Firm Heterogeneity, Sorting and the Minimum Wage

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Abstract

In this paper, we show that firm heterogeneity and labor market sorting can help us understand a number of empirical facts, and aspects related to the political economy of minimum wages. We study a competitive economy with non-transferable utility, and preferences which depend on worker and firm types. Sorting in this environment can be induced by complementarities in productions or forces related to preferences. With firm heterogeneity, minimum wage increases affect workers above the minimum wage threshold, reducing wage inequality, increasing dispersion in firm profits and reducing the size of employment effects. It can also explain why such policies have political support, as workers above the threshold benefit from the policy.

1 Introduction

In this paper, we study the importance of firm heterogeneity and labor market sorting to understand the effects of minimum wages policies, with an emphasis on the effects on inequality. We contrast models with and without firm heterogeneity. We start with a competitive model that features heterogeneous workers but homogeneous firms, which we call the "truncation model". This is a widely used model in the related literature to understand the effect of minimum wage laws. In this model, a rise in minimum wages leads to some low skilled workers to lose their jobs, but nothing happens to workers above the truncation point. The results change dramatically once we introduce an extension to this competitive model: hierarchical firm heterogeneity and labor market sorting. There is an extensive empirical literature that documents that firm heterogeneity is pervasive and strongly related to wage inequality, which makes this is a natural extension to the model. As in David Ricardo's analysis of land rents, differentiation allows firms to earn positive profits (or "rents") in equilibrium. The minimum

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1 This is the model used in DiNardo et al [10] and the "truncation" model described in Lee [20].

2 Bartelsman and Doms [3] survey the literature on firm heterogeneity, and document that even within detailed sectors it is common to find firms three times more productive than others. Regarding the importance for wage inequality, Davis and Haltiwanger [8] and Barth et al [4] perform a within-between firms variance decomposition using US data, and find that the between firms component accounts for over 60% of the cross-sectional wage dispersion, and almost all of the change in the variance during the time period considered. Lazear and Shaw [19] summarizes a similar body of evidence for 9 additional countries.
wage allows workers to capture some of these rents, which affects workers above the truncation point. As we argue below, this helps us understand a number of empirical facts, and aspects related to the political economy of minimum wages.

We study a competitive economy with non-transferable utility, and preferences which may depend on both worker and firm types. Sorting patterns in this environment can arise due to complementarities in productions, as emphasized by Becker [5], or due to forces related to preferences, as emphasized in the theory of equalizing differences (e.g. Rosen [25]). We provide necessary and sufficient condition for assortative matching in this economy and a characterization of equilibrium. Our main results are relative to specifications which feature weak positive sorting, but we also discuss the case with negative sorting. The results do not rely on the assignment featuring perfect sorting (top worker assigned to the top firm, and so on and so forth), and include the case of random matching.

Our main results are as follows. First, a rise in the minimum wage leads to a reduction in wage inequality, whereas at the same time it leads to an increase in the dispersion of firm profits. This is true even after controlling for selection, and only considering workers who earn wages above the minimum wage itself. We follow the related literature and call these effects spillover effects of minimum wages. Several empirical papers have argued that these spillovers are pervasive in actual economies. For example, Lee [20] argues that the 50 − 20 and the 50 − 30 wage differentials in the US increased in response to the minimum wage declines in the 80s. Since minimum wages in the US are binding only at less than the 10th percentile this suggests the presence of spillover effects above the truncation point. The evidence on the implication for firms is more limited, but a recent paper by Draca et al [11] argues that a minimum wage increase in the UK reduced the profit of low wage firms relative to firms who pay well above the minimum wage, which is consistent with our theory. In addition, firm heterogeneity helps us understand the employment effects and the firms provision of fringe benefits in response to changes in minimum wage laws. Increasing firm heterogeneity attenuates the disemployment effects of minimum wages, and it also attenuates the firm’s response in terms of the provision of

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3 The change in inequality is determined by looking at Lorenz curves.
4 We also study the implications for the labor share of income. In the baseline model, the labor share necessarily increases. However, this is not robust to free entry, which we introduce in Section 7.
5 For example, see Lee [20], Autor et al [1], Bosch and Manacorda [6], Teulings [34] and Dickens and Manning [9].
6 Autor et al [1] revisit this result and find smaller spillovers once they control for some biases (e.g. division bias). Moreover, they argue that the measured spillovers that they find may be an artifact of measurement error, but cannot reject the hypothesis that there are true spillovers. Bosch and Manacorda [6] apply the same methodology as Autor et al [1] to Mexican data and find sizable spillovers which cannot be artifacts of measurement error.
7 One alternative is that the results are driven by selection. Lee [20], Autor et al [1], Teulings [34] and Dickens and Manning [9] argue that selection effects are not substantial during this time period. In our theory, we control for selection so that we analyze genuine spillovers.
8 It is worth noting that their treatment group also includes firms who pay minimum wages: it includes firms with average wage smaller or equal to ~ 1.5 times the minimum wage. In the US, the minimum wage is binding at the 8th percentile, whereas workers at the 30th percentile earn 1.5 times the minimum wage. Thus, it is likely that their evidence contains a mix of truncation and spillover effects. They also find that the worker wages increased proportionally more in the low wage firms relative to the high wage firms, with small effects on employment.
fringe benefits. This contrasts with the "truncation" model, where employment effects are the strongest, and firms reduce benefits 1-to-1 in response to a rise in minimum wages. The empirical evidence on both of these effects is mixed, ranging from a zero response to the expected negative effects, which is consistent with out theory.\textsuperscript{9}

Firm heterogeneity and spillovers also helps us understand the political economy of minimum wages. In the "truncation" model, nobody supports increases in minimum wages: the workers who lose their jobs are strictly worse off, and workers above the truncation point are indifferent. On the other hand, with firm heterogeneity and spillovers, all employed workers are in favor of (small) increases in minimum wages. In Section 5.1 we show that the maximum attainable gains can be quite large, with a 10\% minimum wage increase leading to a 9\% increase in wages for workers at the 10th percentile of the wage distribution, 4\% for the 50th percentile and around 2.3\% for the 90th percentile.\textsuperscript{10} Firm owners on the other hand get hurt by the policy, and would oppose it.\textsuperscript{11}

Finally, given the importance of firm profits for our analysis we study two additional entry structures. In our baseline model, firms can decide if they operate or remain idle, but the total mass of jobs is exogenous. In that sense, our comparative statics can be seen as "short-run" responses, before firms have time to adjust the optimal number of jobs.\textsuperscript{12} We show that if entry is deterministic as in a Putty-Clay model (e.g. Solow [31]) then spillovers disappear. However, if entry is risky as in Melitz [22] our implications for wages and profits inequality remain unchanged. The implications for political economy become different, with workers at the bottom of the distribution favoring the policy and workers at the top of the distribution being adversely affected by the policy. From an ex-ante perspective firms are unaffected by the policy.

The rest of the paper is structured as follows. In Section 2, we discuss the related literature. In Section 3, we introduce the "truncation" model and provide the simplest example where the introduction of firm heterogeneity generates minimum wage spillovers. This is instructive because in this simple case it is easy to see the mechanics behind our main results. In Section 4, we introduce the general model, with general production function and preferences. In Section 5, we discuss the implications of the model for inequality and for the political economy of minimum wages. In Section 6 we extend the model to the case where firms pay fringe benefits, and analyze multi-sector extensions, and in Section 7 we analyze two alternative entry structures. Section 8 concludes.

2 Related Literature

This paper relates to many streams of literature. First, it relates to the empirical literature that measures the effects of minimum wages on inequality, employment and

\textsuperscript{9}Neumark and Wascher [24] surveys the literature on employment effects. Simon and Kaestner [30] discuss the evidence on benefits provision.

\textsuperscript{10}This calculation uses hourly wage data from the CPS.

\textsuperscript{11}If each worker owns the firm he works for than the policy is innocuous above the truncation point.

\textsuperscript{12}The empirical literature that tries to measure minimum wage effects typically focus on year-to-year variation. See Neumark and Wascher [24].
other outcomes, which we have discussed this in Section 1.

Second, we have papers with theoretical explanations for the existence of minimum wage spillovers. With the exception of Teulings [33, 34], most explanations rely on non-competitive features of labor markets. Flinn [14] studies the equilibrium of a search model with ex-ante homogeneous workers and heterogeneous match quality, in the presence of a minimum wage constraint. Minimum wages cause spillovers in this model because it affects the threat point of workers when negotiate their wages with firms. One feature of this approach is that spillovers only arise for workers who can potentially earn the minimum wage in their labor market path (because of luck), which limits the scope of these spillovers. Falk, Fehr and Zehnder [13] use laboratory experiments to argue that subjects that have been subject to a temporary minimum wage have their reservation wages permanently increased. They use theories based on the concept of fairness of transactions to interpret their findings.

Two papers by Teulings [33, 34] are the closest to ours. He also introduces a frictionless matching model with two-sided heterogeneity and labor market sorting which generates spillovers above the truncation point. However, the explanation behind the results is very different. In our model, spillovers come from a transfer of resources from firms to workers. In his paper, spillovers come from complementarities between factors of production and changes in factor composition in response to the policy. His main result is that because of aggregation previous estimates of elasticities of complementarity were biased downward, which underestimates potential general equilibrium effects from factor relocation. He finds that once one corrects for this bias the general equilibrium are sizable and can generate substantial spillovers. Our analysis also includes non-transferable utility, patterns of sorting possibly driven by taste (see Rosen [25]), more general functional forms and a more explicit discussion of the role of entry. His analysis includes differentiated producs, whereas in ours all units of production produce final goods.

The paper also relates to the literature of the identification of sorting in labor markets, such as Eeckhout and Kircher [12] and Bagger and Lentz [2]. Eeckhout and Kircher [12] study a special case of the model studied in Section 4, and show that using wage and assignment data one cannot distinguish a model with that features positive sorting from a model that features negative sorting. In Section 5 we show that if the environment is subject to shocks (e.g. a minimum wage change) an economy that features positive sorting responds differently from an economy that has negative sorting, which can be used for identification.

Finally, it relates to papers that use two-sided matching models to understand facts about the distribution of earnings. Tervio [32] and Gabaix and Landier [15] used a special case of our general framework to understand the evolution of CEO pay in the United States. They show that the evolution of CEO pay is highly correlated with the evolution of firm characteristics (e.g. firm size), which comes naturally from the matching model.

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13In our baseline model with positive sorting in Section 4 minimum wages do not affect factor composition. In the model with negative sorting in Section 5.2, and the model with risky free entry in Section 7.2 factor composition changes.
3 Minimum Wage Changes With No Spillovers

We begin our analysis presenting a simple competitive economy that has been widely used to analyze the impact of minimum policies, but in which minimum wage policies generate no spillovers above the truncation point. Then we show that spillovers arise naturally if we add hierarchical firm heterogeneity to the basic competitive model. This is instructive because in this simple formulation it is easy to see the mechanism behind our main results.

There is a measure 1 of workers, each of which has type $x$, which is a skill index that is supposed to include education, experience, unobservable skill, etc. Workers are distributed according to distribution $G(x)$.\textsuperscript{14} In this economy, types $x$ are analogous to the widely used concept of latent wages.\textsuperscript{15} There is a large mass of jobs $J > 1$ in the economy. When a worker and a firm form a match they produce a final good, $F(x) = x$ without loss of generality.\textsuperscript{16} The price of the final good is normalized to 1.

Workers supply labor inelastically and firms maximize profits subject to a competitive wage schedule, $w(x)$. The wage schedule is subject to a minimum wage constraint $w(x) \geq w$. In our theory, the minimum wage is strictly enforced and there is no issue with compliance, which may be a concern in empirical applications. In Section 6.1 we introduce fringe benefits to the model, and firms can adjust those benefits in response to a minimum wage policy.

Since there is a large mass of (identical) entrants, in equilibrium, wages will be such that each firm earns zero profits.\textsuperscript{17} It comes immediately from our assumptions that workers of type $x \geq x_0$ get employed at wage $w(x) = x$, and workers with type $x < x_0$ remain idle, where $x_0 = w$. One thing to note is that there is not a mass point of workers at the minimum wage, something that has been verified to be empirically relevant. We discuss how can we add spikes to the model without affecting our conclusions in Section 6.1.

Let $E = 1 - G(x_0)$ denote the employment in this economy. The effect of an increase in $w$ is as follows:\textsuperscript{18}

$$\frac{dE}{dw} = -g(x_0) > 0$$

$$\frac{dw(x)}{dw} = 0 \text{ if } x > x_0.$$

Thus, a change in minimum wages causes some workers at the margin to lose their jobs, but nothing happens for workers above $x_0$. The strength of this employment

\textsuperscript{14}We do not follow the common normalization in the related literature of having types being Uniform on support $[0, 1]$ because some of the comparative statics are easier to interpret with distributions than the implied modified production function. For example, it is more intuitive to say that minimum wages have small effects if there is a small mass of workers around it’s threshold than to say that it’s because the slope of the production function is steep around the threshold.

\textsuperscript{15}For example, DiNardo et al [10] and Lee [20].

\textsuperscript{16}Since $G(x)$ is arbitrary, the functional form of $F$ is arbitrary.

\textsuperscript{17}The worker supplies labor inelastically, and all jobs are identical. Later, workers and firms will have a choice of where to work.

\textsuperscript{18}Note that $\frac{dx_0}{dw} = 1$. 
effect depends on the mass of workers around the minimum wage threshold (how binding is the minimum wage), \( g(x_0) \). This comparative statics is similar to the one in the model used in DiNardo et al [10], the “truncation” model described in Lee [20].\(^{19}\) We follow the literature and define minimum wage spillovers in the following way.

**Definition 1.** An economy has no minimum wage spillovers if an increase in the minimum wage does not change the wage of workers above the truncation point:

\[
\frac{d w(x)}{d w} = 0, \ x > x_0.
\]

Thus, according to this definition, the economy described above has no spillovers above the truncation point. This does not mean that minimum wages do not affect measures of inequality. For example, if we compute measures of inequality based on percentiles such as the 50-20 log wage differential those may change because of selection: even if the minimum wage is not binding at the 20th percentile low skilled workers are losing their jobs, which is likely to change the type of worker at the 20th and 50th percentiles. Since the effects of selection are already well understood on the related literature (e.g. Heckman [17]), throughout the paper we focus on measures of inequality after controlling for selection.

Another feature of this example is that nobody likes increases in minimum wage in this economy: workers at the threshold, \( x_0 \) lose their jobs and are strictly worse off. Worker above \( x_0 \) remain employed and nothing happens to their wages. Also, since firms earn zero profits they are indifferent. Thus, it is puzzling how such a policy would get approved in the political process.

Next, we show that all these predictions change with one modification to the model: jobs are heterogeneous in productivity, in a hierarchical way.

### 3.1 Firm Heterogeneity and Positive Spillovers

The economy is the same as before, except that the jobs are heterogeneous: there is a mass \( J > 1 \) of jobs, each of which with type \( y \), which is a productivity index, supposed to reflect entrepreneurial talent, differences in the stock of capital, etc. Jobs have distribution \( H(y) \).\(^{20}\) There is an extensive empirical literature that documents that firm heterogeneity is pervasive and strongly related to wage inequality, which makes this a natural extension to the model.\(^{21}\) Through most of our analysis the economy consists of a single sector/occupation (the whole economy), but in Section 6.2 we discuss how our comparative statics remain unchanged in three different multiple sector/occupation extensions. Matches now produce according to technology \( F(x, y) \). Here we study our simplest example, and assume that \( F(x, y) = x + y \).\(^{22}\) We analyze the case with general production as well as general preferences in Section 4.

\(^{19}\)The same holds for the Roy [26] model with symmetric sectors. See Section 6.2.

\(^{20}\)Note that the formulation with univariate types includes the case where worker and firm heterogeneities are multivariate, \( \tilde{x} \in R^W \) and \( \tilde{y} \in R^F \), but these characteristics only enter the utility and production functions via separate indexes, \( x = \varphi(\tilde{x}) \) and \( y = \chi(\tilde{y}) \), as in Chiappori et al [7]. In this case, \( G \) and \( H \) represent the distributions of the indexes.

\(^{21}\)See footnote 2.

\(^{22}\)Given the additive assumption, linearity is without loss of generality.
In this economy with heterogeneous productivity the least productive firms will not produce in equilibrium. This generates a mass of potential entrants which drives the profits of the marginal firm to zero. The equilibrium in this economy has the following features. Workers below $x_0$, and firms below $y_0$ remain idle. Workers of type $x \geq x_0$ earn a wage, and firms of type $y \geq y_0$ earn profits

\[ w(x) = x + w - x_0 \tag{3.1} \]
\[ \pi(y) = y - w + x_0 \tag{3.2} \]

Marginal types are determined by a market clearing condition,

\[ 1 - G(x_0) = J(1 - H(y_0)) \]

and by a zero profits condition at the margin,

\[ \pi(y_0) = 0 \]

Because of the lack of complementarities, the assignment of active agents is arbitrary. Note that because of differentiation the firms above $y_0$ earn positive profits. We discuss profits and entry in further detail in Section 7.

The effects of a minimum wage change in this economy are as follows

\[ \frac{dE}{dw} = -g(x_0) \frac{Jh(y_0)}{Jh(y_0) + g(x_0)} > 0 \]
\[ \frac{dw(x)}{dw} = \frac{g(x_0)}{Jh(y_0) + g(x_0)} \in (0, 1) \quad \text{if } x \geq x_0 \]
\[ \frac{d\pi(y)}{dw} = -\frac{dw(x)}{dw} \in (0, 1) \quad \text{if } y \geq y_0 \]

Similarly as before, some workers at the margin lose their jobs. However, this employment effect is necessarily attenuated by the firm heterogeneity.\(^{24}\) The size of this effect depends positively on the mass of workers around $x_0$ (how binding is the minimum wage), $g(x_0)$, and on the measure of firms around the threshold (the extent of firm heterogeneity), $Jh(y_0)$. If firms are homogeneous, $\frac{dx_0}{w} = 1$ as before, or if $Jh(y_0) \approx 0$ then the employment effect is negligible.\(^{25}\) In addition, the increase in $w$ generates spillovers above the truncation point: for every $x > x_0$ \(\frac{dw(x)}{w} \in (0, 1)\). A rise in minimum wages increases the wages of all workers in the economy by the same amount. The size of this change depends positively on $g(x_0)$, and negatively on the $Jh(y_0)$. Note that if firms are homogeneous the spillover effects vanish, and if $Jh(y_0) \approx 0$ then the spillover effects are at their maximum, $\frac{dw(x)}{w} \approx 1$.\(^{26}\)

Equations 3.1 and 3.2 clearly show the mechanism behind this result: an increase in $w$ transfers resources from the firms to the workers. The increase in worker’s wages is smaller than the minimum wage increase itself because the job destruction at the margin, $\frac{dx_0}{w}$, reduces the effective “outside option” of workers.

This level increase in wages decreases inequality even after controlling for selection. This is because the low wage workers earn proportionally more than the high

\(^{23}\)Note from the first order condition of the firm’s profit maximization that $w'(x) = 1$. Imposing that $w(x_0) = w$ yields Equation 3.1.
\(^{24}\)Note that $g(x_0) \frac{Jh(y_0)}{Jh(y_0) + g(x_0)} < g(x_0)$.
\(^{25}\)With homogeneous firms, $Jh(y_0) \to \infty$, which implies that $\frac{dx_0}{w} = 1$.
\(^{26}\)With homogeneous firms, $Jh(y_0) \to \infty$, which implies that $\frac{dw(x)}{w} = 0$. 
wage workers. The firm’s counterpart to that is that each firm has a level decrease in profits. By a similar argument, inequality in firm profits rises, since the low productivity firm lose proportionally more. We study this with more detail in Section 5.1, by looking at how the Lorenz curve of wages and profits change in response to a minimum wage increase. As discussed in Section 1 there is a body of empirical evidence that suggests that both effects are empirically relevant. Firm heterogeneity also changes the political economy of minimum wages. Now, all employed workers in this economy would favor a small increase in $w$ since their wages would be raised. In Section 5.1 we show that these raises can potentially be substantial. The firm owners on the other hand have their profits reduced, so they oppose the policy. It is worth emphasizing that this is an economy without frictions or externalities. Thus, this policy is necessarily undesirable for efficiency purposes as it lowers total output, and would be justified only for distributional purposes.

In Sections 4, 5, and 7 we generalize the framework in a number of ways, and show that the implications for inequality and the political economy of minimum wages are much more general than this particular case.27

Finally, before introducing the general model we analyze an example with two-sided heterogeneity, but horizontal. This illustrates the importance of hierarchies for our results. Worker have type $x$, firms have type $y$, and both have Uniform $[0, 1]$ distribution. Production is given by $T(|x - y|)$, where $T' < 0$.28

In this example, equilibrium requires self-matching: $\mu(x) = x$, so that $T(|x - \mu(x)|) = T(0), \forall x \in X$. Now, since $J > 1$, for each firm type a mass $\frac{J-1}{J}$ of firms remain unmatched, and the ones who do match earn zero profits. Thus, in equilibrium $w(x) = T(0), \forall x \in X$. Therefore, as long as $T(0) > w$,

$$\frac{dw(x)}{dw} = 0 \text{ for } \forall x \in X.$$ 

It is worth noting that conversely, if workers were on the short side of the market, $J < 1$, then all workers would earn the minimum wage, and as long as $T(0) > w$,

$$\frac{dw(x)}{dw} = 1, \forall x \in X.$$ However, since all workers earn the minimum wage it is still true in a vacuous sense that workers above the truncation point are unaffected.

4 General Model

The model consists of the matching problem of a large population of heterogeneous workers and firms. Heterogeneity in our setting is hierarchical, in a sense that we make precise later. In this economy, sorting patterns between workers and firms can be induced by many economic forces, such as complementarities in production, as emphasized by Becker [5] or forces related to preferences, such as income effects, complementarities between consumption and job amenities, among others, as emphasized in the theory of equalizing differences (e.g. Rosen [25]).

27The spillovers need not be a constant level effect as in this example, and with free entry low skilled workers benefit from the policy, whereas high skilled workers oppose the policy.

28This model is very similar to a frictionless version of the circular matching model of Marimon and Zilibotti [21].
### 4.1 Hierarchical Matching Model

Throughout the Section we use subscripts to denote the derivative of a function with respect to the variable denoted in the subscript. Moreover, we sometimes omit the arguments of functions for purposes of exposition.

There is a measure 1 of workers, each of which has type $x \in X = [x, \pi] \subset \mathbb{R}$. Workers are distributed according to distribution $G(x)$, with corresponding smooth density $g(x)$. There is a mass $J$ of jobs, each of which with types $y \in Y = [y, \bar{y}] \subset \mathbb{R}$. Jobs have distribution $H(y)$, with corresponding smooth density $h(y)$. In order to include the possibility of remaining single in the matching problem I define the sets $\tilde{X} = X \cup \emptyset$ and $\tilde{Y} = Y \cup \emptyset$, where $\emptyset$ denotes the option of remaining single. When a worker and a job engage in production they produce an amount of output $F(x, y)$. We defer making assumptions on $F$, except that it is smooth. All units produce the single final good, which price is normalized to 1.

Workers have preferences $U(x, y, w)$, where $w \in [w, \infty]$ is a payment that the worker receives from the firm in exchange of labor services, and $\bar{w}$ is the mandatory minimum wage in this economy. We consider a range of minimum wage policies $w \in \bar{W} = [\underline{w}, \bar{w}] \subset \mathbb{R}^+$. For now, we assume $U$ is smooth and that $U_w > 0$. In this formulation, workers are allowed to have heterogeneous preferences, which may depend on the job type. Moreover, we do not restrict the setting to the case where $U$ is quasi-linear in $w$, which means that in the most general case utility is non-transferable.

Firms have preferences $\tilde{V}(x, y, \pi)$, where $\pi = F(x, y) - w$ and $\bar{V} \equiv \tilde{V}(x, y, F(x, y) - w)$. We assume that $\tilde{V}$ is smooth, $V_x > 0, V_y > 0$ and $V_w < 0$. The precise sense in which types are hierarchical is that, conditional on wages, higher types generate more value for the firm, which may due to increased output or due to preferences.

We normalize the outside option of workers and firms to 0: $U(x, \emptyset, w(x, \emptyset)) = 0, \forall x \in X$, and $V(\emptyset, y, w(\emptyset, y)) = 0, \forall y \in Y$, where $w(x, \emptyset)$ and $\pi(\emptyset, y)$ respectively denote the payoffs of unmatched worker and firms. Also, we assume that $U(x, y, w) \geq 0, \forall x \in X, y \in Y, w \in \bar{W}$ so that workers prefer working for the minimum wage than remaining unmatched, and $\tilde{V}(\pi, \bar{y}, \bar{w}) > 0$, for $w \in \bar{W}$, in order to guarantee positive gains from matching for a positive measure of individuals. We also assume that $\tilde{V}(x, y, w) < 0, \forall y \in Y, w \in \bar{W}$, in order to guarantee that the minimum wage is binding in equilibrium. Note that the model has a concept of involuntary unemployment, in the sense that the workers below the marginal type would like to work for the minimum wage, but it is unprofitable for firms to hire them. Finally, we assume that $\tilde{V}(x, y, 0) \leq 0, \forall x, y$, in order to guarantee that in equilibrium firms always generate enough output to pay wages. We discuss additional properties of $U$ and $\tilde{V}$ when we discuss assortative matching.

One example that satisfies our requirements is classical marriage market model of Becker [5], and adapted to a labor market setting in Sattinger [28].

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29 See footnotes 14 and 20.
30 We do not need to specify these payoffs.
31 This guarantees that the top agents always participate, and by continuity at least an interval around them also do.
Example 1. Sorting on complementarities: $U(x, y, w) = w - b$ and $V(x, y, w) = F(x, y) - w - k$, where $F_x > 0$, $F_y > 0$, $b \in [-\infty, w]$ is the outside payoff of the worker and $k \geq 0$ is an operating cost for the firm. In addition, let $F_{xy} > 0$.

The assumption that $F_{xy} > 0$ is equivalent to strict supermodularity with smooth functions. This condition assures that the equilibrium features positive assortative matching: the bottom worker matches with the bottom firm, and so on and so forth. We provide necessary and sufficient conditions for assortative matching below.

We look for a competitive equilibrium in this matching economy. Workers and firms take the wage schedule $w(x, y), x \in \tilde{X}, y \in \tilde{Y}$ as given, and choose their optimal partner accordingly. Let the equilibrium assignment be described by the mapping $\mu(x)$, which assigns a firm or $\emptyset$ to each worker, and its inverse $\nu(y)$, which denotes the worker assigned to firms of type $y$. In general, $\mu$ and $\nu$ are correspondences, but in the case of positive (negative) sorting they will be strictly increasing (decreasing) functions. We choose parameters such that the equilibria will have the property that there exist marginal types $x_0$ and $y_0$ such that workers and firms below the marginal type remain single and workers and firms above the marginal type find a pair.

The decision problem of the worker is described by the optimization

$$\max_{y \in \tilde{Y}} U(x, y, w(x, y)).$$

(4.1)

If $x > x_0$ the solution satisfies the first order condition

$$U_y + U_w w_y = 0.$$ (4.2)

Likewise, the firm optimizes

$$\max_{x \in \tilde{X}} V(x, y, w(x, y)).$$

(4.3)

If $y > y_0$ the solution satisfies the first order condition

$$V_x + V_w w_x = 0.$$ (4.4)

Definition 2. A competitive equilibrium with minimum wages consists of a wage schedule $w(x, y)$, marginal types $x_0$ and $y_0$, and assignment correspondences $\mu(x)$ and $\nu(y)$ such that

1. Markets clear: if $x < x_0$ then $\mu(x) = \emptyset$, and if $y < y_0$ then $\nu(y) = \emptyset$. Also, for $x \geq x_0$ let $B(x)$ be the set of all firms assigned to workers whose types lie on the interval $[x, \pi]$. We need

$$\int_{x}^{\pi} dG(x') = J \int_{y' \in B(x')} dH(y') \quad \text{for} \ x \geq x_0$$

(4.5)

2. For $x \geq x_0$ any $y \in \mu(x)$ solves Problem 4.1, and for $y \geq y_0$ any $x \in \nu(y)$ solves Problem 4.3. Firms below the marginal type prefer to stay single.

3. Minimum wage constraints are satisfied: $w(x, y) \geq \underline{w}, \ x \in \tilde{X}, y \in \tilde{Y}$.

4. If $x_0 > \xi, w(x_0, \mu(x_0)) = \underline{w}.$
Note that according to this definition markets do not clear for types below the marginal type: workers below \( x_0 \) would like to work for the minimum wage \( w \), but no firm is willing to hire them at that price.\(^{32}\) The last condition assures that if there is excess supply, the marginal worker is paid the minimum wage.\(^{33}\)

In addition, it is useful to define two special cases with respect to the matching patterns of workers and firms. In the first, there is perfect assortative matching: the bottom worker matches with the bottom firm and vice versa. In the second, there are no systematic pattern of matching and workers are matched randomly to firms.

**Definition 3.** A competitive equilibrium with positive assortative matching consists of a wage schedule \( w(x, y) \), marginal types \( x_0 \) and \( y_0 \), and assignment functions \( \mu(x) \) and \( \nu(y) \), such that the conditions in Definition 2 are satisfied, plus \( \mu(x_0) = y_0 \), \( \mu(x) = y_0 \), and for \( x > x_0 \), \( y > y_0 \), \( \frac{d\mu}{dx}(x) > 0 \) and \( \nu(y) = \mu^{-1}(y) \).

In the case of negative sorting we have that \( \mu(x_0) = y_0 \), \( \mu(x) = y_0 \), and for \( x > x_0 \), \( y < y_0 \), \( \frac{d\mu}{dx}(x) < 0 \).

**Definition 4.** A competitive equilibrium with random matching consists of a wage schedule \( w(x, y) \), marginal types \( x_0 \) and \( y_0 \), and assignment correspondences \( \mu(x) \) and \( \nu(y) \) such that the conditions in Definition 2 are satisfied, plus for \( x \geq x_0 \) any \( y \geq y_0 \) solves Problem 4.1, and for \( y \geq y_0 \) any \( x \geq x_0 \) solves Problem 4.3.

### 4.2 Sorting, Existence and Uniqueness

In this Section, we describe necessary and sufficient conditions for the existence of positive (negative) assortative matching equilibrium, and the random matching equilibrium. Throughout the section, we restrict attention to differentiable equilibria.

**Proposition 1.** A necessary condition for equilibria with positive assortative matching is that

\[
-\frac{V_{xy}}{V_w} + \frac{V_{yw}}{V_w} \frac{U_y}{U_w} + \left[ V_{wy} - \frac{V_{ww}}{U_w} \right] \frac{V_x}{(V_w)^2} + \frac{U_{yx}}{U_w} - \frac{U_{yw}}{U_w} \frac{V_x}{V_w} \geq 0
\]

holds along the equilibrium path. Moreover, if this condition holds with strict inequality at all values, then the only equilibria that exists are positively assorted. If we repeat the statement with the opposite inequalities then the same holds true, but for negative sorting. Finally, if the condition holds with strict equality everywhere then all the equilibria feature random matching.

**Proof.** See Appendix 10.1. \( \square \)

\(^{32}\)Recall the assumption that \( U(x, y, w) \geq 0, \forall x \in X, y \in Y, w \in W \).

\(^{33}\)This rules out trivial equilibria, like \( w(x, y) = F(x, y) \), \( x \in X, y \in Y, x_0 = 0 \) and \( y_0 = 0 \), with everybody idle.
Since our sorting condition, Equation 4.6, has so many components it is difficult to interpret it directly. Instead, we provide a number of examples that satisfy it’s requirements, which illustrates the many ways that sorting can arise in our model. It can be easily verified that in the Becker marriage model, Example 1, Equation 4.6 reduces to \( F_{xy} \geq 0 \). Also, we can show that in the more general case with concavity (or convexity), where \( U = u(w) \), where \( w' > 0 \) and \( w'' \leq 0 \) and \( V = v(F(x,y)-w) \) where \( v' > 0 \) and \( v'' \leq 0 \), the sorting condition still only relies on complementarities in production: \( F_{xy} \geq 0 \). Next, we describe three additional examples that yield positive assortative matching, where sorting is driven by preferences, as discussed in Rosen [25].

**Example 2.** Job amenities: \( U = u(y,w) \), where \( u_y > 0 \), \( u_{yw} \geq 0 \), \( u_w > 0 \) and \( u_{ww} \leq 0 \), and \( V = F(x,y)-w \), where \( F_x > 0 \), \( F_y > 0 \) and \( F_{xy} \geq 0 \).

In this economy workers have non-pecuniary tastes for high type jobs (e.g. working conditions), and either their marginal utility of consumption is decreasing or (and) consumption is complementary to job amenities (e.g. good jobs are located in nicer areas, which increases the marginal utility of consumption). In either case, even if \( F_{xy} = 0 \), it is optimal for the high type workers (the high wage ones) to “buy” the high quality jobs, which generates positive assortative matching. Sorting on income effects has been explored in Weiss [35], and consumption-amenities complementarities have been studied in Sattinger [27].

**Example 3.** Job disamenities: \( U = u(x,y) + w \), where \( u > 0 \), \( u_y < 0 \), \( u_{yx} \geq 0 \) and \( V = F(x,y)-w \), where \( F_x > 0 \), \( F_y > 0 \) and \( F_{xy} \geq 0 \).

In this case more the more productive jobs have a displeasure element to them (perhaps because of long hours or a competitive environment). However, this distaste element influences less the high skill workers, which induces positive sorting. One motivation for the correlation between low distaste and productivity is that workers who have a lower distaste for high type jobs have a stronger incentive to accumulate human capital, which makes them more productive. Note that the firm’s counterpart of this mechanism is the term \( V_{xy} \), which may also induce sorting due to complementarities in preferences.

**Example 4.** Taste for workers: \( U = w \) and \( V = v(x,\pi) \) where \( \pi = F(x,y)-w \), \( F_x > 0 \), \( F_y > 0 \), \( F_{xy} \geq 0 \), \( v_\pi > 0 \), \( v_x \geq 0 \), \( v_{x\pi} \geq 0 \) and \( v_{\pi\pi} \leq 0 \).

This is the firm’s counterpart of Example 2. Here, firm’s have a non-pecuniary taste for high skilled workers, and because of either decreasing marginal utility of profits (perhaps due to credit constraints) or complementarities between profits and taste it is optimal for the top firm to “buy” the top workers.

Finally, we use the economy studied in Section 3.1 as an example that features random sorting. In this case, Equation 4.6 equals to zero everywhere.

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**Footnotes:**

34 In this case, Equation 4.6 reduces to \( F_{xy} - u_{wuw} F_y \frac{w_{wy} F_x}{w_{wy} F_x} + u_{ww} F_x \geq 0 \).

35 One example of utility that satisfies our requirement is \( U = e^{-y(F_x-x)} + w \).

36 In this case, Equation 4.6 reduces to \( F_{xy} + u_{xy} \geq 0 \).

37 In this case, Equation 4.6 reduces to \( F_{xy} + \frac{v_{x\pi} F_y}{v_x} - \frac{v_{\pi\pi} F_y v_x}{v_x} \geq 0 \).
Example 5. Two-sided heterogeneity and random sorting: \( U = w \) and \( V = x + y - w \).

We conclude the section with a proof of existence and uniqueness of equilibrium in case Equation 4.6 is satisfied with strict positive inequality everywhere, or the case with weak inequality and job attributes do not affect worker utility. This equilibrium characterization applies to Examples 1 to 5, including the one with random sorting. Most of our comparative static results will be relative to this characterization.

**Proposition 2.** If our sorting condition, Equation 4.6, is satisfied with strict positive inequality everywhere then the equilibria features positive sorting, the marginal types \( \tilde{x}_0 \) and \( \tilde{y}_0 \), and the assignment function \( \tilde{\mu}(x) \) are unique, and the wage and profit functions are unique on the equilibrium path: \( \tilde{w}(x) \equiv w(x, \tilde{\mu}(x)), x \geq \tilde{x}_0 \) and \( \tilde{\pi}(y) \equiv F(\tilde{\nu}(y), y) - \tilde{w}(\tilde{\nu}(y), y), y \geq \tilde{y}_0 \). We can compute the equilibrium as follows:

1. \( \tilde{x}_0 \) and \( \tilde{y}_0 \) solve the system
   \[
   1 - G(\tilde{x}_0) = J [1 - H(\tilde{y}_0)] \quad (4.7)
   \]

2. \( \tilde{\mu}(x) \) solves the differential equation, with boundary condition \( \tilde{\mu}(\tilde{x}) = \overline{\mu} \):
   \[
   \tilde{\mu}'(x) = \frac{\partial(x)}{\partial(\mu(x))} \quad \text{for } x \geq \tilde{x}_0 \quad (4.8)
   \]

3. For \( x \geq \tilde{x}_0 \)
   \[
   \tilde{w}(x) = w - \int_{\tilde{x}_0}^{x} \frac{V_x(x', \tilde{\mu}(x'), \tilde{w}(x'))}{V_w(x', \tilde{\mu}(x'), \tilde{w}(x'))} dx' \quad (4.9)
   \]

4. For \( y \geq \tilde{y}_0 \),
   \[
   \tilde{\pi}(y) = F(\tilde{\nu}(y), y) - \tilde{w}(\tilde{\nu}(y)). \quad (4.10)
   \]

Finally, if Equation 4.6 is satisfied with weak positive inequality, and \( U(x, y, w) = u(x, w) \), where \( u_w > 0 \), then there is always an equilibrium where the marginal types are \( \tilde{x}_0 \) and \( \tilde{y}_0 \), the assignment function satisfies \( \tilde{\mu}(x) \in \mu(x), x \geq \tilde{x}_0 \), and on the equilibrium path wages and profits equal \( \tilde{w}(x) \) and \( \tilde{\pi}(y) \), where the variables with tilde are constructed as described above. Note that in this case the actual assignment does not need to be \( \tilde{\mu}(x) \).

**Proof.** See Appendix 10.2.

\[ \square \]

### 5 The effect of a Minimum Wage Increase

Next, we introduce our main result, the comparative static of the positive sorting equilibrium with respect to a minimum wage increase.

**Proposition 3.** Suppose that the conditions of Proposition 2 are satisfied, and the variables with tilde be determined as in Proposition 2. Let \( V^0 = V(\tilde{x}_0, \tilde{y}_0, \tilde{w}(\tilde{x}_0)) \), and \( V^0_x, V^0_y \) and \( V^0_w \) be defined analogously. A change in minimum wages has the following effects in this economy.
1. \( \frac{d\tilde{\mu}(x)}{dx} = 0 \) for \( x > \tilde{x}_0 \)

2. \( \frac{dE}{dw} = g(\tilde{x}_0) \frac{V^0_w Jh(\tilde{y}_0)}{V^0_x Jh(\tilde{y}_0) + V^0_y g(\tilde{x}_0)} \leq 0 \)

3. If \( x > \tilde{x}_0 \)
   \( \frac{d\tilde{w}(x)}{dw} = \frac{d\tilde{w}(\tilde{x}_0)}{dw} \int^{\tilde{x}_0}_{x_0} \frac{V^0_w V^0_x - V^0_x V^0_y}{V^0_w} dx' > 0 \)
   where \( \frac{d\tilde{w}(\tilde{x}_0)}{dw} = \frac{V^0_y g(\tilde{x}_0)}{Jh(\tilde{y}_0) V^0_x + g(\tilde{x}_0) V^0_y} \in (0, 1] \)

4. If \( y > y_0 \)
   \( \frac{d\tilde{\pi}(y)}{dw} = -\frac{d\tilde{w}(\tilde{y}(y))}{dw} \in [-1, 0) \)

Proof. See Appendix 10.3.

The first thing to note is that the perfect sorting assignment does not change above \( \tilde{x}_0 \), \( \frac{d\tilde{\mu}(x)}{dx} = 0 \). Some worker-firm matches at the margin are dissolved, but above those types hierarchical sorting is still optimal and unchanged. It is worth noting that the actual assignment does not need to feature perfect positive sorting in order for this to be true. In fact, in Example 5 assignment is arbitrary.

Similarly as in Section 3.1, minimum wage increases might decrease employment. This effect depends positively the mass of workers around \( \tilde{x}_0 \) (how binding is the minimum wage policy), \( g(\tilde{x}_0) \), and on the amount of firms around the threshold (the extent of firm heterogeneity), \( Jh(\tilde{y}_0) \).

The Proposition also shows that a rise in minimum wages increases the wage of all workers in the economy. This increase may be increasing or decreasing in \( x \), depending on the term \( V^0_w V^0_x - V^0_x V^0_y \). We show in Section 5.1 that if \( V^0_w V^0_x - V^0_x V^0_y \leq 0 \) this change unambiguously reduces log-wage inequality in the economy. Most of our examples satisfy this property. The effect at the margin, \( \frac{d\tilde{w}(\tilde{x}_0)}{dw} \), is very similar to the one studied in Section 3.1: it depends positively on \( g(\tilde{x}_0) \), and negatively \( Jh(\tilde{y}_0) \). Note that employment changes are not necessary in order for this effect to be large. In particular, if \( h(\tilde{y}_0) \approx 0 \) there are no employment losses and \( \frac{d\tilde{w}(\tilde{x}_0)}{dw} = 1 \), it’s maximum attainable value. Conversely, when firms are homogeneous employment effects are at it’s maximum and wages are unaffected above the truncation point. Finally, this change reduces firm profits by the same amount it increases worker wages. We discuss profits and entry in detail in Section 7.

We proceed to characterize this comparative statics in our examples.

38 In Section 3.1 the employment effect was always positive because \( J > 1 \).

39 As in Section 3.1, type distributions can be normalized so that \( V^0_y = 1 \) and \( V^0_y = 1 \), so that we discuss our results in terms of distributions. Alternatively, we could normalize types such that \( g(\tilde{x}_0) = 1 \) and \( Jh(\tilde{y}_0) = 1 \) and switch the statement to slopes of the firm preferences instead of type distributions.
Corollary 1.  
\[ V(x, y, w) = v(y, F(x, y) - w) \]
\[ V_w = -v_y \]
\[ V_x = v_x F_x \]
\[ V_{ww} = v_{xx} \]
\[ V_{wx} = -v_{x\pi} F_x \]
\[ V_w V_{xw} - V_{ww} V_x = v_x v_{\pi\pi} F_x - v_{\pi\pi} v_x F_x \]

Corollary 2.  
In Examples 1 to 3 and 5 the effects of minimum wage changes are as described in Proposition 3, plus for \( x \geq x_0 \)
\[ \frac{d w(x)}{d w} = \frac{V^0_w}{J_h(y_0) + V^0_w} \in (0, 1] \].

Proof. As argued before, these examples satisfy the sorting condition with strict inequality. Moreover, they also feature \( V_{xw} = 0 \) and \( V_{ww} = 0 \), which implies that the effect in wages is constant. \( \square \)

In most of our examples examples, raising minimum wages increases the level of wages for everyone in the economy by the same amount, just as in the economy studied in Section 3.1. Our only example that can display non-constant minimum wage effects is Example 4.

Corollary 3. Take Example 4, specializing to the case of CRRA utility: \( v(x, \pi) = x^{\alpha} \pi^{\frac{\gamma - \sigma}{1 - \sigma}} \) where \( \alpha \in \mathbb{R} \), and \( \sigma > 0 \). The effects of minimum wage changes are as described in Proposition 3. Moreover, if \( x > x_0 \),
1. If \( \alpha = 0 \) (no taste for workers) then \( \frac{d w(x)}{d w} \) is constant and equal to the value at \( x_0 \).
2. If \( \alpha > 0 \) (positive preference for skilled workers) and \( \sigma > 1 \) then \( \frac{d w(x)}{d w} \) is positive and strictly decreasing in \( x \).
3. Otherwise, \( \frac{d w(x)}{d w} \) is positive strictly increasing in \( x \).

Proof. Here, \( V_w V_{xw} - V_{ww} V_x = \frac{\alpha x^{2\alpha - 1} \pi^{-2\sigma}}{1 - \sigma} \). Thus, its sign depends on the ratio \( \frac{\alpha}{1 - \sigma} \), which establishes the claim. \( \square \)

5.1 Inequality and Political Economy of Minimum Wages

Throughout this Section, we make the assumptions in Proposition 2, and that \( V_w V_{xw} - V_{ww} V_x \leq 0 \) on the equilibrium path. As previously discussed, this implies that \( \frac{d w(x)}{d w} \) non-increasing in \( x \). Most of our examples satisfy this property.

\[ \text{\footnote{Note that if } \alpha > 0 \text{ (} \alpha < 0 \text{) denotes that employers value positively (negatively) high skill workers in a non-pecuniary way.} } \]
One useful way to characterize how inequality changes in this economy is to see how the Lorenz curve changes. Let \( p \) denote the \( p^\text{th} \) percentile of the wage distribution in the economy. The Lorenz curve for wages in this economy is given by

\[
L^w(p) = \frac{\int_0^p w_{p'} dp'}{\int_0^1 w_{p'} dp'}
\]

(5.1)

where \( w_p \) denotes the value of wage at percentile \( p \). Likewise, we can define \( L^\pi(p) \) the same way, with \( p \) referring to the percentile of the distribution of profits among jobs, and \( \pi_p \) the firm profits at that percentile. It is well known that if the Lorenz curve increases at every percentile then inequality goes down, according to several measures like Gini, Theil, Atkinson, and others coefficients.\(^{41}\) Conversely, if the Lorenz curve moves down then inequality unambiguously increases.

As discussed in Section 3, the increase in \( w \) may cause some workers to lose their jobs. As a result, because of selection the workers at different percentiles of the distribution may change, which affects \( L^w(p) \). Since this effect is already well understood in the related literature, we compute the change in the Lorenz curve after controlling for selection.\(^{42}\) Since we are looking at small changes, the relevant distribution is \( G(x) - G(x_0) \), for \( x > x_0 \).

**Proposition 4.** Suppose that the assumptions in Proposition 2 are satisfied, and \( V_{ww} V_{xw} - V_{ww} V_x \leq 0 \) on the equilibrium path. After controlling for selection, we have that for \( p \in (0, 1) \), \( \frac{dL^w(p)}{dw} > 0 \) and \( \frac{dL^\pi(p)}{dw} < 0 \). Furthermore, among surviving matches, the labor share of income increases.

**Proof.** Since we are controlling for selection, the percentiles are unaffected by the minimum wage change. Differentiating Equation 5.1 with respect to \( w \) we obtain

\[
\frac{dL^w(p)}{dw} = \frac{\int_0^p \frac{dw_p}{\pi_x} dp' - \int_0^1 \frac{dw_p}{\pi_x} dp' L^w(p)}{\int_0^1 w_{p'} dp'} \\
\geq \left( \int_0^1 \frac{dw_p}{\pi_x} dp' \right) \left( \frac{p - L^w(p)}{\int_0^1 w_{p'} dp'} \right) > 0
\]

where the first inequality uses the fact that \( \frac{dw(x)}{dx} \) is non-increasing in \( x \), and the second the fact that \( p - L(p) > 0 \) if the wage distribution lacks mass points.\(^{43}\) The proof can be adapted to show that \( \frac{dL^\pi(p)}{dw} < 0 \), using the fact that \( \frac{d\pi(y)}{dw} = -\frac{d\pi(\nu(y))}{dw} \).

Finally, since \( \int_0^1 \frac{dw_p}{\pi_x} dp' > 0 \), and \( \int_0^1 \frac{\pi_x}{\pi_x} dp' < 0 \), for surviving matches the labor share of income goes up.

Therefore, inequality in wages unambiguously goes down in response to a minimum wage increase. This is easy to understand, since the same level increase in wages for everyone implies a larger proportional increase for the lower percentiles.

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\(^{41}\) See Sen [29].

\(^{42}\) For example, see Heckman [17].

\(^{43}\) This is an implication from our assumptions about smooth distributions.
which decreases inequality. The firm’s counterpart of this is that dispersion in firm-level profits unambiguously go up, while wage inequality is going down. These are testable implications that comes naturally from this theory. As discussed in Section 1, several empirical papers have argued that these spillover effects are pervasive in actual economies. The evidence on the firm effects is more limited, but the results from Draca et al. [11] are consistent with our theory.

Another testable implication of this comparative statics is that the labor share of income goes up with minimum wage increases. However, unlike our predictions about inequality, this prediction is not robust to the introduction of entry in the model, as we do in section 7.

In addition to that, it is useful to see how individuals at different percentiles of the income distribution are affected by this minimum wage increase. First, it is useful to convert the level increase in wages to logs.

\[
\frac{d\log(w_p)}{d\log(w)} = \frac{w}{w_p} \frac{dw(x_p)}{dw} \quad \text{for } x_p > x_0
\]

where \(x_p = G^{-1}(p[1 - G(x_0)] + G(x_0))\). Now, since \(\frac{dw(x)}{dw} \in (0, 1] \) and it is non-decreasing in \(x\), it is useful to compute the maximum minimum wage effect as \(\frac{dw(x_0)}{dw} = 1, x \geq x_0\). In this case, \(\frac{d\log(w_p)}{d\log(w)} = \frac{w}{w_p}\). Note that this does not depend on the parameters from the model, and can be computed using data on empirical wage distributions. Figure 5.1 displays the maximum gain from a minimum wage increase using data on log-wages from the CPS for the year of 1990. As we can see, the gains are potentially quite large, with a 10% minimum wage increase leading to a 9% increase in wages at the 10th percentile, 4% at the 50th percentile and around 2.3% for the 90th percentile.

These results can help explain the political economy of minimum wages. In the model without firm heterogeneity presented in Section 3, the workers who lose their jobs are unambiguously worse off and workers above the threshold are unaffected by the minimum wage increase. Thus, some workers are against this policy, and no one defends it. On the other hand, in the case with firm heterogeneity and positive spillovers, all the employed workers are affected: those who lose their jobs are still worse off, but now the entire mass of employed workers are better off. As seen in Figure 5.1, these gains are potentially large for most of the wage earners. Thus, in the model, the mass of workers in favor of the change vastly outnumber the marginal workers who lose their jobs, which could explain why there would be strong support for this policy among the workforce.

It is worth emphasizing two things about this argument. First, this is an economy without frictions or externalities, which implies that this policy necessarily decreases the total output in this economy. Thus, this policy is not desirable from an efficiency perspective, and could only be justified if used for equity purposes. Second, this is an economy where firms generate positive profits, and as seen in Proposition 3,

\footnote{Note that this statement only includes worker-firm pairs that are still profitable after the minimum wage change. Including matches that are terminated skews the result towards a smaller increase in the labor share of profits, since the marginal match has zero profits.}

\footnote{Empirical estimates of the disemployment effect of minimum wages are mixed, ranging from the expected negative effect to small positive effects. See Neumark and Wascher [24].}
The figure shows the percentage increase in wages in response to a 1% increase in minimum wages, at different percentiles, in the case where \( \frac{\Delta w(x)}{\Delta W} = 1 \), \( \forall x \). We use data for all the US population with the CPS MORG for 1990. The definition of wages is either the self-declared hourly rate of pay, or the weekly wage divided by the usual weekly hours, excluding imputations.

Figure 5.1: Maximum Increase in Wages

\[
\frac{d\bar{\pi}(y)}{dw} = -\frac{d\bar{\pi}(\bar{\nu}(y))}{dw},
\]

firms lose as much as workers gain. Therefore, to understand the full political economy of the problem we would need to assign firm ownership in the model. Of course, if workers own the firm they work for, this policy does not affect agents above the truncation point. On the other hand, if firms are owned by capitalists (non-workers) then all workers would favor the policy, whereas the capitalists would oppose it. We discuss this further in Section 7.

5.2 Negative Assortative Matching

Through this section we maintained the assumption that condition 4.6 was satisfied with either strict or weak positive inequality, which implies positive assortative matching.\(^{46}\) Here, we discuss the case where the inequality is negative, which implies negative assortative matching. To simplify the analysis, we specialize our economy to the specification in Example 1: \( V(x, y, w) = F(x, y) - w \), where \( F_x > 0, F_y > 0, F_{xy} < 0 \) and \( U(x, y, w) = w \), and focus on specifications where \( (x_0, y_0) > (x, y) \).

We have seen from Proposition 1 that if \( F_{xy} < 0 \) then the equilibrium features negative assortative matching. Thus, \( \mu(x_0) = \bar{\nu}, \mu(x) = y_0 \), and for \( x > x_0, y > y_0, \mu'(x) < 0 \). The characterization of equilibrium is similar to the one in proposition 2, \(^{46}\)We reiterate that our results do not rely on the actual allocation exhibiting perfect positive sorting. Example 5 satisfies our assumptions and features random sorting.
except optimality at the margin becomes

$$F(\bar{x}, y_0) = w(\bar{x})$$  \hspace{1cm} (5.3)

i.e., the marginal firm hires the top worker and makes zero profits. Moreover, the measure consistency condition, Equation 4.5, becomes

$$G(x) - G(x_0) = J[1 - H(\mu(x))] .$$  \hspace{1cm} (5.4)

Using the measure consistency condition, Equation 4.9, and a technical condition on $$F(\bar{x}, y)$$, then $$x_0$$ and $$y_0$$ can be uniquely determined.\(^{47}\)

Differentiating Equation 5.4 with respect to $$w$$ we obtain

$$\frac{dJ}{dw} = Jh(\mu(x)) = \frac{dx_0}{dw} g(x_0) > 0.$$  \hspace{1cm} (5.5)

An increase in minimum wages destroys marginal jobs, which leads to all workers getting a job upgrade. This is unlike the positive (or random) sorting economy, where assignment remained unchanged. Finally, differentiating Equation 4.9 with respect to $$w$$ we obtain that \(^{48}\)

$$\frac{dx(x)}{dw} = F_y(\bar{x}, y_0) \frac{dy_0}{dw} - \int_{x}^{\infty} \frac{d\mu(x')}{dw} F_{xy}(x, \mu(x)) dx.$$  \hspace{1cm} (5.6)

Since $$F_{xy} < 0$$, and $$\frac{d\mu(x')}{dw} > 0$$, $$\frac{dx(x)}{dw}$$ is always positive, and is strictly decreasing in $$x$$.

Given this characterization, we can use Proposition 4 to conclude that inequality in wages unambiguously decreases in this economy, whereas inequality in profits unambiguously goes up. Furthermore, for surviving matches the labor share of income goes up. Thus, the predictions for inequality and political economy are similar to the case with positive sorting.\(^{49}\) However, the implications for assignment are necessarily different: with positive sorting assignment remains unchanged, whereas with negative sorting workers unambiguously get upgraded (and firms downgraded) in their job assignment.

It is useful to contrast our results to the results in Eeckhout and Kircher \cite{12}. In that paper, they show that in the assignment model in Example 1, using data on wages and assignment, one cannot distinguish between positive and negative sorting. One maintained assumption in their analysis is that the parameters of the model remain unchanged in the sample period. Our results show that if there are shocks to the economy (in our case the minimum wage change), then we can distinguish between positive and negative sorting using data on wages and assignment alone. In particular, in this example, $$\frac{dw(x)}{dw}$$ does not depend on $$x$$ with positive sorting, and is strictly decreasing with negative, and assignment is unchanged with positive sorting, and workers get upgraded (firms downgraded) with negative sorting.

\(^{47}\)We need that $$F(\bar{x}, y) - \left( w + \int_{x}^{\infty} F_x(x', \mu(x')) dx' \right) < 0$$, where $$x^*$$ solves $$1 - G(x^*) = J \left[ 1 - H(y) \right]$$.

\(^{48}\)Here, we also use the differentiated Equation 5.3.

\(^{49}\)Under the assumptions in Example 1, the implications for the Lorenz curve are similar, but $$\frac{dw(x)}{dw}$$ is constant under positive sorting, and is strictly decreasing with negative sorting.
6 Extensions

6.1 Spikes, Benefits and Compliance

One undesirable feature of the models studied thus far is that none of them feature spikes at the minimum wage: there is a measure 0 of individuals at the truncation point, $x_0$. One way to introduce spikes in a competitive framework is by introducing fringe benefits: the firm can pay the worker either via wages or other forms of compensation such as health insurance. This is an old idea and has been pointed out by Meyer and Wise [23], among others. One argument against this mechanism is that the “truncation” model implies that firms reduce the provision of benefits 1-to-1 in response to a minimum wage increase, whereas the empirical evidence on this effect is mixed. As we discuss below, for the same reason that firm heterogeneity generates spillovers, it also attenuates this effect, which helps explaining the empirical evidence.

Our model is an extension of the model of Section 4 of the paper, in the case where firms are profit maximizers: $\tilde{V}(x, y, \pi) = \pi$, and the assumptions of Proposition 2 are satisfied.\footnote{Note that corollary 2 is also satisfied.} Everything is identical, except that workers receive a compensation package, $\omega = w + B$, where $w$ denotes wage compensation, and $B$ denotes the dollar value of benefits. Note that only $w$ is subject to minimum wage regulations. We make the simplifying assumption that benefits and wages are perfect substitutes from the perspective of the worker. From the perspective of the firm, profits become $\pi = F - (w(1+\epsilon) + B)$, where $\epsilon \approx 0$ is an infinitesimal tax that the firm has to pay to compensate workers using wages. We assume that $B \in \mathbb{R}$. Note that $B$ is allowed to be negative. Because of this tax, firms always prefer to remunerate workers using benefits.

Let $\omega = w + B$ be the effective minimum wage of this economy. We consider a range of effective minimum wages $\omega \in \mathcal{W}^B = [\omega_L, \omega_H]$. Note that since $B$ is allowed to be negative, effective minimum wages can lower than actual minimum wages, which means that the policy is less binding than it seems. Then, replacing $\omega$ for $w$, and $\omega$ for $w$, the characterization of the equilibrium is as in Proposition 2, and the comparative statics with respect to $w$ is as in Corollary 2.\footnote{Note that the comparative statics with respect to $w$ is identical to the one with respect to $\omega$.} This yields the equilibrium compensation policy $\omega(x), x \geq x_0$.

This describes the patterns of total compensation in this economy. However we are interested in wages, so we need to translate our results from total compensation back to wages. The marginal worker in this economy earns compensation $\omega(x_0) = w + B$. His wage equals to the minimum wage $w$. Recall that firms prefer to pay first using benefits, since these are not subject to taxes. This implies that there is a range of workers $[x_0, x_B]$, such that $w(x) = w$, and $B(x) = \omega(x) - w$, where $x_B$ is such that $\overline{B} = \omega(x_B) - w$. This implies that a positive mass of workers, $G(x_B) - G(x_0)$ earns the minimum wage. of Finally, if $x > x_B$, then $B(x) = \overline{B}$ and $w(x) = \omega(x_B) - \overline{B}$.\footnote{One implication of our assumptions is that jobs just above the minimum wage gets the full amount of benefits, $\overline{B}$, which is counter-factual. This is an artifact of our simplifying assumption that benefits and wages are perfect substitutes in the utility function (e.g. workers can eat health insurance), and that this is not experienced in practice.} Therefore, the comparative statics with respect to wages is
as follows
\[
\frac{dw(x)}{dx} = \begin{cases} 
\frac{1}{J_h(y_0)V^0_y} & \text{if } x \in (x_0, x_B) \\
\frac{g(x_0)V^0_y}{J_h(y_0)V^0_y + g(x_0)V^0_y} & \text{if } x > x_B \\
0 & \text{if } x > x_B
\end{cases}
\]

Therefore, workers at the minimum wage spike benefit from the full minimum wage increase, whereas workers above the spike benefit from the spillover effects, which are identical to the ones in Corollary 2. By Proposition 4, the implications for inequality and political economy of wages remain unchanged.\footnote{This refers both to compensation and wages.}

Finally, we can compute the implications of the model for benefit provision of the firms:
\[
\frac{dB(x)}{dx} = \begin{cases} 
\frac{-J_h(y_0)V^0_y}{J_h(y_0)V^0_y + g(x_0)V^0_y} & \text{if } x \in (x_0, x_B) \\
0 & \text{if } x > x_B
\end{cases}
\]

Thus, in response to a rise in \(w\) firms may reduce the provision of benefits for minimum wage jobs. Note that this depends negatively on the extent of firm heterogeneity, \(J_h(y_0)\): if \(h(y_0) \approx 0\) there are negligible employment losses, \(\frac{dw(x)}{dx} \approx 1\), it’s maximum attainable value, and benefits are not affected \(\frac{dB(x)}{dx} \approx 0\). Conversely if firms are homogeneous then employment losses are at it’s maximum, there are no spillovers and firms reduce benefits 1-to-1 with minimum wage increases \(\frac{dB(x)}{dx} \approx 1\).

### 6.2 Multiple Sectors/Occupations

Throughout the paper we assumed that the economy consists of a single sector or occupation with heterogeneous firms and workers. In this Section we discuss how our results change if the economy consisted of multiple sectors/occupation categories. This is important because sectoral heterogeneity has been acknowledged to be important in several contexts (as in trade), and it is likely that minimum wage policies have an asymmetric direct impact across sectors. For example, in a high skill occupation, like the set of lawyers, very few workers earn the minimum wage. We consider three extreme assumptions about sector heterogeneity, and show that in the three cases, qualitatively, our comparative statics are identical to our baseline scenario.

First, we consider the case with strictly hierarchical sectors. The model identical to the one in Section 4, except that we partition the support of firm types in \(S\) hierarchical categories: \(Y = \bigcup_{s=1}^S [y^s, y^{s+1}], \) where \(y^{s+1} = y^s\).\footnote{Of course, if we extend the argument and have a large number of sectors, then firms and sectors are indistinguishable.} This is very similar to the hierarchical sorting model of Groes, Kircher and Manovskii [16]. The equilibrium characterization and comparative statics are identical to the ones in Sections 4 and 5. It is worth noting that in this case, only the least productive active sector has workers at the threshold \(x_0\), and workers in all sectors above that earn wages strictly higher providing benefits has a constant monetary cost. Relaxing either assumption would change this prediction.\footnote{In that paper they also have imperfect information. Another difference is that for each sector productivity is collapsed at a single value.}
than \( w \) (e.g. no lawyer or CEO earns the minimum wage). Nonetheless, the minimum wage increase affects wages of workers in all sectors. This is because, even if no workers at a sector earn the minimum wage, the marginal worker at that sector is indifferent between working at that sector and the one below, and the minimum wage may be binding on the sector below. Figure 6.1 illustrates an example with three categories.

Second, we consider an islands economy. In this model, worker and firm productivities are specific to a single sector, which implies that the economy consists of a set of heterogeneous islands, each of which operating independently. Thus, each island \( s \) has heterogeneous workers with distribution \( G^s \) and preferences \( U^s \), firms with measure \( J^s \), distribution \( H^s \) and preference \( V^S \). Since each island has its own independent matching market the characterization and comparative statics is just as in Sections 4 and 5 of the paper. Therefore, in every sector, an increase in \( w \) would lead to some matches would be terminated, wage inequality would go up, profits inequality would go down, and the labor share would increase. Note that the strength of these effects would be asymmetric across sectors. As discussed before, the wage spillover effects are strong when there are many workers around the threshold, high \( g^s (x^s_0) \), and firm heterogeneity is high, low \( h^s (y^s_0) \).

Third, we consider the case of perfectly horizontal sector heterogeneity. The model is an extension of a Roy model with \( S \) symmetric sectors, where within each sector there are heterogeneous firms, which produce output specific of that sector. In this case, workers have multivariate type \( \hat{x} \in \mathbb{R}^S \), which represents the skill of that worker specific to sector \( s \) tasks. \( \hat{x} \) has distribution \( \hat{G} \), with density \( \hat{g} \). In each sector there is a distribution of firm types, \( y \), with measure \( \hat{J} = J \) and distribution \( \hat{H} = H \), where the equality comes from the symmetry assumption. When a worker \( \hat{x} \) chooses to work in sector \( s \) with firm \( y \) the match yields output \( F(x^s, y) \), where \( x^s \) denotes the \( sth \) entry of \( x \). We assume that \( F \) is increasing in both arguments, and \( F_y > 0 \).

Workers and firms have preferences identical as in Example 1, \( U(x^s, y, w) = w \) and \( F(x, y, w) = F(x^s, y) - w \).

In this formulation, the wage schedule is \( w^s (x^s, y) \), being sector specific but only depending on the skill of the worker at that particular sector. Conditional on choosing to work on sector \( s \), let \( x^s \) be the job assigned to this worker. Because of symmetry, \( w^s (x^s, y) = w(x^s, y) \) and \( x^s(x^s) = \mu(x^s) \). The worker’s sectoral choice decision is given by \( \max_{s \in \{1, \ldots, S\}} w(x^s) \), where \( w(x^s) = w(x^s, \mu(x^s)) \). Since in the matching equilibrium \( w'(x^s) > 0 \), the worker’s choice is to go to the sector that he is the most productive at. The worker’s decisions imply a measure of workers in each sector:

\[
N^s = \int_{x^s \geq \mu(x^s)} \hat{G} (\hat{x}) d\hat{G} (\hat{x})
\] (6.1)
We also have an endogenous distribution of worker skills in each sector

\[ G^s(x) = \frac{\int_{\max(\tilde{x}), x_s \leq \tilde{x}} \hat{G}(\tilde{x}) \, \hat{d} \tilde{x}}{N^s} \]

Because of symmetry, \( N^s = N \) and \( G^s(x) = G(x) \).

Within each sector, the equilibrium in this economy is identical to the positive sorting equilibrium described in Proposition 2. In addition Equations 6.1 and 6.2 determine the endogenous distribution of worker types in each sector, already taking into account the worker’s optimal sectoral choice. One thing to note is that because of symmetry, the minimum wage change does not affect the choice of sector. It is easy to show that the comparative statics with respect to \( w \) is identical to the the single sector case: \( \frac{d\mu(x)}{dw} = 0, \frac{dx}{dw} \geq 0 \) and \( \frac{dw(x)}{dx} \in (0, 1] \). Therefore, the implications for inequality, labor share and political economy are identical as in the single sector case. We describe the details of the comparative statics in Appendix 9.1.

7 Entry

As demonstrated in Section 5, increases in minimum wages increase the wages of all workers in this economy. Likewise, they decrease the profits of firms by the same amount. Therefore, this effect is driven by a transfer of profits from firms to the workers. Our model is an extension of the differential rents model analyzed by Sattinger [28] and many others. Like in the classical analysis of land rents by David Ricardo, firms earn positive profits (or “rents”) because of differentiation: no two firms are exactly alike, so the workers are not able to extract the full marginal output from a match.56 The minimum wage allows workers to extract some of those rents from the firms.

One maintained assumption thus far is that the mass of firms in the economy is exogenous: each production unit can decide if they operate or remain idle, but the amount and density of jobs is exogenously given. One can interpret our comparative static results as “short-run” responses, before firms have time to adjust the optimal number of jobs.57 We proceed to analyze cases where the mass of jobs is determined endogenously.

The implication of free entry for our results depend if entry is risky or not. In order to outline that, we study two environments with free entry. The first is a Putty-Clay model: entry is deterministic, and firms can create jobs of specific types by paying a type-specific cost.58 In this case minimum wage changes have no spillovers above the truncation point. In the second environment, entry is stochastic as in Melitz [22]: firms pay a fixed cost and take a draw from the distribution of job productivity. In this case spillovers arise, and low skill workers benefit from this change, whereas high skill workers necessarily lose with this change. The implications for wages and profits

56Note that this argument does not rely on worker heterogeneity. We need worker heterogeneity in order to have a non-degenerate wage distribution.
57See footnote 12.
58For example, Solow [31] studies an environment with this feature.
inequality are the same as in Section 5.1. We see this uncertainty in the entry decision as a reduced form for the entry, learning and selection dynamics of firms over their lifecycle (e.g. Jovanovic [18]). As discussed in Melitz [22], this model matches the pervasive empirical fact that entrants are less productive on average than surviving firms.

We can adapt the definition of equilibrium to this environment with entry. This can be seen as a two-stage game. First, entry decisions determine $J$ and $H(y)$. Second, given $J$ and $H(y)$ workers and firms participate in a competitive matching market, similarly as before.

**Definition 5.** A competitive equilibrium with minimum wages and entry consists of a wage schedule $w(x, y)$, marginal types $x_0$ and $y_0$, assignment correspondences $\mu(x)$ and $\nu(y)$, mass of firms $J$ and distribution $H(y)$ such that conditions 1-4 of Definition 2 are satisfied, and

5. Entrants make zero profits in equilibrium.

Furthermore, to focus on the role of entry, we specialize our economy to the specification in Example 1: $V(x, y, w) = F(x, y) - w$, where $F_x > 0$, $F_y > 0$, $F_{xy} > 0$ and $U(x, y, w) = w$.

### 7.1 Deterministic Entry: No Spillovers

The first entry structure that we analyze is a Putty-Clay structure as in Solow [31]: firms know the quality of the jobs they create with certainty, upon paying a cost. The environment is similar to the one studied in Section 4, except for the firm side of the economy. Firms can create jobs of type $y$ by paying a cost $C(y) > 0, y \in Y$, and there is a large mass of potential entrants in this economy. We assume that $C' > 0$, and $C'' > 0$. Since there is free entry for every firm type, the free entry condition becomes

$$F(x, y) = w(x, y) + C(y), x \geq x_0, y \in \mu(x) \quad (7.1)$$

It follows that

**Proposition 5.** Assume that $F_{xy} > 0$, $F_{yy} \leq 0$, $F_y(x, y) \geq C'(y)$. $F_y(x, y) \leq C'(y)$ and $F_y(x, y) \geq w + C(y), y \in Y, w \in W$. Then the unique equilibrium with entry, $x_0, y_0, \mu(x), w(x), J$ and $H(y)$, satisfies the following properties

1. $x_0$ and $y_0$ solve the system

$$F(x_0, y_0) = w + C(y_0) \quad (7.2)$$

$$F_y(x_0, y_0) = C'(y_0) \quad (7.3)$$

2. $\mu(x)$ and $w(x)$ are given by equations

$$F_y(x, \mu(x)) = C'(\mu(x)) \quad (7.4)$$

$$w(x) = w + \int_{x_0}^{x} F_x(x', \mu(x')) dx' \quad (7.5)$$
3. $J = 1 - G(x_0)$ and $H(y) = \frac{f^{\nu(x)}g(x')dx'}{\mu(x)}$.

Proof. First, since the free entry condition holds along the equilibrium path, $(x, \mu(x))$, we can differentiate it with respect to $x$, combine it with first order conditions 4.2 and 4.4, and rearrange,

$$F_y(x, \mu(x)) = C'(\mu(x))$$

this equation implicitly determines $\mu(x)$. Note that our assumptions imply that $\mu'(x) > 0$. At the margin, this equation holds together with the free entry condition, which allows us to compute $x_0$ and $y_0$. Our assumptions on $C$ and $F$ assure that $x_0$ and $y_0$ are unique. Given $x_0, y_0$ and $\mu(x)$, we can retrieve $w(x)$ by integrating Equation 4.4 over the equilibrium path and, $J$ and $H(y)$ from the measure consistency conditions.

Now, we are ready to determine the effect of a minimum wage change in this economy.

Proposition 6. If $F_{xy} > 0$ then

1. $$\frac{dw(x)}{dw} = 0 \quad \text{for } x > x_0$$

2. $$\frac{dx_0}{dw} = \frac{1}{F_x(x_0, y_0)} > 0$$

3. $$\frac{dw(x)}{dw} = 0 \quad \text{for } x > x_0$$

Proof. $\frac{dw(x)}{dw} = 0$ because $\mu(x)$ is the only endogenous object in Equation 7.4. The second and third claims comes from differentiating Equations 7.2, 7.3 and 7.5 with respect to $w$.

Therefore, with Putty-Clay entry changes in minimum wage policy have no effects on wages above the truncation point. The reason this happens is because for each firm type entry affects profits such that they are just enough to cover the cost of entry $C(y)$. Since the equilibrium assignment remains unchanged it is not possible anymore to transfer profits from the worker to the firm, and as a result wages must remain unchanged.

7.2 Entry With Uncertainty: Positive Spillovers

The second entry structure that we study has uncertainty as in Melitz [22]. The environment is similar to the one studied in Section 4, except for the firm side of the economy. Firms pay a fixed cost $K$ and take a draw from the exogenous job productivity distribution $H(y), y \in Y$, with corresponding smooth density $h(y)$. To assure existence of equilibrium, we assume that $F(\pi, y) - w > K, y \in Y, w \in W$. In this case, the measure of jobs $J$ will be endogenous. Once jobs are created, the equilibrium on the matching market is similar to the one described before: firms (workers) below $y_0$ ($x_0$) remain idle, whereas the ones above the threshold participate in the matching market.
Proposition 7. If $F_{xy} > 0$ then the unique equilibrium with positive sorting and entry, $\tilde{J}, \tilde{x}_0, \tilde{y}_0, \tilde{\mu}(x), \tilde{\nu}(x)$ and $\tilde{\pi}(y)$ satisfies conditions 1-4 of Proposition 2, and

5. Entrants earn zero profits in expectation

$$\int_{\tilde{y}_0}^\infty \tilde{\pi}(y') dH(y') = K \quad (7.6)$$

Proof. See appendix 10.4.

Now, we are ready to determine the effect of a minimum wage change in this economy.

Proposition 8. If $F_{xy} > 0$ then

1. $\frac{dJ}{dw} < 0$, and $\frac{d\tilde{x}}{dw} > 0$.

2. $\frac{d\mu(x)}{dw} < 0$ for $x \in [\tilde{x}_0, \bar{x})$.

3. $\frac{d\tilde{\mu}(x)}{dw}$ is strictly decreasing in $x$. Moreover, there exists $\hat{x} \in (x_0, \bar{y})$ such that $\frac{d\tilde{\mu}(x)}{dw} = 0$, $\frac{d\tilde{\mu}(x)}{dw} > 0$ if $x < \hat{x}$, and $\frac{d\tilde{\mu}(x)}{dw} < 0$ if $x > \hat{x}$.

4. $\frac{d\tilde{\nu}(x)}{dw} = \frac{d\tilde{\nu}(x)}{dw}$ for $y \geq y_0$.

5. $\int_{\tilde{y}_0}^\infty \frac{d\tilde{\pi}(y')}{dw} dH(y') = 0 \quad (7.7)$

Proof. See Appendix 10.5.

The first thing to note is that, unlike the previous economies, workers get downgraded to inferior jobs $\frac{d\mu(x)}{dw} < 0$. Conversely, firms are substituting towards better workers. This happens because raising the minimum wages reduces the profitability of jobs, which induces less job creation: $\frac{dJ}{dw} < 0$. On the firm side, there are fewer jobs of all types, whereas on the worker side only the marginal workers are driven out from the market. This implies that all workers (except for $\bar{x}$) get assigned to worse jobs. This downgrading effect affect high skill workers more than low skill ones, which cause $\frac{d\tilde{\nu}(x)}{dw}$ to be strictly decreasing in $x$. Finally, since average profits need to remain unchanged, $\frac{d\tilde{\mu}(x)}{dw}$ necessarily starts positive and end up negative. Figure 7.1 illustrates this comparative statics in a particular example.

In terms of the implications for inequality and political economy of minimum wage, things change as follows.

Proposition 9. Suppose that $F_{xy} > 0$. After controlling for selection, we have that for $p \in (0, 1)$, $\frac{dL^0_x(p)}{dp} > 0$ and $\frac{dL^y(p)}{dp} < 0$.

Furthermore, for surviving matches, the labor share of income remains unchanged.
Proof. First, we can use the results from Proposition 8 to show that
\[ \int_{y_0}^y \frac{dw}{dw} dp' = 0. \]
Second, differentiating 5.1 with respect to \( w \) we obtain, for \( p \in (0, 1) \)
\[ \frac{dL^w(p)}{dw} = \frac{\int_0^p \frac{dw}{dw} dp' - \int_0^1 \frac{dw}{dw} dp' L^w(p)}{\int_0^1 w' dp'} > 0 \]
where the second equality uses the fact that \( \int_0^1 \frac{dw}{dw} dp' = 0 \), and \( \frac{dw}{dw} \) is strictly decreasing in \( p \). A similar derivation yields \( \frac{dL^w(p)}{dw} < 0 \), \( p \in (0, 1) \). Finally, for surviving matches, \( \int_{y_0}^y \frac{d\pi'(y')}{dG} dG(y') = \int_0^1 \frac{d\pi'(y')}{dG} dp' = 0 \), which implies that the labor share of income remains unchanged.

Thus, similarly as before, wage inequality unambiguously goes down, whereas the dispersion in profits increases. However, the implications for the political economy change: workers with \( x < \hat{x} \) favor the policy, and workers with \( x > \hat{x} \) oppose the policy. Likewise, the capitalists who hire low skill workers oppose this policy, whereas the ones who hire high-skilled workers benefit from the policy, since they can hire these workers at lower wages. Alternatively, we can say that from an ex-ante perspective firms are unaffected by the policy, but workers are affected as described. Finally, the last significant difference is that, unlike in the environment without entry, the labor share of income remains unchanged: firms respond to the minimum wage increase by creating fewer jobs, which increases profits of the most profitable matches, until expected profits from entry are the same as before.\(^{59}\)

\(^{59}\)As before, this refers to surviving matches. If we include matches that are destroyed, the labor share goes down, since the marginal matches accrue zero profits.
8 Conclusion

In this paper, we studied the importance of firm heterogeneity and labor market sorting to understand the effects of minimum wages policies. We showed that under very general conditions the effects of the policy are unambiguous: a rise in minimum wages decrease the inequality in wages, and increases the inequality in profits, including for workers who earn wages above the minimum wage. This also helps understand the political economy of minimum wages: in this model all workers would favor (small) increases in minimum wages. All this contrasts to the model without firm heterogeneity, where there are no spillovers above the truncation point, and there is no political support for the policy.

9 Appendix 1

9.1 Comparative Statics With Multiple Sectors

INCOMPLETE

10 Appendix 2: Proofs

10.1 Proof of Proposition 1

First order conditions 4.2 and 4.4 hold on the equilibrium path \( \{ x, \mu (x) \}_{x \geq x_0} \) or conversely, \( \{ \nu (y), y \}_{y \geq y_0} \). If we total differentiate 4.2 with respect to \( x \) and 4.4 with respect to \( y \) we obtain

\[
U_y + U_{yw} w_x + [U_{wx} + U_{ww} w_x] w_y + U_w w_{xy} = -\mu' (x) \frac{d[U_y + U_{wy}]}{dy} \tag{10.1}
\]

and

\[
V_{xy} + V_{yw} w_x + [V_{wy} + V_{ww} w_y] w_x + V_w w_{yx} = -\nu' (y) \frac{d[V_x + V_{wx}]}{dx} \tag{10.2}
\]

if we divide both sides of 10.1 by \( U_w \) and 10.2 by \( -V_w \), substitute \( w_x \) and \( w_y \) using the first order conditions, and add the two equations we obtain

\[
-\frac{V_y}{V_w} + \frac{V_{yw}}{V_w} \frac{U_y}{U_w} + \left[ \frac{V_{wy}}{V_w} - \frac{U_{wy}}{U_w} \right] \frac{V_x}{(V_w)^2} + \frac{U_{wx}}{U_w} - \frac{U_{ww} V_x}{U_w V_w} - \left[ \frac{U_{wx} - U_{ww} V_x}{V_w} \right] \frac{U_y}{(U_w)^2} = -\mu' (x) \frac{d[U_y + U_{wy}]}{dy} + \nu' (y) \frac{d[V_x + V_{wx}]}{dx} \tag{10.3}
\]

Note the left hand side of the expression is our sorting condition, 4.6, that the sign of \( \mu' (x) \) is the same as the sign of \( \nu' (\mu (x)) \) at \( x \), and the terms in brackets on the right hand side need to be smaller or equal to zero. Therefore, if \( \mu' (x) > 0 \) then the left
hand side needs to be non-negative. Moreover, if 4.6 is strictly positive everywhere, then it must be the case that for \( x \geq x_0 \), \( \mu'(x) > 0 \), and at least one second order condition needs to be strictly negative. The same proof can be adapted to the case of negative assortative matching, which we omit.

Finally, we need to prove that if Equation 4.6 holds with strict equality then the equilibrium features random matching. To do that we need to show that workers and firms above the marginal types are indifferent between any pairings. Suppose by contradiction that worker \( x \) strictly prefers firm \( y^* \) over other firms around it's neighborhood. That means that at \( x \), \( \frac{d[U_{\nu_0}+U_{\omega_0}]}{dy} < 0 \). Let's label \( \mu(x) = y^* \). The previous result implies that at \( x \), \( \mu'(x) \neq 0 \) and \( \nu'(y) \neq 0 \) with the same sign. Since \( \frac{d[V_0+V_{\omega_0}]}{dx} \leq 0 \), if we repeat the same calculations leading to Equation 10.3 we arrive at our sorting condition being different from zero, which is a contradiction. We can build an analogous contradiction argument to the case where firms strictly prefer workers.

### 10.2 Proof of Proposition 2

First, in Proposition 1 we have already established that for \( x \geq x_0 \), \( \mu \) is a function and \( \mu'(x) > 0 \). This implies that \( \mu(x_0) = y_0 \). Next, we show that \( x_0 \) and \( y_0 \) are unique. Our assumptions on preferences imply that \( x_0 > \overline{x} \). There are two mutually exclusive options for \( y_0 \). First, we may have \( y_0 = y \), and \( x_0 > \overline{x} \) determined by \( 1 - G(x_0) = J \), which is optimal if \( V(x_0,y,w) \geq 0 \). Alternatively, if \( V(x_0,y,y) < 0 \), \( x_0 \) and \( y_0 \) can be uniquely determined by the system of equations 4.7, with equality in the second equation. To show that, first assume that \( J \geq 1 \). Since \( G \) and \( H \) are distributions, we can use the top equation to implicitly define the strictly increasing function \( y = \phi(x), x \in X \).

Now, define \( M(x) \equiv V(x,\phi(x),w) \). Our assumptions imply that \( M(x) < 0 \), \( M(\overline{x}) > 0 \). \( M \) is continuous and strictly increasing in \( x \). Thus, \( x_0 \) can be computed by solving \( M(x_0) = 0 \). The argument can be adapted to the case where \( J \leq 1 \). We label our solution \( \overline{x}_0 \) and \( \overline{y}_0 \).

Now given \( \overline{x}_0 \) and \( \overline{y}_0 \), and the fact that the equilibrium is positively assorted \( \bar{\mu}(x) \) can be uniquely determined using Equation 4.8. Note that by construction, the implied assignment satisfies the measure consistency condition, Equation 4.5. Given \( \overline{x}_0 \) and \( \bar{\mu}(x) \), then \( \overline{w}(x) \) can be uniquely determined by integrating Equation 4.4 over \( x \) along the equilibrium path.

Finally, we have the case where the sorting condition is satisfied with weak equality and \( U(x,y,w) = u(x,w) \). Take any wage function \( w(x,y) \) such that first order conditions 4.2 and 4.4 are satisfied for the perfect sorting assignment, \( (x,\bar{\mu}(x)) \), \( x \geq \overline{x}_0 \). Following the same steps as in the proof of Proposition 1, from Equation 10.3, we know that the positive sorting equilibrium satisfies the second order condition of both worker and firm problems. Given this wage schedule, any assignment \( \mu \) that satisfies worker and firm optimizations, plus the measure consistency condition, , is a valid

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60 Recall that we assumed that \( V(x,y,w) < 0 \), \( \forall y \in Y; w \in W \).

61 Recall that workers prefer the minimum wage over remaining unmatched.

62 Note that his is only possible if \( J < 1 \).

63 i.e. \( \phi \) is defined by \( 1 - G(x) = J [1 - H(\phi(x))] \).
one. The perfect sorting assignment is an example of such assignment. Finally, since \( \tilde{\mu}(x) \) is in the optimality set, for any \( y \in \mu(x) \), \( u(x, w(x, y)) = u(x, w(x, \mu(x))) \), which implies that \( w(x, y) = w(x, \tilde{\mu}(x)) \equiv \tilde{w}(x), \forall x \geq \tilde{x}_0 \).

10.3 Proof of Proposition 3

First, to show that \( \frac{d\tilde{\mu}(x)}{dw} = 0 \), note that neither the boundary condition \( \tilde{\mu}(x) = y \) nor the differential equation 4.8 depend on endogenous variables other than \( \tilde{\mu}(x) \).

Second, we compute \( \frac{d\tilde{x}_0}{dw} \). In the case where \( \tilde{y}_0 = y \) we have that \( 1 - G(\tilde{x}_0) = J \).

In that case, \( \frac{d\tilde{x}_0}{dw} = 0 \). In the alternative case, both Equations in 4.7 are satisfied to equality. Differentiating both with respect to \( w \), we obtain

\[
\begin{align*}
\frac{d\tilde{x}_0}{dw} g(\tilde{x}_0) &= \frac{d\tilde{y}_0}{dw} J h(\tilde{y}_0) \\
V_x \frac{d\tilde{x}_0}{dw} + V_y \frac{d\tilde{x}_0}{dw} &= 0
\end{align*}
\]

Combining both equations we obtain our result.

Third, \( \frac{d\tilde{w}(x)}{dw} \) can be obtained by differentiating Equation 4.9 with respect to \( w \):

\[
\frac{d\tilde{w}(x)}{dw} = 1 + \frac{d\tilde{x}_0}{dw} V_x^0 \frac{d\tilde{w}(x)}{dw} + \int_{x_0}^{x} \frac{d\tilde{w}(x')}{dw} V_x V_{xw} - V_{ww} V_x \frac{d\tilde{x}_0}{dw} dx'
\]

where we already dropped terms that contained \( \frac{d\tilde{\mu}(x)}{dw} \). If we combine this with the result about \( \frac{d\tilde{x}_0}{dw} \) we obtain \( \frac{d\tilde{w}(x)}{dw} \). Now, for \( x > x_0 \), if we define \( H(x) \equiv \frac{d\tilde{w}(x)}{dw} \), and differentiate it with respect to \( x \) this equation becomes

\[
H'(x) = H(x) \frac{V_w V_{xw} - V_{ww} V_x}{V_x^2}
\]

Integrating this equation over \([x_0, x]\) determines \( \frac{d\tilde{w}(x)}{dw} \).

Finally, to determine \( \frac{d\tilde{w}(y)}{dw} \) we differentiate Equation 4.10 with respect to \( w \):

\[
\frac{d\tilde{\pi}(y)}{dw} = \frac{d\tilde{\pi}(y)}{dw} \left[ F_x (\tilde{\nu}(y), y) - \tilde{w}(\tilde{\nu}(y)) \right] - \frac{d\tilde{w}(\tilde{\nu}(y))}{dw}
\]

where the last equality uses the fact that \( \frac{d\tilde{\pi}(y)}{dw} = 0 \).

10.4 Proof of Proposition 7

Proof. Conditional on \( J \), the matching market works as in Section 4, and we know from Proposition 2 that there is an unique positive sorting equilibrium, which we denote \( x_0(J), y_0(J), \mu(x|J), w(x|J) \) and \( \pi(y|J) \). In what follows, we omit the dependence on \( J \) for exposition. It remains to show that there is a unique \( J \) such that...
Equation 7.6 is satisfied with equality. First, note that
\[
\frac{d\pi(y)}{dJ} = \frac{d\nu(y)}{dw} \left[ F_x(\nu(y), y) - w_x(\nu(y), y) \right]
\]
\[
= - \frac{dw(\nu(y))}{dJ}
\]
where the second inequality uses the envelope condition with respect to the firm's optimization.

Second, the marginal conditions in this economy are
\[
1 - G(x_0) = J \left[ 1 - H(y_0) \right]
\]
\[
F(x_0, y_0) \geq w
\]
where the second equation is satisfied with equality if \( y_0 > y \). Differentiating these
with respect to \( J \), we obtain
\[
\frac{dx_0}{dJ} \left[ I_{y_0 > y} J h(y_0) \frac{F_x(x_0, y_0)}{F_y(x_0, y_0)} + 1 \right] = - \left[ 1 - H(y_0) \right] < 0
\]
(10.5)
where \( I_{y_0 > y} \) is an indicator function that takes value 1 if \( y_0 > y \). This implies that
\[
\frac{dx_0}{dJ} < 0.
\]

Third, the measure condition becomes, for \( x \in (x_0, x) \)
\[
1 - G(x) = J \left[ 1 - H(\mu(x)) \right].
\]
(10.6)
Differentiating if with respect to \( J \) at \( (x_0, x) \), we obtain
\[
\frac{d\mu(x)}{dw} = \frac{[1 - H(\mu(x))]}{J h(\mu(x))} > 0.
\]

Finally, differentiating Equation 4.9 with respect to \( w \) we obtain
\[
\frac{dw(x)}{dw} = 1 - \frac{dx_0}{dw} F_x(x_0, y_0) + \int_{x_0}^{x} \frac{d\mu(x')}{dw} F_{yx}(x', \mu(x')) \, dx' > 0
\]
where the last inequality comes from our previous results. Note that this implies that expected profits from entry decrease:
\[
\int_{y_0}^{\bar{y}} \frac{d\pi(y')}{dJ} dH(y') - \pi(y_0) h(y_0) =
\]
\[- \int_{y_0}^{\bar{y}} \frac{dw(\nu(y'))}{dJ} dH(y') < 0
\]
where the first equality uses the fact that \( \pi(y_0) = F(x_0, y_0) - w = 0 \)

Finally, from 10.4 we know that if \( J \to \infty, y_0 \to \bar{y} \), which implies that \( \int_{y_0}^{\bar{y}} \pi(y') dH(y') = 0 \). On the other hand, if \( J \to 0, x_0 \to \bar{x} \) and since \( w(x_0) = w \), our assumptions assure that average profits cover the entry costs.
10.5 Proof of Proposition 8

First, differentiating Equation 4.10 with respect to \( w \) we obtain

\[
\frac{d\pi(y)}{dw} = \frac{d\nu(y)}{dw} \left[ F_x(\nu(y), y) - w_x(\nu(y), y) \right] - \frac{dw(\nu(y), y)}{dw}
\]

where the second inequality uses the envelope condition with respect to the firm’s optimization.

Second, differentiating the free entry condition we obtain

\[
\int_{y_0}^{y} \frac{d\pi(y')}{dw} dH(y') - \frac{dy_0}{dw} \pi(y_0) h(y_0) = \int_{y_0}^{y} \frac{d\pi(y')}{dw} dH(y') = -\int_{y_0}^{y} \frac{d\nu(y)}{dw} dH(y') = 0
\]

(10.7)

where the first equality uses the fact that \( \pi(y_0) = F(x_0, y_0) - w = 0 \).

Third, with regards to the the marginal types, \( x_0 \) and \( y_0 \) we have two options. First, if \( y_0 = y \), then

\[
-\frac{dx_0}{dw} g(x_0) = \frac{dJ}{dw} \quad (10.8)
\]

Otherwise, \( y_0 > y \), and both Equations 10.4 are satisfied with equality. Differentiating these with respect to \( w \), we obtain

\[
\frac{dx_0}{dw} F_x(x_0, y_0) + \frac{dy_0}{dw} F_y(x_0, y_0) = 1
\]

\[
\frac{dy_0}{dw} \left[ \frac{F_y(x_0, y_0)}{F_x(x_0, y_0)} + Jh(y_0) \right] = \frac{g(x_0)}{F_x(x_0, y_0)} + \frac{dJ}{dw} [1 - H(y_0)] \quad (10.9)
\]

Fourth, differentiating Equation 4.9 with respect to \( w \) we obtain

\[
\frac{dw(x)}{dw} = 1 - \frac{dx_0}{dw} F_x(x_0, y_0) + \int_{x_0}^{x} \frac{d\mu(x')}{dw} F_{yx}(x', \mu(x')) dx' \quad (10.10)
\]

Fifth, differentiating the measure consistency, Equation 10.6, with respect to \( w \) at \((x_0, \pi)\), we obtain

\[
\frac{dJ}{dw} [1 - H(\mu(x))] = \frac{d\mu(x)}{dw} Jh(\mu(x)).
\]

This implies that the sign of \( \frac{d\mu(x)}{dw} \) is identical to the sign of \( \frac{dJ}{dw} \). Suppose by contradiction that \( \frac{dJ}{dw} \geq 0 \). This implies that \( \frac{d\mu(x)}{dw} \geq 0 \). From Equations 10.8, 10.9 and 10.10 this would imply that \( \frac{dw(x)}{dw} > 0, x_0 \geq 0 \), which contradicts Equation 10.7.
Therefore, \( \frac{dJ}{dw} < 0 \), which implies that \( \frac{d\mu(x)}{dw} < 0 \), \( x \in [x_0, x] \). From Equation 10.10 we can see that \( \frac{dw(x)}{dw} \) is strictly decreasing in \( x \). Equation 10.7 says that the average \( \frac{dw(x)}{dw} \) is zero, which implies that \( \frac{dw(x)}{dw} \) is positive for low values of \( x \) and negative for high values of \( x \).

References


