{SP}^r$ denote the set of generalized second-price auctions with $A_k(b) = b$ for any $k \in \mathcal{K}$. For any $m \in \mathcal{M}_{SP}^A$, we let $m[\mathfrak{r}]$ denote the reserve price in the auction and $m[A_i^I]$ the bid distortion of incumbent i for any $i \in \mathcal{I}$.

Remark 6.1 *The virtual pivotal mechanism m_{β,X_S}^{VP} corresponds to the generalized second-price auction in $\mathcal{M}_{SP}^A$ characterized by $m_{\beta,X_S}^{VP}[\mathfrak{r}] = X_S$ and $m_{\beta,X_S}^{VP}[A_i^I] = \widetilde{x}_i^I$ for any $i \in \mathcal{I}$.*

From the perspective of potential entrants, a mechanism $m \in \mathcal{M}_{SP}^A$ is equivalent to a standard second-price auction with the reserve $m[\mathfrak{r}]$ and where conditional on z and for any $i \in \mathcal{I}$, the valuation distributions of the incumbents are no longer $F_i^I(.|z)$ but are rather replaced by $F_i^I((m[A_i^I])^{-1}(.)|z)$ which denotes the distribution of the variable $m[A_i^I](X)$ where the variable X is drawn according to $F_i^I(.|z)$. This results from the fact that their bids are not distorted for $m \in \mathcal{M}_{SP}^A$. For a given $m \in \mathcal{M}_{SP}^A$, let $\widetilde{F}_m^{(1:N\cup I)}(x) = E_Z[\prod_{k=1}^K [F_k(x|Z)]^{n_k} \cdot \prod_{i=1}^I F_i^I((m[A_i^I])^{-1}(x)|Z)]$ denote the CDF of the highest new (or distorted) bid among the bidders under truthful bidding given that the realization of the profile of entrants is N. When there are no distortions, i.e. when $m \in \mathcal{M}_{SP}^r$, then note we have $\widetilde{F}_m^{(1:N\cup I)} = F^{(1:N\cup I)}$. On the whole, for any $m \in \mathcal{M}_{SP}^A$, the expected ex ante utility of a group k buyer is given by $u_k(\mu, m) = \sum_{N\in\mathbb{N}^K} P(N|\mu) \cdot V_{k,N_{+k}}(m)$ where

$$V_{k,N}(m) = \int_{m[\mathfrak{r}]}^{\infty} (\widetilde{F}_m^{(1:N_{-k}\cup I)}(x) - \widetilde{F}_m^{(1:N\cup I)}(x))dx. \tag{31}$$

The problem is somehow the same as before from the perspective of entrants, up to the twist that the CDFs of the valuation of the incumbents are now possibly distorted. Analogously to the expected welfare function $W_N(m, X_S; \sigma(m))$, we can define a notion of 'distorted welfare' for any $m \in \mathcal{M}_{SP}^A$ (which guarantees truthful bidding), denoted by $\widetilde{W}_N(m, X_S$, where incumbents' valuations have been substituted by their distorted valuations and the seller's reservation value X_S by the reserve price $m[\mathfrak{r}]$. Formally, we define

$$\widetilde{W}_N(m, X_S) := \int \max\left\{m[\mathfrak{r}], \max_{j=1,\dots,|N|}\{x_j^N\}, \max_{i\in\mathcal{I}}\{x_i^{\mathcal{I}}\}\right\} d[G_N(x^{\mathcal{I}}, s^N, s_0)]. \tag{32}$$

We have also

$$\widetilde{W}_N(m, X_S) := m[\mathfrak{r}] \cdot \widetilde{F}_m^{(1:N\cup I)}(m[\mathfrak{r}]) + \int_{m[\mathfrak{r}]}^{\infty} x d[\widetilde{F}_m^{(1:N\cup I)}(x)].$$

The fundamental property of the pivotal mechanism (8) translates now into

$$\widetilde{W}_{N_{+k}}(m, X_S) - \widetilde{W}_N(m, X_S) = V_{k,N_{+k}}(m) \text{ for any } k \in \mathcal{K}, \tag{33}$$

for any $m \in \mathcal{M}_{SP}^A$.

For any $m \in \mathcal{M}_{SP}^A$, we let $\widetilde{TW}(\mu, m, X_S) := \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot \widetilde{W}_N(m, X_S) - \sum_{k=1}^K \mu_k \cdot C_k$ denote the total expected (ex ante) distorted welfare.

From (33), entrants obtain the incremental surplus they generate where the surplus is defined according to the distorted valuations. We get then after a simple calculation that

$$\frac{\partial \widetilde{TW}(\mu, m, X_S))}{\partial \mu_k} = u_k(\mu, X_S) - C_k. \tag{34}$$

For any $m \in \mathcal{M}_{SP}^A$, we let $\widetilde{J}(m, X_S) := \{\mu \in R_+^K |$ for each $k \in \mathcal{K}, \widetilde{TW}(\mu, m, X_S)) \underset{(resp. \leq)}{=} 0$ if $\mu_k \underset{(resp. =)}{>} 0\}$ and $\widetilde{J}^{MAX}(m, X_S) := \operatorname{Arg\,max}_{\mu \in R_+^K} \widetilde{TW}(\mu, m, X_S))$. Any local maximum of the function $\mu \to \widetilde{TW}(\mu, m, X_S))$ belongs to $\widetilde{J}(m, X_S)$. In particular, we have $\widetilde{J}^{MAX}(m, X_S) \subseteq \widetilde{J}(m, X_S)$.

On the whole, we have from (34)

$$\widetilde{J}^{MAX}(m, X_S) \subseteq \widetilde{J}(m, X_S) = M(X_S). \tag{35}$$

It is straightforward that the way we prove that the function $\mu \to TW(\mu, X_S, X_S)$ is concave applies to the function $\mu \to \widetilde{TW}(\mu, m, X_S)$. We obtain thus a generalized version of Lemma 3.1.

Lemma 6.2 *For any $X_S \in \mathbb{R}_+$ and $m \in \mathcal{M}_{SP}^A$, $\mu \to \widetilde{TW}(\mu, m, X_S)$ is concave on R_+^K. As a corollary, we have $\widetilde{J}^{MAX}(m, X_S) = \widetilde{J}(m, X_S) = M(X_S)$.*

Remark 6.2 For the virtual pivotal mechanism, namely when $m = m_{\beta, X_S}^{VP}$, then the terms $\widetilde{W}_N(m, X_S)$, $\widetilde{TW}(\mu, m, X_S)$, $\widetilde{J}(m, X_S)$ and $\widetilde{J}^{MAX}(m, X_S)$ are equal to $\overline{W}_N(m, X_S)$, $\overline{TW}(\mu, m, X_S)$, $\overline{J}(m, X_S)$ and $\overline{J}^{MAX}(m, X_S)$.

Proof of Proposition 4.2

The proof is almost the same as the one of Proposition 3.2 up to the twist that we are dealing with 'distorted' environments.

From a classic calculation (see Myerson 1981), we have from A2 that for any equilibrium

$$\begin{aligned} V_i^I(x, m; \mu^*(m), \sigma^*(m)) = \quad & V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)) \\ & + \sum_{N \in \mathbb{N}^K} P(N|\mu^*(m)) \cdot \int_{\underline{x}_i}^x \int Q_{i,N}^I(x^{\mathcal{I}}, s^N; \sigma^*(m)) d[G_{-i,N}(x_{-i}^{\mathcal{I}}, s^N)] dx_i^{\mathcal{I}} \end{aligned} \tag{36}$$

for any $x \geq \underline{x}_i$ and $m \in \mathcal{M}$ and then

$$u_i^I(\mu^*(m), m; \sigma^*(m)) = \begin{array}{l} V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)) \\ + \sum_{N \in \mathbb{N}^K} P(N|\mu^*(m)) \cdot \int \frac{1-F_i^I(x_i^{\mathcal{I}})}{f_i^I(x_i^{\mathcal{I}})} Q_{i,N}^I(x^{\mathcal{I}}, s^N; \sigma^*(m)) d[G_N(x^{\mathcal{I}}, s^N)]. \end{array} \tag{37}$$

Note that the participation constraints at the auction stage reduce to

$$V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)) \geq 0. \tag{38}$$

for $i \in \mathcal{I}$, while the incentive compatibility constraints require that

$$x_i^{\mathcal{I}} \to \sum_{N \in \mathbb{N}^K} P(N|\mu^*(m)) \cdot \int Q_{i,N}^I(x^{\mathcal{I}}, s^N, s_0; \sigma^*(m)) d[G_{-i,N}(x_{-i}^{\mathcal{I}}, s^N, s_0)] \quad \text{is non-decreasing} \tag{39}$$

for any $i \in \mathcal{I}$, a constraint that will not be binding next thanks to our 'regularity' assumption.

Combining the utility maximization conditions (3) and (4), the mechanism chosen by the seller in any equilibrium solves the maximization program:

$$\max_{m \in \mathcal{M}} TW(\mu^*(m), m, X_S; \sigma^*(m)) - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot u_i^I(\mu^*(m), m; \sigma^*(m)). \tag{40}$$

or equivalently from (37), the program (11).

From Remark 6.2, we can apply all our results (in particular Lemma 6.2) to the virtual pivotal mechanism.

For a given $\widetilde{\mu} \in \overline{J}^{MAX}(m_{\beta,X_S}^{VP}, X_S)$, we let $m^* = m_{\beta,X_S}^{VP}$ and $\mu^*(m_{\beta,X_S}^{VP}) = \widetilde{\mu}$ and we pick any value in $M(m)$ for $m \in \mathcal{M} \setminus \{m_{\beta,X_S}^{VP}\}$ (which is possible since $M(m) \neq \emptyset$). For any $m \in \mathcal{M}$, bidders' strategies $\sigma^*(m)$ are defined in order to guarantee the equilibrium conditions. In particular, $\sigma^*(m)$ corresponds to truthful bidding in generalized second-price auctions. We have then $\overline{TW}(\mu^*(m), m, X_S; \sigma^*(m)) \leq \overline{TW}(\mu^*(m), m_{\beta,X_S}^{VP}, X_S) \leq \overline{TW}(\widetilde{\mu}, m_{\beta,X_S}^{VP}, X_S)$ for any $m \in \mathcal{M}$. From (38) and (14), this implies that $\overline{TW}(\mu^*(m), m, X_S; \sigma^*(m)) - \sum_{i=1}^{I} V_i^I(\underline{x}_i, m; \mu^*(m), \sigma^*(m)) \leq \overline{TW}(\widetilde{\mu}, m_{\beta,X_S}^{VP}, X_S) - \sum_{i=1}^{I} V_i^I(\underline{x}_i, m_{\beta,X_S}^{VP}; \widetilde{\mu})$ or equivalently

$$u_S(\mu^*(m), m, X_S; \sigma^*(m)) \leq u_S(\widetilde{\mu}, m_{\beta,X_S}^{VP}, X_S)$$

for any $m \in \mathcal{M}$. On the whole, we obtain that (m^*, μ^*, σ^*) is an equilibrium.

Consider a given equilibrium (m^*, μ^*, σ^*). From (3) and the participation constraints (38) and

(14), we have $\overline{TW}(\mu^*(m^*), m^*, X_S; \sigma^*(m^*)) \geq \overline{TW}(\mu^*(m^{VP}_{\beta,X_S}), m^{VP}_{\beta,X_S}, X_S) = \max_{\mu \in R^K_+} \overline{TW}(\mu, m^{VP}_{\beta,X_S}, X_S)$ where the latter equality results from Lemma 6.2. Since $\overline{TW}(\mu, m, X_S; \sigma^*(m)) \leq \overline{TW}(\mu, m^{VP}_{\beta,X_S}, X_S)$ for any μ and m, this further implies that $\overline{TW}(\mu^*(m^*), m^*, X_S; \sigma^*(m^*)) = \overline{TW}(\mu^*(m^*), m^{VP}_{\beta,X_S}, X_S)$, $\mu^*(m) \in \overline{J}^{MAX}(m^{VP}_{\beta,X_S}, X_S)$ and $V^I_i(\underline{x}_i, m^*; \mu^*(m^*), \sigma^*(m^*)) = 0$ for any $i \in \mathcal{I}$. From (17), the equality $\overline{TW}(\mu^*(m^*), m^*, X_S; \sigma^*(m^*)) = \overline{TW}(\mu^*(m^*), m^{VP}_{\beta,X_S}, X_S)$ implies that $\overline{W}_N(m^*, X_S; \sigma^*(m^*)) = \overline{W}_N(m^{VP}_{\beta,X_S}, X_S)$ for any N such that $P(N|\mu^*(m^*)) > 0$ or equivalently that the good is assigned with probability one to the agents with the highest distorted valuation for any N such that $P(N|\mu^*(m^*)) > 0$. Any assignment where the good is given to the agent with the highest distorted valuation can be implemented with the pivotal mechanism provided that the breaking rule is well-specified (remember that we do not exclude that tie occur with a positive probability). On the whole we have shown that the equilibrium (m^*, μ^*, σ^*) is equivalent to an equilibrium where the virtual pivotal mechanism is proposed and where the equilibrium participation rate is $\mu^*(m^*)$.

Proof of Proposition 5.1

Since we assume from A3 that the CDFs F^I_i, $i \in \mathcal{I}$, and F do not depend on z, we drop the variable z from the notation. For any $i \in \mathcal{I}$, we have:

$$u^I_i(\mu, r) = \int_r^\infty e^{-\mu(1-F(x))} \prod_{\substack{i'=1 \\ i' \neq i}}^{I} F^I_{i'}(x)(1 - F^I_i(x))dx, \tag{41}$$

$$\frac{\partial u^I_i(\mu, r)}{\partial r} = -e^{-\mu(1-F(r))} \prod_{\substack{i'=1 \\ i' \neq i}}^{I} F^I_{i'}(r)(1 - F^I_i(r)) \leq 0$$

and

$$\frac{\partial u^I_i(\mu, r)}{\partial \mu} = -\int_r^\infty e^{-\mu(1-F(x))} \prod_{\substack{i'=1 \\ i' \neq i}}^{I} F^I_{i'}(x)(1 - F^I_i(x))(1 - F(x))dx \leq 0.$$

The expected payoff of the seller when there are no entrants is then given by

$$u_S(0, r, X_S) = \int_0^\infty \max\{X_S, x\} dF^{(1:I)}(x) - \sum_{i=1}^{I} (1 - \beta^I_i) \cdot u^I_i(0, r).$$

From A5, the optimal reserve $r^M_I(X_S)$ in the case where there are no entrants is characterized by $\frac{\partial u_S(0, r^M_I(X_S), X_S)}{\partial r} = 0$ or equivalently by

$$X_S + \sum_{i=1}^{I} (1 - \beta^I_i) \cdot \frac{\prod_{\substack{i'=1 \\ i' \neq i}}^{I} F^I_{i'}(r^M_I(X_S))(1 - F^I_i(r^M_I(X_S)))}{f^{(1:I)}(r^M_I(X_S))} = r^M_I(X_S). \tag{42}$$

From A5 and A4, we have $X_S \in (\underline{x}, \overline{x})$ and $\beta_i^I < 1$. Combined with (42) this implies that $r_I^M(X_S) > \underline{x}$.

For the entrants, we let $u(\mu, r) \equiv u_1(\mu, r)$ and we have similarly

$$u(\mu, r) = \int_r^{\infty} e^{-\mu(1-F(x))} \prod_{i=1}^{I} F_k(x)(1 - F(x))dx, \tag{43}$$

$$\frac{\partial u(\mu, r)}{\partial r} = -e^{-\mu(1-F(r))} \prod_{i=1}^{I} F_k(r)(1 - F(r)) \leq 0 \tag{44}$$

and

$$\frac{\partial u(\mu, r)}{\partial \mu} = -\int_r^{\infty} e^{-\mu(1-F(x))} \prod_{i=1}^{I} F_k(x)(1 - F(x))^2 dx \leq 0. \tag{45}$$

If $u(0, r) = 0$, then $u(\mu, r) = 0$ for any μ and $M(r) = \{0\}$. If $u(0, r) > 0$, then $F(r) < 1$ and $\frac{\partial u(\mu,r)}{\partial \mu} \leq -(1 - F(r))u(\mu, r) < 0$ and the arbitrage condition (4) is thus satisfied by a unique solution, i.e. $M(r)$ is a singleton. On the whole we have thus that $\mu^*(r)$ is uniquely defined. From (45), $\mu^*(.)$ is nonincreasing. We note that $\frac{\partial u(\mu,r)}{\partial r} = 0$ if $r < \underline{x}$ (since $I > 0$). We have thus $\mu^*(r) = \mu^*(0)$ for any $r \leq \underline{x}$. From Lemma 3.1, we have also $\mu^*(r) \in J^{MAX}(r, r)$ which implies that $X_S \in \operatorname{Arg\,max}_{r \in R_+} TW(\mu^*(r), r, X_S)$.

If $\mu^*(r) = 0$ for any $r \in R_+$, then we are back to a framework with exogenous entry and the optimal reserve price is $r_I^M(X_S) > X_S$. We now consider that $\mu^*(0) > 0$ and let $\widehat{r} := \sup\{r \in R_+ | \mu^*(r) > 0\}$. Note that $\widehat{r} \in (\underline{x}, \overline{x})$.

If $r < \underline{x}$ or $x > \widehat{r}$, then $\frac{d\mu^*(r)}{dr} = 0$. If $r \in (\underline{x}, \widehat{r})$, then we have $\frac{\partial V(\mu^*(r),r)}{\partial \mu} < 0$ and $\frac{\partial V(\mu^*(r),r)}{\partial r} < 0$. From (4), we obtain finally that

$$\frac{d\mu^*(r)}{dr} = -\frac{e^{-\mu^*(r)(1-F(r))} \prod_{k=1}^{K-1} F_k(r)(1 - F(r))}{\int_r^{\infty} e^{-\mu^*(r)(1-F(x))} \prod_{k=1}^{K-1} F_k(x)(1 - F(x))^2 dx} < 0. \tag{46}$$

For any $r \in (\underline{x}, \widehat{r})$, we have

$$\begin{aligned} \frac{du_i^I(\mu^*(r), r)}{dr} &= \frac{\partial u_i^I(\mu^*(r), r)}{\partial r} + \frac{d\mu^*(r)}{dr} \cdot \frac{\partial u_i^I(\mu^*(r), r)}{\partial \mu} \\ &= -e^{-\mu^*(r)(1-F(r))} \prod_{i=1}^{I} F_i^I(r) \cdot (1 - F(r)) \cdot \\ & \underbrace{\left[\frac{(1 - F_i^I(r))}{F_i^I(r)(1 - F(r))} - \frac{\int_r^{\infty} e^{-\mu^*(r)(1-F(x))} \prod_{i=1}^{I} F_i^I(x) \frac{(1-F_i^I(x))}{F_i^I(x)} (1 - F(x))dx}{\int_r^{\infty} e^{-\mu^*(r)(1-F(x))} \prod_{i=1}^{I} F_i^I(x) \cdot (1 - F(x)) \cdot (1 - F(x))dx}\right]}_{>0} \end{aligned} \tag{47}$$

where the last inequality is a corollary of the fact that the function $x \rightarrow \frac{(1-F_k(x))}{F_k(x)(1-F(x))}$ is decreasing on $(\underline{x}, \overline{x})$ (Assumption A6). Then we obtain that $\frac{d\left[\sum_{i=1}^{I}(1-\beta_i^I)\cdot u_i^I(\mu^*(r),r)\right]}{dr} < 0$ for any $r \in (\underline{x}, \widehat{r})$. Since $X_S \in \operatorname{Arg\,max}_{r\in R_+} TW(\mu^*(r), r, X_S)$, we obtain finally that $r^* > X_S$. Note that $r^* = r_I^M(X_S)$ if $\mu^*(X_S) = 0$.

It remains to show that $r^* \leq r_I^M(X_S)$. We have

$$u_S(\mu, r, X_S) = TW(\mu, r, X_S) - \sum_{i=1}^{I}(1-\beta_i^I)\cdot u_i^I(\mu, r) = \sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[W_{N_n}(r, X_S) - C - \sum_{i=1}^{I}(1-\beta_i^I)\cdot V_{i,N_n}^I(r)\right], \tag{48}$$

with $N_n = (n)$ for any $n \in \mathbb{N}$. We have then

$$\begin{aligned}
&\frac{\partial[TW(\mu,r,X_S) - \sum_{i=1}^{I}(1-\beta_i^I)\cdot u_i^I(\mu,r)]}{\partial r} = \sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[(X_S - r)\cdot f^{(1:N_n\cup I)}(r) + \sum_{i=1}^{I}(1-\beta_i^I)\cdot[F(r)]^n \prod_{\substack{i'=1\\ i'\neq i}}^{I} F_{i'}^I(r)(1-F_i^I(r))\right] \\
&= \sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[(X_S - r)\cdot([F(r)]^n f^{(1:I)}(r) + n[F(r)]^{n-1}f(r)F^{(1:I)}(r)) + [F(r)]^n\cdot\sum_{i=1}^{I}(1-\beta_i^I)\cdot\prod_{\substack{i'=1\\ i'\neq i}}^{I} F_{i'}^I(r)(1-F_i^I(r))\right] \\
&= \sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[[F(r)]^n\cdot\underbrace{\frac{\partial u_S(0,r,X_S)}{\partial r}}_{<0 \text{ if } r>r_I^M(X_S)} + \underbrace{(X_S - r)\cdot n[F(r)]^{n-1}f(r)F^{(1:I)}(r)}_{\leq 0 \text{ if } r\geq X_S}\right]
\end{aligned} \tag{49}$$

On the whole, we obtain then that $\frac{\partial u_S(\mu,r,X_S)}{\partial r} < 0$ if $r > r_I^M(X_S)$. We have finally

$$\begin{aligned}
\frac{du_S(\mu^*(r), r, X_S)}{dr} = &\overbrace{\frac{\partial u_S(\mu^*(r), r, X_S)}{\partial r}}^{<0 \text{ if } r>r_I^M(X_S)} \\
&+ \underbrace{\frac{d\mu^*(r)}{dr}}_{\leq 0}\cdot\left[\frac{\partial TW(\mu^*(r), r, X_S)}{\partial \mu} - \sum_{i=1}^{I}(1-\beta_i^I)\cdot\underbrace{\frac{u_i^I(\mu^*(r), r)}{\partial \mu}}_{\leq 0}\right]
\end{aligned} \tag{50}$$

To conclude the proof, it remains to show that $\frac{\partial TW(\mu^*(r),r,X_S)}{\partial \mu} \leq 0$ if $r > r_I^M(X_S)$. We have

$$\begin{aligned}
\frac{\partial TW(\mu, r, X_S)}{\partial \mu} &= \sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[W_{N_{n+1}}(r, X_S) - W_{N_n}(r, X_S)\right] - C \\
&\leq \sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[W_{N_{n+1}}(r, X_S) - W_{N_n}(r, X_S) - V_{1,N_n}(r)\right] \\
&= -\sum_{n=0}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\left[(r - X_S)F^{(1:N_n\cup I)}(r)(1-F(r))\right] \leq 0
\end{aligned}$$

where the first inequality comes from (4) and the last equality comes from (26).