The Simple Structure of Top Trading Cycles in School Choice: A Continuum Model*

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Preliminary Version - Comments Welcome

Abstract

Many cities determine the assignment of students to schools through a school choice mechanism which calculates an assignment based on student preferences and school priorities. The prominent Top Trading Cycles (TTC) mechanism is strategy-proof and Pareto efficient, but the combinatorial description of TTC makes it non-transparent to parents and difficult to analyze for designers. We give a tractable characterization of the TTC mechanism for school choice: the TTC assignment can be simply described by n^2 cutoffs, where n is the number of schools, and these cutoffs can be easily observed after running the mechanism. We define TTC in a continuum model, in which these thresholds can be directly calculated in closed form by a differential equation.

Our continuum model allows us to compute comparative statics, and show that changes in the priority structure can have non-trivial indirect effects on the allocation. We also apply this model to solve for optimal investment in school quality, compare mechanisms, and help explain empirical findings about the relation between the TTC assignment and the Deferred Acceptance assignment. To validate the continuum model we show that it gives a good approximation for strongly converging economies. Our analysis draws on an interesting connection between continuous trading procedures and continuous time Markov chains.

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1 Introduction

School choice mechanisms are commonly used to incorporate student preferences and priorities when assigning students to school. Since Abdulkadiroğlu and Sönmez (2003) first formulated the school choice problem as a mechanism design problem, the design of school choice mechanisms has received much attention in both the academic and policy-making spheres. In their seminal paper, Abdulkadiroğlu and Sönmez (2003) suggest two mechanisms, the Deferred Acceptance (DA) mechanism, introduced by Gale and Shapley (1962), and the Top Trading Cycles (TTC) mechanism, originally presented by Shapley and Scarf (1974) and attributed to David Gale. In recent years, many school districts have redesigned their school choice mechanisms, mostly choosing to implement the DA mechanism (Pathak and Sönmez, 2013; IIPSC, 2017). In contrast, the TTC mechanism has not been widely implemented, despite being strategy-proof and having superior efficiency properties.

Why did school districts choose not to implement the efficient and strategy-proof TTC mechanism? Based on his experience in designing many school choice programs, Pathak (2016) asserts that the difficulty of explaining TTC caused school districts to favor DA, as it was easier to explain.[1] The commonly used description of the TTC mechanism requires following the algorithm's calculation of the assignment through a simulation of a series of trades between students. It is challenging to convey this description of TTC to parents, and as a result it is challenging to convince parents that the mechanism is strategy-proof and cannot be gamed. In addition, Abdulkadiroğlu (2013) reports that parents react badly to the language of "trade", which raises connotations of strategic behavior and gaming. He therefore uses the name "Efficient Transfer Mechanism" when referring to TTC in conversations with parents, in order to avoid complicating the discussion.

Similar issues arise after the TTC assignment is published, because it is hard for parents to verify that TTC was run correctly. After New Orleans implemented the TTC mechanism, administrators faced many inquiries from students who did not get admitted into desired schools, and found those inquiries challenging to handle (Pathak, 2016). In contrast, under DA it is simpler for administrators to address such inquiries, as they can answer that the school filled with applicants who had higher priority at that school. Under TTC such an answer is not valid, and there may be lower ranked applicants who did get admitted into the school.[2] The unsatisfied higher-ranked applicants may

[1] In addition, Boston and NYC were early school redesigns that set a precedent in favor of DA.

[2] Such lower ranked applicants may get into the school by having high priority at another school and

naturally be concerned that the mechanism did not run correctly, and administrators do not currently have a simple way to explain why the assignment is correct.[3]

A related problem is that, from the algorithmic description of TTC, it can be hard to predict how changes in prioritize and preferences would affect the TTC assignment. While TTC allows the school district to control the allocation by adjusting the priority structure, there is no transparent relation between the priority structure and the resulting TTC assignment. As a result, system designers and school districts often have to rely on simulations to assess the TTC assignment.[4]

This paper aims to address these issues by providing a tractable and intuitive characterization of the TTC mechanism. We show that if there are n schools the TTC assignment is simply described using n^2 admission thresholds, or cutoffs, which can be easily obtained after the TTC mechanism has been run. For each pair of schools b, c the cutoff p_b^c is the minimal priority for school b required to get a seat in school c. The assignment of each student is the most preferred school they can get a seat at, using their priority from any school (see Figure 2). In particular, the cutoff structure allows administrators to explain the assignment of a student without relying on the algorithmic description of TTC.

These admission thresholds can also be tractably calculated in the continuum model of Azevedo and Leshno (2016). We define the TTC mechanism in the continuum model through trade balance equations, which stipulate balance conditions that must be satisfied by any combination of trade cycles. These trade balance equations allow us to track the process of the algorithm without having to track the individual cycles cleared. The cutoffs $\{p_b^c\}$ are directly calculated as a solution to a differential equation. This formu-

participating in a trade cycle.

[3]Although unsatisfied students may form blocking pairs with their desired schools, there is no strategic concern that blocking pairs will match to each other outside the mechanism. This is because of two differences between school choice and two-sided settings like the medical match. First, in Boston and other districts school cannot directly admit students without getting approval from the district's public school administration. Second, the priorities are often decided by school zone and lotteries, and are not controlled by the school. In that case, schools do not have a preference to admit higher priority students. Pathak (2016) writes that:

> "I believe that the difficulty of explaining TTC, together with the precedent set by New York and Boston's choice of DA, are more likely explanations for why TTC is not used in more districts, rather than the fact that it allows for justified envy, while DA does not."

More details can be found in the discussion in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005).

[4]Abdulkadiroğlu, Pathak, and Roth (2009) report simulation results evaluating the outcome of TTC in NYC and Boston. Pathak (2016) reports a comparison of TTC and DA given a specific lottery draw.

lation is tractable, and allows us to give closed form solutions for the TTC assignment in parameterized settings.

Using the framework, we study optimal investment in school quality when students are assigned through TTC. We first provide comparative statics as to how the TTC assignment changes as a school becomes more popular.[5] The marginal effect of an increase in a school's quality on welfare can be decomposed into two components. The first is the positive effect of improvement in the utility of students assigned to the school. Under school choice there is an additional term that arises from the change in composition of students assigned to each school, which can be negative. For example, students may choose schools according to idiosyncratic preferences when all school have equal quality, but not if some school have higher quality. We derive the TTC assignment in closed form as a function of school quality, and give policies for optimal investment in quality under budget constraints. We provide a parameterized setting where we solve for the optimal investment in school quality under TTC.

This framework gives us tools to study the design of priorities for school choice. We find that, under TTC, the priority structure is "bossy" in the sense that a change in the relative priority among top priority students can change the assignment of low priority students, without changing the assignment of any high priority student. One technical implication is that it is not possible to determine the cutoffs directly through a supply-demand equation, as additional information is required to determine the unique TTC assignment. Another implication of this result is that the choice of tie-breaking between high priority students can affect the allocation of low priority students. We characterize the range of possible assignments generated by TTC after changes to relative priority of high-priority students, and show that a small change to the priorities will only change the allocation of a few students.

We also examine the Clinch and Trade mechanism by Morrill (2015b), which can be computed by running TTC on an appropriately modified priority structure. The priority structure determines the sequence of trade cycles performed by TTC, and we can change priorities to have TTC implement the same trade cycles as the Clinch and Trade mechanism. In this formulation, the fact that Clinch and Trade gives a different allocation from TTC can be seen as an example of the bossiness of priorities. This

[5] Hatfield, Kojima, and Narita (2016) explore changes in school quality in the discrete setting, and found that it is possible for a school to be assigned lower priority students when it becomes more popular. Using the continuum framework, we are able to calculate the magnitude of the effect as well as the composition of affected students.

formulation also allows us to compare the two mechanisms and demonstrate that TTC may result in fewer blocking pairs.

To establish the validity of our continuum framework, we provide several technical results. The continuum model is an extension of the standard discrete model in that any discrete economy can be naturally embedded into the continuum framework, and the definition of the continuum TTC is consistent with the discrete TTC definition. Thus the cutoff characterization can be applied to both the discrete and continuum TTC models. We also show that the TTC assignment changes continuously with small perturbations to the economy. This implies that our continuum framework can be also interpreted as a limit economy.

While the continuum model simplifies much of the analysis, it also imposes some complications. Because schools point to a mass of students, it is not immediate how the algorithm should clear trading cycles. Therefore, instead of defining the TTC mechanism through sequential clearing of trading cycles, we define the continuum algorithm through the trade balance conditions. Using these conditions, we derive the marginal trade balance equations, the solutions of which describe the run of the TTC algorithm over a short interval. A connection to Markov chain theory allows us to show that the marginal trade balance equations always have a solution, and to find which trades the mechanism will conduct. This gives a differential equation that characterizes the run of the algorithm while schools have remaining capacity. Following this curve until capacity is filled gives the cutoffs and the TTC assignment.

1.1 Related Literature

Abdulkadiroğlu and Sönmez (2003) first introduced school choice as a mechanism design problem and suggested the TTC mechanism as a solution with several desirable properties. Since then, TTC has been considered for use in a number of school choice systems. Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discusses how the city of Boston debated between using DA or TTC for their school choice systems, and ultimately chose to use DA. Abdulkadiroğlu, Pathak, and Roth (2009) compare the outcomes of DA and TTC for the NYC public school system, and shows that TTC gives higher student welfare. Kesten (2006) also study the relationship between DA and TTC, and shows that they are equivalent mechanisms if and only if the priority structure is acyclic.

Axiomatic characterizations of TTC were given by Abdulkadiroğlu, Che, and Ter-

cieux (2010), Dur (2012)and Morrill (2013, 2015a). These characterizations explore the properties of TTC, but do not provide another method for calculating the TTC outcome. Ma (1994),Pápai (2000) and Pycia and Ünver (2015) give characterizations of more general classes of Pareto efficient and strategy-proof mechanisms which include TTC.

Several variations of TTC have been studied in the literature. Morrill (2015b) introduces the Clinch and Trade mechanism, which differs from TTC in that it identifies students who are guaranteed admission to their first choice and assigns them immediately without implementing a trade. Hakimov and Kesten (2014) introduce Equitable TTC, a variation of TTC that aims to reduce inequity. In Section 5.2 we use our model to analyze such variations of TTC and compare their assignments. Other variations of TTC can also arise from the choice of tie-breaking rules. Ehlers (2014) shows that any fixed tie-breaking rule satisfies weak efficiency, Alcalde-Unzu and Molis (2011); Jaramillo and Manjunath (2012) and Saban and Sethuraman (2013) give specific variants of TTC that are strategy-proof and efficient. The continuum model allows us to characterize the possible outcomes from different tie-breaking rules.

Several papers study properties of TTC directly from the algorithmic description and using stochastic methods. Che and Tercieux (2015a,b) study the properties of TTC in a large market where the number of items grows as the market gets large. Hatfield, Kojima, and Narita (2016) study the incentives for schools to improve their quality under TTC and find that a school may be assigned some less preferred students when it improves its quality.

This paper is contributes to the growing literature that use continuum models in market design (Avery and Levin, 2010; Abdulkadiroğlu, Che, and Yasuda, 2015; Ashlagi and Shi, 2015; Che, Kim, and Kojima, 2013; Azevedo and Hatfield, 2015). Our description of the continuum economy uses the setup of Azevedo and Leshno (2016), who characterize stable matchings in terms of cutoffs that satisfy a supply and demand equation. Our results from Section 5.2 imply that the TTC cutoffs depend on the entire distribution and cannot be computed from simple supply and demand equations.

Our finding that the TTC assignment can be represented in terms of cutoffs parallels the role of prices in competitive markets. Dur and Morrill (2016) show that the outcome of TTC can be expressed as the outcome of a competitive market where there is a price for each priority position at each school, and agents may buy and sell exactly one priority position. Their characterization explains the connection between TTC and competitive

markets, but it relies on a high-dimensional set of prices and does not provide a method of directly calculating these prices without running TTC. He, Miralles, Pycia, Yan, et al. (2015) propose a pseudo-market approach for discrete allocation problems, and in particular show that the Hylland-Zeckhauser pseudo-market mechanism (Hylland and Zeckhauser, 1979) can be extended to allow for priorities. Miralles and Pycia (2014) show that any Pareto efficient assignment of discrete goods without transfers can be decentralized through prices.

1.2 Organization of the Paper

Section 2 gives a description of the TTC mechanism in the discrete model and presents our characterization of the TTC outcome in terms of cutoffs. Section 3 presents the continuum model and gives an informal description of the TTC mechanism in the continuum. Section 4 formally defines the TTC model in the continuum and presents our main results. In Section 5 we explore several applications: quantifying the effects of improving school quality and solving for optimal investment, showing the "bossiness" of the TTC priorities, and comparing the TTC assignment with the DA assignment. Omitted proofs can be found in the appendix.

2 TTC in School Choice

In this section, we describe the standard model for the TTC mechanism in the school choice literature, and outline some of the properties of TTC in this setting. In the ensuing discussion, we will refer to schools as colleges and index them with the letter c.

Let $\mathcal{S}$ be a finite set of students, and let $\mathcal{C}$ be a finite set of colleges. Each college $c \in \mathcal{C}$ has a finite capacity $q_c > 0$. Each student $s \in \mathcal{S}$ has a strict preference ordering $\succ^s$ over colleges, and we let $Ch^s(C) = \arg\max_{\succ^s} \{C\}$ denote s's most preferred college out of the set C. Each college $c \in \mathcal{C}$ has a strict priority ordering $\succ_c$ over students. To simplify notation, we assume that all students and colleges are acceptable, and that there are more students than available seats at colleges.[6]

A *feasible allocation* is $\mu : \mathcal{S} \to \mathcal{C} \cup \{\emptyset\}$ where $|\mu^{-1}(c)| \leq q_c$ for every $c \in \mathcal{C}$. If $\mu(s) = c$ we say that s is assigned to c, and we use $\mu(s) = \emptyset$ to denote that the student s

[6]This is without loss of generality, as we can introduce auxiliary students and schools that represent being unmatched.

is unassigned. As there is no ambiguity, we let $\mu(c)$ denote the set $\mu^{-1}(c)$ for $c \in \mathcal{C} \cup \{\emptyset\}$. A *discrete economy* is $E = (\mathcal{C}, \mathcal{S}, \succ^{\mathcal{S}}, \succ_{\mathcal{C}}, q)$ where $\mathcal{C}$ is the set of colleges, $\mathcal{S}$ is the set of students, $q = \{q_c\}_{c \in \mathcal{C}}$ is the capacity of each college, and $\succ^{\mathcal{S}} = \{\succ^s\}_{s \in \mathcal{S}}$, $\succ_{\mathcal{C}} = \{\succ_c\}_{c \in \mathcal{C}}$.

The Top Trading Cycles algorithm (TTC) calculates an allocation by creating a virtual exchange for priorities. The algorithm runs in discrete steps as follows.

Algorithm 1 (Top Trading Cycles)**.** *Initialize unassigned students $S = \mathcal{S}$, available colleges $C = \mathcal{C}$, capacities $\{q_c\}_{c \in C}$ and partial allocation μ.*

- *While there are still unassigned students and available colleges:*
  - *Each available college $c \in C$ tentatively offers a seat by pointing to its highest priority remaining student.*
  - *Each student $s \in S$ that was tentatively offered a seat points to his most preferred remaining college.*
  - *Select at least one trading cycle, that is, a list of students $s_1, \ldots, s_\ell, s_{\ell+1} = s_1$ such that s_i points to the college pointing to s_{i+1} for all i, or equivalently s_{i+1} was offered a seat at s_i's most preferred college. Assign all students in the cycles to the college they point to.*[7]
  - *Remove the assigned students from S, reduce the capacity of the colleges they are assigned to by 1, and remove colleges with no remaining capacity from C.*

TTC satisfies a number of desirable properties. An allocation μ is *Pareto efficient for students* if no group of students can improve by swapping their allocations, and no individual student can improve by swapping their allocation for an unassigned object. A mechanism is *Pareto efficient* if it always produces a Pareto efficient allocation. A mechanism is *strategy-proof for students* if reporting preferences truthfully is a dominant strategy . It is well known that the Top Trading Cycles mechanism, as used in the school choice setting, is both Pareto efficient and strategy-proof for students (Abdulkadiroğlu and Sönmez, 2003). Moreover, when type-specific quotas must be imposed, Top Trading Cycles can be easily modified to meet quotas while still maintaining constrained Pareto efficiency and strategy-proofness (Abdulkadiroğlu and Sönmez, 2003).

[7] Such a trading cycle must exist, since every vertex in the pointing graph with vertex set $S \cup C$ has out-degree 1.

2.1 A Cutoff Characterization of Top Trading Cycles

Our first main contribution is that the TTC allocation can be simply characterized by n^2 cutoff students $\{s_b^c\}$, one for each pair of colleges.

Theorem 2. *The TTC allocation can be given by*

$$\mu(s) = Ch^s\left(\{c \mid s \succeq_b s_b^c \text{ for some } b\}\right),$$

where s_b^c is the worst ranked student at college b that traded a seat at college b for a seat at college c.

Theorem 2 allows us to give an intuitive explanation to individual students for why TTC placed them in a certain school. A student s is assigned to their favorite college c at which there is a college b that prefers them to the cutoff student s_b^c.[8] If a student does not receive a seat at a desired college, it is because they do not have sufficiently high priority at *any* college. Moreover, the cutoff students s_b^c can be easily determined after the mechanism has been run.

However, Theorem 2 does not explain how the cutoff students $\{s_b^c\}$ change with changes in school priorities or student preferences. We therefore introduce the continuum model for TTC which allows us to directly calculate the cutoffs and do comparative statics. We omit the direct proof of Theorem 2, as it follows from Theorem 6 and Proposition 16.

3 The Continuum Model for TTC

We model the school choice problem with a continuum of students and finitely many colleges, as in Azevedo and Leshno (2016). There is a finite set of colleges denoted by $\mathcal{C} = \{1, \ldots, n\}$, with each college $c \in \mathcal{C}$ having capacity to admit a $q_c > 0$ mass of students. A student $\theta \in \Theta$ is given by $\theta = \left(\succ^\theta, r^\theta\right)$. The student's strict preferences over colleges is $\succ^\theta$, and we let $Ch^\theta(C) = \arg\max_{\succ^\theta}\{C\}$ denote θ's most preferred college out of the set C. The priorities of colleges over students are captured by the vector $r^\theta \in [0,1]^C$. We say that r_c^θ is the rank of student θ at college c. Colleges prefer students with *higher ranks*, that is $\theta \succ^c \theta'$ if and only if $r_c^\theta > r_c^{\theta'}$.

[8] We can formulate this as a condition on the percentile rank of student s at every college. Let $\hat{p}_b^c$ be the percentile of s_b^c in the priority list of college b. A student s is assigned to their favorite college c at which there is a college b for which their percentile is higher than $\hat{p}_b^c$.

Definition 3. A *continuum economy* is given by $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ where $q = \{q_c\}_{c \in C}$ is the vector of capacities of each college, and η is a probability measure over Θ.

Without loss of generality, we make the following assumptions for the sake of tractability. First, we assume that all students and colleges are acceptable and normalize the mass of students to be $\eta(\Theta) = 1$. Second, as r_c^θ carry only ordinal information we can normalize a student's rank to be equal to his percentile rank in the college's preferences. That is, for any $c \in \mathcal{C}$, $x \in [0,1]$ we have that $\eta(\{\theta : r_c^\theta \leq x\}) = x$. Third, we assume there is an excess of students, that is, $\sum_{c \in \mathcal{C}} q_c < 1$.

We make the following assumption for technical reasons, but it is not without loss of generality. For example, it is violated when all colleges share the same priorities over students.[9]

Assumption 4. *The measure η admits a density ν. That is for any subset of students $A \subseteq \Theta$*

$$\eta(A) = \int_A \nu(\theta) d\theta.$$

Furthermore, ν is piecewise Lipschitz continuous everywhere except on a finite grid,[10] *bounded from above, and bounded from below away from zero on its support.*[11]

An immediate consequence of this assumption is that a college's indifference curves are of η-measure 0. That is, for any $c \in \mathcal{C}$, $x \in [0,1]$ we have that $\eta(\{\theta : r_c^\theta = x\}) = 0$. This is analogous to colleges having strict preferences in the standard discrete model. In school choice, it is common for schools to have coarse priorities, and to refine these using a tie-breaking rule. Our economy $\mathcal{E}$ captures the strict priorities that result after applying the tie-breaking rule.

As in the discrete model, an *allocation* is a mapping $\mu : \Theta \to \mathcal{C} \cup \{\emptyset\}$ specifying the assignment of each student. An allocation μ is *feasible* if it respects capacities, that is, for each college $c \in \mathcal{C}$ we have that $\eta(\mu(c)) \leq q_c$. To exclude two different allocations that differ only on a set of zero measure we require that the assignment is right continuous, that is, for any sequence of student types $\theta^k = (\succ, r^k)$ and $\theta = (\succ, r)$, with r^k converging

[9] We can incorporate an economy where two colleges have perfectly aligned priories by considering them as a combined single school in the trade balance equations. The capacity constraints still consider the capacity of each school separately.

[10] A grid $G \subset \Theta$ is given by a finite set of grid points $D = \{d_1, \ldots, d_L\} \subset [0,1]$ as $G = \{\theta | \exists c \text{ s.t. } r_c^\theta \in D\}$. Equivalently, ν is Lipschitz continuous on $\Theta \setminus G$, which is a collection of hypercubes.

[11] That is, there exists $M > m > 0$ such that for every $\theta \in \Theta$ either $\nu(\theta) = 0$ or $m \leq \nu(\theta) \leq M$.

to r, and $r_c^k \geq r_c^{k+1} \geq r_c$ for all k, c, we can find some large K such that $\mu(\theta^k) = \mu(\theta)$ for $k > K$.

We give an informal description of the TTC algorithm here, and formally describe and characterize the algorithm in Section 4. In the continuum model the TTC algorithm runs in continuous time indexed by t, starting with $t = 0$.

Algorithm 5 (Continuum Top Trading Cycles)**.** *Initialize unassigned students $S = \Theta$, available colleges $C = \mathcal{C}$, capacities $\{q_c\}_{c \in C}$ and partial allocation μ.*

- *At time t, if there are still unassigned students and available colleges:*
  - *Each available college $c \in C$ tentatively offers a seat by pointing to the measure 0 set of remaining students with highest priority at c.*
  - *Each student $s \in S$ that was tentatively offered a seat points to their most preferred remaining college.*
  - *Select at least one trading cycle, that is, a list of sets of students $S_1, \ldots, S_\ell, S_{\ell+1} = S_1$ such that $s_i \in S_i$ points to the college pointing to S_{i+1} for all i, or equivalently each student in S_{i+1} was offered a seat at each student in S_i's most preferred college. Assign all students in the cycles to the college they point to.*
  - *Remove the assigned students from S, reduce the capacity of the colleges, and remove any colleges with capacity $q_c = 0$.*

We remark that there are several challenges in making sure the algorithm is well defined. Each cycle is of zero measure, but we need to reduce college capacities. We also need to guarantee that aggregation of many cycles assigns to a school a set of students that has mass equal to the amount of seats offered by the school. A college will generally point to a zero measure set that includes more than one student type. This implies each school may be involved in multiple cycles at a given point, a type of multiplicity that leads to non-unique TTC allocations in the discrete setting. In the following we give a formal definition of the algorithm, and show that it has a well defined unique outcome.

4 Characterization of the TTC Assignment

In the continuum model, as in Theorem 2, the TTC allocation can also be simply characterized by n^2 cutoffs $\{p_b^c\}$, one for each pair of colleges.

Theorem 6. *The TTC allocation can be given by*

$$\mu(\theta) = Ch^{\theta}\left(\left\{c \,:\, r_b^{\theta} \geq p_b^c \text{ for some } b\right\}\right),$$

where p_b^c is the worst rank at college b that is traded for a seat at college c.

A student θ is assigned to her favorite college c at which her score at some college b exceeds the cutoff required to trade into c. In addition, the TTC allocation can be computed as the solution to a set of differential equations which can be calculated from the problem primitives.

Theorem 7. *In the continuum model, the TTC cutoffs p_b^c can be calculated as the solutions to a differential equation*

$$p_b^c = \gamma_b(t^{(c)}),$$

where γ tracks the progress of the algorithm and $t^{(c)}$ is the run-out time of college c. The equation $\gamma'(t) = d(\gamma(t))$ defines γ, where $d(x)$ is calculated from the marginal distribution of students with rank $r^{\theta} = x$. The run-out times $t^{(c)}$ are calculated from γ and the capacity constraints.

The rest of this section provides the necessary framework and proves these results, and is structured as follows. In Section 4.1, we provide the definitions and framework for proving our main results. In Section 4.2 we prove Theorem 6. In Section 4.3, we prove Theorem 7. Finally in Section 4.4, we show that our definition of continuum TTC is a generalization of discrete TTC and prove a convergence result.

4.1 Defining the TTC Algorithm Using Trade Balance Equations

In this section, we show that the TTC algorithm in the continuum model can be understood and defined via a curve in $[0,1]^{\mathcal{C}}$ that we call a *TTC path,* and a set of equations that we term *trade balance equations* and *capacity equations* that must be satisfied by all valid TTC paths. We will begin with some observations to motivate our formal definition of the continuum TTC algorithm. It is worth remarking that although our observations are used to motivate our framework for TTC in the continuum model, they are also,

unless otherwise specified, valid for both the discrete and continuum models, and can be used to shed insight on both models.

The first observation is that we may track the progression of the algorithm by recording the student each college is pointing to. In the continuum model, since the set of students is given by a distribution over $\Theta \times [0,1]^{\mathcal{C}}$, and since colleges do not discriminate based on Θ, we may think of this as tracking the progression of the algorithm via a curve in $[0,1]^{\mathcal{C}}$.

Formally, denote $\gamma(t) : [0,T] \to [0,1]^{\mathcal{C}}$ to be the *TTC path*, where $\gamma_c(t)$ is the rank of the student(s) school c points to at time t.[12]

The second observation is that it will be useful to divide the run of the algorithm into discrete *rounds* indexed by $\ell = 1, \ldots, L$, corresponding to times when the set of available colleges remains constant. We start the algorithm at time $t = 0$ in round $\ell = 1$, and each round ℓ ends when some college exhausts its capacity. We denote the set of colleges that still available capacity in round ℓ by $\mathcal{C}^{(\ell)}$, where $\mathcal{C}^{(\ell+1)} \subsetneq \mathcal{C}^{(\ell)}$, $\mathcal{C}^{(1)} = \mathcal{C}$ and $\mathcal{C}^{(L+1)} = \phi$. Denote the time interval corresponding to round ℓ by $[t^{(\ell-1)}, t^{(\ell)}]$, with $t^{(\ell-1)} < t^{(\ell)}$, $t^{(0)} = 0$ and $t^{(L)} = T$. We refer to $\left\{\left(\mathcal{C}^{(\ell)}, t^{(\ell)}\right)\right\}_{\ell}$ as a *run-out sequence*.

We now introduce the necessary definitions and notation to characterize the run of the TTC algorithm within a round. One way to calculate $\gamma(\cdot)$ in the discrete model is by an iterative process of identifying and implementing trade cycles. The continuum model allows a simpler characterization in terms of *trade balance equations* instead of trading cycles.

Consider an available school $c \in \mathcal{C}^{(l)}$. Denote the set of students who were offered a seat by college c before time t by[13]

$$\mathcal{T}_c(\gamma; t) \stackrel{def}{=} \left\{\theta \in \Theta \mid \exists \tau \leq t \text{ s.t. } r_c^{\theta} = \gamma_c(\tau) \text{ and } r^{\theta} \leq \gamma(\tau)\right\}.$$

That is, a student θ was offered a seat at college c if at some time $\tau \leq t$ the college was pointing to students with rank r_c^{θ}, and student θ was still unassigned at time τ. To use the discrete terminology, we may think of the student as being pointed to by college c before step k in the algorithm. Denote the set of students who were assigned a seat at college $c \in \mathcal{C}^{(\ell)}$ before time $t \in [t^{(\ell-1)}, t^{(\ell)}]$ by[14]

[12] If $\mathcal{S}(t)$ denotes the set of students that are still unassigned at time t, then $\mathcal{S}(t) = \left\{\theta \mid r^{\theta} \leq \gamma(t)\right\}$ and $\gamma_c(t) = \sup\left\{r_c^{\theta} \mid \theta \in \mathcal{S}(t)\right\}$.

[13] Note that $\mathcal{T}_c(\gamma; t)$ includes students who were offered a seat in the college in previous rounds.

[14] Note that $\mathcal{T}^c(\gamma; t)$ includes students who were assigned to c during a previous round, because if $\ell_1 < \ell_2$ we have that $\mathcal{C}^{(\ell_2)} \subset \mathcal{C}^{(\ell_1)}$ and therefore $c = Ch_{\theta}\left(\mathcal{C}^{(\ell_1)}\right) \Rightarrow c = Ch_{\theta}\left(\mathcal{C}^{(\ell_2)}\right)$.

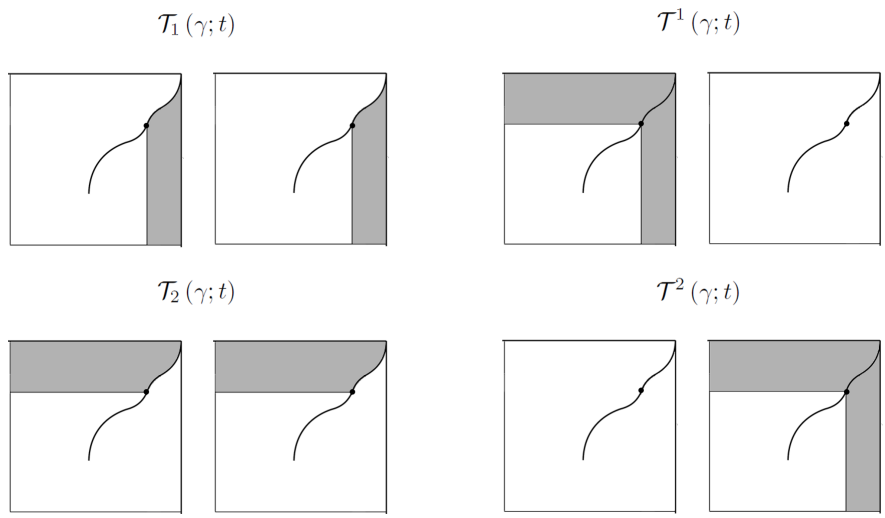

Figure 1: An illustration of the sets $\mathcal{T}^c(\gamma;t)$ and $\mathcal{T}_c(\gamma;t)$ for $c = 1, 2$. In each square the horizontal axis corresponds to the student's rank at college 1 and the vertical axis to the rank at college 2. The left square includes students who prefer college 1 and the right square includes students who prefer college 2. The curved line is γ the TTC path, and the point is at $\gamma(t)$.

$$\mathcal{T}^c(\gamma;t) \stackrel{def}{=} \left\{\theta \in \Theta \mid r^\theta \not\leq \gamma(t) \text{ and } Ch_\theta\left(\mathcal{C}^{(\ell)}\right) = c\right\}.$$

That is, a student was assigned somewhere by time t if there exists a college c' such that $r^\theta_{c'} > \gamma_{c'}(t)$, and the student θ is assigned to c only if c is $\theta's$ most preferred available college.

Our third and main observation is that a process of cycle clearing imposes a simple condition on the sets $\mathcal{T}^c(\gamma;t)$ and $\mathcal{T}_c(\gamma;t)$. Any cycle that is cleared has the same amount of students offered a seat at a college and the amount of students assigned to the college. Therefore, at any time t and for any college c the total amount of seats offered by college c is students equal to the amount of students assigned to c plus the amount of seats that were offered but not claimed. In the continuum model the the amount of seats offered but not claimed is of η measure-0.[15] It follows that if γ is a TTC path, at every time t it must satisfy the *trade balance equations* for every college c that is available at time t:

[15] A student can have a seat that is offered but not claimed in one of two ways. The first is the seat is offered at time t and not yet claimed by a trade. The second is that the student that got offered two or more seats at the same time $\tau \leq t$ (and was assigned through a trade involving only one seat). Both of these sets of students are of η measure-0 under our assumptions.

$$\eta\left(\mathcal{T}^c\left(\gamma;t\right)\right)=\eta\left(\mathcal{T}_c\left(\gamma;t\right)\right). \qquad (\textit{trade balance equations}) \quad (1)$$

We will show in Section 4.3 that the trade balance equations fully characterize the run of the TTC algorithm within a round.

It remains to characterize the stopping time of each round. Informally, the end of a round is determined by the first time some college fills its capacity. Formally, the stopping time $t^{(\ell)}$ and next available set $\mathcal{C}^{(\ell+1)}$ are determined by the equation $\mathcal{C}^{(\ell+1)} \subsetneq \mathcal{C}^{(\ell)}$ and the *capacity equations*:[16]

$$\begin{aligned} \eta\left(\mathcal{T}^c\left(\gamma;t^{(\ell)}\right)\right)&=q_c \quad \forall c\notin\mathcal{C}^{(\ell+1)},\\ \eta\left(\mathcal{T}^c\left(\gamma;t^{(\ell)}\right)\right)&<q_c \quad \forall c\in\mathcal{C}^{(\ell+1)}. \end{aligned} \qquad (\textit{capacity equations}) \quad (2)$$

Using the trade balance equations (1) and capacity equations (2), we can formally define the TTC mechanism for a continuum economy.

Definition 8. Given a continuum economy $\mathcal{E}=(\mathcal{C},\Theta,\eta,q)$ we say that a weakly decreasing function $\gamma:[0,T]\to[0,1]^C$ is a *valid TTC path* if it satisfies the *trade balance equations* (1) for all times t and satisfies the *capacity equations* (2) for some run-out sequence $\left\{\left(\mathcal{C}^{(\ell)},t^{(\ell)}\right)\right\}_{\ell=1..L}$.

In addition, we normalize the algorithm running time so that γ is continuous, piecewise differentiable with $\|\frac{d\gamma(t)}{dt}\|_1=1$ a.e., and wlog γ_c is constant on $[t^{(\ell-1)},1]$ for $c\notin\mathcal{C}^{(\ell)}$.

Note that if γ is a valid TTC path, then there is a unique run-out sequence $\left\{\left(\mathcal{C}^{(\ell)},t^{(\ell)}\right)\right\}_{\ell=1..L}$ that satisfies the capacity equations (1) for γ. Thus a valid TTC path γ gives a complete description of the run of the TTC algorithm, and allows us to describe the resulting TTC allocation. The set of students who are assigned to c is $\mathcal{T}^c\left(\gamma;t(c)\right)$, where

$$\begin{aligned} t(c)&\stackrel{def}{=}\max\left\{t^{(\ell)}\mid c\in\mathcal{C}^{(\ell)}\right\}\\ &=\sup\left\{\tau\mid\eta\left(\mathcal{T}^c\left(\gamma;\tau\right)\right)<q_c\right\} \end{aligned}$$

denotes the time that college c fills its capacity.[17] This allows us to formally define the TTC mechanism for a continuum economy.

[16] The trade balance equations hold even if students find some colleges unacceptable.

[17] This means that for all colleges c, γ_c is constant on $[t(c),1]$.

Definition 9. Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy. The *TTC assignment for* $\mathcal{E}$ is given by $\mu(c) = \mathcal{T}^c(\gamma; t(c))$ where γ is a valid TTC path for $\mathcal{E}$.

In the following sections we give a simpler characterization of the TTC allocation. We finish this section with our first result, which states that the TTC assignment is well defined and independent of the choice of the TTC path.

Proposition 10. *The TTC assignment fo*r a continuum economy $\mathcal{E}$ *is well defined. That is,*

(i) there exists a valid TTC path γ;

(ii) if γ and γ' are both valid TTC paths for $\mathcal{E}$ then they give the same assignment.

In 4.3, we prove part (i) of Proposition 10 by explicitly constructing a valid TTC path γ. We prove part (ii) of Proposition 10 in Appendix B.4.

4.2 A Simple Description of the TTC Assignment using Cutoffs

In this section, we show that the TTC assignment can be simply and succinctly described by n^2 cutoffs as follows.[18] Let $c, b \in \mathcal{C}$ be two colleges, and consider a student who wishes to use his priority for college b to get a seat in college c. For that, the student needs to receive a seat at college b at a time when c is still available. Denote the lowest b priority of a student that is offered a seat at b when college c has not filled its capacity by

$$\begin{aligned} p_b^c &\stackrel{def}{=} \min\left\{\gamma_b\left(t^{(\ell)}\right) \mid \ell \in \{1 \dots L\} \text{ s.t. } c \in \mathcal{C}^{(\ell)}\right\} \\ &= \inf\left\{\gamma_b(\tau) \mid \tau \in [0,1] \text{ s.t. } \eta\left(\mathcal{T}^c(\gamma;\tau)\right) < q_c \text{ and } \eta\left(\mathcal{T}^b(\gamma;\tau)\right) < q_b\right\}, \end{aligned}$$

and we refer to p_b^c as the required b priority to get a seat in c. Note that p_b^c depends on both c and b and can be fully calculated from the values of the TTC path at the end of the rounds $\left\{\gamma\left(t^{(\ell)}\right)\right\}_{\ell=1,\dots,L}$.

Given cutoffs $\boldsymbol{p} = \{p_b^c\}_{b,c\in\mathcal{C}}$, we say that the set of colleges student θ can afford via their priority at college b is

$$B_b(\theta, \boldsymbol{p}) \stackrel{def}{=} \left\{c \in \mathcal{C} \mid r_b^\theta \geq p_b^c\right\}.$$

[18] All formal definitions in this section will be given in the continuum model, but can be analogously defined in the discrete model as well.

We say that the set of colleges student θ can afford is

$$B(\theta, \boldsymbol{p}) \stackrel{def}{=} \cup_b B_b(\theta, \boldsymbol{p}) = \left\{c \in \mathcal{C} \mid \exists b \in \mathcal{C} \text{ s.t. } r_b^\theta \geq p_b^c\right\},$$

that is, a student can afford college c if there is some college b for which his r_b^θ priority is high enough to trade for a seat in c. We will sometimes refer to $B(\theta, \boldsymbol{p})$ as the *budget set* of student θ. We use this budget set structure to prove Theorem 6.

Proof of Theorem 6. Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy, and let $\boldsymbol{p} = \left\{p_c^b\right\}_{c,b \in \mathcal{C}}$ be cutoffs derived from a TTC path. Note that p_c^b is the worst rank at college b that is traded for a seat at college c. We show that the TTC assignment of a student θ is given by

$$\begin{aligned}\mu(\theta) &= Ch^\theta\left(B(\theta, \boldsymbol{p})\right) \\ &= Ch^\theta\left(\left\{c \in \mathcal{C} \mid \exists b \in \mathcal{C} \text{ s.t. } r_b^\theta \geq p_b^c\right\}\right).\end{aligned}$$

$B(\theta, \boldsymbol{p})$ is the set of available colleges when the TTC algorithm reaches student θ. To see this, observe that the TTC algorithm reaches student θ at the smallest time t such that $\gamma_b(t) = r_b^\theta$ for some b. If $c \notin B(\theta, \boldsymbol{p})$, then $r_b^\theta < p_b^c$ and college c has already filled its capacity. Observe that if $r_{b_1}^\theta = \gamma_{b_1}(t_1) \geq p_{b_1}^c$ and $r_{b_2}^\theta = \gamma_{b_2}(t_2) < p_{b_2}^c$ we must have that $t_1 < t_2$ because $\eta\left(\mathcal{T}^c(\gamma; t)\right)$ is monotonically increasing in t. Therefore, if $c \in B(\theta, \boldsymbol{p})$ the algorithm reaches θ at time t such that $\gamma_b(t) = r_b^\theta \geq p_b^c$, and college c is still available to student θ. Finally, TTC assigns students to their most preferred available school. □

This shows that TTC assignment has a simple structure. If there are $n = |\mathcal{C}|$ colleges we can describe the allocation using n^2 assignment thresholds. We remark that the need for n^2 assignment thresholds contrasts with the cutoff characterization of the deferred acceptance mechanism, which only requires n assignment thresholds Azevedo and Leshno (2016). An example for two schools is given in Figure 2. The example shows that the allocation TTC cannot be described by only n thresholds. In Section 5.2 we show that, unlike DA, we cannot identify the TTC cutoffs solely from the demand given these cutoffs.

We further find that the budget sets derived from the cutoffs have a nested structure. If a college c is in a student's budget set because of their priority at b, then every college c' that runs out after c is also in their budget set because of their priority at b. This is formally stated in the following proposition and illustrated in Figure 3.

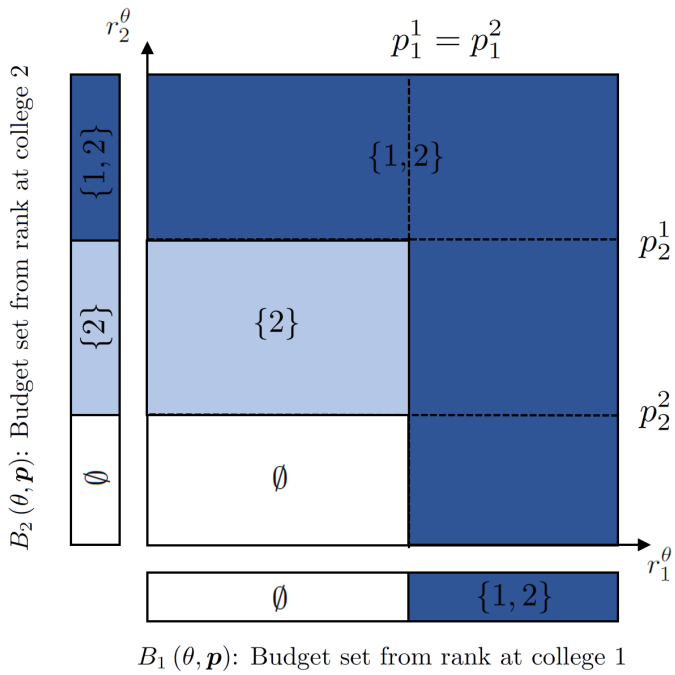

Figure 2: An example of the TTC assignment. Students' ranks at college 1 are given by the horizontal axis and ranks at college 2 by the vertical axis. The budget sets along the axis for college c list $B_c(\theta, \boldsymbol{p})$, the colleges that enter the student's budget set because of their rank at college c. Each student is assigned to his most preferred college out of the union of the budget sets from both axes, $B(\theta, \boldsymbol{p}) = \cup_c B_c(\theta, \boldsymbol{p})$.

Proposition 11. *Assume that colleges run out in the order* $1, 2, \ldots, n$. *For each student* θ *and college* b, *the set of colleges* $B_b(\theta, \boldsymbol{p})$ *student* θ *can afford via their priority at college* b *under the TTC cutoffs* $\boldsymbol{p}$ *is of the form*

$$B_b(\theta, \boldsymbol{p}) = \mathcal{C}^{(c)} = \{c, c+1, \ldots, n\}$$

for some $c \leq b$. *Moreover* $B(\theta, \boldsymbol{p}) = \cup_b B_b(\theta, \boldsymbol{p}) = \mathcal{C}^{(c)}$ *for some* c.

Proof. We note that the TTC cutoffs $\boldsymbol{p} = \left\{p_c^b\right\}_{c,b \in \mathcal{C}}$ are points on the TTC path γ, and γ is weakly decreasing in every coordinate. Since $t^{(1)} \leq t^{(2)} \leq \cdots \leq t^{(n)}$, it follows that $p_b^c \geq p_b^{c'}$ for all $c < c'$. Hence if $c \in B_b(\theta, \boldsymbol{p})$, then $r_b^\theta \geq p_b^c$ and so $r_b^\theta \geq p_b^c \geq p_b^{c'}$ and $c' \in B_b(\theta, \boldsymbol{p})$. This implies that if $c \leq c'$ for all $c' \in B_b(\theta, \boldsymbol{p})$ and $c \in B_b(\theta, \boldsymbol{p})$ then $B_b(\theta, \boldsymbol{p}) = \mathcal{C}^{(c)}$. The structure for $B(\theta, \boldsymbol{p})$ follows by taking unions, and also independently from the fact that $B(\theta, \boldsymbol{p})$ is the set of available colleges when the TTC algorithm reaches student θ. $\square$

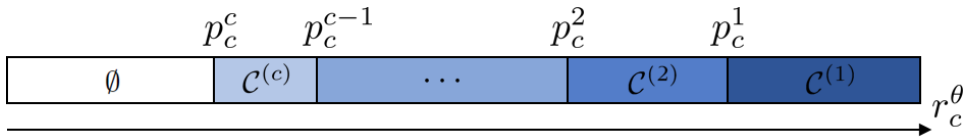

Figure 3: The colleges $B_c(\theta, \boldsymbol{p})$ that enter a student's budget set because of their rank at college c, assuming that colleges run out in the order $1, 2, \ldots, n$. Students may use their rank at college c to obtain a budget set of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(c)}$. The cutoffs p_c^b are weakly decreasing in b, and equal for all $b \geq c$, $p_c^c = p_c^{c+1} = \cdots = p_c^n$.

4.3 Calculating the TTC Assignment: A Differential Equation Characterization

In this section, we show that the TTC assignment can be computed using a differential equation. We first show that the gradient of a valid TTC must be a solution to marginal trade balance equations, which are linear equations involving the marginal density of η. The TTC path within each round is a solution to the differential equation given by these gradients. We then trace the TTC path until the capacity constraints bind, calculate the cutoffs and derive the TTC assignment.

We briefly motivate each of these steps by identifying their counterparts in the discrete TTC mechanism. The valid gradients are the continuum analogue to valid trading cycles. When there are multiple valid gradients, the choice of a gradient is analogous to the selection of trading cycles to clear. The progression of the algorithm is captured through the TTC path, and increments of the algorithm and stopping are governed by the marginal trade balance equations and the capacity equations. Note that the TTC allocation is unique (Proposition 10) and any choice of gradients yields the same allocation.

Discrete TTC	$\rightarrow$	**Continuum TTC**	**Expression**	**Equation**
Cycle	$\rightarrow$	Valid gradient d	d	$d = Hd$
Cycle selection	$\rightarrow$	Valid gradient selection	$d(x)$	$d(x) = H(x)\, d(x)$
Algorithm progression	$\rightarrow$	TTC Path	$\gamma(t)$	$\gamma'(t) = d(\gamma(t))$

We first show how to obtain an equation for the gradient of a valid TTC path at a given point $x \in [0,1]^{\mathcal{C}}$. Consider the incremental step of a TTC path γ from $\gamma(t) = x$ in

the direction $\boldsymbol{d} = \gamma'(t)$. Define[19]

$$\widetilde{H}_c^b(x) \stackrel{def}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta \left(\left\{ \theta \in \Theta \mid r^\theta \in [x - \varepsilon \cdot e^c, x) \text{ and } Ch_\theta \left(\mathcal{C}^{(\ell)} \right) = b \right\} \right),$$

that is, $\widetilde{H}_c^b(x)$ is the marginal density of students who want college b that will get an offer from college c, restricting only to students who are unassigned when $\gamma(t) = x$. Then by taking the trade balance equations over an incremental step we get the *marginal trade balance equations*

$$\sum_b d_b \cdot \widetilde{H}_b^c(x) = \sum_b d_c \cdot \widetilde{H}_c^b(x) \quad \forall c \in \mathcal{C} \tag{3}$$

where $\gamma(t) = x$ and $d_c = \gamma_c'(t)$.[20] The LHS is the marginal mass of students who want c and will be assigned to c, and the RHS is the marginal mass of students who will be offered a seat at c.

We can simplify the equation (3) by defining the matrix of relative marginal densities $H(x)$. Let $v_c = \sum_b \tilde{H}_c^b(x)$ be the measure of marginal students that will get an offer from college c, and let $\bar{v} = \max_b v_b$. For $\bar{v} > 0$, define the matrix $H(x)$ to have (b, c)-th entry[21]

$$H_b^c(x) \stackrel{def}{=} \frac{1}{\bar{v}} \tilde{H}_b^c(x) + \mathbf{1}_{b=c} \cdot \left(1 - \frac{v_c}{\bar{v}}\right).$$

This normalization makes $H(x)$ to be a (right) stochastic matrix, and allows us to equivalently write the marginal trade balance equations in vector form.

Definition 12. For for $x \in [0,1]^{\mathcal{C}}$ we say $\boldsymbol{d}$ is a *valid direction from* x if $\boldsymbol{d}$ has all non-positive entries and solves

$$\begin{aligned} \boldsymbol{d} \cdot H(x) &= \boldsymbol{d} \\ \|\boldsymbol{d}\|_1 &= 1 \end{aligned} \quad \text{marginal trade balance equations} \tag{4}$$

The first equation is equivalent to (3) and the second equation is a normalization of

[19] We use $e^c \in \mathbb{R}^n$ to denote the vector whose c-th coordinate is 1 and all other coordinates are 0.

[20] We prove this in Appendix B.7.

[21] If $v_c = 0$ then the marginal students for college c were already assigned through an offer from another college. In this case $H_b^c(x) = \mathbf{1}_{b=c}$, and we can choose a direction $d = e^c$ to move college c to point to students with lower r_c^θ. It is possible that $\bar{v} = 0$, that is $v_c = 0 \ \forall c$. In that case $H(x) = I$ and we can choose any d.

$\boldsymbol{d}$.

We now show how to calculate the TTC path as a solution to a differential equation. Suppose that $\boldsymbol{d}(\cdot) : [0,1]^{\mathcal{C}} \to \mathbb{R}^n$ is a piecewise Lipschitz continuous function whose value at x is a solution to $d(x) \cdot H(x) = \boldsymbol{d}(x), \|\boldsymbol{d}(x)\|_1 = 1$. The function $\boldsymbol{d}(\cdot)$ gives a possible direction $\boldsymbol{d}(x)$ for a TTC path γ that reaches $\gamma(t) = x$. To find a valid TTC path, we start with $\gamma(0) = \mathbf{1}$ and move in the direction given by $\boldsymbol{d}(\cdot)$ until we fill the capacity of some college. Once we complete a round we remove the colleges that filled their capacity and use the reduced problem to calculate the next round.

If η has full support[22] then $H(x)$ is irreducible[23] at every $x \in [0,1]^{\mathcal{C}}$, and there is a unique solution $\boldsymbol{d}(\cdot)$ to equations (4). Furthermore, solving for $\boldsymbol{d}(\cdot)$ is as simple as inverting a matrix, and $\boldsymbol{d}(\cdot)$ is piecewise Lipschitz continuous.

Theorem 13. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy such that η has full support. Then there exists a unique valid TTC path γ. Within each round $\gamma(\cdot)$ is given by*

$$\frac{d\gamma(t)}{dt} = \boldsymbol{d}\left(\gamma(t)\right)$$

where $d(x)$ is the unique valid direction from $x = \gamma(t)$ that satisfies (4).

For the case where η does not have full support, we appeal to a connection to Markov chain theory, which we present briefly here, and more fully in Appendix B.4.1. Fix x, and note that $H(x)$ is a stochastic matrix. We reinterpret $H(x)$ as transition probabilities of a Markov chain whose states are $\mathcal{C}$, and $\boldsymbol{d}$ as a probability distribution over $\mathcal{C}$. Under this interpretation, the marginal trade balance equations (4) say that the total probability flow out of state c is equal to the total probability flow into state c. In other words, $\boldsymbol{d}$ is a solution to (4) if and only if it is a stationary probability distribution of the Markov chain.

We can therefore derive the set of solutions to equations (4) using well known results about Markov chains. We restate them here for completeness. Given $H(x)$, a recurrent communication class is a subset $K \subseteq \mathcal{C}$, such that $H(x)$ restricted to rows and columns with coordinates in K is an irreducible matrix, and $H_c^b(x) = 0$ for every $c \in K$ and $b \notin K$. There exists at least one recurrent communication class, and two different communication classes have an empty intersection. We say that we restrict the equations (4) to a subset

[22] We say that η has full support if for every open set $A \subset \Theta$ we have $\eta(A) > 0$.

[23] A matrix is irreducible if it cannot be transformed by a relabeling of its rows and columns to a block upper triangular matrix.

K to mean $\sum_{b \in K} \boldsymbol{d}_b H_b^c(x) = \boldsymbol{d}_b \ \forall c \in K$ and $\|\boldsymbol{d}\|_1 = 1$. We refer the reader to any standard stochastic processes textbook (e.g. Karlin and Taylor 1981) for a proof of the following result.

Lemma 14. *Fix $H(x)$ and let $\mathcal{K}(x)$ be the set of recurrent communication classes. The set of solutions to the marginal trade balance equations (4) is the set of convex combinations of $\left\{d^K\right\}_{K \in \mathcal{K}(x)}$, where d^K is the unique solution to equations (4) restricted to K.*

We thus find that there is always at least one solution to the marginal trade balance equations (4), but it may not be unique. However, the Markov chain and recurrent communication class structure also gives some intuition as to the proof of part (ii) of Theorem 10, which states that the TTC allocation is unique. Lemma (14) shows that having multiple possible valid direction in the continuum is parallel to having multiple possible trade cycles in the discrete model. That is, the set of possible valid directions can be decomposed into convex combinations of mutually exclusive trades. Hence the only multiplicity in choosing valid TTC directions is whether to implement one set of trades before the others, or implement them in parallel at various relative rates. If we implement these trades in a different order we will have a different TTC path, but as in the discrete TTC, all these paths will result in the same allocation.

Therefore, we can give the following recipe for calculating the TTC path. First, we construct $\boldsymbol{d}(x)$ to be the unique valid direction from x whose support is minimal under some well behaved order, and we arbitrarily use the shortlex order.[24] Using $\boldsymbol{d}$ we construct a valid TTC path γ that follows the direction $\boldsymbol{d}$. The path γ clears trades according to the shortlex order and results in the same allocation as any other valid TTC path.

Theorem 15. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy. Then there is a valid TTC path $\gamma(\cdot)$ such that in any round γ is given by*

$$\frac{d\gamma(t)}{dt} = \boldsymbol{d}\left(\gamma(t)\right)$$

where $d(x)$ is the valid direction from x with minimal support under the shortlex order.

[24] The shortlex order over subsets of a set is a total ordering that orders subsets first by cardinality, and then by their smallest elements (Sipser, 2012). We order colleges $c_i \in \mathcal{C}$ by their indices. For example, if the set of valid directions is the set of convex combinations of $d_1 = \left[-\frac{1}{2}, 0, \frac{1}{2}\right]$ and $d_2 = [0, 1, 0]$, then we select $d(x) = [0, 1, 0]$.

4.4 Consistency with the Discrete TTC Model

In this section, we show that the continuum TTC model generalizes the standard discrete TTC model, and that the continuum TTC allocation can be used to approximate the TTC allocation on sufficiently similar economies.

To show that the continuum TTC model generalizes the discrete TTC model, we map each instance of TTC on a discrete economy into the continuum model, and show that the two produce equivalent allocations. Informally, to perform this mapping, we think of a discrete economy as a continuum economy by representing each student by a point in $\mathbb{R}^{\mathcal{C}}$ and then 'smearing' each of these points to put a finite upper bound on the density. We then run TTC on this continuum economy, and assign a student to a college if their 'smeared' point is fully assigned to the college.

Formally, consider a discrete economy $E = (\mathcal{C}, \mathcal{S}, \succ^C, \succ^S, q)$ with colleges $\mathcal{C}$, students $\mathcal{S}$, college preferences $\succ^C$, student preferences $\succ^S$, college quotas q and $N = |\mathcal{S}|$ students. We map this to the continuum economy $\mathcal{E} = (\mathcal{C}, \Theta, \eta, \frac{q}{N})$ defined as follows. For each student $s \in \mathcal{S}$ and each college $c \in \mathcal{C}$, define $r_c^s = |\{s' \in \mathcal{S} : s' \succeq^c s\}|$ to be the rank of s at c. We identify each student $s \in \mathcal{S}$ with the N-dimensional cube $I^s = \succ^s \times \prod_{c \in \mathcal{C}} \left(1 - \frac{r_c}{N}, 1 - \frac{r_c - 1}{N}\right]$ of student types in the continuum economy. Define η to be the measure with constant density $\frac{1}{N} \cdot N^N$ on $\cup_s I^s$, and density 0 everywhere else. Let $\mu_d : \mathcal{S} \to \mathcal{C}$ be the allocation given by discrete TTC on the discrete economy E, let $\mu : \Theta \to \mathcal{C}$ be the allocation given by continuum TTC on the continuum embedding $\mathcal{E}$, and let $\hat{\mu}_d : \mathcal{S} \to \mathcal{C}$ be the allocation on the discrete economy E defined in terms of the continuum allocation μ as follows:

$$\hat{\mu}_d(s) = c \qquad \Leftrightarrow \qquad \forall \theta \in I^s \quad \mu(\theta) = c\,.$$

The following proposition shows that this embedding of a discrete economy in the continuum model gives a TTC allocation that is consistent with discrete TTC.

Proposition 16. *The outcome of TTC in the continuum embedding gives the same assignment as TTC on the discrete model, $\mu_d = \hat{\mu}_d$.*

This result validates the informal intuition provided in Section 4.3 that the continuum TTC process is analogous to the standard discrete TTC algorithm, and shows that it provides a strict generalization to a larger class of economies. Intuitively, we may view the continuum TTC process as performing the same assignments as the discrete TTC process,

continuously and in fractional amounts instead of in discrete steps. See Appendix A for an example of an embedding of a discrete economy.

Next, we show that we can use a continuum economy to approximate sufficiently similar economies by proving that the TTC allocations for strongly convergent sequences of economies are also convergent. Specifically, in the full support setting, if a sequence of economies converges in total variation to a limit economy, then the TTC allocations also converge.

Theorem 17. *Consider two continuum economies $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ and $\tilde{\mathcal{E}} = (\mathcal{C}, \Theta, \tilde{\eta}, q)$, where the measures η and $\tilde{\eta}$ have a total variation distance ε. Suppose also that both measures have full support. Then the TTC allocations in these two economies differ on a set of students of measure $O(\varepsilon|C|^2)$.*

In Section 5.2, we show that changes to the priorities of a set of high priority students can affect the final allocation of other students in a non-trivial manner. This raises the question of what the magnitude of these effects are, and whether the TTC mechanism is robust to small perturbations in student preferences or college priorities. Our convergence result implies that the magnitude of the effects of perturbations is proportional to the total variation distance of the two economies, and suggests that the TTC mechanism is fairly robust to small perturbations in preferences.

5 Applications

5.1 Optimal Investment in School Quality

In this section, we examine the investment in school quality when students are assigned through the TTC mechanism. School financing has been subject to major reforms, and empirical evidence suggests that increased financing has substantial impact on school quality (Hoxby, 2001; Cellini, Ferreira, and Rothstein, 2010; Jackson, Johnson, and Persico, 2016; Lafortune, Rothstein, and Schanzenbach, 2016). Under school choice, changes in school quality will affect student preferences over schools, and therefore change the assignment of students to schools. Moreover, changes in quality may change students' budget sets (the set of schools a student chooses from), and will therefore affect how much choice students can exert. When students have heterogeneous preferences (Abdulkadiroğlu, Agarwal, and Pathak, 2015; Hastings, Kane, and Staiger, 2009) welfare

will depend on whether students can choose a school for which they have an idiosyncratically high preference.

We first provide more general comparative statics on how an increase in school quality affects the TTC assignment. Using a stylized model we examine how the optimal investment in school quality varies between school choice and neighborhood assignment.[25] Omitted proofs and derivations can be found in Appendix C.1.

5.1.1 Model with quality dependent preferences and comparative statics

We enrich our model to allow student preferences to depend on school quality investments. An economy with quality dependent preferences is given by $\mathcal{E} = (\mathcal{C}, \mathcal{S}, \eta, q)$, where $\mathcal{C} = \{1, 2, \ldots, n\}$ is the set of schools and $\mathcal{S}$ is the set of student types. A student $s \in \mathcal{S}$ is given by $s = (u^s(\cdot \mid \cdot), r^s)$, where $u^s(c \mid \delta)$ is the utility of student s for school c given the quality of each school $\delta = \{\delta_c\}$, and r^s_c is the student's rank at school c. We assume $u^s(c \mid \cdot)$ is differentiable, increasing in δ_c and non-increasing in δ_b for any $b \neq c$. The measure η over $\mathcal{S}$ specifies the distribution of student types. School capacities are $q = \{q_c\}$, where $\sum q_c < 1$.

For a fixed school quality δ, we denote the induced economy by $\mathcal{E}_\delta = (\mathcal{C}, \Theta, \eta_\delta, q)$, where η_δ is the induced distribution over Θ.[26] We assume that for any δ the induced η_δ has a Lipschitz continuous non-negative density ν_δ that is bounded below on its support and depends smoothly on δ. We denote the TTC allocation given δ by μ_δ, and the associated cutoffs by $\{p^c_b(\delta)\}_{c \in \mathcal{C}}$. We omit the dependence on δ when it is clear from context.

The following proposition describes how the TTC allocation μ_δ changes when we slightly increase δ_ℓ, making school ℓ more popular. We consider changes that do not change the strict order of school run-out times and without loss of generality assume that colleges run out in the order $1, 2, \ldots, n$, or $p^c_b(\delta) > p^{c+1}_b(\delta)$ for all $b > c$. This is equivalent to assuming that schools are numbered in order of their run-out times.

Proposition 18. *Suppose $\mathcal{E} = (\mathcal{C}, \mathcal{S}, \eta, q)$ with δ induces an economy $\mathcal{E}_\delta$ such that the TTC cutoffs have a strict runout order $p^c_b(\delta) > p^{c+1}_b(\delta)$ for all $b > c$. Suppose $\hat{\delta}$ has*

[25] A related question is whether the competition between schools arising from school choice gives incentives for schools to improve their quality. This problem was studied by Hatfield, Kojima, and Narita (2016), who showed that under TTC it is possible that a school receives lower ranked students if its quality improves.

[26] To make student preferences strict we arbitrarily break ties in favor of school with lower index. We assume students receives $-\infty$ utility from being unassigned, so all schools are acceptable.

higher school ℓ quality $\delta_\ell \leq \hat{\delta}_\ell$, the same quality $\delta_b = \hat{\delta}_b$ for $b \neq \ell$, and $\mathcal{E}_{\hat{\delta}}$ has the same runout order, i.e. $p_b^c\left(\hat{\delta}\right) \geq p_b^{c+1}\left(\hat{\delta}\right)$. Then when we change from δ to $\hat{\delta}$ the cutoffs $p_b^c(\cdot)$ change as follows:

- *p_ℓ^c decreases for all $c < \ell$, i.e., it becomes easier to trade into more popular schools using priority at ℓ.*
- *If $\ell = 1$, p_b^1 increases for all b, i.e., it becomes harder to trade into ℓ.*
- *If $\ell = n = 2$, p_2^2 increases and p_b^1 decreases for all b, i.e., it becomes easier to trade into 1.*
- *In all other cases, the cutoff p_b^c can either increase or decrease.*

Figure 4 illustrates the effect of improving the quality of school $\ell = 2$ when $\mathcal{C} = \{1, 2\}$. Notice that small changes in the cutoffs can result in individual students' budget sets growing or shrinking by more than one school. In general, if the TTC cutoffs change slightly then there will be students whose budget set switches between $\mathcal{C}^{(b)}$ and $\mathcal{C}^{(c)}$ for every pair of schools $b \neq c$.

With additional structure we can avoid the ambiguous comparative statics in Proposition 18. Consider the logit economy where students' utilities for each school c are randomly distributed as a logit with mean δ_c, independently of priorities and utilities for other schools. That is, utility for school c is given by $u^s(c \mid \delta) = \delta_c + \varepsilon_{cs}$ with η chosen so ε_{cs} are i.i.d. EV shifted to have mean 0 (McFadden, 1973). Schools have uncorrelated uniform priorities over the students. This model allows us to capture a fixed utility term δ_c that can be impacted by investment together with heterogeneous idiosyncratic taste shocks. Under the logit economy we have closed form expressions for the TTC cutoffs, given in Proposition 19, which allow us to describe the comparative statics.

Proposition 19. *Under the logit economy with fixed qualities δ the TTC cutoffs p_b^c for $b \geq c$ are given by*

$$p_b^c = \left(\frac{\prod_{c' < c} p_{c'}^{c-1} - \rho_c \pi_c}{\prod_{c' < c} p_{c'}^{c'}} \right)^{\pi^{b|c}} \tag{5}$$

where $\pi^{b|c}$ is the probability that a student chooses school b given budget set $\mathcal{C}^{(c)}$, $\rho_c = \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}$ is the relative residual capacity for school c, $\pi_c = \sum_{c' \geq c} e^{\delta_{c'}}$ normalizes ρ_c for

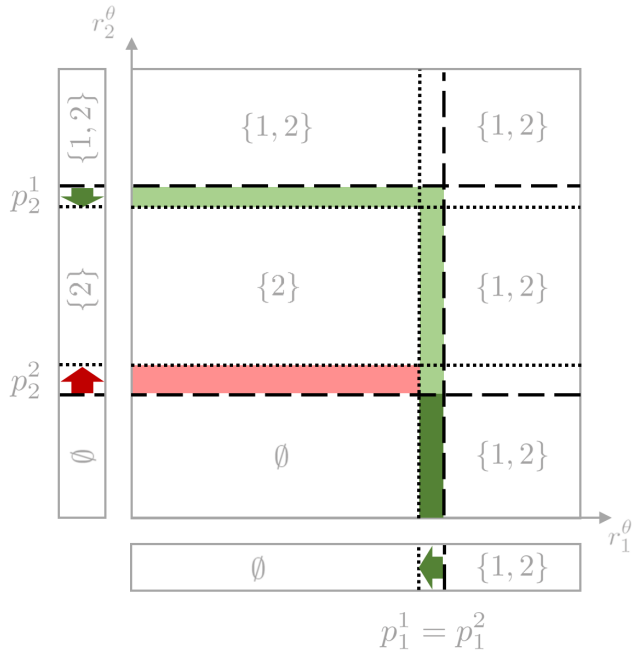

Figure 4: The effect of an increase in the quality of school 2 on TTC cutoffs and budget sets. The dashed lines indicate the initial TTC cutoffs, and the dotted lines indicate the TTC cutoffs given the increased quality of school 2. $p_1^1 = p_1^2$ decreases, p_2^1 decreases and p_2^2 increases. Students in the colored sections receive a different budget set before and after the change. Students in the dark green section improve to have the budget set $\{1,2\}$ instead of $\emptyset$, students in the light green section improve to $\{1,2\}$ from $\{2\}$, and students in the red section receive a worse budget set, going from $\{2\}$ to $\emptyset$.

when the set of available schools is $\mathcal{C}^{(c)}$, and the schools are indexed in the run-out order $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \cdots \leq \frac{q_n}{e^{\delta_n}}$.

Figure 5 illustrates how the TTC cutoffs change with an increase in the quality of school ℓ. Using equation (5), we derive closed form expressions for $\frac{dp_b^c}{d\delta_\ell}$, which can be found in Appendix C.1. The TTC cutoffs for more popular schools decrease with δ_ℓ, that is, $\frac{dp_b^c}{d\delta_\ell} < 0$ for $c < \ell$, and almost all the TTC cutoffs p_b^ℓ for school ℓ increase with δ_ℓ.

5.1.2 Optimal investment in school quality

Suppose a social planner wishes to select optimal quality levels δ for economy $\mathcal{E}$. The social welfare of students given quality levels δ and assignment μ is

$$U(\delta) = \int_{s \in S} u^s(\mu(s) \mid \delta)\, d\eta.$$

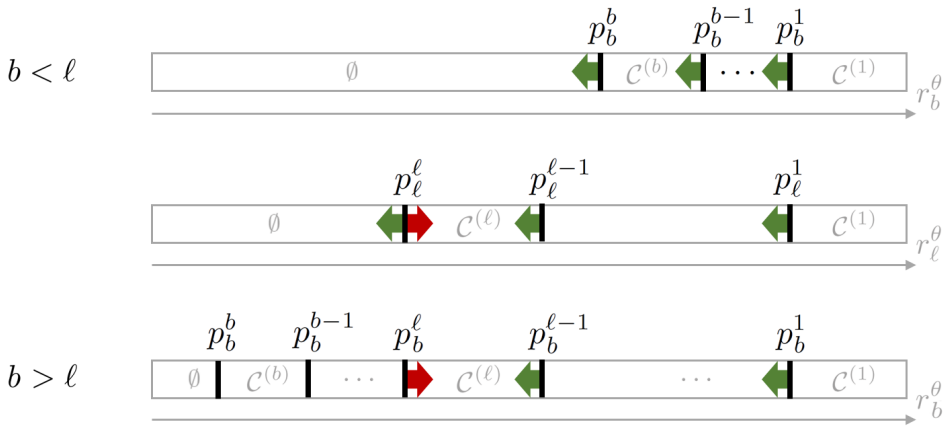

Figure 5: The effects of changing the quality δ_ℓ of school ℓ on the TTC cutoffs p_b^c. If $c < \ell$ then $\frac{dp_b^c}{d\delta_\ell} < 0$ for all $b \geq c$, so it becomes easier to get into the more popular schools. If $c > \ell$ then $\frac{dp_b^c}{d\delta_\ell} = 0$. If $c = \ell$ then $\frac{dp_b^c}{d\delta_\ell} = \frac{dp_b^\ell}{d\delta_\ell} > 0$ for all $b > \ell$, and p_ℓ^ℓ may increase or decrease depending on the specific problem parameters. Note that although p_b^c and p_ℓ^c look aligned in the picture, in general it does not hold that $p_b^c = p_\ell^c$ for all b.

The social planner's objective is to maximize $U(\delta) - \sum \kappa_c(\delta_c)$ where $\sum \kappa_c(\delta_c)$ is the cost of quality level δ.

First consider investment under neighborhood assignment μ_{NH}, which assigns each student to a fixed school regardless of quality and preferences and fills the capacity of each school. Social welfare for the logit economy is

$$U_{NH}(\delta) = \sum_c q_c \cdot \delta_c,$$

because $\mathbb{E}[\varepsilon_{cs}] = 0$. The marginal welfare gain for increasing δ_ℓ is $\frac{dU_{NH}}{d\delta_c} = q_\ell$, as an increase in the school quality benefits each of the q_ℓ students assigned to school ℓ. The optimal investment solves$\kappa_\ell'\left(\delta_\ell^{NH}\right) = q_\ell$ for each ℓ.

When the assignment is generated by TTC we can express student social welfare by considering the budget set formulation of TTC. Assume the schools are indexed in the run-out order given some fixed δ. A student who is offered the budget set $\mathcal{C}^{(c)} = \{c, \ldots, n\}$ is assigned to the school $b = \arg\max_{c \in \mathcal{C}^{(c)}} \{\delta_c + \varepsilon_{cs}\}$, and the logit distribution implies that their utility is $U^c = \ln\left(\sum_{c' \geq c} e^{\delta_{c'}}\right)$ (Small and Rosen, 1981). Let N^c be the mass of agents with budget set $\mathcal{C}^{(c)}$. Social welfare under the TTC assignment given quality δ simplifies to

$$U_{TTC}(\delta) = \sum_c N^c \cdot U^c.$$

This expression has an intuitive interpretation. Under school choice students can select schools for which they have favorable idiosyncratic taste shocks, and these shocks will be higher on average for students with a bigger budget set. Thus, student welfare depends not only on the final assignment of students, but also on the extent to which students can choose their assignment. The marginal effect of increasing δ_ℓ thus has an additional term which corresponds to the effect on students' budget sets.

Proposition 20. *For the logit economy, the increase in social welfare under TTC $U_{TTC}(\delta)$ from a marginal increase in δ_ℓ is given by*

$$\frac{dU_{TTC}}{d\delta_\ell} = q_\ell + \sum_{c \leq \ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c.$$

Under neighborhood assignment $\frac{dU_{NH}}{d\delta_c} = q_\ell$.

The following examples show how the optimal investment under TTC can differ from the optimal investment under neighborhood assignment.

The intuition for Proposition 20 is that a marginal change $d\delta_\ell$ in the quality of school ℓ will have two effect. It will change the utility of the q_ℓ students assigned to ℓ by $d\delta_\ell$, which is the effect under neighborhood assignment. The second term captures the additional effect of the change in the number of students offered each budget set. This term can be negative, as an increase in the quality of a school can reduce the number of students who are offered a budget set that includes the school. The following example illustrates this.

Example 21. Consider a logit economy with two schools and $q_1 = q_2 = 1/4$, and assume the planner is constrained to choose quality levels δ such that $\delta_1 + \delta_2 = 2$ and $\delta_1, \delta_2 \geq 0$. Under neighborhood assignment $U_{NH} = 0.5$ for any choice of δ.

Under TTC assignment the unique optimal quality is $\delta_1 = \delta_2 = 1$, yielding $U_{TTC} = \frac{1}{2} \cdot (1 + \ln(2)) \approx 0.85$, and each student is assigned to the school for which he has higher idiosyncratic taste. If $\delta_1 \neq \delta_2$ welfare is lower under TTC since less students choose the school for which they have higher idiosyncratic taste, either because they prefer the higher quality school or because their budget set only includes the lower quality school. For instance, given $\delta_1 = 2$, $\delta_2 = 0$ welfare is $U_{TTC} = \left(\frac{1}{4} + \frac{1}{4e^2}\right) \log(1 + e^2) \approx 0.60$.

Under Deferred Acceptance (DA) assignment the unique optimal quality is also $\delta_1 = \delta_2 = 1$, yielding $U_{DA} \approx 0.56$. This is strictly lower than the welfare under TTC because under DA offers many students a budget set of only one school.

In Example 21 the planner would maximize welfare by making all schools of equal quality to induce students to choose the school that gives them a higher idiosyncratic term. When schools differ in their quality there will be less benefit from sorting students according to their idiosyncratic preferences. When schools are of different sizes, under neighborhood assignment the planner would like to invest more in bigger schools that benefit more students. Under TTC it is optimal to make the schools of different quality to equalize relative demand and allow each student a choice between both schools.

Example 22. Consider a logit economy with two schools and $q_1 = 1/4$, $q_2 = 1/8$, and assume the planner is constrained to choose quality levels δ such that $\delta_1 + \delta_2 = 2$ and $\delta_1, \delta_2 \geq 0$. Under neighborhood assignment welfare is maximized by $\delta_1 = 2$, $\delta_2 = 0$, yielding $U_{NH} = 0.5$. Under TTC assignment the unique optimal quality is $\delta_1 = 1 + \frac{1}{2}\ln(2)$, $\delta_2 = 1 - \frac{1}{2}\ln(2)$, yielding $U_{TTC} = \frac{3}{8}\log\left(\frac{3e}{\sqrt{2}}\right) \approx 0.66$. Given $\delta_1 = 2$, $\delta_2 = 0$ welfare is $U_{TTC} \approx 0.60$. Under DA assignment the unique optimal quality is $\delta_1 = 2$, $\delta_2 = 0$, yielding $U_{DA} \approx 0.52$.

The optimal quality levels under TTC in Example 22 imply that students have a 2/3 chance of preferring school 1, and therefore all assigned students are offered a choice between both schools. Increasing δ_1 further (and decreasing δ_2) would increase welfare holding the assignment fixed, but would result in lower welfare because of deteriorating the sorting of students to schools.

Finally, consider a central school board with a fixed amount of capital K to invest in the n schools, and the cost of quality δ_c is the convex function $\kappa_c(\delta_c) = e^{\delta_c}$.[27] Using Proposition 20 we solve for optimal investment in school quality. We find that social welfare is maximized when the amount invested in each school is proportional to the number of seats at the school.

Proposition 23. *Social welfare is uniquely maximized when the amount κ_c invested in school c is proportional to the capacity q_c, that is, $\kappa_c = e^{\delta_c} = \frac{q_c}{\sum_b q_b} K$.*

[27] Note that κ_c is the total school funding. This is equivalent to setting student utility of school c to be to $\log(\kappa_c) = \log(\kappa_c/q_c) + \log(q_c)$, which is the log of the per-student funding plus a fix school utility that is larger for bigger schools.

Under optimal investment, the resulting TTC assignment is such that every student receives their top choice school, but schools with higher capacity have better quality than schools with lower capacity. This allows the TTC mechanism to offer assigned students a choice between all schools. Moreover, every assigned student has the same ex ante expected utility, regardless of their ranking.

5.2 Bossiness of the TTC Priorities and Optimal Tie-Breaking

To better understand the role of priorities in the TTC mechanism, we examine how the TTC assignment changes with changes in the priority structure. Notice that any student θ whose favorite college is c and who is within the q_c highest ranked students at c is guaranteed admission to c. In the following example, we consider changes to the relative priority of such highly ranked students and find that these changes can have an impact on the allocation of other students, without changing the allocation of any student whose priority changed.

Example 24. The economy $\mathcal{E}$ has two colleges $1, 2$ with capacities $q_1 = q_2 = q$. A measure $1/2$ of students prefer college 1, and a measure $1/2$ of students prefer college 2. Student ranks at each college are uniformly distributed on $[0, 1]$ independently for each college and independently of preferences. The TTC algorithm ends after a single round, and the resulting allocation is given by $p_1^1 = p_1^2 = p_2^1 = p_2^2 = \sqrt{1-2q}$.

Consider the set of students $\left\{\theta \mid r_c^\theta \geq m \ \forall c\right\}$ for some $m > 1 - q$. Any student in this set is assigned to his top choice, regardless of his rank. Suppose we construct an economy $\mathcal{E}'$ by arbitrarily changing the rank of students within the set, subject to the restriction that their ranks must remain in $[m, 1]^2$.[28] The range of possible TTC cutoffs for $\mathcal{E}'$ is given by[29] $p_1^1 = p_1^2$, $p_2^1 = p_2^2$ where

$$p_1^1 \in [\underline{p}, \bar{p}]$$
$$p_2^2 = \frac{1-2q}{p_1^1}$$

for $\underline{p} = \sqrt{(1-2m+2m^2)(1-2q)}$ and $\bar{p} = \sqrt{\frac{1-2q}{1-2m+2m^2}}$. Figure 6 illustrates the range of possible TTC cutoffs for $\mathcal{E}'$ and the economy $\bar{\mathcal{E}}$ for which TTC obtains the extreme cutoffs.

[28] The remaining students still have ranks distributed uniformly on the complement of $[1-r, 1]^2$.
[29] The derivation can be found in Appendix C.2.

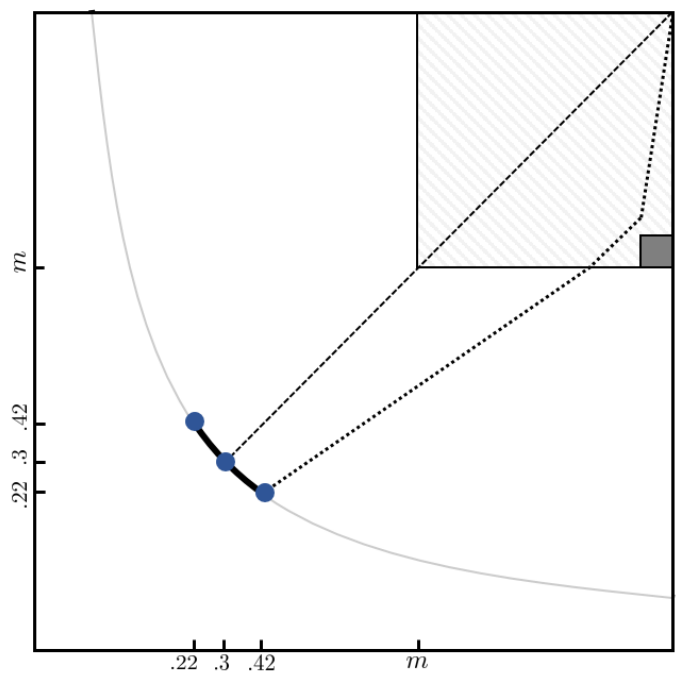

Figure 6: The range of possible TTC cutoffs in example 24 with $q = 0.455$ and $m = 0.6$. The points depict the TTC cutoffs for the original economy and the extremal cutoffs for the set of possible economies $\mathcal{E}'$, with the range of possible TTC cutoffs for $\mathcal{E}'$ given by the bold curve. The dashed line is the TTC path for the original economy. The shaded squares depict the changes to priorities that generate the economy $\bar{\mathcal{E}}$ which has the extremal cutoffs. In $\bar{\mathcal{E}}$ the priority of all top ranked students is uniformly distributed within the smaller square. The dotted line depicts the TTC path for $\bar{\mathcal{E}}$, which results in cutoffs $p_1^1 = \sqrt{\frac{1-2q}{1-2m+2m^2}} \approx 0.42$ and $p_2^2 = \sqrt{(1-2q)(1-2m+2m^2)} \approx 0.22$.

Example 24 has several implications. First, it implies that it is not possible to directly compute TTC cutoffs from student demand. The set of cutoffs such that student demand is equal to school capacity (depicted by the grey curve in Figure 6) are the cutoffs that satisfy $p_1^1 = p_1^2$, $p_2^1 = p_2^2$ and $p_1^1 p_2^2 = 1 - 2q$. Under any of these cutoffs the students in $\left\{\theta \mid r_c^\theta \geq m \ \forall c\right\}$ have the same demand, but the resulting TTC outcomes are different. It follows that the mechanism requires more information to determine the allocation. However, the changes in TTC outcomes are small if $1 - m$ is small, which is implied by Theorem 17.

A second implication is that the TTC priorities may be 'bossy' in the sense that changing the relative priority of high priority students can affect the allocation of other students, even when all high priority students receive the same assignment. Notice that in all the economies considered in Example 24, we only changed the relative priority within the set $\left\{\theta \mid r_c^\theta \geq m \ \exists c\right\}$, and all these students were always assigned to their top

choice. However, these changes resulted in a different allocation for low priority students. For example, if $q = 0.455$ and $m = 0.4$, a student θ with priority $r_1^\theta = 0.35$,$r_2^\theta = 0.1$ could possibly receive his first choice or be unassigned depending on the choice of $\mathcal{E}'$. Such changes to priorities may naturally arise when there are many indifferences in student priorities, and tie-breaking is used. Since priorities are bossy, the choice of tie-breaking between high-priority students can have indirect effects on the allocation of low priority students.

Example 24 can also be used to compare TTC with the Clinch and Trade (C&T) mechanism, introduced by Morrill (2015b). The C&T mechanism identifies students whose favorite school is c and have priority $r_c^\theta \geq 1 - q$ and allows them to immediately "clinch" and be assigned to c without doing a trade. Morrill (2015b) gives an example where the C&T allocation has fewer blocking pairs than the TTC allocation. The fact that allowing students to clinch can change the allocation can be interpreted as another example for the bossiness of priorities under TTC: we can equivalently implement C&T by running TTC on a changed priority structure where students who clinched at school c have higher rank at c than any other student.[30] The following example builds on Example 24 and shows that C&T may produce more blocking pairs than TTC.

Example 25. Economy $\mathcal{E}_1$ is the same as $\bar{\mathcal{E}}$, except that school 2 rank is redistributed among students with $r_2^\theta \leq \underline{p}$ so that students with $r_1^\theta \geq \bar{p}$ have higher school 2 rank.[31] The C&T allocation for $\mathcal{E}_1$ is given by $p_1^1 = p_2^2 = 0.3$, while TTC gives $p_1^1 = \bar{p}$ and $p_2^2 = \underline{p}$ (and under both $p_1^1 = p_1^2$, $p_2^1 = p_2^2$). Under TTC unmatched students will form blocking pairs only with school 2, while under C&T all unmatched students will form a blocking pair with either school.

5.3 Comparing Top Trading Cycles and Deferred Acceptance

Both TTC and Deferred Acceptance (DA) (Gale and Shapley, 1962) are strategyproof, but differ in that TTC is efficient whereas DA is stable. In theory, the choice between the mechanisms requires a trade-off between efficiency and stability (this trade-off is evident in Example 21). Kesten (2006); Ehlers and Erdil (2010) show the two mechanisms are

[30] For brevity, we abstract away from certain details of C&T mechanism that are important when not all schools run out at the same round.

[31] Specifically, select $\ell_1 < \ell_2$. Among students with $r_2^\theta \leq \underline{p}$ and $r_1^\theta \geq \bar{p}$ the school 2 rank is distributed uniformly in the range $[\ell_2, \underline{p}]$. Among students with $r_2^\theta \leq \underline{p}$ and $r_1^\theta < \bar{p}$ the school 2 rank is distributed uniformly in the range $[0, \ell_1]$. Within each range r_1^θ and $r_2^\theta$are still independent. See Figure 9 for an illustration.

equivalent only under strong conditions that are unlikely to hold in practice. However, Pathak (2016) evaluates the two mechanisms on application data from school choice in New Orleans and Boston, and reports that the two mechanisms produced similar outcomes. Pathak (2016) conjectures that neighborhood priority leads to correlation between student preferences and school priorities that may explain the similarity between the TTC and DA allocations.

We consider a simple model with neighborhood priority to evaluate the effect of the resulting correlation between student preferences and school priorities.[32] There are n neighborhoods, each with one school and a mass q of students. Schools have capacities $q_1 \leq \cdots \leq q_n = q$, and each school gives priority to students in their neighborhood. For each student, the neighborhood school is their top ranked choice with probability α. With probability $1 - \alpha$ the students ranks the neighborhood school in position k drawn uniformly at random from $\{2, 3, \ldots, n\}$. Student preference ordering over non-neighborhood schools are drawn uniformly at random. This model supports the conjecture of Pathak (2016), as the proportion of students whose assignments are the same under both mechanisms scales linearly with the probability of preference for the neighborhood school α.

Proposition 26. *The proportion of students who have the same assignments under TTC and DA is given by*

$$\alpha \frac{\sum_i q_i}{nq}.$$

Proof. We use the methodology developed in Section 4 and in Azevedo and Leshno (2016) to find the TTC and DA allocations respectively. Students with priority are given a lottery number uniformly at random in $\left[\frac{1}{2}, 1\right]$, and students without priority are given a lottery number uniformly at random in $\left[0, \frac{1}{2}\right]$, where lottery numbers at different colleges are independent. For all values of α, the TTC cutoffs are given by $p_j^i = p_i^j = 1 - \frac{q_i}{2q}$ for all $i \leq j$, and the DA cutoffs are given by $p_i = 1 - \frac{q_i}{2q}$. The derivations of the cutoffs can be found in Appendix C.3.

The students who have the same assignments under TTC and DA are precisely the students at neighborhood i whose ranks at school i are above $1 - \frac{q_i}{2q}$, and whose first choice school is their neighborhood school. This set of students comprises an $\alpha \frac{\sum_i q_i}{nq}$ fraction of

[32] Che and Tercieux (2015b) also show that when there are a large number of schools with a single seat per school and preferences are random both DA and TTC are asymptotically efficient and stable and give asymptotically equivalent allocations. As Example 21 shows, these results do not hold when there are many students and a few large schools.

the entire student population, which scales proportionally with the correlation between student preferences and school priorities. □

6 Discussion

The cutoff characterization of Top Trading Cycles suggest the following way of communicating the outcome of TTC to students and their families. For each pair of schools b, c, calculate p_b^c to be the (percentile) b-rank of the lowest b-ranked student who traded a seat at b for a seat at c. Publish the n^2 cutoff ranks p_b^c, and declare that students can choose to be admitted to college c if there exists some college b for which they are above the cutoff p_b^c. We hope that this method of communicating TTC will make the mechanism more palatable to students and their parents, and facilitate a more informed comparison with the Deferred Acceptance mechanism, which also has a cutoff structure. The DA cutoff structure is simpler in that it only requires n cutoffs, while our examples show that the TTC allocation cannot be described by only n cutoffs.

The model assumes for simplicity that all students and schools are acceptable. It can be naturally extended to allow for unacceptable students or schools by erasing from student preferences any school that they find unacceptable or that finds them unacceptable. Type-specific quotas can be incorporated, as in Abdulkadiroğlu and Sönmez (2003), by adding type-specific capacity equations and erasing from the preference list of each type all the schools which do not have remaining capacity for their type.

In many school choice systems, indifferences in school priority are broken using tie-breaking lotteries. Our model can be used to calculate the TTC outcome given a tie-breaking rule. In Section 5.2 we characterize all the possible TTC outcomes for a class of tie-breaking rules, and find that the choice of tie-breaking rule can have significant effect on the allocation. We leave the problem of determining the optimal choice of tie-breaking lottery for future research.

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A Example: Embedding a discrete economy in the continuum model

Consider the discrete economy $E = \left(\mathcal{C}, \mathcal{S}, \succ^{\mathcal{S}}, \succ^{\mathcal{C}}, q\right)$ with two colleges and six students, $\mathcal{C} = \{1, 2\}$, $\mathcal{S} = \{a, b, c, u, v, w\}$. College 1 has capacity $q_1 = 4$ and 2 has capacity $q_2 = 2$. The college preferences and student preferences are given by

$$\begin{aligned} 1 &: a \succ u \succ b \succ c \succ v \succ w, \\ 2 &: a \succ b \succ u \succ v \succ c \succ w, \\ a, b, c &: 1 \succ 2, \\ u, v, w &: 2 \succ 1. \end{aligned}$$

In Figure 8, we display three TTC paths for the continuum embedding $\mathcal{E}$ of the discrete economy E. The first path γ_{all} corresponds to clearing all students in recurrent communication classes, that is, all students in the maximal union of cycles in the pointing graph. The second path γ_1 corresponds to taking $K = \{1\}$ whenever possible. The third path γ_2 corresponds to taking $K = \{2\}$ whenever possible. We remark that the third path gives a different first round cutoff point p^1, but all three paths give the same allocation.

A.1 Calculating the TTC paths

We first calculate the TTC path in the regions where the TTC paths are the same.

In what follows, we will let $\tilde{H}$ be the matrix with (i, j)th entry $\tilde{H}_i^j$, the marginal density of students who want college j and get an offer from college i. Let H be the matrix with (i, j)th entry $\frac{1}{\overline{v}}\tilde{H}_i^j + \mathbf{1}_{i=j}\left(1 - \frac{v_i}{\overline{v}}\right)$, where $v_i = \sum_j \tilde{H}_i^j$ is the ith row sum of $\tilde{H}$, and $\overline{v} = \max_i v_i$ is the largest row sum of $\tilde{H}$, as defined in Section 4.3.

At every point (x_1, x_2) with $\frac{5}{6} < x_1 \leq x_2 \leq 1$ the $\tilde{H}$ matrix is $\begin{bmatrix} x_2 - \frac{5}{6} & 0 \\ x_1 - \frac{5}{6} & 0 \end{bmatrix}$, so $v_1 = x_2 - \frac{5}{6} > v_2$ and $H = \begin{bmatrix} 1 & 0 \\ \frac{6x_1 - 5}{6x_2 - 5} & \frac{6x_2 - 6x_1}{6x_2 - 5} \end{bmatrix}$. Hence $d = [-1, 0]$ is the unique (non-positive) gradient satisfying $dH = d$ and $d\mathbf{1} = 1$, and the TTC path is defined uniquely for $t \in \left[0, \frac{1}{6}\right]$ by $\gamma(t) = (1 - t, 1)$. This section of the TTC path starts at $(1, 1)$ and ends at $\left(\frac{5}{6}, 1\right)$.

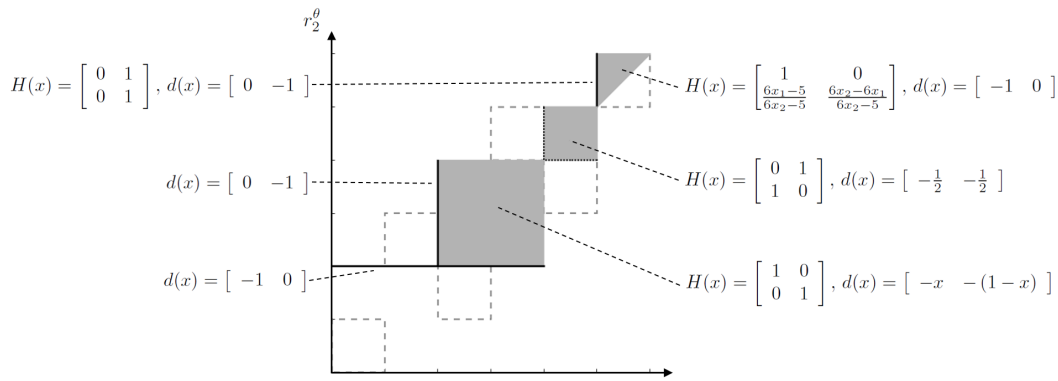


Figure 7: The matrices $H(x)$ and valid directions $d(x)$ for the continuum embedding $\mathcal{E}$. The borders of the squares corresponding to the students are drawn using dashed grey lines.

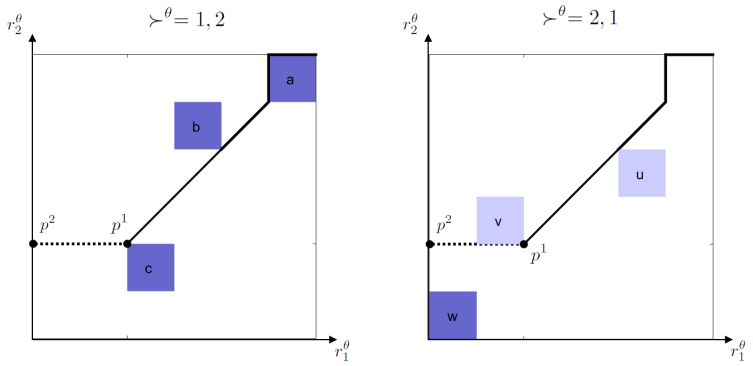
At every point $\left(\frac{5}{6}, x_2\right)$ with $\frac{5}{6} < x_2 \leq 1$ the $\tilde{H}$ matrix is $\begin{bmatrix} 0 & \frac{1}{6} \\ 0 & 0 \end{bmatrix}$, so $\overline{v} = v_1 = \frac{1}{6}, v_2 = 0$ and hence $H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Hence $d = [0, -1]$ is the unique (non-positive) gradient, and the TTC path is defined uniquely for $t \in \left[\frac{1}{6}, \frac{1}{3}\right]$ by $\gamma(t) = \left(\frac{5}{6}, \frac{7}{6} - t\right)$. This section of the TTC path starts at $\left(\frac{5}{6}, 1\right)$ and ends at $\left(\frac{5}{6}, \frac{5}{6}\right)$.

At every point (x_1, x_2) with $\frac{2}{3} < x_1, x_2 \leq \frac{5}{6}$ the $\tilde{H}$ matrix is $\begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix}$, and so $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $d = \left[-\frac{1}{2}, -\frac{1}{2}\right]$ is the unique gradient, the TTC path is defined uniquely to lie on the diagonal $\gamma_1(t) = \gamma_2(t)$, and this section of the TTC path starts at $\left(\frac{5}{6}, \frac{5}{6}\right)$ and ends at $\left(\frac{2}{3}, \frac{2}{3}\right)$.

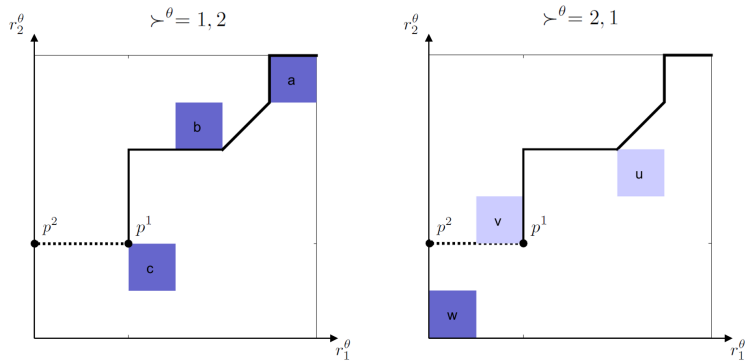
Finally, at every point $\left(x_1, \frac{1}{3}\right)$ with $0 < x_1 \leq \frac{2}{3}$, the measure of students assigned to college c_1 is at most 3, and the measure of students assigned to college c_2 is 2, so c_2 is unavailable. Hence, from any point $\left(x_1, \frac{1}{3}\right)$ the TTC path moves parallel to the x_1 axis.

We now calculate the various TTC paths where they diverge.

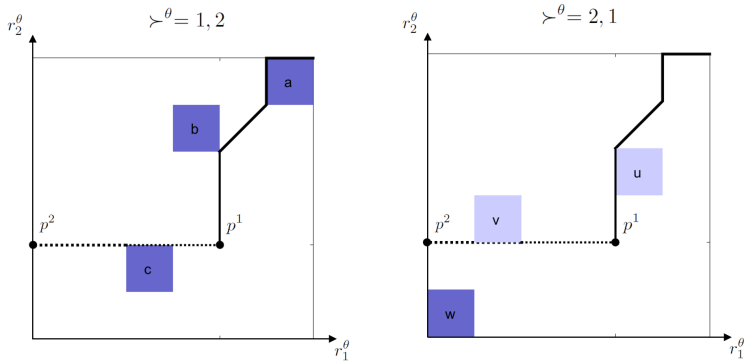
At every point $x = (x_1, x_2)$ with $\frac{1}{2} < x_1, x_2 \leq \frac{2}{3}$ the $\tilde{H}$ matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (i.e. there are no marginal students), and so $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, at every point $x = (x_1, x_2)$



TTC path γ_{all} clears all students in recurrent communication classes.



TTC path γ_1 clears all students who want college 1 before students who want college 2.



TTC path γ_2 clears all students who want college 2 before students who want college 1.

Figure 8: Three TTC paths and their cutoffs and allocations for the discrete economy in example A. In each set of two squares, students in the left square prefer college 1 and students in the right square prefer college 2. The first round TTC paths are solid, and the second round TTC paths are dotted. The cutoff points p^1 and p^2 are marked by filled circles. Students shaded light blue are assigned to college 1 and students shaded dark blue are assigned to college 2.

with $\frac{1}{3} < x_1, x_2 \leq \frac{1}{2}$ the $\tilde{H}$ matrix is $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$, and so $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, at every point $x = (x_1, x_2)$ with $\frac{1}{3} < x_1 \leq \frac{1}{2}$ and $\frac{1}{2} < x_2 \leq \frac{2}{3}$, the $\tilde{H}$ matrix is $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix}$ so again $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The same argument with the coordinates swapped gives that $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ when $\frac{1}{2} < x_1 \leq \frac{2}{3}$ and $\frac{1}{3} < x_2 \leq \frac{1}{2}$. Hence in all these regions, both colleges are in their own recurrent communication class, and any vector d satisfies $dH = d$. Finally, at every point $x = \left(\frac{1}{3}, x_2\right)$ with $\frac{1}{3} < x_2 \leq \frac{2}{3}$ the $\tilde{H}$ matrix is $\begin{bmatrix} 0 & 6x_2 - 2 \\ 0 & 0 \end{bmatrix}$, and so $H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $d = [0, -1]$ is the unique (non-positive) gradient, and the TTC path is parallel to the y axis.

The first path corresponds to taking $d = \left[-\frac{1}{2}, -\frac{1}{2}\right]$, the second path corresponds to taking $d = [-1, 0]$ and the third path corresponds to taking $d = [0, -1]$. The first path starts at $\left(\frac{2}{3}, \frac{2}{3}\right)$ and ends at $\left(\frac{1}{3}, \frac{1}{3}\right)$ where college 2 fills. The third path starts at $\left(\frac{2}{3}, \frac{2}{3}\right)$ and ends at $\left(\frac{2}{3}, \frac{1}{3}\right)$ where college 2 fills. Finally, when $x = \left(\frac{1}{3}, x_2\right)$ with $\frac{1}{3} < x_2 \leq \frac{1}{2}$, the $\tilde{H}$ matrix is $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and so $H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Hence $d = [0, -1]$ is the unique gradient, and the second TTC path starts at $\left(\frac{1}{3}, \frac{1}{2}\right)$ and ends at $\left(\frac{1}{3}, \frac{1}{3}\right)$ where college 2 fills. All three paths continue until $\left(0, \frac{1}{3}\right)$, where college 1 fills.

Note that all three paths result in the same TTC allocation, which assigns students a, b, c, w to college 1 and u, v to college 2. All three paths assign the students assigned before p^1 (students a, u, b, c for paths 1 and 2 and a, u, b for path 3) to their top choice college. All three paths assign all remaining students to college 1.

B Proofs

B.1 Definitions and Notation

Throughout this section, if $\underline{x}, \overline{x}$ are vectors, then we let $(\underline{x}, \overline{x}] = \{x : x \not\leq \underline{x} \text{ and } x \leq \overline{x}\}$ denote the set of vectors that are weakly smaller than $\overline{x}$ along every coordinate, and strictly larger than $\underline{x}$ along some coordinate. Let $K \subseteq \mathcal{C}$ be a set of colleges. For all vectors x, we let $\pi_K(x)$ denote the projection of x to the coordinates indexed by colleges in K.

In our proofs, we will be comparing allocations TTC paths across rounds. Therefore we need to expand all our previous definitions that include the notion of a top choice college so that they also specify the set of colleges available to a student. We do so below.

Recall that a *valid TTC path* γ is a weakly decreasing function $\gamma : [0,T] \to [0,1]^{\mathcal{C}}$ that satisfies the *trade balance equations* for all times t, and satisfies the *capacity equations* for some run-out sequence $\left\{\left(\mathcal{C}^{(\ell)}, t^{(\ell)}\right)\right\}_{\ell=1..L}$. For brevity, for each t we will let $\ell(t)$ denote the round being run at $\gamma(t)$, that is, $\ell(t)$ is the unique round ℓ that satisfies $t^{(l-1)} \leq t < t^{(l)}$.

Let us now incorporate information about the set of available colleges. We denote by

$$\Theta^{c|C} = \{\theta \in \Theta | \, Ch_\theta(C) = c\}$$

the set of students whose top choice in C is c, and denote by $\eta^{c|C}$ the measure of these students. That is, for $S \subseteq \Theta$, let $\eta^{c|C}(S) := \eta\left(S \cap \Theta^{c|C}\right)$. In an abuse of notation, for a set $A \subseteq [0,1]^{\mathcal{C}}$, we will often let $\eta(A)$ denote $\eta\left(\left\{\theta \in \Theta \,|\, r^\theta \in A\right\}\right)$, the measure of students with ranks in A, and let $\eta^{c|C}(A)$ denote $\eta\left(\left\{\theta \in \Theta^{c|C} \,|\, r^\theta \in A\right\}\right)$, the measure of students with ranks in A whose top choice college in C is c.

For a set of colleges C and individual colleges $b, c \in C$, let

$$\begin{aligned}\tilde{H}_b^{c|C}(x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta\left(\left\{\theta \in \Theta \;\mid r^\theta \in \left[x - \varepsilon \cdot e^b, x\right) \text{ and } Ch_\theta(C) = c\right\}\right) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta\left(\left\{\theta \in \Theta^{c|C} \;\mid r^\theta \in \left[x - \varepsilon \cdot e^b, x\right)\right\}\right)\end{aligned}$$

be the marginal density of students pointed to by college b at the point x whose top choice college in C is c.

Let $H^C(x)$ be the $|C| \times |C|$ matrix with (b,c)th entry

$$H^C(x)_{b,c} = \frac{1}{\overline{v}} \tilde{H}_b^{c|C}(x) + \mathbf{1}_{b=c}\left(1 - \frac{v_c}{\overline{v}}\right),$$

where $v_c = \sum_{d \in C} \tilde{H}_c^{d|C}(x)$ is the row sum of $\tilde{H}(x)$, and $\overline{v} = \max_c v_c$ is the maximum row sum.

Let $M^C(x)$ be the Markov chain with state space C, and transition probability from state b to state c equal to

$$H^C(x)_{b,c}.$$

We remark that such a Markov chain exists, since $H^C(x)$ is a (right) stochastic matrix for each pair C, x.

We will also need the following definitions. For a matrix H and sets of indices I, J we let $H_{I,J}$ denote the submatrix of H with rows indexed by elements of I and columns indexed by elements of J. Recall that, by Assumption 4, the measure η is defined by a probability density ν that is right-continuous, piecewise Lipschitz continuous with points of discontinuity on a finite grid. Let the finite grid be the set of points $\{x \mid x_i \in D_i \forall i\}$, where the D_i are finite subsets of $[0,1]$. Then there exists a partition $\mathcal{R}$ of $[0,1]^{\mathcal{C}}$ into hyperrectangles such that for each $R \in \mathcal{R}$ and each face of R, there exists an index i and $y_i \in D_i$ such that the face is contained in $\{x \mid x_i = y_i\}$.

The following notion of continuity will be useful, given this grid-partition. We say that a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ is *right-continuous* if $f(x) = \lim_{y \geq x} f(y)$, where x, y are vectors in $\mathbb{R}^n$ and the inequalities hold coordinate-wise. For an $m \times n$ matrix A, let $\mathbf{1}(A)$ be the $m \times n$ matrix with entries

$$\mathbf{1}(A)_{ij} = \begin{cases} 1 & \text{if } A_{ij} \neq 0, \\ 0 & \text{if } A_{ij} = 0. \end{cases}$$

We will also frequently make use of the following lemmas.

Lemma 27. *Let γ be a TTC path. Then γ is Lipschitz continuous.*

Proof. By assumption, γ is normalized so that $\|\frac{d\gamma(t)}{dt}\|_1 = 1$ a.e., and so since $\gamma(\cdot)$ is monotonically decreasing, for all c it holds that $\gamma_c(\cdot)$ has bounded derivative and is Lipschitz with Lipschitz constant L_c. It follows that $\gamma(\cdot)$ is Lipschitz with Lipschitz constant $\max_c L_c$. □

Lemma 28. *Let $C \subseteq \mathcal{C}$ be a set of colleges, and let D be a region on which $H^C(x)$ is irreducible for all $x \in D$. For each x let $A(x)$ be given by replacing the nth column of $H^C(x) - I_C$ with the all ones vector $\mathbf{1}$.*[33] *Then the function $f(x) = \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A(x)^{-1}$ is piecewise Lipschitz continuous in x.*

Proof. It suffices to show that the function which, for each x, outputs the matrix $A(x)^{-1}$ is piecewise Lipschitz continuous in x.

[33] I_C is the identity matrix with rows and columns indexed by the elements in C.

Now

$$\tilde{H}_b^{c|C}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\theta\,:\, r^\theta \geq x, r^\theta \not\geq x_b + \varepsilon \cdot e_b,\, c \succ^\theta C} \nu(\theta)\, d\theta,$$

where $\nu(\cdot)$ is bounded below on its support and piecewise Lipschitz continuous, and the points of discontinuity lie on the grid. Hence $\tilde{H}_b^{c|C}(x)$ is Lipschitz continuous in x for all b, c, and $\sum_d \tilde{H}_c^{d|C}(x)$ nonzero and hence bounded below. Further, since $H^C(x)_{b,c} = \frac{1}{\overline{v}(x)} \tilde{H}_b^{c|C}(x) + \mathbf{1}_{b=c}\left(1 - \frac{v_c(x)}{\overline{v}(x)}\right)$, where $v_c(x) = \sum_d \tilde{H}_c^{d|C}(x)$ and $\overline{v}(x) = \max_d v_d(x)$, this implies that $H^C(x)_{b,c}$ is bounded above and piecewise Lipschitz continuous in x, and therefore so is $A(x)$. Finally, since $H^C(x)$ is an irreducible row stochastic matrix for each $x \in D$, it follows that $A(x)$ is full rank and continuous. This is because when $H^C(x)$ is irreducible, $H^C(x) - I_C$ has strictly negative diagonal entries and weakly positive off-diagonal entries, and every choice of $n-1$ columns of $H^C(x) - I_C$ gives an independent set whose span does not contain the all ones vector $\mathbf{1}_C$. Therefore if we let $A(x)$ be given by replacing the nth column in $H^C(x) - I_C$ with $\mathbf{1}_C$, then $A(x)$ has full rank.

Since $A(x)$ is full rank and continuous, in each piece $det(A(x))$ is bounded away from 0, and so $A(x)^{-1}$ is piecewise Lipschitz continuous, as required. □

B.2 Proof of Theorem 13

We prove the following slightly more general theorem.

Theorem 29. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy such that $H(x)$ is irreducible for all x. Then there exists a unique valid TTC path γ. Within each round $\gamma(\cdot)$ is given by*

$$\frac{d\gamma(t)}{dt} = \boldsymbol{d}(\gamma(t))$$

where $d(x)$ is the unique valid direction from $x = \gamma(t)$ that satisfies equations (4).

Moreover, if we let $A(x)$ be obtained from $H(x) - I$ by replacing the nth column with the all ones vector $\mathbf{1}$, then

$$d(x) = \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A(x)^{-1}.$$

Proof. The main steps of the proof are as follows. We first show that $d(\cdot)$ is unique by solving equations (4) explicitly. This gives a closed form expression for $d(\cdot)$ which is Lipschitz continuous. The existence and uniqueness of $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = \boldsymbol{d}(\gamma(t))$ follows by invoking Picard-Lindelöf. Finally, we verify that the solution $\gamma(\cdot)$ is a valid

TTC path.

Consider the equations (4),

$$\begin{aligned} d(x)\,H(x) &= d(x) \\ \|d(x)\|_1 &= 1. \end{aligned}$$

When $H(x)$ is irreducible, $H(x) - I$ has strictly negative diagonal entries and weakly positive off-diagonal entries, and every choice of $n-1$ columns of $H(x) - I$ gives an independent set whose span does not contain $\mathbf{1}$. Therefore if we let $A(x)$ be given by replacing the nth column in $H(x) - I$ with $\mathbf{1}$, then $A(x)$ has full rank, and the equations (4) are equivalent to

$$\begin{aligned} d(x)\,A(x) &= \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix}, \\ \text{i.e. } d(x) &= \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A(x)^{-1}. \end{aligned}$$

Next, we invoke Lemma 28 to show that $d(x)$ is Lipschitz continuous, and so it follows from the Picard-Lindelöf theorem that there exists a unique function $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = d(\gamma(t))$. Since all valid TTC paths must satisfy the differential equation, it suffices to show that the unique solution $\gamma(\cdot)$ is a valid TTC path, that is, it satisfies the trade balance equations (1) and capacity equations (2). This is easily shown to be true by integrating over the marginal trade balance equations (4). □

B.3 Proof of Theorem 15

Proof of Theorem 15. The main steps of the proof are as follows. We first show that if we take $d(x)$ to be the valid direction from x with minimal support under the shortlex order, then $d(\cdot)$ is piecewise Lipschitz continuous. The existence and uniqueness of $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = \boldsymbol{d}(\gamma(t))$ follows by invoking Picard-Lindelöf. Finally, we verify that the solution $\gamma(\cdot)$ is a valid TTC path.

Let C be the set of available colleges. Fix a point x, and consider the set of vectors d such that $dH^C(x) = d$. We invoke the following theorem, whose proof we defer to Section B.4.

Theorem 30. *Let C be the set of available colleges, and let $\mathcal{K}(x)$ be the set of subsets $K \subseteq C$ for which $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C\setminus K}$ is the zero matrix. Then the equation $d = dH^C(x)$ has a unique solution d^K that satisfies $||d^K = 1||$ and supp $\left(d^K\right) \subseteq$*

K. Moreover, if $||d|| = 1$ and d is a solution to the equation $d = dH^C(x)$, then d is a convex combination of the vectors in $\{d^K\}_{K \in \mathcal{K}(x)}$.

It follows that if $d(x)$ is the valid direction from x with minimal support under the shortlex order, then $d(x) = d^{K(x)}$ for the element $K(x) \in \mathcal{K}(x)$ that is the smallest under the shortlex ordering. As the density $\nu(\cdot)$ defining $\eta(\cdot)$ is Lipschitz continuous, it follows that $\mathcal{K}(\cdot)$ and $K(\cdot)$ are piecewise constant. Hence we may invoke Lemma 28 to conclude that $d(\cdot)$ is piecewise Lipschitz within each piece, and hence piecewize Lipschitz in $[0,1]^{\mathcal{C}}$.

Since $d(\cdot)$ is piecewise Lipschitz, it follows from the Picard-Lindelöf theorem that there exists a unique function $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = d(\gamma(t))$. Since all valid TTC paths must satisfy the differential equation, it suffices to show that the unique solution $\gamma(\cdot)$ is a valid TTC path, that is, it satisfies the trade balance equations (1) and capacity equations (2). This is easily shown to be true by integrating over the marginal trade balance equations (4), which are equivalent to $d(x) H^C(x) = d(x)$ for all x. □

B.4 Proof of Proposition 10

B.4.1 Connection to Continuous Time Markov Chains & General Solutions for the Gradient

In Section 4.3, we showed that if a TTC path has gradient $d(x)$ at x, then $d(x)$ is a solution to the equation $d(x) H(x) = d(x)$. We also appealed to a connection with Markov chain theory to provide a method for solving for all the possible values of $d(x)$. Specifically, we showed, in Lemma 14, that if $\mathcal{K}(x)$ is the set of recurrent communication classes of $H(x)$, then the set of valid directions $d(x)$ is identical to the set of convex combinations of $\left\{d^K\right\}_{K \in \mathcal{K}(x)}$, where d^K is the unique solution to equations (4) restricted to K. We present the relevant definitions, results and proofs here in full.

Let us first present some definitions from Markov chain theory.[34] A square matrix P is a *right-stochastic matrix* if all the entries are non-negative and each row sums to 1. A *probability vector* is a vector with non-negative entries that add up to 1. Given a right-stochastic matrix P, the *Markov chain with transition matrix P* is the Markov chain with state space equal to the column/row indices of P, and a probability P_{ij} of moving to state j in one time step, given that we start in state i. Given two states i, j

[34] See standard texts such as Karlin and Taylor (1975) for a more complete treatment.

of a Markov chain with transition matrix P, we say that states i and j *communicate* if there is a positive probability of moving to state i to state j in finite time, and vice versa.

For each Markov chain, there exists a unique decomposition of the state space into a sequence of disjoint subsets $C_1, C_2, \ldots$ such that for all i, j, states i and j communicate if and only if they are in the same subset C_k for some k. Each subset C_k is called a *communication class* of the Markov chain. A Markov chain is *irreducible* if it only has one communication class. A state i is *recurrent* if, starting at i and following the transition matrix P, the probability of returning to state i is 1. A communication class is recurrent if it contains a recurrent state.

The following proposition gives a characterization of the stationary distributions of a Markov chain.

Proposition 31. *Suppose that P is the transition matrix of a Markov chain. Let $\mathcal{K}$ be the set of recurrent communication classes of the Markov chain with transition matrix P. Then for each recurrent communication class $K \in \mathcal{K}$, the equation $\pi = \pi P$ has a unique solution π^K such that $||\pi^K|| = 1$ and $supp\left(\pi^K\right) \subseteq K$. Moreover, the support of π^K is equal to K. In addition, if $||\pi|| = 1$ and π is a solution to the equation $\pi = \pi P$, then π is a convex combination of the vectors in $\{\pi^K\}_{K \in \mathcal{K}}$.*

We refer the reader to any standard stochastic processes textbook (e.g. Karlin and Taylor (1975)) for a proof of this result.

To make use of this proposition, define at each point x and for each set of colleges C a Markov chain $M^C(x)$ with transition matrix $H^C(x)$. We will relate the valid directions $d(x)$ to the recurrent communication classes of $M^C(x)$, where C is the set of available colleges. We will need the following notation and definitions. Given a vector v indexed by $\mathcal{C}$, a matrix Q with rows and columns indexed by C and subsets $K, K' \subseteq C$ of the indices, we let v_K denote the restriction of v to the coordinates in K, and we let $Q_{K,K'}$ denote the restriction of Q to rows indexed by K and columns indexed by K'. In the Markov chain with transition matrix P and state space C, a set $K \subseteq C$ is a recurrent communication class of the Markov chain if and only if the matrix $P_{K,K}$ is irreducible and $P_{K,C\setminus K}$ is the zero matrix.

The following lemma characterizes the recurrent communication classes of the Markov chain $M^C(x)$ using the properties of the matrix $H^C(x)$, and can be found in any standard stochastic processes text.

Lemma 32. *Let C be the set of available college at a point x. Then a set $K \subseteq C$ is a recurrent communication class of the Markov chain $M^C(x)$ if and only if $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C\setminus K}$ is the zero matrix.*

Proposition 31 and Lemma 32 allow us to characterize the valid directions $d(x)$.

Theorem 33. *Let C be the set of available colleges, and let $\mathcal{K}(x)$ be the set of subsets $K \subseteq C$ for which $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C\setminus K}$ is the zero matrix. Then the equation $d = dH^C(x)$ has a unique solution d^K that satisfies $||d^K = 1||$ and $supp\left(d^K\right) \subseteq K$, and its projection onto its support K has the form*

$$\left(d^K\right)_K = \left[\begin{array}{cc} \mathbf{0}^T & 1 \end{array}\right] A_K^C(x)^{-1},$$

where $A_K^C(x)$ is the matrix obtained by replacing the $(|K|-1)$th column of $H^C(x)_{K,K} - I_K$ with the all ones vector $\mathbf{1}_K$.

Moreover, if $||d|| = 1$ and d is a solution to the equation $d = dH^C(x)$, then d is a convex combination of the vectors in $\{d^K\}_{K\in\mathcal{K}(x)}$.

Proof. Proposition 32 shows that the sets K are precisely the recurrent sets of the Markov chain with transition matrix $H(x)$. Hence uniqueness of the d^K and the fact that d is a convex combination of d^K follow directly from Proposition 31. The form of the solution d^K follows from Theorem 29. □

This has the following interpretation. Suppose that there is a unique recurrent communication class K, such as when η has full support. Then there is a unique infinitessimal continuum trading cycle of students, specified by the unique direction d satisfying $d = dH(x)$. Moreover, students in the cycle trade seats from every college in K. Any college not in K is blocked from participating, since there is not enough demand to fill the seats they are offering. When there are multiple recurrent communication classes, each of the d^K gives a unique infinitessimal trading cycle of students, corresponding to those who trade seats in K. Moreover, these trading cycles are disjoint. Hence the only multiplicity that remains is to decide the order, or the relative rate, at which to clear these cycles. We will show in the next section that, as in the discrete setting, the order in which cycles are cleared does not affect the final allocation.

B.4.2 Proof of Uniqueness

In this section, we prove part (ii) of Proposition 10, that any two valid TTC paths give the same allocation. In other words, we show that the TTC allocation is unique.

The intuition for the result is the following. The connection to Markov chains shows that having multiple possible valid direction in the continuum is parallel to having multiple possible trade cycles in the discrete model. Hence the only multiplicity in choosing valid TTC directions is whether to implement one set of trades before the others, or to implement them in parallel at various relative rates. We can show that the set of cycles is independent of the order in which cycles are selected, or equivalently that the sets of students who trade with each other is independent of the order in which possible trades are executed. It follows that any pair of valid TTC paths give the same final allocation.

We remark that the crux of the argument is similar to the argument used to show that discrete TTC gives a unique allocation. However, the lack of discrete cycles and the ability to implement sets of trades in parallel both complicate the argument and lead to a rather technical proof.

The formal proof proceeds in a number of steps. We first produce a rectangular subdivision $\mathcal{R}'$ of the space $[0,1]^{\mathcal{C}}$ such that the sets of colleges that are involved in trading cycles is constant on each rectangle $R \in \mathcal{R}'$. We then formally define *cycles* in the continuum setting, and define a partial order over the cycles corresponding to the order in which cycles can be cleared under TTC. We then define the set of cycles $\Sigma(\gamma)$ associated with a valid TTC path γ. Finally, we show that the sets of cycles associated with two valid TTC paths γ and γ' are the same, $\Sigma(\gamma) = \Sigma(\gamma')$. This last step is the most involved, and hence will be presented in a number of steps.

Recall that

$$\tilde{H}_b^{c|C}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta \left(\left\{ \theta \in \Theta \mid r^\theta \in \left[x - \varepsilon \cdot e^b, x \right) \text{ and } Ch_\theta(C) = c \right\} \right)$$

is the marginal density of students pointed to by college b at the point x whose top choice college in C is c, and $H^C(x)$ is the $|C| \times |C|$ matrix with (b,c)th entry

$$H^C(x)_{b,c} = \frac{1}{\overline{v}} \tilde{H}_b^{c|C}(x) - \mathbf{1}_{b=c} \left(1 - \frac{v_c}{\overline{v}} \right),$$

where $v_c = \sum_{b \in C} \tilde{H}_c^{b|C}(x)$ is the row sum of $\tilde{H}(x)$, and $\overline{v} = \max_c v_c$ is the maximum row sum.

We begin with some observations about $\tilde{H}_c^{b|C}(\cdot)$ and $H^C(\cdot)_{bc}$. For all $b, c \in C$ the function $\tilde{H}_c^{b|C}(\cdot)$ is right-continuous on $[0,1]^{\mathcal{C}}$, Lipschitz continuous on R for all $R \in \mathcal{R}$ and uniformly bounded away from zero on its support. Hence $\mathbf{1}\left(\tilde{H}_c^{b|C}(\cdot)\right)$ is constant on R for all $R \in \mathcal{R}$. It follows that $H^C(\cdot)_{bc}$ is also right-continuous, and Lipschitz continuous on R for all $R \in \mathcal{R}$. Moreover, there exists some finite rectangular subpartition $\mathcal{R}'$ of $\mathcal{R}$ such that for all $C \subseteq \mathcal{C}$ the function $\mathbf{1}\left(H^C(\cdot)\right)$ is constant on R for all $R \in \mathcal{R}'$.

Definition 34. The partition $\mathcal{R}'$ is the minimal rectangular subpartition of $\mathcal{R}$ such that for all $C \subseteq \mathcal{C}$ the function $\mathbf{1}\left(H^C(\cdot)\right)$ is constant on R for all $R \in \mathcal{R}'$.

We now translate this to a result about recurrent communication classes of a matrix. Recall that for a square matrix Q with rows and columns indexed by C, and subsets $D, D' \subseteq C$ of the index set, $Q_{D,D'}$ denotes the restriction of Q to the rows indexed by D and columns indexed by D'.

For each $x \in [0,1]^{\mathcal{C}}$ and $C \subseteq \mathcal{C}$, let $\mathcal{K}^C(x)$ be the set of recurrent communication classes of the Markov chain $M^C(x)$. The following result is an immediate corollary of Proposition 32, since $\mathbf{1}\left(H^C(\cdot)\right)$ is constant on R for every $R \in \mathcal{R}'$, and irreducibility of a matrix Q and whether it is a zero matrix depend only on $\mathbf{1}(Q)$.

Lemma 35. *$\mathcal{K}^C(\cdot)$ is constant on R for every $R \in \mathcal{R}'$.*

For each $K \in \mathcal{K}^C(x)$, let $d^K(x)$ be the unique vector satisfying $d = d\tilde{H}^C(x)$, which exists by Theorem 33.

We now move to defining formally the notion of a (non-infinitessimal) cycle in the continuum setting.

Definition 36. A *(continuum) cycle* $\sigma = (K, \underline{x}, \overline{x})$ is a set $K \subseteq \mathcal{C}$ and a pair of vectors $\underline{x} \leq \overline{x}$ in $[0,1]^{\mathcal{C}}$. A continuum cycle is *valid* for sets of available colleges $\{C(x)\}_{x \in [0,1]^{\mathcal{C}}}$ if $K \in \mathcal{K}^{C(x)}(x)$ for all $\underline{x} < x \leq \overline{x}$.

Intuitively, a cycle is defined by two time points in a run of TTC (given by the difference between two nested hyperrectangles), which gives a set of students, and the set of colleges they most desire, and a cycle is valid if the set of colleges involved is a recurrent communication class of the associated Markov chains. We remark that a cycle is valid only with respect to sets of available colleges, where we specify a set of available colleges at each point x satisfying $\underline{x} < x \leq \overline{x}$.

We define a partial order over continuum cycles that specifies when one cycle must clear before another. To develop some intuition for why such an ordering exists, consider

an instance of TTC in the discrete setting with one college c wth capacity 2 and two students s_1, s_2. Consider the cycles $\sigma_i = c \to s_i \to c$ for $i = 1, 2$. If college c prefers student s_1 to student s_2, then if both cycles σ_1, σ_2 clear, then σ_1 must clear before σ_2. In general, some cycles must clear before other cycles, since they involve higher priority students or colleges. We extend this idea to the continuum setting as follows.

Definition 37. Let θ be a student type and $\sigma = (K, \underline{x}, \overline{x})$, $\sigma' = (K', \underline{x}', \overline{x}')$ be cycles. We say that a *student θ is in cycle σ* if $r^\theta \in (\underline{x}, \overline{x}]$[35], and a *college c is in cycle σ* if $c \in K$.

The cycle σ *blocks* the cycle σ', denoted by $\sigma \rhd \sigma'$, if at least one of the following hold:

(Blocking student) There exists a student θ in σ' who prefers a school in K to all those in K', that is, there exists θ and $c \in K \setminus K'$ such that $c \succ^\theta c'$ for all $c' \in K'$.

(Blocking college) There exists a college in σ' who prefers a positive measure of students in σ to all those in σ', that is, there exists $c \in K'$ such that $\eta\left(\theta \,|\, \theta \text{ in } \sigma,\, r_c^\theta > \overline{x}'_c\right)$.[36]

Let us now define the set of cycles associated with a run of TTC. Intuitively, an infinitessimal cycle is a minimal set of students that trades their seats at a given time, and a cycle is given by aggregating these infinitessimal cycles over some period of time. We make this formal below.

Let γ be a TTC path with run-out sequence $\left\{\left(\mathcal{C}^{(\ell)}, t^{(\ell)}\right)\right\}_\ell$. We first define the set of times that we aggregate over to form cycles, and then formally define the cycles. For each set of colleges $K \subseteq \mathcal{C}$ and each round ℓ , let $\tau^{(\ell)}(K, \gamma)$ be the set of times in $\left[t^{(\ell-1)}, t^{(\ell)}\right)$ when K is a recurrent communication class for $\tilde{H}^{\mathcal{C}^{(l)}}(\gamma(t))$. Since γ is continuous and weakly decreasing, it follows that $\tau^{(\ell)}(K, \gamma)$ is the finite disjoint union of intervals of the form $\left[\underline{t}, \overline{t}\right)$. Let $\mathcal{I}\left(\tau^{(\ell)}(K, \gamma)\right)$ denote the set of intervals in this disjoint union. We may assume that for each interval τ, $\gamma(\tau)$ is fully contained in some hyperrectangle $R \in \mathcal{R}'$.[37] Intuitively, each cycle in the TTC path γ will correspond to some time interval $\tau \in \mathcal{I}\left(\tau^{(\ell)}(K, \gamma)\right)$, and will be the set of students that trade their seats in K.

Consider a time interval $\tau = \left[\underline{t}, \overline{t}\right) \in \mathcal{I}\left(\tau^{(\ell)}(K, \gamma)\right)$. We define the cycle $\sigma(\tau) = (K, \underline{x}(\tau), \overline{x}(\tau))$ as follows. Intuitively, we want to define it simply as $\sigma(\tau) = \left(K, \gamma\left(\underline{t}\right), \gamma\left(\overline{t}\right)\right)$, but in order to minimize the dependence on γ, we define the endpoints $\underline{x}(\tau)$ and $\overline{x}(\tau)$ of the interval of ranks to be as close together as possible, while still describing the same

[35] Recall that since $r^\theta, \underline{x}$ and $\overline{x}$ are vectors, this is equivalent to saying that $r^\theta \not\leq \underline{x}$ and $r^\theta \leq \overline{x}$.

[36] We note that it is necessary but not sufficient that $\overline{x}_c > \overline{x}'_c$.

[37] This is without loss of generality, since if $\gamma(\tau)$ is not contained we can simply partition τ into a finite number of intervals $\cup_{R \in \mathcal{R}'} \gamma^{-1}(\gamma(\tau) \cap R)$, each contained in a hyperrectangle in $\mathcal{R}'$.

set of students (up to a set of η-measure 0). Formally, consider the set

$$\cup_{c\in K}\mathcal{T}^{c}\left(\gamma;\overline{t}\right)\setminus\mathcal{T}^{c}\left(\gamma;\underline{t}\right)$$

of students who are assigned in round ℓ during the time interval τ and whose top choice available school is in K. Define

$$\begin{aligned}\underline{x}\left(\tau\right) &= \max\left\{x:\gamma\left(\underline{t}\right)\leq x\leq\gamma\left(\underline{t}\right),\,\eta\left(\theta:Ch_{\theta}\left(\mathcal{C}^{(\ell)}\right)\in K,\,r^{\theta}\in\left(\underline{x}\left(\tau\right),\gamma\left(\overline{t}\right)\right]\right)=0\right\},\\ \overline{x}\left(\tau\right) &= \min\left\{x:\gamma\left(\underline{t}\right)\leq x\leq\gamma\left(\underline{t}\right):\eta\left(\theta:Ch_{\theta}\left(\mathcal{C}^{(\ell)}\right)\in K,\,r^{\theta}\in\left(\gamma\left(\underline{t}\right),\underline{x}\left(\tau\right)\right]\right)=0\right\},\end{aligned}$$

to be the points chosen to be maximal and minimal respectively such that the set of students allocated by γ during the time interval τ has the same η-measure as if $\gamma\left(\underline{t}\right)=\underline{x}\left(\tau\right)$ and $\gamma\left(\overline{t}\right)=\overline{x}\left(\tau\right)$. In other words, the set

$$\left(\cup_{c\in K}\mathcal{T}^{c}\left(\gamma;\overline{t}\right)\setminus\mathcal{T}^{c}\left(\gamma;\underline{t}\right)\right)\setminus\left\{\theta:Ch_{\theta}\left(\mathcal{C}^{(\ell)}\right)\in K,\,r^{\theta}\in\left(\underline{x}\left(\tau\right),\overline{x}\left(\tau\right)\right]\right\}$$

has η-measure 0.

In a slight abuse of notation, if $\sigma=\sigma\left(\tau\right)$ we will let $\underline{x}\left(\sigma\right)$ denote $\underline{x}\left(\tau\right)$ and $\overline{x}\left(\sigma\right)$ denote $\overline{x}\left(\tau\right)$.

Definition 38. The *set of cycles cleared by* $TTC\left(\gamma\right)$ in round ℓ, denoted by $\Sigma^{(\ell)}\left(\gamma\right)$, is given by

$$\Sigma^{(\ell)}\left(\gamma\right):=\bigcup_{K\subseteq\mathcal{C}^{(\ell)}}\bigcup_{\tau\in\mathcal{I}\left(\tau^{(\ell)}(K,\gamma)\right)}\sigma\left(\tau\right).$$

The *set of cycles cleared by* $TTC\left(\gamma\right)$, denoted by $\Sigma\left(\gamma\right)$, is the set of cycles cleared by $TTC\left(\gamma\right)$ in some round ℓ,

$$\Sigma\left(\gamma\right):=\bigcup_{\ell}\Sigma^{(\ell)}\left(\gamma\right).$$

For any cycle $\sigma\in\Sigma\left(\gamma\right)$ and time t we say that the cycle σ is *clearing at time* t if $\gamma\left(t\right)\not\leq\underline{x}\left(\sigma\right)$ and $\gamma\left(t\right)\not>\overline{x}\left(\sigma\right)$. We say that the cycle σ *is cleared at time* t or *finishes clearing at time* t if $\gamma^{(l)}\left(t\right)\leq\underline{x}\left(\sigma\right)$ with at least one equality. We remark that for any TTC path γ there may be multiple cycles clearing at a time t, each corresponding to a different recurrent set. For any TTC path γ the set $\Sigma\left(\gamma\right)$ is finite.

We first show that, given a TTC path γ, we can define available sets $C\left(x\right)$ at each point $x\in\left[0,1\right]^{\mathcal{C}}$ that are consistent with the available sets along the TTC path γ, such that every cycle $\sigma\in\Sigma\left(\gamma\right)$ is a valid cycle with respect to these available sets. We note

that this is a non-trivial exercise, since a run of the TTC algorithm gives available sets only at points x on the TTC path γ, and we are defining available sets for all $x \in [0,1]^{\mathcal{C}}$.

Lemma 39. *Let γ be a TTC path with run-out sequence $\left\{\left(\mathcal{C}^{(\ell)}, t^{(\ell)}\right)\right\}_{\ell}$, and let $\Sigma(\gamma)$ be the set of cycles cleared by $TTC(\gamma)$. For each $x \in [0,1]^{\mathcal{C}}$, let $t(x) = \max\{t\ :, \gamma(t) \geq x\}$, and let $C(x) = C^{(\ell)}$ if and only if $t(x) \in \left[t^{(\ell-1)}, t^{(\ell)}\right)$. Then*

- *$C(\gamma(t))$ is the set of available schools at time t for all $t \in \left[0, t^{(|\mathcal{C}|)}\right]$, that is, $\forall \ell, t$ s.t. $t \in \left[t^{(\ell-1)}, t^{(\ell)}\right]$, $C(\gamma(t)) = \mathcal{C}^{(\ell)}$; and*
- *every $\sigma \in TTC(\gamma)$ is a valid cycle for the sets of available colleges $C(x)$.*

Proof. The first claim holds almost trivially, since if $x = \gamma(t)$ with $t \in \left[t^{(\ell-1)}, t^{(\ell)}\right]$, then $t(x) = t$ and so $C(\gamma(t)) = \ell$ by definition. It remains to show that every $\sigma \in TTC(\gamma)$ is a valid cycle for the sets of available colleges $C(x)$.

Fix x such that $\underline{x} \leq x \leq \overline{x}$, and let $y \in \gamma([0,1])$ be the point such that $y = \gamma(t(x))$. Then, by definition, $C(x) = C(y) = C$. Moreover, y is in the image of γ, and since $\underline{x} \leq x, y \leq \overline{x}$ and the cycle σ was clearing at time t it holds that K is a recurrent communication class of $M^C(y)$. By Lemma 32 it follows that $H^C(y)_{K,K}$ is irreducible and $H^C(y)_{K,C\setminus K}$ is the zero matrix.

We want to prove that K is a recurrent communication class of $M^C(x)$. By Lemma 32 it suffices to show that $H^C(x)$ is irreducible and $H^C(x)$ is the zero matrix. Now since $x, y \in R$ for some rectangle $R \in \mathcal{R}'$, it follows from the definition of $\mathcal{R}'$ that $\mathbf{1}\left(H^C(x)\right) = \mathbf{1}\left(H^C(y)\right)$. Since, for a given matrix A, irreducibility and being the zero matrix are properties that can be identified using the matrix $\mathbf{1}(A)$, it follows that K is a recurrent communication class of $M^C(x)$. □

Fix two TTC paths γ and γ'. Our goal is to show that they clear the same sets of cycles, $\Sigma(\gamma) = \Sigma(\gamma')$, or equivalently that $\Sigma(\gamma) \cup \Sigma(\gamma') = \Sigma(\gamma) \cap \Sigma(\gamma')$. We will do this by showing that for every cycle $\sigma \in \Sigma(\gamma) \cup \Sigma(\gamma')$, if all cycles in $\Sigma(\gamma) \cup \Sigma(\gamma')$ that block σ are in $\Sigma(\gamma) \cap \Sigma(\gamma')$, then $\sigma \in \Sigma(\gamma) \cap \Sigma(\gamma')$.

We first show that this is true in a special case, which can be understood intuitively as the case when the cycle σ appears in the pointing graph during the run of $TTC(\gamma)$ and also appears in the pointing graph during the run of $TTC(\gamma')$. In terms of the continuum model, if $\sigma = (K, \underline{x}, \overline{x})$, then having σ appear in $TTC(\gamma)$ corresponds to γ passing through $\underline{x}$ and K being a recurrent communication class of the Markov chain at $\underline{x}$, and having σ appear in $TTC(\gamma')$ corresponds to γ' passing through some point $\underline{x}'$

with $\underline{x}_K = \underline{x}'_K$ and K being a recurrent communication class of the Markov chain at $\underline{x}$'. We make these ideas formal in the following lemma.

Lemma 40. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy, and let γ and γ' be two TTC paths for this economy. Let $K \subseteq \mathcal{C}$ and time $\underline{t}$ be such that at time $\underline{t}$, γ has available colleges C, γ' has available colleges C', the paths are at the same point when projected onto the coordinates K, $\gamma(\underline{t})_K = \gamma'(\underline{t})_K$, and K is a recurrent communication class of $M^C(\gamma(\underline{t}))$ and of $M^{C'}(\gamma'(\underline{t}))$. Suppose that for all colleges $c \in K$ and cycles $\sigma' \rhd \sigma$ involving college c, if $\sigma' \in \Sigma(\gamma)$, then σ' is cleared in $TTC(\gamma')$, and vice versa. Suppose also that cycle $\sigma = (K, \underline{x}, \overline{x})$ is cleared in $TTC(\gamma)$, where $\gamma(\underline{t}) = \underline{x}$, but at most measure 0 of σ has been cleared by time t in $TTC(\gamma')$. Then σ is also cleared in $TTC(\gamma')$.*

Proof. We define the 'interior' of the cycle σ by $X = \{x : \underline{x}_c \leq x_c \leq \overline{x}_c \, \forall c \in K, \, x_{c'} \geq \underline{x}_{c'} \, \forall c' \notin K\}$. We first show that in either run of TTC, if a point on the TTC path is in the interior of the cycle (i.e. $x \in X$) then K is a recurrent communication class of the Markov chain for the set of available colleges at that point. Precisely, if $\gamma(u) \in X$ and the set of available colleges at time u in $TTC(\gamma)$ is D, then K is a recurrent communication class of $M^D(\gamma(u))$, and similarly if $\gamma'(u) \in X$ and the set of available colleges at time u in $TTC(\gamma)$ is D', then K is a recurrent communication class of $M^{D'}(\gamma'(u))$. The former claim follows from the fact that σ is cleared in $TTC(\gamma)$, $\sigma \in \Sigma(\gamma)$. It remains to show that the latter is true.

In order to show that K is a recurrent communication class with a given set of available colleges D' at a point x, by Lemma 32 it suffices to show that $H^{D'}(x)_{K,K}$ is irreducible and $H^{D'}(x)_{K,D'\setminus K}$ is the zero matrix. We will use the the fact that K is a recurrent communication class of $M^{C'}(\gamma'(u))$. Since $\gamma'(t)$ and $\gamma'(u)$ are both in the same rectangle R for some $R \in \mathcal{R}'$, it holds that $\mathbf{1}\left(H^{C'}(\gamma'(t))\right) = \mathbf{1}\left(H^{C'}(\gamma'(u))\right)$.

We first examine the difference between C' and D', and the resulting differences between $H^{C'}(\gamma'(t))$ and $H^{D'}(\gamma'(u))$. As $TTC(\gamma')$ progresses from $\gamma'(t)$ to $\gamma'(u)$, some colleges clear, changing the set of available colleges. If $C' = D'$, it follows that $\mathbf{1}\left(H^{C'}(\gamma'(t))\right) = \mathbf{1}\left(H^{C'}(\gamma'(u))\right) = \mathbf{1}\left(H^{D'}(\gamma'(u))\right)$. If $C' \neq D'$ then $C' \supsetneq D'$. The matrices $H^{C'}(\gamma'(u))_{K,C'\setminus K}$ and $H^{D'}(\gamma'(u))_{K,D'\setminus K}$ are given by the measures of people pointed to by colleges in K whose top choice college out of the available set C' (respectively D') is not in K. Since K is a recurrent communication class of $M^{C'}(\gamma'(u))$, it follows that $\mathbf{1}\left(H^{C'}(\gamma'(u))_{K,C'\setminus K}\right) = 0$, so for all students their top choice out of C' is in K. This means that their top choice out of D' is also in K, and so $\mathbf{1}\left(H^{D'}(\gamma'(u))_{K,D'\setminus K}\right) = 0$.

The matrices $H^{C'}(\gamma'(u))_{K,K}$ and $H^{D'}(\gamma'(u))_{K,K}$ are given by the measures of people pointed to by colleges in K whose top choice college out of the available set C' (respectively D') is in K. Since $\mathbf{1}\left(H^{C'}(\gamma'(u))_{K,C\setminus K}\right) = 0$, it follows that all students' top choice out of C' is in K, so all students' top choice colleges are the same irrespective of whether C' or D' is the set of available colleges. Hence $H^{C'}(\gamma'(u))_{K,K} = H^{D'}(\gamma'(u))_{K,K}$ and both matrices are irreducible. Hence, whether $C' = D'$ or $C' \supsetneq D'$, it holds that $H^{D'}(\gamma'(u))_{K,K}$ is irreducible and $H^{D'}(\gamma'(u))_{K,D'\setminus K}$ is the zero matrix, and so K is a recurrent communication class of $M^{D'}(\gamma'(u))$.

We now invoke Theorem 33 to show that in each of the two paths, all the students in the cycle σ clear with each other. In other words, there exists a time t such that $\gamma(t) = \overline{x}_c \forall c \in K$, and similarly there exists a time t' such that $\gamma'(t')_c = \overline{x}'_c \forall c \in K$.

The argument is as follows. While the path γ is in the 'interior' of the cycle, that is $\gamma(t) \in X$, it follows from Theorem 33 that the projection of the gradient of γ to K is a rescaling of some vector $d^K(\gamma(t))$, where $d^K(\cdot)$ depends on $H(\cdot)$ but not on γ. Similarly, while $\gamma'(t') \in X$, it holds that the projection of the gradient of γ' to K is a rescaling of the vector $d^K(\gamma'(t'))$, for the same function $d^K(\cdot)$. Hence if we take the section of γ in the 'interior' and project it to the coordinates in K, and similarly take the section of γ' in the 'interior' and project it to the coordinates in K, then the two projections are identical. In other words, if $\pi_K(x)$ is the projection of a vector x to the coordinates indexed by colleges in K, then $\pi_K(\gamma(\gamma^{-1}((\underline{x},\overline{x}]))) = \pi_K(\gamma'(\gamma'^{-1}((\underline{x},\overline{x}])))$.

Let us denote these projections by $\tilde{\gamma} := \pi_K(\gamma(\gamma^{-1}((\underline{x},\overline{x}]))) \text{ and } \tilde{\gamma}' = \pi_K(\gamma'(\gamma'^{-1}((\underline{x},\overline{x})))$ respectively. Now K is a recurrent communication class at time t during $TTC(\gamma)$ for any $t \in \gamma^{-1}(\underline{x},\overline{x}]$, and similarly at any time t' during $TTC(\gamma')$ for any $t' \in (\gamma')^{-1}(\underline{x},\overline{x}]$. Suppose $\pi_K(\gamma(t)) = \pi_K(\gamma'(t'))$. Then this implies that for all colleges $c \in K$, the same measure of students are assigned to K from time $\underline{t}$ to t under $TTC(\gamma)$, as from time $\underline{t}$ to t' under $TTC(\gamma')$.

Recall that we have assumed that for all colleges $c \in K$ and cycles $\sigma' \rhd \sigma$ involving college c, if $\sigma' \in \Sigma(\gamma)$, then σ' is cleared in $TTC(\gamma')$, and vice versa. This implies that for all $c \in K$, the measure of students assigned to c from time 0 to $\underline{t}$ under $TTC(\gamma)$ is the same as the measure of students assigned to c from time 0 to $\underline{t}$ under $TTC(\gamma')$.

Since $TTC(\gamma)$ clears σ the moment it exits the interior, this implies that $TTC(\gamma')$ also clears σ the moment it exits the interior. □

We are now ready to prove that the TTC allocation is unique. As the proof takes several steps, we separate it into sections for readability.

Proof of uniqueness. Let γ and γ' be two TTC paths. Denote the sets of cycles associated with $TTC(\gamma)$ and $TTC(\gamma')$ respectively by $\Sigma = \Sigma(\gamma)$ and $\Sigma' = \Sigma(\gamma')$. Since the set of cycles of a TTC mechanism define the allocation up to a set of students of η-measure 0, it suffices to show that $\Sigma = \Sigma'$.

Let $\sigma = (K, \underline{x}, \overline{x})$ be a cycle in $\Sigma \cup \Sigma'$ such that the following assumption holds:

Assumption 41. *For all $\tilde{\sigma} \rhd \sigma$ it holds that either $\tilde{\sigma}$ is in both Σ and Σ' or $\tilde{\sigma}$ is in neither.*

We show that if σ is in either Σ and Σ', it is in both Σ and Σ'. Since Σ and Σ' are finite sets, this will be sufficient to show that $\Sigma = \Sigma'$. Without loss of generality we may assume that $\sigma \in \Sigma$.

We give here an intuitive overview of the proof. Let $\Sigma_{\rhd\sigma} = \{\tilde{\sigma} \in \Sigma : \tilde{\sigma} \rhd \sigma\}$ denote the set of cycles that are comparable to σ and cleared before σ in $TTC(\gamma)$. Assumption (41) about σ implies that $\Sigma_{\rhd\sigma} \subseteq \Sigma'$. We will show that this implies that no students in σ start clearing under $TTC(\gamma')$ until all the students in σ have the same top available school in $TTC(\gamma')$ as when they clear in $TTC(\gamma)$, or in discrete terms, that when students in σ start clearing under $TTC(\gamma')$, the cycle σ appears in the pointing graph. We will then show that once some of the students in σ start clearing under $TTC(\gamma')$ then all of them start clearing. It then follows from Lemma 40 that σ clears under both $TTC(\gamma)$ and $TTC(\gamma')$.

Let ℓ denote the round of $TTC(\gamma)$ in which σ is cleared. We define the times in $TTC(\gamma)$ and $TTC(\gamma')$ when all the cycles in $\Sigma_{\rhd\sigma}$ are cleared, by

$$\begin{aligned}
\bar{t}_{\rhd\sigma} &= \min\left\{t : \gamma(t) \leq (\tilde{\underline{x}}) \text{ for all } \tilde{\sigma} = \left(\tilde{K}, \tilde{\underline{x}}, (\tilde{\overline{x}})\right) \in \Sigma_{\rhd\sigma} \text{ and } H(\gamma(t)) \neq \mathbf{0}\right\}, \\
\bar{t}'_{\rhd\sigma} &= \min\left\{t : \gamma'(t) \leq (\tilde{\underline{x}}) \text{ for all } \tilde{\sigma} = \left(\tilde{K}, \tilde{\underline{x}}, (\tilde{\overline{x}})\right) \in \Sigma_{\rhd\sigma} \text{and } H(\gamma'(t)) \neq \mathbf{0}\right\}.
\end{aligned}$$

We define also the times in $TTC(\gamma)$ when σ starts to be cleared and finishes clearing,

$$\underline{t}_\sigma = \max\{t : \gamma(t) \geq \overline{x}\}, \; \bar{t}_\sigma = \min\{t : \gamma(t) \leq \underline{x}\}$$

and the times in $TTC(\gamma')$ when students in σ start to be cleared and finish clearing,

$$\underline{t}'_\sigma = \max\{t : \gamma'(t) \geq \overline{x}\}, \; \bar{t}_\sigma = \min\{t : \gamma'(t) \leq \underline{t}\}.$$

Let C denote the set of available colleges in $TTC(\gamma)$ at time $\underline{t}_\sigma$, and let C' denote

the set of available colleges in $TTC(\gamma')$ at time $\underline{t}'_\sigma$. We remark that part of the issue, carried over from the discrete setting, is that these times when $\Sigma_{\triangleright\sigma}$ stop clearing and σ starts clearing might not match up. In particular, other incomparable cycles could clear during, before or after these times. In the continuum model, there may also be sections on the TTC curve at which no college is pointing to a positive density of students. However, all the issues in the continuum case can be addressed using the intuition from the discrete case. We show that under $TTC(\gamma')$ all the students in σ eventually point to their top choice college in σ, and only a zero η-measure set of students in σ clear before this occurs.

We begin by showing that the cycle σ is in the pointing graph of $TTC(\gamma)$ from the time when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared to the time when σ starts to be cleared, and similarly for $TTC(\gamma')$. We split the analysis of the pointing graphs into sections, first showing that this is true for colleges under $TTC(\gamma)$ and $TTC(\gamma')$ respectively, and then for students.

We first show that in $TTC(\gamma)$, from the time when all the cycles in $\Sigma_{\triangleright\sigma}$ are cleared to the time when σ starts to be cleared, the students pointed to by colleges in K remain constant (up to a set of η-measure 0).

In $TTC(\gamma)$, let $\tilde{\Theta}$ denote the set of students cleared in time $\left[\overline{t}_{\triangleright\sigma}, \underline{t}_\sigma\right)$ who are preferred by some college in $c \in K$ to the students in σ, that is, θ satisfying $r_c^\theta > \overline{x}_c$. Then $\eta\left(\tilde{\Theta}\right) = 0$.

Proof. The idea is that if this set has positive measure, there must be a cycle $\tilde{\sigma}$ containing a positive η-measure of such students. Then $\tilde{\sigma}$ is comparable to σ and, by assumption (41), must be cleared before σ, contradicting that $\tilde{\sigma}$is cleared after time $\overline{t}_{\triangleright\sigma}$. We present this argument formally below.

Suppose $\eta\left(\tilde{\Theta}\right) > 0$. Then, since there are a finite number of cycles in $\Sigma(\gamma)$, there exists some cycle $\tilde{\sigma} = \left(\tilde{K}, \underline{\tilde{x}}, (\tilde{\overline{x}})\right) \in \Sigma(\gamma)$ containing a positive η-measure of students in $\tilde{\Theta}$. Then $\tilde{\sigma}$ is clearing at some time in $\left[\overline{t}_{\triangleright\sigma}, \underline{t}_\sigma\right)$, so $\tilde{\sigma} \neq \sigma$ by the definition of $\underline{t}_\sigma$. Moreover, since $\tilde{\sigma}$ contains a positive η-measure of students in $\tilde{\Theta}$, it holds that there exist $t_1, t_2 \in \left[\overline{t}_{\triangleright\sigma}, \underline{t}_\sigma\right)$ and a college $c \in K$ for which $\underline{\tilde{x}}_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq (\tilde{\overline{x}})_c$. Hence

$$\tilde{\overline{x}}_c \leq \gamma(\underline{t}_\sigma)_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \tilde{\overline{x}}_c,$$

so $\tilde{\sigma} \rhd \sigma$ and must be cleared before σ. But

$$(\tilde{\underline{x}})_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \gamma\left(\bar{t}_{\rhd\sigma}\right)_c,$$

so $\tilde{\sigma}$ is not cleared before $\bar{t}_{\rhd\sigma}$, contradicting the definition of $\bar{t}_{\rhd\sigma}$. □

We also show that in $TTC(\gamma')$, from the time when all the cycles in $\Sigma_{\rhd\sigma}$ are cleared to the time when some students in σ start to be cleared, the students pointed to by colleges in K remain constant (up to a set of η-measure 0).

In $TTC(\gamma')$, let $\tilde{\Theta}$ denote the set of students cleared in time $\left[\bar{t}'_{\rhd\sigma}, \underline{t}'_{\sigma}\right)$ who are preferred by some college in $c \in K$ to the students in σ, that is, θ satisfying $r_c^{\theta} > \overline{x}_c$. Then $\eta\left(\tilde{\Theta}\right) = 0$.

Proof. The idea is that if this set has positive measure, then there must be a cycle $\tilde{\sigma}$ containing a positive η-measure of such students. Any such cycle is smaller than σ and in $\Sigma' \setminus \Sigma$, contradicting assumption (41) on σ. We present this argument formally below.

Suppose $\eta\left(\tilde{\Theta}\right) > 0$. Then, since there are a finite number of cycles in $\Sigma(\gamma')$, there exists some cycle $\tilde{\sigma} = \left(\tilde{K}, \tilde{\underline{x}}, (\tilde{\overline{x}})\right) \in \Sigma(\gamma')$ containing a positive η-measure of students in $\tilde{\Theta}$. As in the proof of (B.4.2), $\tilde{\sigma}$ is clearing at some time in $\left[\bar{t}'_{\rhd\sigma}, \underline{t}'_{\sigma}\right)$, so $\tilde{\sigma} \neq \sigma$ by the definition of $\underline{t}'_{\sigma}$. Moreover, since $\tilde{\sigma}$ contains a positive η-measure of students in $\tilde{\Theta}$, it holds that there exist $t_1, t_2 \in \left[\bar{t}'_{\rhd\sigma}, \underline{t}'_{\sigma}\right)$ for which $\tilde{\underline{x}}_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq (\tilde{\overline{x}})_c$. Hence

$$\tilde{\overline{x}}_c \leq \gamma'(\underline{t}'_{\sigma})_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \tilde{\overline{x}}_c,$$

so $\tilde{\sigma} \rhd \sigma$ and must be cleared before σ. Moreover,

$$(\tilde{\underline{x}})_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \gamma\left(\bar{t}'_{\rhd\sigma}\right)_c,$$

so it follows from the definition of $\bar{t}'_{\rhd\sigma}$ that $\tilde{\sigma} \notin \Sigma_{\rhd\sigma}$.

Since we have assumed that $\tilde{\sigma} \in \Sigma'$, it follows that $\tilde{\sigma} \in \Sigma' \setminus \Sigma$, contradicting assumption (41) on σ. □

We now show that in $TTC(\gamma)$ (and $TTC(\gamma')$), from the time when all the cycles in $\Sigma_{\rhd\sigma}$ are cleared to the time when some students in σ start to be cleared, the colleges pointed to by students in σ remain constant . We do this by showing that any colleges

that students in σ prefer to their favorite college in K become unavailable before all the cycles in $\Sigma_{\rhd\sigma}$ are cleared.

Let $\sigma = (K, \underline{x}, \overline{x}) \in \Sigma$ satisfy Assumption 41. Suppose there is a college c that some student in σ prefers to all the colleges in K. Then college c is unavailable in $TTC(\gamma)$ at any time $t \geq \bar{t}_{\rhd\sigma}$, and unavailable in $TTC(\gamma')$ at any time $t \geq \bar{t}'_{\rhd\sigma}$.

Proof. The idea is that college c is unavailable after all the cycles in $\Sigma_{\rhd\sigma}$ are cleared, which is $\bar{t}_{\rhd\sigma}$ in $TTC(\gamma)$ and $\bar{t}'_{\rhd\sigma}$ in $TTC(\gamma')$.

We know that c is unavailable at time $\underline{t}_\sigma$ in $TTC(\gamma)$. Suppose that college c is available in $TTC(\gamma')$ after all the cycles in $\Sigma_{\rhd\sigma}$ are cleared. Then there exists some cycle $\tilde{\sigma}$ clearing in time $\tilde{t} \in \left(\bar{t}_{\rhd\sigma}, \underline{t}_\sigma\right)$ in $TTC(\gamma)$ involving college c. But this means that $\tilde{\sigma} \rhd \sigma$ so $\tilde{\sigma} \in \Sigma_{\rhd\sigma}$. Hence the measure of students in $\Sigma_{\rhd\sigma}$ assigned to college c is q_c. The result follows from the definitions of $\bar{t}_{\rhd\sigma}$ and $\bar{t}'_{\rhd\sigma}$ as the times when all cycles in $\Sigma_{\rhd\sigma}$ are cleared. □

Claims (B.4.2), (B.4.2) and (B.4.2) show that the pointing graph is constant up until the moment when students in the cycle σ start clearing. We now show that σ is a cycle in the pointing graph at the moment when students in the cycle σ start clearing under both $TTC(\gamma)$ and $TTC(\gamma')$. We formalize this in the continuum model by considering the coordinates of the path γ at the time $\underline{t}_\sigma$ when the cycle σ starts clearing, and showing that, for all coordinates indexed by colleges in K, this is equal to $\overline{x}$.

$\pi_K(\gamma(\underline{t}_\sigma)) = \pi_K(\overline{x})$.

Proof. The minimality of $\overline{x}$ implies that $\gamma(\underline{t}_\sigma)_c \geq \overline{x}_c$ for all $c \in K$. Suppose that there exists some $c \in K$ such that $\gamma(\underline{t}_\sigma)_c > \overline{x}_c$. Since σ starts clearing at time $\underline{t}_\sigma$, for all $\varepsilon > 0$ college c must point to a non-zero measure in σ over the time period $[\underline{t}_\sigma, \underline{t}_\sigma + \varepsilon]$. The set of students that college c points to in this time is a subset of those with score r_c^θ satisfying $\gamma(\underline{t}_\sigma)_c \geq r_c^\theta \geq \gamma(\underline{t}_\sigma + \varepsilon)_c$, where continuity of $\gamma(\cdot)$ and the assumption that $\gamma(\underline{t}_\sigma)_c > \overline{x}_c$ implies that $\gamma(\underline{t}_\sigma + \varepsilon)_c > \overline{x}_c$ for sufficiently $\varepsilon > 0$. But the set of θ cleared in σ with $r_c^\theta > \overline{x}_c$ has η-measure 0, which is a contradiction. □

We now show that the coordinates of the path γ' at time $\underline{t}'_\sigma$ indexed by colleges in K are equal to the corresponding coordinates of $\overline{x}$. In the discrete case, this part of the proof follows almost immediately from the fact that σ is a cycle in the pointing graph of $TTC(\gamma)$, as shown in (B.4.2), and the fact that all cycles that come before σ under $\rhd$ are cleared in both $TTC(\gamma)$ and $TTC(\gamma')$, and so the pointing graph restricted to students

and colleges in $\sigma$are identical under both $TTC(\gamma)$ and $TTC(\gamma')$. In the continuum, we will have to work a little harder to show that this is true, but the idea of the proof is the same.

$\pi_K(\gamma'(\underline{t}'_\sigma)) = \pi_K(\overline{x})$.

Proof. The minimality of $\overline{x}$ implies that $\gamma'(\underline{t}'_\sigma)_c \geq \overline{x}_c = \gamma(\underline{t}_\sigma)_c$ for all $c \in K$. Since we cannot assume that σ is the cycle that is being cleared at time $\underline{t}'_\sigma$ in $TTC(\gamma')$, the argument is more complicated than that for the previous claim and relies on the fact that K is a recurrent communication class in $TTC(\gamma)$, and that all cycles comparable to σ are already cleared in $TTC(\gamma')$. As already stated, the underlying concept is very simple in the discrete model, but is complicated in the continuum by the definition of the TTC path in terms of specific points, as opposed to measures of students, and the need to account for sets of students of η-measure 0. The idea will be to link the existence of positive measures of students pointed to by colleges, as measured by the entries of the matrix H, to the coordinates of $\gamma'(\underline{t}'_\sigma)$ and $\gamma(\underline{t}_\sigma)$.

Let $K_=$ be the set of coordinates in K at which equality holds, $\gamma'(\underline{t}'_\sigma)_c = \gamma(\underline{t}_\sigma)_c$, and let $K_>$ be the set of coordinates in K where strict inequality holds, $\gamma'(\underline{t}'_\sigma)_c > \gamma(\underline{t}_\sigma)_c$. If $K_>$ is empty then the claim holds and $\pi_K(\gamma'(\underline{t}'_\sigma)) = \pi_K(\overline{x})$. The rest of this proof will be dedicated to showing that $K_>$ is empty. The idea is the following. Under $TTC(\gamma')$, at time $\underline{t}'_\sigma$, every college in $K_=$ points to a non-zero measure of students who point to other colleges in K, and some college in $K_=$ is involved in a cycle clearing at time $\underline{t}'_\sigma$. Moreover, since the two TTC paths have the same c- coordinates for all $c \in K_=$, if, at time $\underline{t}_\sigma$in $TTC(\gamma)$ a college $c \in K_=$ points to a non-zero measure of students whose top choice is in $K_>$, then the same is true at time $\underline{t}'_\sigma$ in $TTC(\gamma')$. (This is the part of the argument that looks at the entries of the H matrices.) However, at time $\underline{t}'_\sigma$ in $TTC(\gamma')$ every college in $K_>$ points to a zero measure of students, which contradicts the trade balance equations for cycles clearing at time $\underline{t}'_\sigma$.

Recall that C is the set of available colleges in $TTC(\gamma)$ at time $\underline{t}_\sigma$, and C' is the set of available colleges in $TTC(\gamma')$ at time $\underline{t}'_\sigma$. We prove formally the above results about $K_=$ and $K_>$. Note that if $c \in K_=$ and $c' \in K_>$, by assumption, $\gamma'(\underline{t}'_\sigma)_c = \gamma(\underline{t}_\sigma)_c = \overline{x}_c$ and $\gamma'(\underline{t}'_\sigma)_{c'} > \gamma(\underline{t}_\sigma)_{c'} = \overline{x}_{c'}$. Note also that since some students in σ are being cleared in $TTC(\gamma')$ at time $\underline{t}'_\sigma$, there exists a coordinate $c \in K$ for which equality holds, so $K_=$ is nonempty.

Suppose that $c' \in K_>$. Then there exists $\varepsilon > 0$ such that in $TTC(\gamma')$, the set of students pointed to by college c' in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ has η-measure 0.

The proof of (B.4.2) is as follows. Since $c' \in K_>$ it holds that $\gamma'(\underline{t}'_\sigma)_{c'} > \overline{x}_{c'}$, and since γ' is continuous, for sufficiently small ε it holds that $\gamma'(\underline{t}'_\sigma + \varepsilon)_{c'} > \overline{x}_{c'}$. Hence the set of students that college c' points to in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ is a subset of those with score $r^\theta_{c'}$ satisfying $\gamma'(\underline{t}'_\sigma)_{c'} \geq r^\theta_{c'} \geq \gamma'(\underline{t}'_\sigma + \varepsilon)_{c'} \overline{x}_{c'}$. Suppose $\tilde{\sigma}$ is a cycle clearing some of these students. Since $\tilde{\sigma}$ is cleared after $\Sigma_{\triangleright\sigma}$ and before σ, assumption (41) on σ implies that $\tilde{\sigma}$ is incomparable to σ. If $\tilde{\sigma}$ contains a positive measure of students pointed to by college c', then $\tilde{\sigma}$ blocks σ via the blocking college c'. Hence all cycles clear at most measure 0 of the students that college c' points to in the time interval $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$, and since there are a finite number of cycles this set has measure 0.

Suppose that $c \in K_=$. Then for all sufficiently small ε, the set of students pointed to by advancing the cutoff for college c by ε in $TTC(\gamma)$ contains all but a 0 η-measure set of students pointed to by advancing the cutoff for college c by ε in $TTC(\gamma')$. That is,

$$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)]$$

has η-measure 0.

Proof: We rely mostly on the fact that $\gamma(\underline{t}_\sigma) = \gamma'(\underline{t}'_\sigma) = \overline{x}_c$, so that the set $A = (\gamma(\underline{t}_\sigma) - \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)]$ is a subset of the slice $x : \overline{x}_c - \varepsilon < x_c \leq \overline{x}_c$, and if $x \in A$ then $x_{c'} \leq \gamma'(\underline{t}'_\sigma)_{c'}$ for all $c' \in \mathcal{C}$. The intuition is that the set of students in A that in the cycle σ has η-measure 0, since $A \subseteq (\gamma(\underline{t}_\sigma) - \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)]$, and the set of students in A that are not in the cycle σ has η-measure 0, since $x_{c'} \leq \gamma'(\underline{t}'_\sigma)_{c'}$.

Specifically, suppose $\varepsilon < \overline{x}_c - \underline{x}_c$. Then at most η-measure 0 of the students in

$$(\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)]$$

are not cleared by cycle σ. Hence at most η-measure 0 of the subset

$$(\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) + \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)]$$

is not in the cycle σ.

We now consider the measure of the subset that is in cycle σ. Since $x_{c'} \leq \gamma'(\underline{t}'_\sigma)_{c'}$ for all $c' \in \mathcal{C}$, it follows that all students in A are cleared under $TTC(\gamma')$ by time $\underline{t}'_\sigma$, and by the definition of $\underline{t}'_\sigma$ it follows that the set of students in A that are also in cycle σ has η-measure 0. Hence the η-measure of A is 0.

Suppose that $c \in K_=$, $b \in C$ and $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$. Then for sufficiently small ε, the

set of students in σ whose scores are in the set

$$(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)]$$

has $\eta^{b|C}$-measure of $\Omega(\varepsilon)$.

Proof: The idea is to consider the set of student in σ whose scores are in the set $(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$, show that this set of students has $\eta^{b|C}$-measure of $\Omega(\varepsilon)$, and that it differs from the measure of the set that we want by a set of η-measure 0.

Since $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$, it follows that

$$\widetilde{H}_c^b(x) \doteq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta\left(\left\{\theta \in \Theta \mid r^\theta \in (\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)] \text{ and } Ch_\theta\left(\mathcal{C}^{(\ell)}\right) = b\right\}\right) > 0$$

and hence

$$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$$

has $\eta^{b,C}$-measure $\Omega(\varepsilon)$ for sufficiently small ε.

Moreover, at most η-measure 0 of the students in $(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$ are not in the cycle σ. Finally, $(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)] \supseteq (\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus A$, where $A = ((\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)] \setminus (\gamma'(\underline{t}'_\sigma) + \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)])$. Hence the $\eta^{b|C}$-measure of students in σ who are in $(\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e^c, \gamma'(\underline{t}'_\sigma)]$ is at least

$$\begin{aligned} & \Omega(\varepsilon) - \eta^{b|C}(A) \\ = \; & \Omega(\varepsilon) \text{ (by (B.4.2))} \end{aligned}$$

Suppose that $c \in K_=$, $b \in K$ and $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$. Then $H^{C'}(\gamma'(\underline{t}'_\sigma))_{cb} > 0$.

Proof: Since every $H^C(\gamma'(\underline{t}'_\sigma))_{cb}$ is a positive multiple of $\tilde{H}_c^{b|C}(\gamma'(\underline{t}'_\sigma))$, it suffices to show that

$$\tilde{H}_c^{b|C'}(\gamma'(\underline{t}'_\sigma)) > 0.$$

Since $H^C(\gamma(\underline{t}_\sigma))_{cb} > 0$, it follows from (B.4.2) that there is an $\eta^{b|C}$-measure $\Omega(\varepsilon)$ of students in σ with ranks in

$$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$$

Suppose for the sake of contradiction that

$$\tilde{H}_c^{b|C'}\left(\gamma'\left(\underline{t}'_\sigma\right)\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\eta\left(\left\{\theta\in\Theta \mid r^\theta \in \left(\gamma'\left(\underline{t}'_\sigma\right)-\varepsilon\cdot e^c, \gamma'\left(\underline{t}'_\sigma\right)\right] \text{ and } Ch_\theta\left(C'\right)=b\right\}\right)=0.$$

Then for sufficiently small ε it holds that

$$\eta^{b|C'}\left(\left(\gamma'\left(\underline{t}'_\sigma\right)-\varepsilon\cdot e^c, \gamma'\left(\underline{t}'_\sigma\right)\right]\right) = o\left(\varepsilon\right).$$

Hence there is an η-measure $\Omega\left(\varepsilon\right)$ of students in σ with ranks in $\left(\gamma'\left(\underline{t}'_\sigma\right)-\varepsilon\cdot e^c, \gamma'\left(\underline{t}'_\sigma\right)\right]$ whose top choice college in C is b, but whose top choice college in C' is not b. Let one such student in σ be of type $\theta\in\Theta^{b|C}\setminus\Theta^{b|C'}$. Let college b' be such that the student chooses b' out of C', that is, $\theta\in\Theta^{b'|C'}$. Since $b'\in C'$ it is available in $TTC\left(\gamma'\right)$ at time $\underline{t}'_\sigma$, and since we have shown that no student in σ prefers a college in $C'\setminus K$ to all the colleges in K it holds that $b'\in K$. Moreover, since $\theta\in\Theta^{b|C}$ by construction, it holds that θ prefers college b to all other colleges in K, so $b=b'$. Finally, we have assumed that $\theta\notin\Theta^{b|C'}$, so $b\neq b'$. This gives the required contradiction.

We are now ready to prove (B.4.2). Recall that $K=K_=\cup K_>$, where $K_=$ is nonempty and it suffices to prove that $K_>$ is empty. Suppose for the sake of contradiction that $K_>$ is nonempty.

Consider the colleges K' involved in a cycle at time $\underline{t}'_\sigma$ in $TTC\left(\gamma'\right)$. It follows from the definition of $\underline{t}'_\sigma$ that $K'\cap K_=$ is nonempty. Moreover, if $c\in K'\cap K_=$ and $H^{C'}\left(\gamma'\left(\underline{t}'_\sigma\right)\right)_{cc'}>0$ then $c'\in K'$.

Let $c\in K'\cap K_=$. Since $K=K_=\cup K_>$ is a recurrent communication class of $H^{C'}\left(\gamma\left(\underline{t}_\sigma\right)\right)$, it holds that there exists a chain $c=c_0-c_1-c_2-\cdots-c_n$ such that $H^{C'}\left(\gamma\left(\underline{t}_\sigma\right)\right)_{c_ic_{i+1}}>0$ for all $i<n$, $c_i\in K_=$ for all $i<n$, and $c_n\in K_>$. Since K' is a recurrent communication class, it follows that $c_i\in K'$ for all $i\leq n$. Hence c_n is involved in a cycle at time $\underline{t}'_\sigma$. But since $c_n\in K_>$, there exists $\varepsilon>0$ such that in $TTC\left(\gamma'\right)$, the set of students pointed to by college c_n in the time interval $\left[\underline{t}'_\sigma, \underline{t}'_\sigma+\varepsilon\right]$ has η-measure 0, which is a contradiction. Hence we have shown that $\gamma'\left(\underline{t}'_\sigma\right)_K=\gamma\left(\underline{t}_\sigma\right)_K$. □

We have shown in (B.4.2) that for our chosen $\sigma=\left(K,\underline{x},\overline{x}\right)$, it holds that $\gamma\left(\underline{t}_\sigma\right)_K=\gamma'\left(\underline{t}'_\sigma\right)_K=\underline{x}_K$. Invoking Lemma 40 shows that σ is cleared under both $TTC\left(\gamma\right)$ and $TTC\left(\gamma'\right)$. Hence $\Sigma=\Sigma'$, as required. □

B.5 Proof of Proposition 16

In this section, we show that given a discrete economy, the outcome of TTC in the continuum embedding gives the same assignment as TTC on the discrete model, $\mu_d = \hat{\mu}_d$.

The intuition behind this result is that TTC is essentially performing the same assignments in both models, with discrete TTC assigning students to colleges in discrete steps, and continuum TTC assigning students to colleges continuously, in fractional amounts. Moreover, the structure of the embedding implies that continuum TTC assigns at most one student to a given college at any point in time, and does not begin assigning a new student until to a given college until one student has been completely assigned. Hence, by restricting continuum TTC to the discrete time steps when individual students are fully assigned, we obtain the same outcome as discrete TTC.

Proof. The formal proof is as follows. We construct a discrete cycle selection rule ψ and TTC path γ such that TTC on the discrete economy E with cycle selection rule ψ gives the same allocation as $TTC(\gamma)$. Since the assignment of discrete TTC is unique Shapley and Scarf (1974), and the assignment in the continuum model is unique (Proposition 10), this proves the theorem.

The discrete cycle selection rule ψ is defined as follows. At each step of discrete TTC, take all available cycles in the pointing graph.

The TTC path γ is defined as follows. We first define $d(x)$. At each point x, let C be the set of available colleges, let $K(x)$ be the set of all students in recurrent communication classes of $H(x)$, and let $d(x) = \frac{1}{|K(x)|}$ if $c \in K$ and 0 otherwise. Let X be the set of points x such that x_c is a multiple of $\frac{1}{N}$ for all $c \notin K(x)$. Note that each student's cube has equal density, and the number of cubes intersecting any axis-parallel hyperplane is at most one, so for all $x \in X$ the entries of the matrix $H(x)$ are all 0's and $\frac{1}{N}$'s. We remark that this means that $NH(x)$ is the adjacency matrix of the pointing graph (where college b points to college c if some student pointed to by b wants c). Hence $K(x)$ and the colleges desired by the students in $K(x)$ form a maximal union of cycles in the pointing graph when the set of available colleges is C and the set of available students is those whose cubes contain part of $\{\theta : r^\theta \geq x\}$. It follows that $d(x) = d(x)H(x)$ for all $x \in X$.

Now consider the TTC path γ satisfying $\gamma'(t) = d(\gamma(t))$. We show that $\gamma(t) \in X$ for all t. The path starts at $\gamma(0) = 0$. Moreover, at any time t, if $\gamma(t) \in X$ then the derivative of the TTC path $d(\gamma(t))$ points along the diagonal in the projection onto the

coordinates K, and is 0 along all other coordinates. Hence $\gamma(t) \in X$ for all t.

Let $t_1, t_2, \ldots$ be the discrete set of times when a student s is first fully assigned, that is $\{t_i\} = \cup_s \left\{\inf \left\{t \mid \exists c \in C \text{ s.t. } \gamma_c(t) \leq r_c^\theta \ \forall \theta \in I^s\right\}\right\}$.

We note that for every two students s, s' and college c it holds that the projections I_c^s and $I_c^{s'}$ of I^s and $I^{s'}$ onto the cth coordinates are non-overlapping, i.e. either for all $\theta \in I^s, \theta' \in I^{s'}$ it holds that $r_c^\theta < r_c^{\theta'}$, or for all $\theta \in I^s, \theta' \in I^{s'}$ it holds that $r_c^\theta > r^{\theta'}{}_c$. Since all the capacities are multiples of $\frac{1}{N}$, it holds that for all t_i, every student s is either fully assigned or fully unassigned, $\exists c \in C$ s.t. $\gamma_c(t) \leq r_c^\theta \ \forall \theta \in I^s$ or $\gamma_c(t) \geq r_c^\theta \forall \theta \in I^s, c \in C$. Moreover, since all the capacities are multiples of $\frac{1}{N}$, it follows that colleges fill at a subset of the set of times $\{t_i\}$.

In other words, we have shown that for every i, if S is the set students who are allocated a seat at time t_i, then $S \cup \mu(S)$ are the agents in the maximal union of cycles in the pointing graph at t_{i-1}. Hence γ finishes clearing the cubes corresponding to the same set of cycles at t_i as ψ does in step i. It also follows that every student s who is fully assigned is fully assigned to exactly one college, that is if $\mu(\theta) \neq \emptyset$ for some $\theta \in I^s$ then $\exists c$ s.t. $\forall \theta \in I^s \ \mu(\theta) = c$. Hence $\hat{\mu}_d = \mu_d$. □

B.6 Proof of Theorem 17

To prove Theorem 17, we will want some way of comparing two TTC paths γ and $\tilde{\gamma}$ obtained under two continuum economies differing only in their measures η and $\tilde{\eta}$. Intuitively, we want to pick points on the paths such that there exists a college c where the number of seats offered by college c is less under γ than $\tilde{\gamma}$, but the number of students who are offered some seat and want college c is more under γ than $\tilde{\gamma}$. Since these are difficult to compare under different measures, we instead focus on the ranks of students who are offered seats by college c, and the ranks of students who are offered some seat and want college c. The two conditions that we want then correspond exactly with there being a college c and an interval τ in which, if we re-parametrize so that $\gamma_c(t) = \tilde{\gamma}_c(t)$ for all $t \in \tau$, then $\gamma_b(t) \leq \tilde{\gamma}_b(t)$ for all b and for all $t \in \tau$. We formally define this notion below.

Definition 42. Let γ and $\tilde{\gamma}$ be increasing continuous functions from $[0,1]$ to $[0,1]^{\mathcal{C}}$ with

$\gamma(0) = \tilde{\gamma}(0)$. Then $\gamma(t)$ *dominates* $\tilde{\gamma}(t)$ *via college* c if

$$\begin{aligned} \gamma_c(t) &= \tilde{\gamma}_c(t), \text{ and} \\ \gamma_b(t) &\leq \tilde{\gamma}_b(t) \text{ for all } b \in \mathcal{C}. \end{aligned}$$

We remark that we (somewhat unintuitively) require $\gamma(\cdot) \leq \tilde{\gamma}(\cdot)$, since more students are offered seats under γ than γ' and higher ranks give more restrictive sets. We also say that γ *dominates* $\tilde{\gamma}$ *via college* c *at time* t. If γ and γ' are TTC paths, we can interpret this as college c being more demanded under γ, since with the same rank at c, in γ students are competitive with more ranks at other colleges b. In other words, high ranks at college c are a more valuable commodity under γ than under γ'.

We now show that any two non-increasing continuous paths γ, γ' starting and ending at the same point can be re-parametrized so that for all t there exists a college $c(t)$ such that γ dominates γ' via college $c(t)$ at time t. We first show that, if $\gamma(0) \leq \tilde{\gamma}(0)$, then there exists a re-parametrization of γ such that γ dominates γ' on some interval starting at 0.

Lemma 43. *Suppose* γ, $\tilde{\gamma}$ *are a pair of non-increasing functions* $[0,1] \to [0,1]^{\mathcal{C}}$ *such that* $\gamma(0) \leq \tilde{\gamma}(0)$. *Then there exist coordinates* c, b, *a time* $\bar{t}$ *and an increasing function* $g : \mathbb{R} \to \mathbb{R}$ *such that* $\gamma_b(g(t)) = \tilde{\gamma}_b(t)$, *and for all* $t \in \left[0, \bar{t}\right]$ *it holds that*

$$\gamma_c(g(t)) = \tilde{\gamma}_c(t) \text{ and } \gamma(g(t)) \leq \tilde{\gamma}(t).$$

That is, if we renormalize the time parameter t of $\gamma(t)$ so that γ and $\tilde{\gamma}$ agree along the cth coordinate, then γ dominates $\tilde{\gamma}$ via college c at all times $t \in \left[0, \bar{t}\right]$, and also dominates via college b at time $\bar{t}$.

Proof. The idea is that if we take the smallest function g such that $\gamma_c(g(t)) = \tilde{\gamma}_c(t)$ for some coordinate c and all t sufficiently small, then $\gamma(g(t)) \leq \tilde{\gamma}(t)$ for all t sufficiently small. The lemma then follows from continuity. We make this precise.

Fix a coordinate c. Let $g^{(c)}$ be the renormalization of γ so that γ and $\tilde{\gamma}$ agree along the cth coordinate, i.e. $\gamma_c\left(g^{(c)}(t)\right) = \tilde{\gamma}_c(t)$ for all $t \in [0, T]$.

For all t, we define the set $\kappa_{>}^{(c)}(t)$ of colleges, or coordinates, along which the γ curve renormalized along coordinate c has larger value at time t than $\tilde{\gamma}$ has at time t, that is,

$$\kappa_{>}^{(c)}(t) = \left\{ b \,|\, \gamma_b\left(g^{(c)}(t)\right) > \tilde{\gamma}_b(t) \right\},$$

and similarly define the sets $\kappa_{<}^{(i)}(t)$ and $\kappa_{=}^{(i)}(t)$ where the renormalized γ curve is smaller than $\tilde{\gamma}$ and equal to $\tilde{\gamma}$ respectively,

$$\begin{aligned}\kappa_{<}^{(c)}(t) &= \left\{b \,|\, \gamma_b\left(g^{(c)}(t)\right) > \tilde{\gamma}_b(t)\right\}, \\ \kappa_{=}^{(c)}(t) &= \left\{b \,|\, \gamma_b\left(g^{(c)}(t)\right) = \tilde{\gamma}_b(t)\right\}.\end{aligned}$$

Since γ and $\tilde{\gamma}$ are continuous, there exists some $\bar{t}^{(c)} > 0$ such that the functions $\kappa_{>}^{(c)}(\cdot), \kappa_{<}^{(c)}(\cdot)$ and $\kappa_{=}^{(c)}(\cdot)$ are constant over the interval $\left(0, \bar{t}^{(c)}\right)$. Let $C_{>}^{(c)} = \kappa_{>}^{(c)}(t)$ for all $t \in \left(0, \bar{t}^{(c)}\right)$ and similar define $C_{<}^{(c)}$ and $C_{=}^{(c)}$.

If $\gamma\left(g^{(c)}(t)\right) = \tilde{\gamma}(t)$ in the interval $\left(0, \bar{t}^{(c)}\right)$, then by continuity we may take $\bar{t} = \bar{t}^{(c)}$ and b to be any other coordinate.

Hence we may assume that for all c, $\gamma\left(g^{(c)}(t)\right) \neq \tilde{\gamma}(t)$ in the interval $\left(0, \bar{t}^{(i)}\right)$, so at least one of $C_{>}^{(c)}$and $C_{<}^{(c)}$ is nonempty for all c. Let $\bar{t} = \min_{c \in \mathcal{C}} \bar{t}^{(c)}$.

Suppose that colleges c and b satisfy that $b \in C_{>}^{(c)}$. We claim that $g^{(b)}(t) > g^{(c)}(t)$ for all $t \in \left(0, \bar{t}\right)$. This is because γ is increasing and

$$\begin{aligned}\gamma_b\left(g^{(b)}(t)\right) &= \tilde{\gamma}_b(t) \text{ (for all } t \text{ by the definition of } g^{(b)}) \\ &> \gamma_b\left(g^{(c)}(t)\right) \text{ (since } b \in C_{>}^{(c)}\text{, and hence } b \in \kappa_{>}^{(c)}(t) \text{ for } t \in \left(0, \bar{t}^{(c)}\right)).\end{aligned}$$

Suppose that $C_{>}^{(c)} \neq \emptyset$ for all $c \in \mathcal{C}$. Then for all c, there exists b such that $g^{(b)}(t) > g^{(c)}(t)$ for all $t \in \left(0, \bar{t}\right)$, which is impossible since there are a finite number of elements $c \in \mathcal{C}$ and hence a finite number of values $g^{(c)}(t)$. Hence $C_{>}^{(c)} = \emptyset$ for some coordinate c.

Let c be a coordinate for which $C_{>}^{(c)} = \emptyset$. Then $C_{<}^{(c)}$ is nonempty, and $\kappa_{<}^{(c)}\left(\bar{t}^{(c)}\right) \neq C_{<}^{(c)}$. Hence by continuity there exists $b \in C_{<}^{(c)}$, such that $b \in \kappa_{=}^{(c)}\left(\bar{t}^{(c)}\right)$, and the coordinates c, b, time $\bar{t}^{(c)}$ and function $g^{(c)}$ satisfy the required conditions. □

We are now ready to show that there exists a re-parametrization of γ such that γ always dominates $\tilde{\gamma}$ via some college.

Lemma 44. *Suppose γ, $\tilde{\gamma}$ are a pair of non-increasing functions $[0,1] \to [0,1]^{\mathcal{C}}$ such that $\gamma(0) = \tilde{\gamma}(0) = 1$ and $\gamma(1) = \tilde{\gamma}(1) = 0$. Then there exists an increasing function $g : \mathbb{R} \to \mathbb{R}$ such that for all $t \in [0,1]$, there exists a college $c(t)$ such that $\gamma(g(t))$ dominates $\tilde{\gamma}(t)$ via college $c(t)$.*

Proof. Fix a coordinate c. Let $g^{(c)}$ be the renormalization γ so that γ and $\tilde{\gamma}$ agree along

the cth coordinate, i.e. $\gamma_c\left(g^{(c)}(t)\right) = \tilde{\gamma}_c(t)$ for all $t \in [0,T]$. Let $A^{(c)}$ be the set of times t such that $\gamma\left(g^{(c)}(t)\right)$ dominates $\tilde{\gamma}(t)$. By continuity, $A^{(c)}$ is closed. Consider the set $B^{(c)}$ which we define to be the closure of the interior of $A^{(c)}$. Notice that, since $A^{(c)}$ is closed, it contains $B^{(c)}$. Moreover, since the interior of $A^{(c)}$ is open, it is a countable union of open intervals, and hence $B^{(c)}$ is a countable union of disjoint closed intervals. We show that $\cup_{c\in\mathcal{C}} B^{(c)} = [0,1]$, which shows that $\cup_{c\in\mathcal{C}} A^{(c)} = [0,1]$.

Suppose that $\cup_{c\in\mathcal{C}} B^{(c)} \neq [0,1]$. Then there exists some college c and points $\underline{t} < \bar{t}$ such that $\gamma\left(g^{(c)}(\underline{t})\right)$ dominates $\tilde{\gamma}(\underline{t})$ via college c, and for all b there is no interval τ in $\left[\underline{t},\bar{t}\right]$ such that $\gamma(g^{(b)}(t))$ dominates $\tilde{\gamma}(t)$ via college b for all $t \in \tau$. But this contradicts Lemma 43.

We now construct a function g that satisfies the required properties as follows. Write $[0,1] = \cup_n\{\tau_n\}$ as a countable union of closed intervals such that any pair of intervals intersects at most at their endpoints, and each interval τ_n is a subset of $B^{(c)}$ for some c. For each τ_n fix some $c = c(n)$ so that $\tau \subseteq B^{(c)}$.

We now define g. If $t \in \tau_n \subseteq B^{(i)}$, let $g(t) = g^{(c)}(t)$. Then by definition $\gamma(g(t))$ dominates $\tilde{\gamma}(t)$ via college $c(t) = c(n)$. Now g is defined on all of $[0,1]$ since $\cup_{c\in\mathcal{C}} B^{(c)} = [0,1]$. Moreover g is well-defined since if t is in two different intervals τ_n, τ_m, then domination via $c(n)$ implies that $\gamma\left(g^{(c(m))}(t)\right) \geq \tilde{\gamma}(t) = \gamma\left(g^{(c(n))}(t)\right)$ and domination via $c(m)$ implies that $\gamma\left(g^{(c(n))}(t)\right) \geq \tilde{\gamma}(t) = \gamma\left(g^{(c(n))}(t)\right)$ so $\gamma\left(g^{(c(m))}(t)\right) = \gamma\left(g^{(c(n))}(t)\right)$ and we can pick one value for g that satisfies all required properties. This completes the proof. □

Consider two continuum economies $\mathcal{E} = (\mathcal{C},\Theta,\eta,q)$ and $\tilde{\mathcal{E}} = (\mathcal{C},\Theta,\tilde{\eta},q)$, where the measures η and $\tilde{\eta}$ satisfy the assumptions given in Section (3), namely the given normalization, an excess of students, and piecewise Lipshitz continuity (Assumption 4). Suppose also that the measure η and $\tilde{\eta}$ have total variation distance ε and have full support. Let γ be a TTC path for economy $\mathcal{E}$ with run-out sequence $\left\{\left(\mathcal{C}^{(\ell)},t^{(\ell)}\right)\right\}_\ell$, and let $\tilde{\gamma}$ be a TTC path for economy $\tilde{\mathcal{E}}$ with run-out sequence $\left\{\left(\tilde{\mathcal{C}}^{(\ell)},\tilde{t}^{(\ell)}\right)\right\}_\ell$. Consider any college c and any points $x,\tilde{x}$ such that x is cleared in the first round of $TTC(\gamma)$, i.e. $\gamma^{-1}(t) \in \left[0,t^{(1)}\right]$, $\tilde{x}$ is cleared in the first round of $TTC(\tilde{\gamma})$, i.e. $\tilde{\gamma}^{-1}(t) \in \left[0,\tilde{t}^{(1)}\right]$ and $x_c = \tilde{x}_c$. We show that the set of students allocated to college c when running $TTC(\gamma)$ up to x differs from the set of students allocated to college c when running $TTC(\tilde{\gamma})$ up to $\tilde{x}$ by a set of measure $O(\varepsilon|\mathcal{C}|)$.

Proposition 45. *Suppose that paths γ, $\tilde{\gamma}$ are TTC paths in one round of the continuum*

economies $\mathcal{E}$ and $\tilde{\mathcal{E}}$ respectively, where the set of available colleges is the same in these rounds of $TTC(\gamma)$ and $TTC(\gamma')$. Suppose also that γ and $\tilde{\gamma}$ end at x and $\tilde{x}$ respectively, where $x_b = \tilde{x}_b$ for some $b \in \mathcal{C}$ and $x_c \leq \tilde{x}_c$ for all $c \in \mathcal{C}$. Then for all $c \in C$, the difference in allocation of students to college c with ranks better than x under $\mathcal{E}$ and ranks better than $\tilde{x}$ under $\tilde{\mathcal{E}}$ is $O(\varepsilon |C|)$.

Proof. By Lemma 44, we may assume without loss of generality that γ and $\tilde{\gamma}$ are parametrized such that $x = \gamma(1)$, $\tilde{x} = \tilde{\gamma}(1)$ and for all times $t \leq 1$ there exists a college $c(t)$ such that $\gamma(t)$ dominates $\tilde{\gamma}(t)$ via college $c(t)$.

Let $\tau_c = \{t \leq 1 : c(t) = c\}$ be the times when γ dominates $\tilde{\gamma}$ via college c. We remark that, by our construction in Lemma 44, we may assume that τ_c is the countable union of disjoint closed intervals, and that if $c \neq c'$ then τ_c and $\tau_{c'}$ have disjoint interiors.

Let $\tau = [\underline{t}, \overline{t}]$ be an interval. Recall that $\mathcal{T}_c(\gamma; \tau) = \mathcal{T}_c(\gamma; \overline{t}) \setminus \mathcal{T}_c(\gamma; \underline{t})$ is the set of students who were offered a seat by college c at some time $t \in \tau$.

If $\tau = \cup_n \tau_n$ is a union of disjoint closed intervals, we define $\mathcal{T}_c(\gamma; \tau) = \mathcal{T}_c(\gamma; \tau_n)$ to be the set of students who were offered a seat by college c at some time $t \in \tau$, and $\mathcal{T}^{c|C}(\tau, \gamma) = \cup_n \mathcal{T}^{c|C}(\tau_n, \gamma)$ to be the set of students who were assigned to a college c at some time $t \in \tau$, given a set of available colleges C. Since γ is a TTC path for $\mathcal{E}$ and $\tilde{\gamma}$ is a TTC path for $\tilde{\mathcal{E}}$, the following trade balance equations hold,

$$\eta(\mathcal{T}_c(\gamma; \tau_c)) = \eta\left(\mathcal{T}^{c|C}(\gamma; \tau_c)\right) \text{ for all } c \in C. \tag{6}$$

$$\tilde{\eta}(\mathcal{T}_c(\tilde{\gamma}; \tau_c)) = \tilde{\eta}\left(\mathcal{T}^{c|C}(\tilde{\gamma}; \tau_c)\right) \text{ for all } c \in C. \tag{7}$$

Since γ dominates $\tilde{\gamma}$ via college c at all times $t \in \tau_c$, we have that

$$\mathcal{T}_c(\gamma; \tau_c) \subseteq \mathcal{T}_c(\tilde{\gamma}; \tau_c). \tag{8}$$

Moreover, by the choice of parametrization, $\cup_c \tau_c = [0, 1]$ and so, since $x \leq \tilde{x}$,

$$\cup_c \mathcal{T}(\gamma; \tau_c) \supseteq \cup_c \mathcal{T}(\tilde{\gamma}; \tau_c). \tag{9}$$

Hence

$$
\begin{aligned}
& \eta\left(\mathcal{T}^{c|C}(\gamma;\tau_c)\setminus\mathcal{T}^{c|C}(\tilde{\gamma};\tau_c)\right) \\
= \;& \eta\left(\mathcal{T}^{c|C}(\gamma;\tau_c)\right)-\eta\left(\mathcal{T}^{c|C}(\tilde{\gamma};\tau_c)\right) \text{ (by (9))} \\
\leq \;& \eta\left(\mathcal{T}^{c|C}(\gamma;\tau_c)\right)-\tilde{\eta}\left(\mathcal{T}^{c|C}(\tilde{\gamma};\tau_c)\right)+\varepsilon \text{ (since } \eta,\tilde{\eta} \text{ have total variation } \varepsilon) \\
= \;& \eta\left(\mathcal{T}_c(\gamma;\tau_c)\right)-\tilde{\eta}\left(\mathcal{T}_c(\tilde{\gamma};\tau_c)\right)+\varepsilon \text{ (by (6) and (7))} \\
\leq \;& \eta\left(\mathcal{T}_c(\gamma;\tau_c)\right)-\eta\left(\mathcal{T}_c(\tilde{\gamma};\tau_c)\right)+2\varepsilon \text{ (since } \eta,\tilde{\eta} \text{ have total variation } \varepsilon) \\
\leq \;& 2\varepsilon \text{ (by (8))},
\end{aligned}
$$

that is,

$$\eta\left(\mathcal{T}^{c|C}(\gamma;\tau_c)\setminus\mathcal{T}^{c|C}(\tilde{\gamma};\tau_c)\right)\leq 2\varepsilon. \tag{10}$$

Also, for all colleges $b\neq c$, since η has full support, it holds that

$$\eta\left(\mathcal{T}^{c|C}(\gamma;\tau_b)\setminus\mathcal{T}^{c|C}(\tilde{\gamma};\tau_b)\right)\leq\frac{M}{m}\eta\left(\mathcal{T}^{b,C}(\gamma;\tau_b)\setminus\mathcal{T}^{b,C}(\tilde{\gamma};\tau_b)\right). \tag{11}$$

Hence

$$
\begin{aligned}
& \eta\left(\mathcal{T}^{c|C}(\gamma;1)\setminus\mathcal{T}^{c|C}(\tilde{\gamma};1)\right) \\
= \;& \eta\left(\mathcal{T}^{c|C}(\gamma;1)\right)-\eta\left(\mathcal{T}^{c|C}(\tilde{\gamma};1)\right) \text{ (by (9))} \\
= \;& \eta\left(\mathcal{T}^{c|C}(\gamma;\cup_b\tau_b)\right)-\eta((\mathcal{T}(\tilde{\gamma};\cup_b\tau_b)) \\
= \;& \sum_{b\in C}\left(\eta(\mathcal{T}^{c|C}(\gamma;\tau_b))-\eta(\mathcal{T}^{c|C}(\tilde{\gamma};\tau_b)\right) \\
\leq \;& \sum_{b\in C}\eta\left(\mathcal{T}^{c|C}(\gamma;\tau_b)\setminus\mathcal{T}^{c|C}(\tilde{\gamma};\tau_b)\right) \\
\leq \;& \sum_{b\in C}\frac{M}{m}\eta\left(\mathcal{T}^{b,C}(\gamma;\tau_b)\setminus\mathcal{T}^{b,C}(\tilde{\gamma};\tau_b)\right) \text{ (by (11))} \\
\leq \;& 2|C|\varepsilon\frac{M}{m} \text{ (by (10))}.
\end{aligned}
$$

That is, given a college c, the set of students assigned to college c with score $r^\theta\not\leq x$ under γ and not assigned to college c with score $r^\theta\not\leq\tilde{x}$ under $\tilde{\gamma}$ has η-measure $O(\varepsilon|C|)$. A similar argument shows that the set of students assigned under $\tilde{\gamma}$ but not γ has $\tilde{\eta}$-measure $O(\varepsilon|C|)$. □

We are now ready to prove Theorem 17.

Proof of Theorem 17. Assume without loss of generality that the colleges are $c_1, c_2, \ldots$, where college c_ℓ reaches capacity in round ℓ of $TTC(\gamma)$. We show by induction on ℓ that for all colleges c, the set of students assigned to c under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ by the end of round ℓ has η-measure $O(\varepsilon\ell|\mathcal{C}|)$. This will prove the theorem.

The base case $\ell = 1$ follows directly from Proposition 45.

We now show the inductive step, proving for $\ell + 1$ assuming true for $1, 2, \ldots, \ell$. For each $1 \leq i \leq \ell$ let $p^{(i)} = \gamma\left(t^{(i)}\right)$ be the cutoffs obtained from $TTC(\gamma)$ in round i, and let $\tilde{t}_i$ be the largest time t such that $\gamma\left(t^{(i)}\right)$ dominates $\tilde{\gamma}(i)$ via some college available in round i of $TTC(\gamma)$. Since the measure of students assigned to c_i under $TTC(\gamma)$ and not to $TTC(\tilde{\gamma})$ by this point is $O(\varepsilon\ell|\mathcal{C}|)$ for all i, and the measure $\tilde{\eta}$ is bounded away from 0, if the two measures are sufficiently close, that is, for sufficiently small ε (dependent on the capacities) it holds that the colleges reach capacity in the order $c_1, c_2, \ldots$ in $TTC(\tilde{\gamma})$ at times $\tilde{t}^{(i)}$. We may invoke Proposition 45 to show that the difference in allocations to college c in time $\left[t^{(\ell)}, t^{(\ell+1)}\right]$ under $TTC(\gamma)$ and in time $\left[\tilde{t}^{(\ell)}, \tilde{t}^{(\ell+1)}\right]$ under $TTC(\gamma')$ is of order $O(\varepsilon|\mathcal{C}|)$, and invoke the inductive hypothesis to show that the difference in allocations to college c in times $\left[0, t^{(\ell)}\right]$ under $TTC(\gamma)$ and in times $\left[0, \tilde{t}^{(\ell)}\right]$ under $TTC(\tilde{\gamma})$ is of order $O(\varepsilon\ell|\mathcal{C}|)$. This completes the induction. □

B.7 Derivation of the Instantaneous Trade Balance Equations (3)

In this section, we show that the instantaneous trade balance equations (3) hold. The idea is the following. The trade balance equations must hold over any time interval within a round, and for small enough intervals $[t, t+\varepsilon]$ we can approximate the set of students assigned over the interval by a simple set, and likewise we can approximate the set of students offered by a simple set. Through these simple sets we turn the trade balance equations into linear equations that depend only on η via the values $\tilde{H}_b^c(x)$. We formalize this below.

For $b, c \in \mathcal{C}$, $x \in [0,1]^{\mathcal{C}}$,$\alpha \in \mathbb{R}$ we define the set [38]

$$T_b^c(x, \alpha) \doteq \left\{\theta \in \Theta \mid r^\theta \in [x - \alpha e^b, x) \text{ and } Ch_\theta\left(\mathcal{C}^{(\ell)}\right) = c\right\}.$$

We may think of $T_b^c(x, \alpha)$ as the set of students in the next α students on college b's

[38] We use the notation $[\underline{x}, \overline{x}) = \{z \in \mathbb{R}^n \mid \underline{x}_i \leq z_i < \overline{x}_i \ \forall i\}$ for $\underline{x}, \overline{x} \in \mathbb{R}^n$, and $e^c \in \mathbb{R}^{\mathcal{C}}$ is a vector whose c-th coordinate is equal to 1 and all other coordinates are 0.

priority list who are unassigned when $\gamma(t) = x$, and want college c. We remark that the sets used in the definition of the $\tilde{H}_b^c(x)$ are precisely the sets $T_b^c(x, \varepsilon)$.

We can use the sets $T_b^c(x, \alpha)$ to approximate expressions involving $\mathcal{T}_c(\gamma; t)$ and $\mathcal{T}^c(\gamma; t)$. Specifically, consider the run of the TTC algorithm in a round ℓ from $\gamma(t) = x$ to $\gamma(t+\tau) = x - \delta$. During the interval $[t, t+\tau]$ the students $\mathcal{T}_c(\gamma; t+\tau) \setminus \mathcal{T}_c(\gamma; t)$ were offered a seat at college c, and the students $\mathcal{T}^c(\gamma; t+\tau) \setminus \mathcal{T}^c(\gamma; t)$ were assigned to college c. We relate these sets to the sets $T_b^d(x, \alpha)$ in the following lemma.

Lemma 46. *Consider the interval $[t, t+\tau]$, and let $\gamma(t) = x$ and $\delta(\tau) = \gamma(t) - \gamma(t+\tau)$. During the interval $[t, t+\tau]$, the set of students who were assigned to college c is*

$$\mathcal{T}^c(\gamma; t+\tau) \setminus \mathcal{T}^c(\gamma; t) = \bigcup_b T_b^c(x, \delta(\tau)_b) = \left\{\theta \in \Theta \mid r^\theta \in [\gamma_b(t+\tau), \gamma_b(t)) \text{ and } Ch_\theta\left(\mathcal{C}^{(\ell)}\right) = c\right\}$$

and the set of students who were offered a seat at college c is

$$\mathcal{T}_c(\gamma; t+\tau) \setminus \mathcal{T}_c(\gamma; t) = \bigcup_d T_c^d(x, \delta(\tau)_c) \biguplus \Delta$$

for some small set $\Delta \subset \Theta$. Further, it holds that $\lim_{\tau \to 0} \frac{1}{\tau} \cdot \eta(\Delta) = 0$, and for any $c \neq c', d \neq d' \in \mathcal{C}$ we have $\lim_{\tau \to 0} \frac{1}{\tau} \cdot \eta\left(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^d(x, \delta(\tau)_{c'})\right) = 0$ and $T_c^d(x, \delta(\tau)_c) \cap T_c^{d'}(x, \delta(\tau)_c) = \phi$.

Proof. The first two equations are easily verified, and the fact that the last intersection is empty is also easy to verify. To show the bound on the measure of Δ, we observe that it is contained in the set $\bigcup_{c'} \cup_d \left(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^d(x, \delta(\tau)_{c'})\right)$, so it suffices to show that $\lim_{\tau \to 0} \frac{1}{\tau} \cdot \eta\left(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^d(x, \delta(\tau)_{c'})\right) = 0$. This follows from the fact that the density defining η is upper bounded by M, so $\eta\left(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^d(x, \delta\tau_{c'})\right) \leq M |\gamma_c(t+\tau) - \gamma_c(t)| |\gamma_{c'}(t+\tau) - \gamma_{c'}(t)|$. Since for all colleges c the function γ_c is continuous and has bounded derivative, it is also Lipschitz continuous, so

$$\frac{1}{\tau}\eta(\Delta) \leq \frac{1}{\tau}\eta\left(T_c^d(x, \delta(\tau)_c) \cap T_{c'}^d(x, \delta\tau_{c'})\right) \leq M L_c L_{c'} \tau$$

for some Lipschitz constants L_c and $L_{c'}$ and the lemma follows. □

We are now ready to write the trade balance equations in terms of the entries of the matrix $H(x)$. In the interval $[t, t+\tau]$, the trade balance equations are given by

$\eta\left(\mathcal{T}^c\left(\gamma;t+\tau\right)\setminus\mathcal{T}^c\left(\gamma;t\right)\right)=\eta\left(\mathcal{T}_c\left(\gamma;t+\tau\right)\setminus\mathcal{T}_c\left(\gamma;t\right)\right)$. Let us take the difference, divide by τ and take the limit as $\tau\to 0$. Then on the left hand side we obtain

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{\tau}\eta\left(\mathcal{T}^c\left(\gamma;t+\tau\right)\setminus\mathcal{T}^c\left(\gamma;t\right)\right)\\
=&\lim_{\tau\to 0}\frac{1}{\tau}\eta\left(\bigcup_b T_b^c\left(x,\delta\left(\tau\right)_b\right)\right)\ \text{(by Lemma 46)}\\
=&\lim_{\tau\to 0}\left[\sum_b\frac{1}{\tau}\eta\left(T_b^c\left(x,\delta\left(\tau\right)_b\right)\right)+O\left(\frac{1}{\tau}\left(\left\|\gamma\left(t\right)-\gamma(t+\tau)\right\|_\infty\right)^2\right)\right]\ \text{(since } \nu \text{ is bounded above)}\\
=&\lim_{\tau\to 0}\left[\sum_b\frac{1}{\tau}\eta\left(T_b^c\left(x,\delta\left(\tau\right)_b\right)\right)\right]\ \text{(since } \gamma \text{ is Lipschitz continuous)}\\
=&\lim_{\tau\to 0}\left[\sum_b\frac{\delta\left(\tau\right)_b}{\tau}\cdot\frac{1}{\delta\left(\tau\right)_b}\eta\left(\left\{\theta\in\Theta\ \mid r^\theta\in\left[x-\delta\left(\tau\right)_b e^b,x\right)\ \text{and } Ch_\theta\left(\mathcal{C}^{(\ell)}\right)=c\right\}\right)\right]\\
=&\sum_b\frac{\partial\gamma\left(t\right)_b}{\partial t}\tilde{H}_b^c(x)
\end{aligned}$$

On the right hand side we obtain

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{\tau}\eta\left(\mathcal{T}_c\left(\gamma;t+\tau\right)\setminus\mathcal{T}_c\left(\gamma;t\right)\right)\\
=&\lim_{\tau\to 0}\left[\sum_d\frac{1}{\tau}\eta\left(T_c^d\left(x,\delta\left(\tau\right)_c\right)\right)+O\left(\frac{1}{\tau}\left(\left\|\gamma\left(t\right)-\gamma(t+\tau)\right\|_\infty\right)^2\right)\right]\ \text{(by Lemma 46)}\\
=&\lim_{\tau\to 0}\left[\sum_d\frac{1}{\tau}\eta\left(T_c^d\left(x,\delta\left(\tau\right)_c\right)\right)\right]\ \text{(since } \gamma \text{ is Lipschitz continuous)}\\
=&\lim_{\tau\to 0}\left[\sum_d\frac{\delta\left(\tau\right)_c}{\tau}\cdot\frac{1}{\delta\left(\tau\right)_c}\eta\left(\left\{\theta\in\Theta\ \mid r^\theta\in\left[x-\delta\left(\tau\right)_c e^c,x\right)\ \text{and } Ch_\theta\left(\mathcal{C}^{(\ell)}\right)=d\right\}\right)\right]\\
=&\frac{\partial\gamma\left(t\right)_c}{\partial t}\sum_d\tilde{H}_c^d(x).
\end{aligned}$$

Hence taking the limit in the trade balance equations gives us the following instantaneous trade balance equations at time t,

$$\sum_b\frac{\partial\gamma\left(t\right)_b}{\partial t}\tilde{H}_b^c(x)=\frac{\partial\gamma\left(t\right)_c}{\partial t}\sum_d\tilde{H}_c^d(x)\ \text{for all } c\in C,$$

or equivalently, if we let $d = \frac{\partial \gamma(t)}{\partial t}$, then

$$\sum_b d_b \cdot \tilde{H}_b^c(x) = \sum_b d_c \cdot \tilde{H}_c^b(x),$$

as required.

Let us now write these equations in terms of the matrix $H(x)$. Recall that $v_c = \sum_b \tilde{H}_c^b(x)$ is the measure of marginal students that will get an offer from college c. We rewrite the instantaneous trade balance equations as follows.

$$\begin{aligned}
\sum_b d_b \cdot \tilde{H}_b^c(x) &= d_c \sum_b \tilde{H}_c^b(x) \\
\sum_b d_b \left(\frac{1}{\overline{v}} \tilde{H}_b^c(x) \right) &= d_c \frac{v_c}{\overline{v}} \\
\sum_b d_b \left(\frac{1}{\hat{v}} \tilde{H}_b^c(x) + \mathbf{1}_{b=c} \left(1 - \frac{v_c}{\overline{v}} \right) \right) &= d_c \left(\frac{v_c}{\overline{v}} + \left(1 - \frac{v_c}{\overline{v}} \right) \right) \\
\sum_b d_b H_b^c(x) &= d_c
\end{aligned}$$

and since this holds for all b, we obtain the matrix equation

$$dH(x) = d.$$

C Applications

C.1 Optimal Investment in School Quality

In this section, we prove Propositions **??** and **??**. We also derive the expressions for the TTC cutoffs, welfare measures and optimality conditions in Section 5.1. We will assume that the total measure of students is 1, and speak of student measures and student proportions interchangeably.

C.1.1 Proof of Propositions ?? and ??

Proof of Proposition **??**. Let θ be a student in $S^{b,c}(\boldsymbol{p}, \hat{\boldsymbol{p}})$. We first note that since $B(\theta, \boldsymbol{p}) = \mathcal{C}^{(b)}$ then $\cup_{c'} B_{c'}(\theta, \boldsymbol{p}) = \mathcal{C}^{(b)}$ and so $r_{c'}^{\theta} \in [p_{c'}^{b}, p_{c'}^{b-1})$ for some c' and $r_{c'}^{\theta} < p_{c'}^{b-1}$

for all c'. Similarly, since $B(\theta, \hat{\boldsymbol{p}}) = \mathcal{C}^{(c)}$ then $\cup_{c'} B_{c'}(\theta, \hat{\boldsymbol{p}}) = \mathcal{C}^{(c)}$ and so $r^{\theta}_{c'} \in [\hat{p}^{c}_{c'}, \hat{p}^{c-1}_{c'})$ for some c' and $r^{\theta}_{c'} < \hat{p}^{c-1}_{c'}$ for all c'.

If $c < b-1$ then $r^{\theta}_{c'} < p^{b-1}_{c'}$ for all c', and so since $r^{\theta}_{c''} \in [\hat{p}^{c}_{c''}, \hat{p}^{c-1}_{c''})$ for some c'' it follows that $\hat{p}^{c}_{c''} < p^{b-1}_{c''}$. If $c'' > c$, we have assumed that $\hat{p}^{c}_{c''} \in \left[p^{c+1}_{c''}, p^{c-1}_{c''}\right]$, and it follows that $p^{c+1}_{c''} < p^{b-1}_{c''}$, so $c+1 > b-1 > c$ which is a contradiction. Hence $c'' \leq c$, and since $\hat{p}^{c}_{c''} \neq \hat{p}^{c-1}_{c''}$ it follows that $c'' = c$. Hence $r^{\theta}_{c} \in [\hat{p}^{c}_{c}, p^{b-1}_{c}) = [\hat{p}^{c}_{c}, p^{c}_{c})$ and $B_c(\theta, \boldsymbol{p}) = \emptyset$, and so $\cup_{c' \neq c} B_{c'}(\theta, \boldsymbol{p}) = B(\theta, \boldsymbol{p}) = \mathcal{C}^{(b)}$.

If $c = b-1$, then $r^{\theta}_{c'} < p^{b-1}_{c'}$ for all c', and so since $r^{\theta}_{c''} \in [\hat{p}^{c}_{c''}, \hat{p}^{c-1}_{c''})$ for some c'' it follows that $\hat{p}^{c}_{c''} < p^{b-1}_{c''} = p^{c}_{c''}$. Hence $r^{\theta}_{c''} \in [\hat{p}^{c}_{c''}, p^{b-1}_{c''}) = [\hat{p}^{c}_{c''}, p^{c}_{c''})$, and in order for the interval $\hat{p}^{c}_{c''}, \hat{p}^{c-1}_{c''}$ to be non-empty it must be that $c'' \geq c$. Now if $c'' = c$ our previous analysis shows that it must be the case that $\cup_{c' \neq c} B_{c'}(\theta, \boldsymbol{p}) = \mathcal{C}^{(b)}$, and if $c'' > c$ it is easy to show that $B(\theta, \boldsymbol{p}) = \mathcal{C}^{(b)}$ and $B(\theta, \hat{\boldsymbol{p}}) = \mathcal{C}^{(c)}$.

The cases $c = b+1$ and $c > b+1$ are similar. □

Proof of Proposition **??**. This proposition follows from taking the limit in Proposition **??**, and by observing that □

C.1.2 TTC Cutoffs

We first calculate the TTC cutoffs for different student choice probabilities by using the TTC paths and trade balance equations. In round 1, the marginals $\tilde{H}^{c}_{b}(x)$ for $b, c \in \mathcal{C}$ at each point $x \in [0,1]$ are given by $\tilde{H}^{c}_{b}(x) = e^{\delta_c} \prod_{c' \neq b} x_{c'}$. Hence $v_b = \sum_c \tilde{H}^{c}_{b}(x) = \left(\sum_b e^{\delta_c}\right) \prod_{c' \neq b} x_{c'} = \prod_{c' \neq b} x_{c'}$, so $v = \frac{\prod_c x_c}{\min_c x_c}$ and the matrix $H(x)$ is given by

$$H_{b,c}(x) = e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} + \mathbf{1}_{b-c}\left(1 - \frac{\min_{c'} x_{c'}}{x_b}\right) = \begin{cases} 1 - \left(1 - e^{\delta_c}\right) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise,} \end{cases}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x) H(x) = d(x)$. To solve for this, we observe that this equation is the same as $d(x)(H(x) - I) = \mathbf{0}$, where I is the n-dimensional identity matrix, and and $[H(x) - I]$ has (b,c)th entry

$$[H(x) - I]_{b,c} = \begin{cases} -\left(1 - e^{\delta_b}\right) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise.} \end{cases}$$

Since this has rank $n-1$, the nullspace is easily obtained by replacing the last column of $H(x) - I$ with ones, inverting the matrix and left multiplying it to the vector $e^{|\mathcal{C}|}$ (the

vector with all zero entries, other than a 1 in the $|\mathcal{C}|$th entry). This yields the valid direction $d(x)$ with cth component

$$d_c(x) = -\frac{e^{\delta_c} x_c}{\sum_b e^{\delta_b} x_b}.$$

We now find a valid TTC path γ using the instantaneous trade balance equations 4. Since the ratios of the components of the gradient $\frac{d_b(x)}{d_c(x)}$ only depend on x_b, x_c and the $\delta_{c'}$, we solve for x_c in terms of x_1, using the fact that the path starts at $(1,1)$. This gives the path γ defined by $\gamma_c\left(\gamma_1^{-1}(x_1)\right) = x_1^{e^{\delta_c - \delta_1}}$ for all c.

Suppose college c_1 is the most demanded college, that is, $\frac{e^{\delta_1}}{q_1} = \max_c \frac{e^{\delta_c}}{q_c}$. Since we are only interested in the changes in the cutoffs $\gamma\left(t^{(1)}\right)$ and not in the specific time, let us assume without loss of generality that $\gamma_1(t) = 1 - t$. Then college c_1 fills at time $t^{(1)} = 1 - \left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_1}}{\sum_{c'} e^{\delta_{c'}}}}$. Hence the round 1 cutoffs are

$$p_b^1 = \left(1 - t^{(1)}\right)^{e^{\delta_b - \delta_1}} = \left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}}. \tag{12}$$

It can be shown by projecting onto the remaining coordinates and using induction that the round i cutoffs are given by

$$p_b^c = \begin{cases} \left(\prod_{c'<c} \frac{1}{p_{c'}^{c'}}\right)^{\frac{e^{\delta_b}}{\sum_{c'\geq c} e^{\delta_{c'}}}} \left(\prod_{c'} p_{c'}^{c-1} - \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'\geq c} e^{\delta_{c'}}}} & \text{if } b \geq c \\ p_b^b & \text{if } b \leq c, \end{cases}$$

where the colleges are indexed such that $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \cdots \leq \frac{q_n}{e^{\delta_n}}$.

C.1.3 TTC Cutoffs - Comparative Statics

We perform some comparative statics calculations. For $b \neq \ell$ it holds that the TTC cutoff p_b^1 for using priority at college b to receive a seat at college 1 is decreasing in δ_ℓ,

$$
\begin{aligned}
\frac{\partial p_b^1}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell}\left[\left(1-\frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}}\right] \\
&= \frac{\partial}{\partial \delta_\ell}\left[e^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}\ln\left(1-\frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)}\right] \\
&= p_b^1\left[\frac{\partial}{\partial \delta_\ell}\left(\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}\right)\ln\left(1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)+\left(\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}\right)\frac{\frac{\partial}{\partial \delta_\ell}\left(1-\frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)}{\left(1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)}\right] \\
&= -p_b^1\left(\frac{e^{\delta_\ell+\delta_b}}{(\Delta^1)^2}\right)\left[-\ln\left(\frac{1}{1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1}\right)+\frac{1}{\left(1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)}-1\right] \\
&< 0
\end{aligned}
$$

since $0 < \frac{1}{\left(1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)} < 1$, where for brevity we define $\Delta^c = \sum_{b\geq c} e^{\delta_b}$.

We can decompose this change as

$$
\begin{aligned}
\frac{\partial p_b^1}{\partial \delta_\ell} &= -p_b^1\left(\frac{e^{\delta_\ell+\delta_b}}{(\Delta^1)^2}\right)\left[\ln\left(1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)\right]-p_b^1\left(\frac{e^{\delta_\ell+\delta_b}}{(\Delta^1)^2}\right)\left[\frac{1}{\left(1-\left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)}-1\right] \\
&< 0,
\end{aligned}
$$

where the first term is the increase in p_b^1 due to the fact that relatively fewer students are pointed to and cleared by college b for every marginal change in rank, and the second term is the decrease in p_b^1 due to the fact that college 1 is relatively less popular now, and so more students need to be given a budget set of $\mathcal{C}^{(1)}$ in order for college 1 to reach capacity.

For $b = \ell$ the TTC cutoff p_ℓ^1 is again decreasing in δ_ℓ,

$$
\begin{aligned}
\frac{\partial p_\ell^1}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell}\left[\left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}}}\right] \\
&= \frac{\partial}{\partial \delta_\ell}\left[e^{\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}} \ln\left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)}\right] \\
&= p_\ell^1\left[\frac{\partial}{\partial \delta_\ell}\left(\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}}\right)\ln\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right) + \left(\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}}\right)\frac{\frac{\partial}{\partial \delta_\ell}\left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)}{\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)}\right] \\
&= -p_b^1\left(\frac{e^{\delta_\ell}\left(\Delta^1 - e^{\delta_\ell}\right)}{\left(\Delta^1\right)^2}\right)\ln\left(\frac{1}{1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1}\right) - p_b^1\left(\frac{e^{2\delta_\ell + \delta_b}}{\left(\Delta^1\right)^2}\right)\left(\frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)} - 1\right) \\
&< 0
\end{aligned}
$$

since both terms are negative.

Similarly, for $c < \ell$ and $b \neq \ell$ the TTC cutoff p_ℓ^c is decreasing in δ_ℓ,

$$
\begin{aligned}
\frac{\partial p_b^c}{\partial \delta_\ell} =& \frac{\partial}{\partial \delta_\ell}\left[\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c}e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'\geq c}e^{\delta_{c'}}}}\right] \\
=& \frac{\partial}{\partial \delta_\ell}\left[e^{\frac{e^{\delta_b}}{\sum_{c'\geq c}e^{\delta_{c'}}}\ln\left(1-\left(\prod_{c'<c}{}^{1}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c}e^{\delta_{c'}}\right)\right)}\right] \\
=& p_b^c\left[\frac{\partial}{\partial \delta_\ell}\left(\frac{e^{\delta_b}}{\sum_{c'\geq c}e^{\delta_{c'}}}\right)\ln\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)\right] \\
&+ p_b^c\left[\left(\frac{e^{\delta_b}}{\sum_{c'\geq c}e^{\delta_{c'}}}\right)\frac{\frac{\partial}{\partial \delta_\ell}\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c}e^{\delta_{c'}}\right)\right)}{\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right] \\
=& p_b^c\left[\left(\frac{-e^{\delta_\ell+\delta_b}}{(\Delta^c)^2}\right)\ln\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)\right] \\
&+ p_b^c\left[\left(\frac{e^{\delta_b}}{\Delta^c}\right)\frac{\left(-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)e^{\delta_\ell}\right)-\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\frac{\partial}{\partial \delta_\ell}\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)}{\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right] \\
=& -p_b^c\left(\frac{e^{\delta_\ell+\delta_b}}{(\Delta^c)^2}\right)\left[-\ln\left(\frac{1}{1-P^c\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c}\right)+\frac{1}{\left(1-P^c\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}-1\right] \\
&- p_b^c\left[\frac{e^{\delta_b}\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\frac{\partial P^c}{\partial \delta_\ell}}{\left(1-P^c\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right] \\
<& 0
\end{aligned}
$$

where $P^c=\prod_{c'<c}\frac{1}{p_{c'}^{c'}}$, since $0<1-P^c\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c<1$ and $\frac{\partial P^c}{\partial \delta_\ell}=P^c\left(\sum_{c'<c}\frac{-\frac{\partial p_{c'}^{c'}}{\partial \delta_\ell}}{p_{c'}^{c'}}\right)>0$ so both terms are negative.

We can decompose this change as follows. Let $P^c=\prod_{c'<c}\frac{1}{p_{c'}^{c'}}$. For $c<\ell$ and $b\geq c$, $b\neq\ell$ it holds that

$$
\begin{aligned}
\frac{\partial p_b^c}{\partial \delta_\ell} =& - p_b^c \left(\frac{e^{\delta_\ell+\delta_b}}{(\Delta^c)^2}\right)\left[\ln\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)\right] \\
& - p_b^c \left(\frac{e^{\delta_\ell+\delta_b}}{(\Delta^c)^2}\right)\left[\frac{1}{\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)} - 1\right] - p_b^c\left[\frac{e^{\delta_b}\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\frac{\partial}{\partial\delta_\ell}P^c}{\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right] \\
<& 0,
\end{aligned}
$$

where the first term is the increase in p_b^c due to the fact that relatively fewer students are pointed to and cleared by college j for every marginal change in rank, and the second and third terms are the decrease in p_b^c due to the fact that colleges 1 through c are relatively less popular now, and so more students need to be given a budget set of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(c)}$ in order for colleges 1 through c to reach capacity.

For $c < \ell$ and $b = \ell$ the TTC cutoff p_ℓ^c is also decreasing in δ_ℓ,

$$
\begin{aligned}
\frac{\partial p_\ell^c}{\partial \delta_\ell} =& \frac{\partial}{\partial \delta_\ell}\left[\left(1 - \left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_\ell}}{\sum_{c'\geq c} e^{\delta_{c'}}}}\right] \\
=& \frac{\partial}{\partial \delta_\ell}\left[e^{\frac{e^{\delta_\ell}}{\sum_{c'\geq c} e^{\delta_{c'}}}\ln\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)}\right] \\
=& p_\ell^c\left[\frac{\partial}{\partial \delta_\ell}\left(\frac{e^{\delta_\ell}}{\sum_{c'\geq c} e^{\delta_{c'}}}\right)\ln\left(1 - \left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)\right] \\
& + p_\ell^c\left[\left(\frac{e^{\delta_\ell}}{\sum_{c'\geq c} e^{\delta_{c'}}}\right)\frac{\frac{\partial}{\partial\delta_\ell}\left(1 - \left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)}{\left(1 - \left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right] \\
=& - p_\ell^c\left[\left(\frac{e^{\delta_\ell}\left(\Delta^c - e^{\delta_\ell}\right)}{(\Delta^c)^2}\right)\ln\left(\frac{1}{1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c}\right)\right] \\
& - p_\ell^c\left(\frac{e^{\delta_\ell}}{\Delta^c}\right)\left[\frac{\left(P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)e^{\delta_\ell}\right) + \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\frac{\partial P^c}{\partial\delta_\ell}}{\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right] \\
<& 0
\end{aligned}
$$

where $P^c = \prod_{c'<c} \frac{1}{p_{c'}^{c'}}$, since $\frac{\partial P^c}{\partial \delta_\ell} = P^c \left(\sum_{c'<c} \frac{-\frac{\partial p_{c'}^{c'}}{\partial \delta_\ell}}{p_{c'}^{c'}} \right) > 0$ and so both terms are negative.

When $c = \ell$, the effects of changing $\delta_\ell$on the cutoffs required the obtain a seat at college ℓ are a little more involved. For $c = \ell$ and $b \neq \ell$,

$$
\begin{aligned}
\frac{\partial p_b^\ell}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell} \left[\left(1 - \left(\prod_{c'<\ell} \frac{1}{p_{c'}^{c'}} \right) \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \left(\sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_b}}{\sum_{c' \geq \ell} e^{\delta_{c'}}}} \right] \\
&= \frac{\partial}{\partial \delta_\ell} \left[e^{\frac{e^{\delta_b}}{\sum_{c' \geq \ell} e^{\delta_{c'}}} \ln \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \left(\sum_{c' \geq \ell} e^{\delta_{c'}} \right) \right)} \right] \\
&= p_b^\ell \left[\frac{\partial}{\partial \delta_\ell} \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \ln \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) + \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right] \\
&= p_b^\ell \left[\left(\frac{e^{\delta_\ell + \delta_b}}{(\Delta^\ell)^2} \right) \ln \left(\frac{1}{1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right] \\
&= p_b^\ell \left(\frac{e^{\delta_b}}{\Delta^\ell} \right) \left[\left(\frac{e^{\delta_\ell}}{\Delta^\ell} \right) \ln \left(\frac{1}{1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \frac{\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)}{\left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right)} \right]
\end{aligned}
$$

where $P_\ell = \prod_{c'<\ell} \frac{1}{p_{c'}^{c'}}$, the first term is positive, and the second term has the same sign as its numerator

$$
\frac{\partial}{\partial \delta_\ell} \left(1 - P_\ell \left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right).
$$

Similarly for $c = \ell$ and $b = \ell$,

$$
\begin{aligned}
\frac{\partial p_\ell^\ell}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell}\left[\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_\ell}}{\sum_{c'\geq \ell} e^{\delta_{c'}}}}\right] \\
&= \frac{\partial}{\partial \delta_\ell}\left[e^{\frac{e^{\delta_\ell}}{\sum_{c'\geq \ell} e^{\delta_{c'}}}\ln\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\left(\sum_{c'\geq \ell} e^{\delta_{c'}}\right)\right)}\right] \\
&= p_\ell^\ell\left[\frac{\partial}{\partial \delta_\ell}\left(\frac{e^{\delta_\ell}}{\Delta^\ell}\right)\ln\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right) + \left(\frac{e^{\delta_\ell}}{\Delta^\ell}\right)\frac{\frac{\partial}{\partial \delta_\ell}\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}{\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}\right] \\
&= p_\ell^\ell\left[-\left(\frac{e^{\delta_\ell}\left(\Delta^\ell - e^{\delta_\ell}\right)}{\left(\Delta^\ell\right)^2}\right)\ln\left(\frac{1}{1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell}\right) + \left(\frac{e^{\delta_\ell}}{\Delta^\ell}\right)\frac{\frac{\partial}{\partial \delta_\ell}\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}{\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}\right] \\
&= p_\ell^\ell\left(\frac{e^{\delta_b}}{\Delta^\ell}\right)\left[-\left(\frac{\Delta^\ell - e^{\delta_\ell}}{\Delta^\ell}\right)\ln\left(\frac{1}{1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell}\right) + \frac{\frac{\partial}{\partial \delta_\ell}\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}{\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}\right]
\end{aligned}
$$

where $P_\ell = \prod_{c'<\ell} \frac{1}{p_{c'}^{c'}}$, the first term is negative, and the second term has the same sign as its numerator

$$
\frac{\partial}{\partial \delta_\ell}\left(1 - P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right).
$$

Since $\frac{\partial}{\partial \delta_\ell}\left(\prod_{b\geq \ell} p_b^\ell\right) > 0$, it follows that $\frac{\partial p_b^\ell}{\partial \delta_\ell} > 0$ for all $b \neq \ell$, and there are regimes in which $\frac{\partial p_\ell^\ell}{\partial \delta_\ell}$ is positive, and regimes where it is negative.

C.1.4 Welfare Expressions

We now derive the welfare expressions corresponding to these cutoffs. Let $\mathcal{C}^{(c)} = \{c, c+1, \ldots, n\}$. Since the colleges are ordered so that $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \cdots \leq \frac{q_n}{e^{\delta_n}}$, it follows that the colleges also fill in the order $1, 2, \ldots, n$.

The proportion of students with budget set $\mathcal{C}^{(1)}$ is given by

$$
1 - \prod_c p_c^1 = 1 - \left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{\sum_i e^{\delta_c}}{\sum_{c'} e^{\delta_{c'}}}} = \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right),
$$

the proportion of students with budget set $\mathcal{C}^{(1)}$ or $\mathcal{C}^{(2)}$ is given by

$$\begin{aligned}1-\prod_c p_c^2 &= 1-p_1^1\left(\prod_b p_b^1-\left(\frac{q_2}{e^{\delta_2}}-\frac{q_1}{e^{\delta_1}}\right)\left(\sum_{b\geq 2}e^{\delta_b}\right)\right)\frac{1}{p_1^1}\\ &= 1-\prod_b p_b^1+\left(\frac{q_2}{e^{\delta_2}}-\frac{q_1}{e^{\delta_1}}\right)\left(\sum_{b\geq 2}e^{\delta_b}\right)\end{aligned}$$

and so the proportion of students with budget set $\mathcal{C}^{(2)}$ is given by

$$\eta\left(\left\{\theta : B\left(\theta,\boldsymbol{p}\right)=\mathcal{C}^{(2)}\right\}\right)=\left(\sum_{b\geq 2}e^{\delta_b}\right)\left(\frac{q_2}{e^{\delta_2}}-\frac{q_1}{e^{\delta_1}}\right).$$

An inductive argument shows that the proportion of students with budget set $\mathcal{C}^{(c)}$ is

$$\eta\left(\left\{\theta : B\left(\theta,\boldsymbol{p}\right)=\mathcal{C}^{(c)}\right\}\right)=\left(\sum_{b\geq c}e^{\delta_b}\right)\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right).$$

Moreover, each such student with budget set $\mathcal{C}^{(c)}$, conditional on their budget set, has expected utility Small and Rosen (1981)

$$\mathbb{E}\left[\max_{c'\in\mathcal{C}^{(c)}}\left\{\delta_b+\varepsilon_{\theta c'}\right\}\right]=\ln\left[\sum_{c'\geq c}e^{\delta_b}\right],$$

and hence the expected social welfare from fixed qualities δ_c is given by

$$\sum_c\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{b\geq c}e^{\delta_b}\right)\ln\left(\sum_{b\geq c}e^{\delta_b}\right)=\sum_c\left(\frac{q_c}{e^{\delta_c}}-\frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\ln\Delta^c.$$

C.1.5 Optimal Investment

Finally, we solve for the social welfare maximixing budget allocation. The reformulated problem (**??**) has objective function

$$U\left(\boldsymbol{x}\right)=\left(\frac{q_1}{X-\sum_i x_i}\right)X\ln X+\left(\frac{q_2}{x_2}-\frac{q_1}{X-\sum_i x_i}\right)\Delta^2\ln\Delta^2+\sum_{i\geq 3}\left(\frac{q_i}{x_i}-\frac{q_{i-1}}{x_{i-1}}\right)\Delta^i\ln\Delta^i,$$

where $\Delta^i = \sum_{j \geq i} x_j$. Taking the derivatives with respect to the budget allocations x_k gives

$$\begin{aligned}
\frac{\partial U}{\partial x_k} &= \left(\frac{q_1}{\left(X - \sum_i x_i\right)^2}\right)\left(X \ln X - \Delta^2 \ln \Delta^2\right) + \left(\frac{q_2}{x_2} - \frac{q_1}{X - \sum_i x_i}\right)\left(1 + \ln \Delta^2\right) \\
&\quad + \left(-\frac{q_k}{(x_k)^2}\right)\left(\Delta^k \ln \Delta^k - \Delta^{k+1} \ln \Delta^{k+1}\right) + \sum_{3 \leq i \leq k} \left(\frac{q_i}{x_i} - \frac{q_{i-1}}{x_{i-1}}\right)\left(1 + \ln \Delta^i\right) \\
&= \left(\frac{q_1}{\left(X - \sum_i x_i\right)^2}\right)\left(X \ln \frac{X}{\Delta^2} - \left(X - \Delta^2\right)\right) + \sum_{2 \leq i < k} \frac{q_i}{x_i} \ln \frac{\Delta^i}{\Delta^{i+1}} + \frac{q_k}{(x_k)^2}\left(x_k - \Delta^{k+1} \ln \frac{\Delta^k}{\Delta^{k+1}}\right),
\end{aligned}$$

where

$$\begin{aligned}
X \ln \frac{X}{\Delta^2} - \left(X - \Delta^2\right) &= X\left(\ln \frac{X}{\Delta^2} - 1 + \frac{\Delta^2}{X}\right) \\
&= X\left(\ln x - 1 + \frac{1}{x}\right) \\
&\geq 0,
\end{aligned}$$

$$\ln \frac{\Delta^i}{\Delta^{i+1}} = \ln\left(1 + \frac{x_i}{x_{i+1} + \cdots + x_n}\right) \geq 0, \text{ and}$$

$$\begin{aligned}
x_k - \Delta^{k+1} \ln \frac{\Delta^k}{\Delta^{k+1}} &= \Delta^{k+1}\left(\frac{x_k}{\Delta^{k+1}} - \ln\left(1 + \frac{x_k}{\Delta^{k+1}}\right)\right) \\
&= \Delta^{k+1}\left(x - \ln x\right) \\
&\geq 0,
\end{aligned}$$

and so $\frac{\partial U}{\partial x_k} \geq 0$ for all k.

Moreover, if $\frac{q_{i-1}}{x_{i-1}} = \frac{q_i}{x_i}$, then defining a new problem with $n-1$ colleges, capacities

$$\tilde{q}_j = \begin{cases} q_j & \text{if } j < i-1 \\ q_{i-1} + q_i & \text{if } j = i-1 \\ q_{j+1} & \text{if } j > i-1 \end{cases}$$

and assigning a budget X by

$$\tilde{x_j} = \begin{cases} x_j & \text{if } j < i-1 \\ x_{i-1} + x_i & \text{if } j = i-1 \\ x_{j+1} & \text{if } j > i-1 \end{cases}$$

leads to a problem with the same objective function, since

$$\begin{aligned} &\left(\frac{q_{i-1}}{x_{i-1}} - \frac{q_{i-2}}{x_{i-2}}\right)\Delta^{i-1}\ln\Delta^{i-1} + \left(\frac{q_i}{x_i} - \frac{q_{i-1}}{x_{i-1}}\right)\Delta^{i}\ln\Delta^{i} + \left(\frac{q_{i+1}}{b_{i+1}} - \frac{q_i}{x_i}\right)\Delta^{i+1}\ln\Delta^{i+1} \\ =&\left(\frac{q_{i-1}}{x_{i-1}} - \frac{q_{i-2}}{x_{i-2}}\right)\Delta^{i-1}\ln\Delta^{i-1} + \left(\frac{q_{i+1}}{b_{i+1}} - \frac{q_i}{x_i}\right)\Delta^{i+1}\ln\Delta^{i+1} \\ =&\left(\frac{q_{i-1}+q_i}{x_{i-1}+x_i} - \frac{q_{i-2}}{x_{i-2}}\right)\Delta^{i-1}\ln\Delta^{i-1} + \left(\frac{q_{i+1}}{b_{i+1}} - \frac{q_{i-1}+q_i}{x_{i-1}+x_i}\right)\Delta^{i+1}\ln\Delta^{i+1}. \end{aligned}$$

Hence if there exists i for which $\frac{q_i}{x_i} \neq \frac{q_{i-1}}{x_{i-1}}$, we may take i to be minimal such that this occurs, decrease each of $x_1, \ldots, x_{i-1}$ proportionally so that $x_1 + \cdots + x_{i-1}$ decreases by ε and increase x_i by ε and increase resulting value of the objective. It follows that the objective is maximized when $\frac{q_1}{x_1} = \frac{q_2}{x_2} = \cdots = \frac{q_n}{x_n}$, i.e. when the money assigned to each college is proportional to the number of seats at the college.

C.2 Bossiness of the TTC priorities and the Set of Feasible Outcomes

We demonstrate how to calculate the TTC cutoffs for the two economies in Figure 6 by using the TTC paths and trade balance equations.

Consider the economy $\mathcal{E}_0$, where the top priority students have ranks uniformly distributed in $[m, 1]^2$. If $x = (x_1, x_1)$ is on the diagonal, then $\tilde{H}_i^j(x) = \frac{x_1}{2}$ for all $i, j \in \{1, 2\}$. Hence

$$H_{i,j}(x) = \frac{1}{x_1}\left(\frac{x_1}{2}\right) + \mathbf{1}_{i=j}\left(1 - \frac{x_1}{x_1}\right) = \frac{1}{2} \quad \forall i, j \in \{1, 2\}$$

and so there is a unique valid direction $d(\vec{x}) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Moreover, $\gamma(t) = \left(\frac{t}{2}, \frac{t}{2}\right)$ satisfies $\frac{d\gamma(t)}{dt} = \boldsymbol{d}(\gamma(t))$ for all t and hence Theorem (15) implies that $\gamma(t) = \left(\frac{t}{2}, \frac{t}{2}\right)$ is the unique TTC path, and the cutoff points $p_b^c = \sqrt{1-2q}$ give the unique TTC allocation.

Consider now the economy $\mathcal{E}_1$, where top priority students have ranks uniformly

distributed in the $\tilde{r} \times \tilde{r}$ square $(1-\tilde{r}, 1] \times (m, m+\tilde{r}]$ for some small $\tilde{r}$.

If x is in $(1-\tilde{r}, 1] \times [m+\tilde{r}, 1]$ then $\tilde{H}_1^j(x) = \frac{1}{2}\left(m + (1-m)\frac{1-m}{\tilde{r}}\right)$ for all j and $\tilde{H}_2^j(x) = \frac{m}{2}$ for all j. Hence $v_1 = m\left(1 + (1-m)\frac{1-m}{\tilde{r}m}\right)$, $v_2 = m$, $v = v_1$. So

$$H(x) = \frac{1}{v_1}\begin{bmatrix} \frac{v_1}{2} & \frac{v_1}{2} \\ \frac{v_2}{2} & v_1 - \frac{v_2}{2} \end{bmatrix}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x)H(x) = d(x)$. Solving for this,

$$\begin{aligned} d(x)(H(x) - I) &= \mathbf{0} \\ d(x)\frac{1}{v_1}\begin{bmatrix} -\frac{v_1}{2} & \frac{v_1}{2} \\ \frac{v_2}{2} & -\frac{v_2}{2} \end{bmatrix} &= \mathbf{0} \\ d(x) &= \frac{1}{v_1+v_2}\begin{bmatrix} -v_2 \\ -v_1 \end{bmatrix} = \frac{1}{2+\frac{r^2}{\tilde{r}(1-r)}}\begin{bmatrix} -1 \\ -1-\frac{r^2}{\tilde{r}(1-r)} \end{bmatrix}. \end{aligned}$$

If x is in $(m, 1-\tilde{r}] \times (m, 1]$ then $\tilde{H}_i^j(x) = \frac{m}{2}$ for all i, j. Hence $v_1 = v_2 = v = m$ and

$$H_{i,j}(x) = 1 + \frac{\tilde{r}}{1-m} + \mathbf{1}_{i=j}\left(1 - \frac{m}{m}\right) = 1 + \frac{\tilde{r}}{1-m} \quad \forall i, j,$$

and so there is a unique valid direction $d(x) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

Finally, if $x = (x_1, x_2)$ is in $[0,1] \setminus (m,1]^2$ then $\tilde{H}_1^j(x) = \frac{1}{2}x_2$ and $\tilde{H}_2^j = \frac{1}{2}x_1$ for all j. Hence $v_1 = x_2, v_2 = x_1$ and $v = \max\{x_1, x_2\}$. So

$$H(x) = \frac{1}{v}\begin{bmatrix} \frac{x_2}{2} + (v - x_2) & \frac{x_2}{2} \\ \frac{x_1}{2} & \frac{x_1}{2} + (v - x_1) \end{bmatrix} = \frac{1}{v}\begin{bmatrix} v - \frac{x_2}{2} & \frac{x_2}{2} \\ \frac{x_1}{2} & v - \frac{x_1}{2} \end{bmatrix}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x)H(x) = d(x)$.

Solving for this,

$$
\begin{aligned}
d(x)\left(H(x)-I\right) &= \mathbf{0} \\
d(x)\frac{1}{v}\begin{bmatrix} -\frac{x_2}{2} & \frac{x_2}{2} \\ \frac{x_1}{2} & -\frac{x_1}{2} \end{bmatrix} &= \mathbf{0} \\
d(x) &= \frac{1}{x_1+x_2}\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}.
\end{aligned}
$$

Hence the TTC path $\gamma(t)$ has gradient $\frac{1}{2+\frac{(1-m)^2}{\tilde{r}m}}\begin{bmatrix} -1 \\ -1-\frac{(1-m)^2}{\tilde{r}m} \end{bmatrix}$ from $(1,1)$ to $\left(1-\tilde{r}, 1-\tilde{r}-\frac{r^2}{1-r}\right)$, gradient $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ from the point $\left(1-\tilde{r}, 1-\tilde{r}-\frac{r^2}{1-r}\right)$ to $\left(m+\frac{r^2}{1-r}, m\right)$ and gradient $\frac{1}{2+\frac{(1-m)^2}{m^2}}\begin{bmatrix} -1-\frac{(1-m)^2}{m^2} \\ -1 \end{bmatrix}$ from $\left(m+\frac{(1-m)^2}{m}, m\right)$ to $\left(\sqrt{\frac{1-2q}{1-2m+2m^2}}, \sqrt{(1-2q)(1-2m+2m^2)}\right) = \left(\overline{p}, \underline{p}\right)$.

Finally, we show that if economy $\mathcal{E}_2$ is given by perturbing the relative ranks of students in $\left\{\theta \mid r_c^\theta \geq m \ \forall c\right\}$, then the TTC cutoffs for $\mathcal{E}_2$ are given by $p_1^1 = p_1^2 = x,\ p_2^1 = p_2^2 = y$ where $x \leq \overline{p} = \sqrt{\frac{1-2q}{1-2m+2m^2}}$ and $y \geq \underline{p} = \sqrt{(1-2q)(1-2m+2m^2)}$. (By symmetry, it follows that $\underline{p} \leq x, y \leq \overline{p}$.) Let γ_1 and γ_2 be the TTC paths for $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Consider the point (x_{bound}, m) on γ_2. The TTC path γ_2 for $\mathcal{E}_2$ has gradient $\frac{1}{x_{bound}+m}\begin{bmatrix} -x_{bound} \\ -m \end{bmatrix}$ from (x_{bound}, m) to (x, y).

Consider the aggregate trade balance equations for students assigned before the TTC path reaches (x_{bound}, m). They stipulate that the measure of students in $[0,m] \times [m,1]$ who prefer college 1 is at most the measure of students who are either perturbed, or in $[x_{bound}, 1] \times [0, m]$, and who prefer college 2. This means that

$$
\begin{aligned}
\frac{1}{2}m(1-m) &\leq \frac{1}{2}\left((1-m)^2 + m(1-x_{bound})\right), \\
mx_{bound} &\leq 1-2m+2m^2 \\
x_{bound} &\leq m + \frac{(1-m)^2}{m}.
\end{aligned}
$$

Hence γ_2 lies above γ_1[39] and so $x \leq \overline{p}$ and $y \geq \frac{1-2q}{\overline{p}} = \underline{p}$.

[39] That is, for each x_1, if (x_1, y_1) lies on γ_1 and (x_1, y_2) lies on γ_2, then $y_2 \geq y_1$.

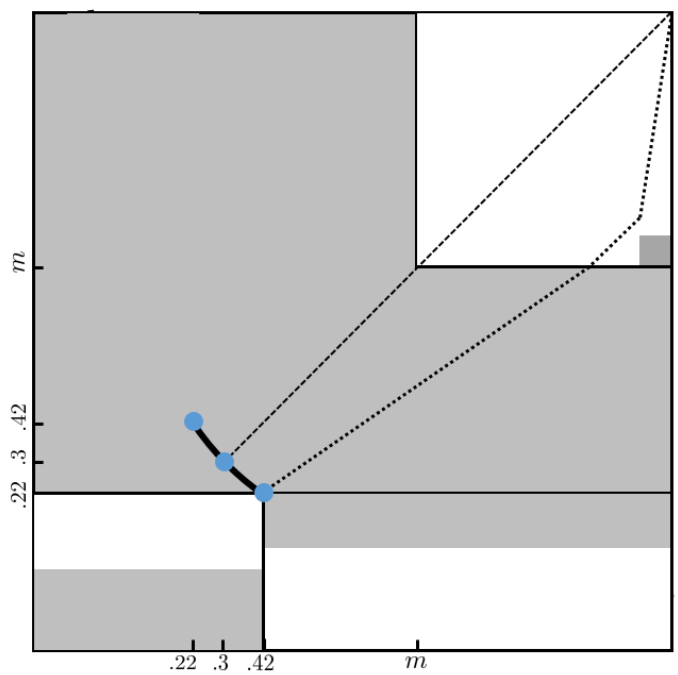


Figure 9: Economy $\mathcal{E}_1$ from Example 25. The black borders partition the space of students into four regions. The density of students is zero on white areas, and constant on each of the shaded areas within a bordered region. In each of the four regions, the total measure of students within is equal to the total area (white and shaded) within the borders of the region.

C.3 Comparing Top Trading Cycles and Deferred Acceptance

In this section, we derive the expressions for the TTC and DA cutoffs given in Section 5.3.

Consider the TTC cutoffs for the neighborhood priority setting. We prove by induction on ℓ that $p_j^\ell = 1 - \frac{q_\ell}{2q}$ for all ℓ, j such that $j \geq \ell$.

Base case: $\ell = 1$.

For each school i, there are measure q of students whose first choice school is i, αq of whom have priority at i and $\frac{(1-\alpha)q}{n-1}$ of whom have priority at school j, for all $j \neq i$.

The TTC path is given by the diagonal, $\gamma(t) = \left(1 - \frac{t}{\sqrt{n}}, 1 - \frac{t}{\sqrt{n}}, \ldots, 1 - \frac{t}{\sqrt{n}}\right)$. At the point $\gamma(t) = (x, x, \ldots, x)$ (where $x \geq \frac{1}{2}$) a fraction $2(1-x)$ of students from each neighborhood have been assigned. Since the same proportion of students have each school as their top choice, this means that the quantity of students assigned to each school i is $2(1-x)q$. Hence the cutoffs are given by considering school 1, which has the smallest capacity, and setting the quantity assigned to school 1 equal to its capacity q_1.

It follows that $p_j^1 = x^*$ for all j, where $2(1-x^*)q = q_1$, which yields

$$p_j^1 = 1 - \frac{q_1}{2q} \text{ for all } j.$$

Inductive step.

Suppose we know that the cutoffs $\left\{p_j^i\right\}_{i,j\,:\,i\leq\ell}$ satisfy $p_j^i = 1-\frac{q_i}{2q}$. We show by induction that the $(\ell+1)$th set of cutoffs $\left\{p_j^{\ell+1}\right\}_{j>\ell}$ are given by $p_j^{\ell+1} = 1 - \frac{q_{\ell+1}}{2q}$.

The TTC path is given by the diagonal when restricted to the last $n-\ell$ coordinates, $\gamma\left(t^{(\ell)}+t\right) = \left(p_1^1, p_2^2, \ldots, p_\ell^\ell, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}, \ldots, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}\right)$.

Consider a neighborhood i. If $i > \ell$, at the point $\gamma(t) = (p_1^1, p_2^2, \ldots, p_\ell^\ell, x, x, \ldots, x)$ (where $x \geq \frac{1}{2}$) a fraction $2\left(p_\ell^\ell - x\right)$ of (all previously assigned and unassigned) students from neighborhood i have been assigned in round $\ell+1$. If $i \leq \ell$, no students from neighborhood i have been assigned in round $\ell+1$.

Consider the set of students S who live in one of the neighborhoods $\ell+1, \ell+2, \ldots, n$. The same proportion of these students have each remaining school as their top choice out of the remaining schools. This means that for any $i > \ell$, the quantity of students assigned to school i in round $\ell+1$ by time t is a $\frac{1}{n-\ell}$ fraction of the total number of students assigned in round $\ell+1$ by time t, and is given by$(n-\ell)\,q\frac{1}{n-\ell} = 2\left(p_\ell^\ell - x\right)q$. Hence the cutoffs are given by considering school $\ell+1$, which has the smallest residual, and setting the quantity assigned to school $\ell+1$ equal to its residual capacity $q_{\ell+1} - q_\ell$. It follows that $p_j^{\ell+1} = x^*$ for all $j > \ell$ where $2\left(p_\ell^\ell - x^*\right)q = q_{\ell+1} - q_\ell$, which yields

$$p_j^{\ell+1} = p_\ell^\ell - \frac{q_{\ell+1}-q_\ell}{2q} = 1 - \frac{q_\ell}{2q} - \frac{q_{\ell+1}-q_\ell}{2q} = 1 - \frac{q_{\ell+1}}{2q} \text{ for all } j > \ell.$$

This completes the proof that the TTC cutoffs are given by $p_j^i = p_i^j = 1 - \frac{q_i}{2q}$ for all $i \leq j$.

Now consider the DA cutoffs. We show that the cutoffs $p_i = 1 - \frac{q_i}{2q}$ satisfy the supply-demand equations.

We first remark that the cutoff at school i is higher than all the ranks of students without priority at school i, $p_i \geq \frac{1}{2}$. Since every student has priority at exactly one school, this means that every student is either above the cutoff for exactly one school and is assigned to that school, or is below all the cutoffs and remains unassigned. Hence there are $q2(1-p_i) = q_i$ students assigned to school i for all i, and the supply-demand equations are satisfied.