Subjective Contingencies and Limited Bayesian Updating*

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Motivated by the observation that the same piece of information may well be interpreted differently by different individuals, we propose a way to incorporate the decision maker's subjective understanding of uncertainty in the revealed preference paradigm. Our decision maker acts as if she operates with a set of subjective contingencies that is coarser than the analyst's state space, and adopts an updating rule that follows the Bayesian spirit but is limited to the contingencies that she has in mind. Our representation results connect the parameters of the model to the agent's preferences and, in particular, permit the analyst to uniquely identify her subjective understanding of uncertainty without fixing it exogenously or placing her in different mental frames.

1 Introduction

1.1 Motivation and Objectives

On many occasions, different individuals understand and process the same piece of information in different ways; while this seems a fairly intuitive observation, it is yet striking that, even if a piece of information is communicated clearly and unambiguously, people may arrive at entirely different conclusions as to what it implies for their decision problems. To exemplify, consider the news that Alexis Tsipras and his party have won

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elections in Greece. This conveys a lot of information to a person who closely follows European events; at the same time, an average US investor would probably have little idea about its implications for the price movements at the opening of the stock exchange market next day. This simple example represents a rather typical scenario in real world problems. In this paper, we suggest that one way to formally reason about this type of situations comes from taking into account that people may have different perceptions of what could happen. Within a revealed preference analysis, our objective is to develop a model of individual decision making that accommodates a subjective view of the relevant uncertainty.

The contribution of this paper is three-fold. First, we modify the classic framework of choice under uncertainty to model situations in which the agent may interpret information subjectively, and her view of uncertainty does not necessarily coincide with the modeler's one. Second, we propose a tractable model that covers the decision maker's preferences and updating rule upon arrival of new information. In accordance with the revealed preference paradigm, we lay a foundation for the identification of a unique subjective state space which reflects the agent's subjective understanding of uncertainty and is, in a certain sense, coarse. Remarkably, our analysis does not operate with exogenous specifications of contingencies that the decision maker understands imperfectly, and is not relying on the ability of the analyst to manipulate her mental picture of the world. Rather, it provides a method to elicit the decision maker's subjective view of uncertainty by observing her choices. Our results show that the model has a certain discipline, and, in particular, that imperfect understanding of the world cannot serve as an explanation for arbitrary deviations from the choice patterns of a standard Bayesian expected utility maximizer. Third, we discuss ways in which individuals can be compared when their understanding of uncertainty comes in question. In the first place, we are interested in defining and characterizing conditions by which one agent can be found to have a better comprehension of uncertainty than another. In addition to that, our model provides room for studying decision maker's attitudes, such as optimism and pessimism, in the cases in which her view of the choice situation is imperfect.

The traditional approach to the study of decision making under uncertainty relies on the concept of a universal state space. As stated in the seminal work of Savage (1954), each state corresponds to "a description of the world, leaving no relevant aspect undescribed." Moreover, the state space is (implicitly) assumed to be commonly known by the decision maker and the analyst. As noted by various scholars,[1] this assumption presumes that the state space is directly observable, whereas, in many decision problems, the specification of the states that the agent has in mind might be a nontrivial task. To accommodate the possibility of a subjective view of uncertainty, we dispense of the above assumption and allow the decision maker to have her own subjective state space, in addition to her beliefs and tastes. In particular, we distinguish between the analyst's view of uncertainty, described by an exogenously given state space, and the decision maker's subjective understanding of uncertainty, which is derived endogenously from testable properties on preferences.

More formally, we consider a simple setup consisting of an ex ante preference and a collection of conditional preferences over *acts*. *Events* and *acts* are part of the primitives of the model: They represent the potential information and actions, respectively, and are directly processed by the decision maker without any reference to the states used by the modeler to describe the uncertainty.

To elaborate on these ideas, consider the following example. Suppose that an entrepreneur rents a part of a warehouse to store arriving goods and thinks about purchasing supplementary insurance coverage. The analyst can ask her to evaluate a contract a that provides 100% protection from damages that result from fire, a contract b that provides 100% protection from damages that result from broken pipes, and a contract c that provides both types of protection. The evaluation can also be done conditionally. Consider an event B that the arriving good consists of musical instruments, which makes the agent particularly concerned about potential water damages because even small amounts of water may deteriorate them. Suppose that the decision maker declares $b \succ_B a$ and $b \sim_B c$ — that is, she strictly prefers b over a and is indifferent

[1]See, e.g., Dekel, Lipman, and Rustichini (1998) and Gilboa (2009, p. 136–137). A related discussion appears also in Dekel, Lipman, and Rustichini (2001).

between b and c. Now, let the analyst model the uncertainty using a state space with four states: There is a possibility of direct damages from fire or broken pipes, also a possibility of damages from water used to extinguish fire in the area adjacent to the entrepreneur's storage, and there may be no damage at all. Hence, the analyst's state space is $\Omega = \{Fire, BrokenPipe, FireWater, NoDamage\}$. Then, he can model contracts as vectors of state-contingent payoffs, $a = (1,0,1,0)$, $b = (0,1,0,0)$, $c = (1,1,1,0)$, and represent event B as $\{BrokenPipe, FireWater, NoDamage\}$. What should he infer from the decision maker's preference $b \sim_B c$? What should he infer if, in addition, he learns that the decision maker express the ex ante preference $e \succ d$, where the contract $e = (0,0,0.9,0)$ provides coverage for 90% of damages resulted from lawful actions of third parties and $d = (0,0,0,0)$ is no contract? From the analyst's perspective, the two preference statements — $b \sim_B c$ and $e \succ d$ — are obviously inconsistent with assigning any probability to the third state — damages from water used to extinguish fire in an adjacent area — and updating in the Bayesian way.

The central feature of the above example is the difficulty in rationalizing the decision maker's choices using the standard Bayesian approach. The source of this difficulty is the analyst's reliance on a parsimonious state space and his own mapping of the verbal descriptions of possibilities and contracts into mathematical objects. As will be seen later, our approach can accommodate the above choices.

1.2 Overview of the Results

Our representation results are based on two keystones. First, the decision maker acts as if she is operating with a set of subjective contingencies, which are, in a certain sense, coarser than the analyst's state space. These contingencies reflect the decision maker's subjective view of uncertainty and constitute her building blocks to evaluate acts and to understand events. Second, when updating — i.e., when she is asked to evaluate acts conditionally — she 'maps' the described event to her subjective contingencies, and then updates her beliefs in a Bayesian-type manner that is limited to the contingencies she has in mind. Remarkably, this updating rule makes the agent's preferences dynamically consistent within the limits of her understanding of uncertainty. More specifically, if the decision maker's preferences are analyzed from the perspective of the analyst's state space, then they will appear to be inconsistent with the standard precepts of Bayesian updating. The reason is that our agent processes information by using her own subjective state space, which is less expressive than the analyst's one. Consequently, her inferences on a given piece of information might be different from those she would make if she had in mind the analyst's state space. Such seemingly erroneous inferences reveal the agent's limited understanding of uncertainty in determining her behavior conditional on events.

Our first result provides a general representation of ex ante and conditional preferences according to which the decision maker evaluates an act by computing its utility level in each feasible contingency and, then, calculates the overall value of the act using an additively separable criterion.

Let us provide some intuition for our representation result through the insurance example discussed previously. There, the agent's choices reveal that she considers only three subjective contingencies: The first one is related to damages as a result of fire (without making a finer distinction between direct and indirect damages), the second one is related to damages from water coming from broken pipes, and the third one corresponds to the case that no damages realize. Formally, the agent holds a set $\mathcal{S} = \{1, 2, 3\}$ of contingencies (viewed as indices) and a corresponding partition $\Pi = \{\{Fire, FireWater\}, \{BrokenPipe\}, \{NoDamage\}\}$ reflecting her understanding of uncertainty. She evaluates acts on each contingency $i \in \mathcal{S}$ by means of a utility function V_i and, then, aggregates evaluations using an additively separable criterion. Ex ante, all contingencies are possible and, therefore, an act f is evaluated according to $f \mapsto \sum_{i\in\mathcal{S}} V_i(f)$. Conditional on the occurrence of some event E, the agent discards contingencies that are not perceived as relevant and evaluates an act f using the criterion $f \mapsto \sum_{i\in\mathcal{S}(E)} V_i(f)$, where $\mathcal{S}(E) = \{i \in \mathcal{S} : \text{“}i \text{ is consistent with } E\text{”}\}$.[2] In the insurance example, conditional on event B, the agent perceives as relevant only contingency $i = 2$ (corresponding to damages from broken pipes) and $i = 3$ (corresponding to no damage) and, hence, evaluates

[2]Motivated by the desire to provide a preview of our representation, we present here only a pseudo criterion. Formal results and discussions will be introduced in Section 3.

contracts b and c using the mapping $f \mapsto V_2(f) + V_3(f)$ for $f = b, c$. As will be discussed with technical details later, the subjective state space is coarse because the states $Fire$ and $FireWater$ constitute a single contingency in the eyes of the decision maker — that is, she fails to further distinguish between direct and indirect consequences of a fire.

A noteworthy feature of our first utility representation is its minimality in structure. In particular, we do not impose restrictions on the functional specification of the utilities V_i's.[3] Here, our aim is to propose a general representation result capable of describing an individual with a subjective state space. By imposing additional behavioral assumptions on preferences, we can derive more structured utility forms for the functions V_i's. Hence, one could think of our first representation as providing a unifying criterion for alternative functional specifications. One of the advantages brought up by such flexibility is to leave room for studying the agent's attitudes — pessimistic or optimistic — towards resolution of uncertainty that she does not fully understand. Inter alia, our first result suggests a link with the literature on ambiguity since, at the ex ante stage, the behavior of our agent may be consistent with ambiguity-sensitive preferences.

Our second result establishes a more structural representation which is, formally, a special case of the first one. In particular, we present a tractable functional specification based on weighted averages to compute the value of an act in each contingency. In this case, the agent's ex ante behavior is consistent with the Subjective Expected Utility model. The two main theorems of the paper provide axiomatic characterizations of these functional forms of preferences in terms of testable properties imposed on preferences with focus on the way the agent performs updating.

As noted earlier, our present model assumes that the agent's understanding of uncertainty is coarser than the analyst's. The opposite situation will require not only modifying the axioms, but also redefining the meaning of a representation of preferences and the sense in which the analyst's model can predict the agent's choice. Our assumption, however, does not affect much a potential use of the model in applications: Indeed, in applied works, the modeler is typically presumed to have a full understanding of the environment. This is especially applicable for retrospective models, for example, the ones

[3]The restrictions on these functions are technical and include, for instance, continuity.

that study the precursors and the evolution of the financial crisis of 2008.

The two functional forms of preferences that we outlined earlier enable a comparative statics analysis along multiple dimensions. First, we study the completeness of the agent's understanding of uncertainty, discuss the way individuals can be compared in this regard, and provide a result connecting it to the parameters of our model. Second, we investigate deeper the situations in which, from the analyst's perspective, the agent erroneously includes or excludes states when updating, and introduce and characterize the concept of being more (or less) prone to inclusion and exclusion errors.

The rest of the paper is organized as follows. Next, we discuss the related literature. Section 2 introduces our setup. In Section 3, we present our general model: the axioms, the representation of preferences, and the theorem linking the two, as well as uniqueness and comparative statics results. In Section 4, we specialize to a more structured model, in which the agent resolves indeterminacy by using weighted averages, and present the corresponding axioms, representation, and the results about properties of the model.

1.3 Related Literature

The general idea of modeling a decision maker with a subjective state space goes back, at least, to Kreps (1979, 1992). In a framework of preferences over menus of alternatives, Kreps postulates preference for flexibility as reflecting uncertainty about future preferences due to the individual's awareness of unforeseen contingencies. His model leads to a natural interpretation of the individual's future preferences as subjective states. By refining Kreps' menu approach, Dekel, Lipman, and Rustichini (2001) deliver conditions for the uniqueness of the state space, whereas Epstein, Marinacci, and Seo (2007) relate an individual's awareness of her coarse state space to ambiguity-sensitive behavior. Nehring (1999) and Epstein and Seo (2009) provide further alternative extensions of Kreps' work. Ahn and Sarver (2013) propose a joint representation of ex ante preference for flexibility, as in Dekel et al. (2001), and ex post random choice behavior, as in Gul and Pesendorfer (2006). Their approach allows to model when the agent overlooks a relevant contingency.

Conceptually, the present work is related to the above literature in that it is also concerned with deriving a subjective state space from observed behavior. However, it differs both in the adopted framework and in the objectives — in particular, we are primarily motivated by modeling how subjective understanding of uncertainty may affect information processing.

Within the literature on ambiguity, Mukerji (1997) and Ghirardato (2001) study a decision maker with a coarse perception of the state space and argue that awareness of her incomplete understanding may lead the agent to hold non-additive beliefs. In particular, Ghirardato (2001) (and, with a different perspective, Mukerji, 1997) captures coarse perception of uncertainty by using multi-valued acts which are assumed to be commonly known to both the analyst and the decision maker. His focus is on developing a procedure that the agent may adopt to cope with her ignorance. In comparison to our approach, the main difference is that both works take the state space as given and focus on the individual decision process in the presence of unforeseen contingencies.

Ahn and Ergin (2010) incorporate framing effects into a model of decision making under uncertainty. They are interested in comparing the agent's preferences across different frames, which are modeled as partitions of the state space and are observed by the analyst. Ahn and Ergin's setting accommodates a behavioral distinction between events which are immune from framing effects — thereby transparent — and events which are overlooked. Hence, their theory lends itself to an interpretation in terms of unforeseen contingencies.

Schipper (2013) and Karni and Vierø (2013) take an axiomatic approach to model unawareness. Schipper (2013) proposes consistency properties on preferences across different awareness structures and focuses on the behavioral implications of unawareness. Taking a different perspective, Karni and Vierø's (2013) model accommodates different sources of unawareness and studies the impact of growing awareness on the agent's choices and beliefs. They consider preferences defined over different sets of conceivable acts, reflecting levels of awareness, and propose representation results that are akin to subjective expected utility but constrained by the level of awareness. An important contribution lies in the proposed updating rule according to which the relative likelihoods of events in the initial space do not vary when awareness expands.

In comparison with the papers discussed above, our main point of departure is that we provide representation results with a subjective state space uniquely derived from properties on observed behavior.

From a methodological perspective, Kochov (2015) also makes a distinction between the analyst's view of uncertainty and the decision maker's subjective understanding of uncertainty. However, the objectives are different. Kochov (2015) introduces a choice-based definition of unforeseen events and develops a model in which the decision maker has preferences over uncertain consumption streams. The gist of his notion is that an event is interpreted as unforeseen if the agent is unable to realize that her payoff is independent of that event because of the particular intertemporal structure of the consumption stream she is facing. In turn, an essential endogenous component of Kochov's representations consists of subjective acts, reflecting how the individual perceives outcomes as depending on foreseen events. We take a different angle by adopting a setup in which time plays no role and study how the decision maker reacts to information arrival, focusing on how limited understanding of uncertainty, as captured by the agent's subjective state space, may lead to a subjective interpretation of information.

A noteworthy feature of our model is that our agent reveals her limited understanding of uncertainty by displaying choices which are inconsistent with Bayes' rule. In the axiomatic decision theory literature, Epstein (2006) provides a foundation for non-Bayesian updating by positing that individuals may be tempted to change their ex ante beliefs about future states after observing the realization of a signal at an interim stage. Ortoleva (2012) generalizes Bayes' rule to capture agents' reactions given the realization of events which were thought to have zero (or small) probability. These works accommodate various well-known findings, such as overreaction or underreaction to arriving information. In our work, deviations from Bayes' rule are not related to overreaction or, more generally, to the agent's potential desire to reconsider her prior belief after receiving a particularly good or bad news, but stem from our departure from the assumption that the decision maker operates with the same state space as the analyst.

In conclusion, we should mention that one can find other types of inquires under the broad subject of information processing. For instance, Dillenberger, Lleras, Sadowski, and Takeoka (2014) model situations in which the analyst is unable to observe the type of information received by the agent. Their approach allows to elicit the signal received by the agent from choice behavior. Alternatively, individuals may be constrained by attention costs, as in the literature on rational inattention (see, e.g., Ellis (2014) and de Oliveira, Denti, Mihm, and Ozbek (2014) for an axiomatic foundation).

2 Setup

We consider a separable metric space X of *outcomes*, a set $\mathcal{H}$ of *acts*, and the set $\Delta(\mathcal{H})$ of all mixtures of acts in $\mathcal{H}$. Acts are simply physical actions, such as the purchase of a certain amount of stocks. An *event* is a description of what can happen. We denote by Σ the collection of all events and we assume that Σ is an algebra and it is finite. Since it is finite, it is uniquely described by its atoms. We denote by Ω the set of all atoms of Σ.

The agent's preferences on $\Delta(\mathcal{H})$ are modeled by an ex ante binary relation $\succsim$ and a collection of conditional preferences $\{\succsim_A\}_{A\in\mathscr{C}}$. The collection $\mathscr{C}$ is a subset of Σ to be discussed below. As usual, we also identify $\succsim_\Omega$ with $\succsim$.

The main departure of the present framework from the standard Anscombe-Aumann setup consists in the fact that the decision maker does not necessarily share the same perspective as the analyst on the resolution of uncertainty. We assume that the analyst views Ω as the state space, in the sense that Ω resolves all possible uncertainty. Hence, from the analyst's perspective, an act $a \in \Delta(\mathcal{H})$ yields a uniquely determined outcome $x \in \Delta(X)$ for each $\omega \in \Omega$. Put differently, the analyst encapsulates acts into state-contingent payoff profiles according to a mapping $o : \Delta(\mathcal{H}) \times \Omega \to \Delta(X)$. Note that the function $o : \Delta(\mathcal{H}) \times \Omega \to \Delta(X)$ reflects the fact that, according to the analyst, Ω is an exhaustive description of all possible states of the world. Therefore, the function o naturally leads us to consider functions of the form $\Omega \mapsto \Delta(X)$.

A *state-contingent act* is a function $f : \Omega \to \Delta(X)$. Let $\mathcal{F}$ denote the set of all state-contingent acts. The following two assumptions clarify the relationship between

$\Delta(\mathcal{H})$ and $\mathcal{F}$. First, we assume that, for each $f \in \mathcal{F}$, there exists an $a \in \Delta(\mathcal{H})$ such that $f = o(a, \cdot)$. That is, the mapping $\Delta(\mathcal{H}) \mapsto \mathcal{F}$ is surjective. This is a rather weak condition ensuring that the set $\mathcal{H}$ is sufficiently rich. Second, we assume that if two acts $a, b \in \Delta(\mathcal{H})$ are mapped into the same state-contingent act $f \in \mathcal{F}$ — i.e., $o(a, \cdot) = f = o(b, \cdot)$ — then the agent is indifferent between a and b. This second condition has an important behavioral content. The intuition is simple: if a and b are mapped into the same state-contingent act f, then a and b yield the same consequence in each state $\omega \in \Omega$ — i.e., a and b are indistinguishable from the analyst's perspective. Then, independently of how coarse the agent's view of uncertainty is, she has no reasons to hold a strict preference between a and b. A strict preference would be justified only if the agent had a finer understanding of uncertainty in comparison with the analyst's understanding, captured by Ω. Hence, the above assumption maintains that the decision maker can only have a coarser view of uncertainty in comparison with the analyst's view.[4] These two assumptions ensure that there is no loss of generality in assuming that the analyst considers directly acts in $\mathcal{F}$.

Given $\succsim$ on $\Delta(\mathcal{H})$, we define a binary relation $\hat{\succsim}$ on $\mathcal{F}$ as

$$f \mathrel{\hat{\succsim}} g \Leftrightarrow a \succsim b,$$

for all $f, g \in \mathcal{F}$, where $a, b \in \Delta(\mathcal{H})$ are such that $f = o(a, \cdot)$ and $g = o(b, \cdot)$. The induced preference relation $\hat{\succsim}$ on $\mathcal{F}$ captures the decision maker's ranking of acts from the perspective of the analyst. In the paper we will use the induced relation $\hat{\succsim}$ on $\mathcal{F}$ and, with little abuse of notation, we will adopt the same notation $\succsim$ to refer to both a binary relation on $\mathcal{F}$ and on $\Delta(\mathcal{H})$.

Let $\mathscr{C} \subseteq \Sigma$ denote the collection of all events (labels) that are *possible* — i.e., events that the decision maker can condition on. It will follow from our axioms that $\Omega \in \mathscr{C}$ and $\varnothing \notin \mathscr{C}$. The analyst does not observe $\mathscr{C}$ directly — he learns about more sets in $\mathscr{C}$ by knowing about some sets in $\mathscr{C}$ and observing the decision maker's choices. This situation is analogous (although more complicated) to the standard theory of Bayesian updating,

[4] Albeit interesting, the situation in which the decision maker has a finer view of uncertainty is beyond the scope of this paper.

in which the analyst first samples the decision maker's ex ante choices, learns from that the sets that the decision maker can condition on, and then can proceed with sampling the conditional choices and verifying the axioms.

3 General Representation of Preferences With Subjective Contingencies

3.1 Axiomatic foundations

This subsection introduces the set of axioms that form the foundations of our model.

Before proceeding to their content, we need to make a general remark. Since the analyst (the modeler) and the decision maker (the agent) have different perspectives on uncertainty, we have to take a stand on whose point of view we adopt in formulating the axioms. The choice of one of the two viewpoints affects not only the way the axioms are stated (and the language to be used for that), but also the way they are interpreted. When formulated from the analyst's viewpoint, they describe properties of the choice behavior that he should expect from a decision maker who may understand the uncertainty and its resolution differently, but otherwise is intelligent, reasonable, and consistent. They do not have to look normatively appealing or even be understood by the decision maker if she were to read them herself. By contrast, when formulated from the decision maker's viewpoint, axioms may represent principles that look reasonable from her perspective and according to which she would like to act to be consistent in her choices. Since our objective is to elicit the agent's state space endogenously, we will adopt the analyst's viewpoint. As a result, our axioms are testable without the need for the analyst to know anything about the decision maker's perspective in advance.

Basic assumptions

We start with the straightforward assumption that the decision maker's unconditional and conditional choices can be summarized by preference relations.

Axiom A1 (Weak Order)**.** *The binary relations* $\succsim$ *and* $(\succsim_E)_{E \in \mathscr{C}}$ *are reflexive, complete, and transitive.*

Next, we list the required technical conditions. We assume, first, that the agent's preferences are continuous. Second, we make an assumption that the space of outcomes have elements that are (subjectively) the best and the worst, i.e. that the space of outcomes is not order-unbounded. This assumption does not play an important role in our results, but simplifies the transition from outcomes to utility levels. Finally, we postulate that the uncertainty faced by the decision maker is not trivial — that there exist at least three distinct informative events.

Axiom A2 (Continuity)**.** *For any* $h \in \mathcal{F}$*, the sets* $\{f \in \mathcal{F} : f \succsim h\}$*,* $\{f \in \mathcal{F} : h \succsim f\}$*,* $\{f \in \mathcal{F} : f \succsim_E h\}$*, and* $\{f \in \mathcal{F} : h \succsim_E f\}$ *for all* $E \in \mathscr{C}$ *are closed.*

Axiom A3 (Best and Worst Outcomes)**.** *There exist* $x_*, x^* \in \Delta(X)$ *such that* $x^* \succsim f$ *and* $f \succsim x_*$ *for all* $f \in \mathcal{F}$*.*

Axiom A4 (Nontriviality)**.** *There exist* $A, B, C \in \mathscr{C}$ *such that* $\succsim_A$*,* $\succsim_B$*, and* $\succsim_C$ *differ from each other and from* $\succsim$*.*

To conclude our list of basic axioms, we make two assumptions about consistency between ex ante and conditional preferences. First, similarly to many other models of choice under uncertainty, we seek separation between "beliefs" and "tastes," and assume that the decision maker's tastes are not affected by information that partially resolves the uncertainty. Therefore, the ranking of constant outcomes remains unchanged under conditional evaluation.

Axiom A5 (Outcome Preference Consistency)**.** *If* $E \in \mathscr{C}$ *and* $x, y \in \Delta(X)$*, then* $x \succsim y \Leftrightarrow x \succsim_E y$*.*

Second, we impose dynamic consistency. Before that, we recall the standard definition of a null event: For any preference relation $\succsim$, an event E is called $\succsim$-null if, for any acts f and g that differ only on E, we have $f \sim g$.

The following notation will be used to formulate Dynamic Consistency and subsequent postulates.

Notation 1. For any $E \in \mathscr{C}$ and $\omega \in \Omega$, we write $P_E(\omega)$ to denote the statement that ω

is not $\succsim_E$-null, i.e., there exist $f \in \mathcal{F}$ and $x, y \in \Delta(X)$ such that $x\{\omega\}f \succ_E y\{\omega\}f$.

Axiom A6 (Dynamic Consistency). *For any $E \in \mathscr{C}$ and $f, g \in \mathcal{F}$ such that $f(\omega) = g(\omega)$ for all $\omega \in \Omega$ such that $\neg P_E(\omega)$, we have $f \succsim_E g \Leftrightarrow f \succsim g$.*

Suppose that the decision maker faces two acts that are identical in states that are ruled out by an event E. Then, ex ante, the decision maker should understand that her choice between these two acts matters only if E occurs, and the axiom maintains that her ex ante and conditional choices coincide. (In settings that are focused on the dynamic aspect of choice, this coincidence represents the willingness of the decision maker to execute plans that were optimal ex ante.)

Information processing axioms

The second group of axioms is concerned with how the decision maker processes information about occurring events.

Consistently with our objective of modeling a decision maker who has no difficulties in understanding the descriptions of events but may not understand fully how they resolve the payoff-relevant uncertainty, we assume that larger events imply wider sets of possibilities.

Axiom A7 (Monotonicity of the Possibility Correspondence). *Suppose that $A \in \mathscr{C}$ and $B \in \Sigma$ are such that $A \subseteq B$. Then, $B \in \mathscr{C}$ and, for any $\omega \in \Omega$, $P_A(\omega) \Rightarrow P_B(\omega)$.*

We also assume that if the decision maker thinks that some state ω is ruled out by the occurrence of an event A, then her preferences do not change if, in addition to the information that A has occurred, she is explicitly told that ω is impossible.

Axiom A8 (Irrelevance of Impossible States). *Suppose that $A \in \mathscr{C}$ and $\omega \in \Omega$ is such that $\neg P_A(\omega)$. Then, $A \setminus \{\omega\} \in \mathscr{C}$ and $\succsim_{A\setminus\{\omega\}} = \succsim_A$.*

To illustrate the content of this and subsequent axioms, it is useful to see why they hold in the standard model of choice under uncertainty characterized by Subjective Expected Utility preferences, full understanding of uncertainty, and Bayesian updating.

As is well known, ex ante and conditional preferences are represented in this case by

$$\begin{cases} V(f) \quad = \sum_{\omega \in \Omega} u(f(\omega))\, p(\omega), \\ V(f \mid E) = \sum_{\omega \in E} u(f(\omega))\, p(\omega) \text{ for } E \in \mathscr{C}, \end{cases} \tag{1}$$

where p is a probability measure on Ω, u is a utility function over outcomes, and the collection of admissible events $\mathscr{C}$ is $\{E \subseteq \Omega : p(E) > 0\}$. In the described benchmark case, the condition $\neg P_A(\omega)$ implies that either $\omega \notin A$ or $p(\omega) = 0$, and, in both cases, the decision maker's preferences conditional on $A \backslash \{\omega\}$ are the same as conditional on A.

The next axiom is symmetric to the previous one: If some state ω is not ruled out by the occurrence of an event A, then the decision maker's preferences do not change if, in addition to the information that A has occurred, she is explicitly told that ω is still possible.

Axiom A9 (Superficity of Possible States)**.** *Suppose that* $A \in \mathscr{C}$ *and* $\omega \in \Omega$ *is such that* $P_A(\omega)$. *Then,* $\succsim_{A \cup \{\omega\}} = \succsim_A$.

In the benchmark case, the above axiom holds trivially because its antedecent implies that $\omega \in A$ and, therefore, $A \cup \{\omega\}$ coincides with A.

Now, we introduce two axioms that impose consistency on decision maker's conditional preferences by setting minimal requirements for her understanding of intersections and unions of events. The first axiom specifies the conditions on events A and B under which they are revealed to be non-contradictory and the joint event $A \cap B$ is admissible.

Axiom A10 (Intersection of Possibilities)**.** *Suppose that* $A, B \in \mathscr{C}$, $P_A(\omega)$ *implies* $\omega \in A$ *for all* $\omega \in \Omega$, *and there exists* $\omega_0 \in \Omega$ *such that* $P_A(\omega_0)$ *and* $P_B(\omega_0)$ *hold. Then,* $A \cap B \in \mathscr{C}$.

The intuition behind this axiom is as follows. Given two admissible events A and B, their intersection may be inadmissible for two reasons. The first one is standard — these events may be (or appear to be) contradictory and the combination of them be (subjectively) impossible. This situation is ruled out by the assumption that there exists a state ω_0 that is possible conditional on both A and B. The second reason is more subtle. From the analyst's point of view, our agent may make erroneous inferences.[5] In

[5] To emphasize, while our decision maker may make inferences that are erroneous in the analyst's

particular, she may be making an error of including extra states when conditioning on A and B, and in this case, the assumption $P_A(\omega_0) \wedge P_B(\omega_0)$ is not a guarantee that the intersection of A and B is admissible (or even nonempty). Hence, our axiom contains an additional requirement that the set A contains all states that the decision maker deems possible after A has occurred, and, therefore, she does not make errors of inclusion at least with respect to A. Given that, the analyst should be confident that the event $A \cap B$ is admissible. Note that, in the benchmark case, the condition concerning errors of inclusion always holds, the conditions $P_A(\omega_0)$ and $P_B(\omega_0)$ imply that $\omega_0 \in A \cap B$ and $p(\omega_0) > 0$, and the consequent of the axiom indeed holds.

The second axiom concerns unions of events and provides conditions under which an event B opens up new possibilities with respect to an event A.

Axiom A11 (Understanding of Unions)**.** *Suppose that $A \in \mathscr{C}$, $B \in \Sigma$, $P_A(\omega)$ holds for all $\omega \in A$, and suppose that there exists $\omega_0 \in \Omega$ such that $P_{A \cup B}(\omega_0)$ but not $P_A(\omega_0)$. Then, $B \in \mathscr{C}$ and $P_B(\omega_0)$ holds.*

The crucial condition in this axiom is that there exists a state ω_0 that is revealed to be possible after the event $A \cup B$ but not after the event A only. However, similar to the previous axiom, this condition alone is not sufficient: It might be that the agent makes the error of excluding some state when conditioning on A; at the same time, while B alone does not contain any additional information, the event $A \cup B$ may give her a different perspective on uncertainty so that she does not make the same error when conditioning on $A \cup B$. This situation is ruled out by the assumption that every state in A is possible conditional on A and, therefore, the agent does not make errors of exclusion with respect to A. With this qualification, the analyst should expect all new possibilities provided by the event $A \cup B$ in comparison with A alone to originate from the event B, and B to be an admissible event. Note that, in the benchmark case, the additional condition that $P_A(\omega)$ holds for all $\omega \in A$ means that all states in A are non-null, the existence of a state ω_0 such that $P_{A \cup B}(\omega_0)$ but not $P_A(\omega_0)$ means that

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view, it does not mean that she is making mistakes on the logical (or computational) levels. Rather, those errors are categorically similar to statistical errors and are unavoidable given the agent's lack of omniscience — more on this in Section 3.4.2.

$\omega_0 \in B$ and $p(\omega_0) > 0$, and the consequent of the axiom indeed holds.

Finally, we assume that the decision maker understands the description of an empty set and, similarly to the standard agent, is not able to rank acts conditional on the empty event.[6]

Axiom A12 (Inadmissibility of Empty Events)**.** $\varnothing \notin \mathscr{C}$.

3.2 The representation

We need one more definition to state the properties of the objects of our representation.

Definition 2. We say that a set $S \subseteq \Omega$ is the *support* of a function $F : \mathcal{F} \to \mathbb{R}$ if

(i) $F(f) = F(g)$ for all $f, g \in \mathcal{F}$ such that $f|_S = g|_S$; and

(ii) for each $\omega \in S$ there exist $f \in \mathcal{F}$ and $x, y \in \Delta(X)$ such that $F(x\{\omega\}f) \neq F(y\{\omega\}f)$.

Now, we are ready to state our main theorem that describes the way the analyst can model the decision maker with a subjective view of the world, and which follows in an if-and-only-if manner from to the behavioral axioms stated earlier.

Theorem 1. *An ex ante preference relation $\succsim$, a collection $\{\succsim_A\}_{A \in \mathscr{C}}$, and $\mathscr{C}$ jointly satisfy Axioms (A1)–(A12) if and only if there exist*

- *a nonempty set of indices (subjective states) $\mathcal{S} = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ with $n \geq 3$;*
- *a collection $\Pi = \{C_1, \ldots, C_n\}$ of nonempty and mutually disjoint subsets of Ω;*
- *a collection of nonconstant continuous functions $V_i : \mathcal{F} \to \mathbb{R}$ for $i \in \mathcal{S}$, with a compact range, and such that C_i is the support of V_i for all $i \in \mathcal{S}$; and*
- *a collection of functions $\sigma_i : 2^{C_i} \to \{0, 1\}$ for $i \in \mathcal{S}$, such that, for all $i \in \mathcal{S}$, $\sigma_i(\varnothing) = 0$, $\sigma_i(C_i) = 1$, and $\sigma_i(A) \geq \sigma_i(B)$ for all $A, B \subseteq C_i$ satisfying $A \supseteq B$*

such that:

[6] It is possible to extend our theory by dropping this assumption. As a main implication, it would allow accommodating a decision maker who makes mistakes of inclusion with respect to all events, including the empty set. However, we do not view this additional level of generalization as particularly relevant for practice.

(i) The collection $\mathscr{C}$ *can be computed as*

$$\mathscr{C} = \{E \subseteq \Omega : \sigma_i(E \cap C_i) = 1 \text{ for at least one } i \in \mathcal{S}\}; \tag{2}$$

(ii) $\succsim$ *is represented by*

$$V(f) = \sum_{i \in \mathcal{S}} V_i(f) \quad \forall f \in \mathcal{F}; \tag{3}$$

(iii) For each $E \in \mathscr{C}$, $\succsim_E$ *is represented by*

$$V(f \mid E) = \sum_{i \in \mathcal{S} : \sigma_i(E \cap C_i) = 1} V_i(f) \quad \forall f \in \mathcal{F}, \tag{4}$$

(iv) $V_i(x) \geq V_i(y) \Leftrightarrow V_j(x) \geq V_j(y) \quad \forall i, j \in \mathcal{S}$ *and* $x, y \in \Delta(X)$.

Theorem 1 provides a general model of preferences of a decision maker who perceives uncertainty subjectively. The main message of the theorem can be summarized as follows.

1. The decision maker's view of uncertainty is captured by a set of contingencies $\mathcal{S}$ — this is reflected in the definition of the objects of the representation.
2. Conditional on an event E having occurred, the decision maker identifies a subset of contingencies that she perceives "consistent" with the received information, and updates her preferences in a Bayesian-like manner — this is captured by the relationship between Equations (4) and (3).
3. When evaluating acts, the decision maker's preferences are additively separable with respect to her subjective contingencies — this is captured by Equation (3).

The first feature of the contingencies that represent the decision maker's mindset is that they are *subjective* — i.e., the decision maker is free to come up with her own set of contingencies, and that is revealed by her choice. By contrast to the literature on framing effects, the agent's perception of uncertainty is not imposed by the analyst or the surrounding, and may well not be known to the analyst before he asks the decision maker to compare some acts. Our proof of the theorem in the Appendix is constructive, so the analyst can, in principle, follow its steps and learn the contingencies representing a decision maker's mindset by asking her to rank sufficiently many acts. In anticipation of the discussion of Subsection 3.3, we also note that the agent's subjective contingencies are identified uniquely.

The second important property of the subjective contingencies described in the theorem is that they correspond to *sets* of the analyst's states — the cells of Π — and, as such, are coarse. The general coarseness of the agent's understanding of uncertainty does not have any notable implications for her ex ante preferences in isolation. Rather, it manifests in the way her preferences are updated. Her coarse and, in a sense, imperfect perception of uncertainty may result in overlooking certain states (from the analyst's state space), as illustrated in the insurance example in the introduction. However, within the bounds of what she understands, she does her best at incorporating the arriving information. Namely, she determines which contingencies remain a possibility and which are ruled out; and, then, similar to the standard Bayesian updating recollected in (1), she computes the conditional value of an act by dropping the additive terms referring to no longer relevant contingencies.

The additive separability of representations (3), as well as (4), is noteworthy because this kind of structure is a precursor to defining the agent's *beliefs* about contingencies. Note that the collection Π may span over a subset of the state space that is strictly smaller than the entire Ω. In this case, as follows from the representation, the remaining states are null and have no impact on the evaluation of acts. In loose terms, the possibilities that are not covered by Π are viewed by the decision maker as having zero probability. Regarding the contingencies in $\mathcal{S}$, at the current level of generality we cannot say much about their relative likelihoods in the decision maker's eyes. One obvious reason is that our functions V_i for $i \in \mathcal{S}$ are not normalized in any way. We can pin down the likelihoods of the contingencies by imposing more structure on the ex ante preferences, and one particular refinement of the model of Theorem 1 is provided in Section 4. We also note that a standard Bayesian expected utility maximizer is a special case of our model, and we elaborate on this in Subsection 3.4.

To illustrate the way the decision maker's behavior gives rise to the representation of Theorem 1, let us review in more detail the insurance example discussed in the introduction.

Recall that there the analyst's state space was $\Omega = \{Fire, BrokenPipe, FireWater, NoDamage\}$ and the decision maker ranked acts $b = (0,1,0,0) \sim_B c = (1,1,1,0)$ and $e = (0,0,0.9,0) \succ d = (0,0,0,0)$, where $B = \{BrokenPipe, FireWater, NoDamage\}$. This decision maker can be modeled as if she has in mind three contingencies $\mathcal{S} = \{1,2,3\}$ with the corresponding partition $\Pi = \{\{Fire, FireWater\}, \{BrokenPipe\}, \{NoDamage\}\}$. Her evaluation functions could be $V_1(f) = \frac{1}{50}f_1 + \frac{1}{100}f_3$, $V_2(f) = \frac{1}{25}f_2$, and $V_3(f) = \frac{93}{100}f_4$ for any act $f \in \mathbb{R}^4$, the "consistency" map σ_1 satisfies $\sigma_1(\{Fire\}) = 1$ and $\sigma_1(\{FireWater\}) = 0$ and all maps σ_i satisfy $\sigma_i(\varnothing) = 0$ and $\sigma_i(C_i) = 1$ for $i = 1,2,3$. It directly follows that $V(e) > V(d)$, so we turn to examining the evaluations of acts conditional on B. Observe that $\sigma_1(\{Fire, FireWater\} \cap B) = 0$, $\sigma_2(\{BrokenPipe\} \cap B) = 1$, $\sigma_3(\{NoDamage\} \cap B) = 1$, and, therefore, $V(f \mid B) = \frac{1}{25}f_2 + \frac{93}{100}f_4$. Then, $V(b \mid B) = \frac{1}{25} = V(c \mid B)$, and we see that our specification of the parameters of the representation is consistent with the preference $b \sim_B c$. To conclude this discussion, note that the assumed rankings of acts necessarily imply that $Fire$ and $FireWater$ states must form a single contingency: Indeed, the event B that makes $Fire$ irrelevant also turns $FireWater$ into a null state. This may be interpreted as a failure to distinguish between direct and indirect consequences of a fire.

3.3 Uniqueness

As typical in axiomatic decision theory, we proceed next to the uniqueness result for the representation of Theorem 1. This exercise is particularly important for our model because the question of uniqueness of subjective state spaces has drawn substantial attention in the decision theory literature in other contexts.

Definition 3. We refer to a quadruple $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, (V_i)_{i\in\mathcal{S}}, (\sigma_i)_{i\in\mathcal{S}})$ that satisfies the conditions of Theorem 1 as a *representation with subjective contingencies.*

Proposition 2. *Suppose that* $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, (V_i)_{i\in\mathcal{S}}, (\sigma_i)_{i\in\mathcal{S}})$ *and* $(\mathcal{S}', (C'_i)_{i\in\mathcal{S}'}, (V'_i)_{i\in\mathcal{S}'}, (\sigma'_i)_{i\in\mathcal{S}'})$ *are two representations with subjective contingencies. Then, they represent the same system* $(\succsim, \{\succsim_A\}_{A\in\mathscr{C}})$ *of binary relations if and only if there exists a bijection* $\pi : \mathcal{S} \to \mathcal{S}'$ *such* *that*

(i) $C'_{\pi_i} = C_i$ *for all* $i \in \mathcal{S}$*;*

(ii) there exist $\alpha > 0$ *and* $\beta_i \in \mathbb{R}$ *for* $i \in \mathcal{S}$*, such that* $V'_{\pi_i} = \alpha V_i + \beta_i$ *for all* $i \in \mathcal{S}$*; and*

(iii) $\sigma'_{\pi_i} = \sigma_i$ *for all* $i \in \mathcal{S}$*.*[7]

In words: First, the set of subjective contingencies (and the corresponding partition of the state space) are identified uniquely. Second, the model also identifies uniquely the rules to determine contingencies that are consistent with an event. Finally, the model identifies, up to a joint positive affine transformation, the collection of mappings from an act to the utility level obtained from this act in each contingency.

3.4 Comparative statics

3.4.1 Understanding of uncertainty

Now we turn to studying the ways in which our decision makers can be compared and the corresponding comparative statics in the representation of their preferences. The starting point of this exercise is a comparison in terms of their understanding of the nature of uncertainty — i.e., their perception of the state space.

Definition 4. We say that a decision maker described by $(\succsim, (\succsim_A)_{A\in\mathscr{C}}, \mathscr{C})$ *fully understands* an event $E \subseteq \Omega$ if

- E is $\succsim$-null; or
- $E \in \mathscr{C}$ and, for all $\omega \in \Omega$,

$$P_E(\omega) \quad \Leftrightarrow \quad P_\Omega(\omega) \text{ and } \omega \in E.$$

Definition 5. We say that one decision maker has a *finer understanding of uncertainty* than another if the first one fully understands all events $E \subseteq \Omega$ that are fully understood by the second one. We also say that two decision makers have *equal understanding*

[7]The uniqueness of the objects of the representation does not depend on $\mathscr{C}$: In fact, our axioms imply that the way the decision maker compares acts fully determines the events that $\mathscr{C}$ may or may not contain. We clarify this point by Lemma 16 in the Appendix.

of uncertainty if the first one fully understands E if and only if the second one fully understands E, for any $E \subseteq \Omega$.

Now, we characterize our concept of full understanding of an event and the corresponding comparative notion.

Proposition 3. *Suppose that* $(\mathcal{S}, \Pi, (V_i)_{i\in\mathcal{S}}, (\sigma_i)_{i\in\mathcal{S}})$ *is a representation with subjective contingencies and* $\succsim$ *is the ex ante preference relation that it represents. Then, the decision maker fully understands* $E \subseteq \Omega$ *if and only if* E *is a union of some elements of the collection* $\Pi \cup \mathcal{N}(\succsim)$*, where* $\mathcal{N}(\succsim) \coloneqq \{\{\omega\} \mid \omega \in \Omega \text{ is } \succsim\text{-null}\}$.

To state this result in different terms, we note that $\Pi \cup \mathcal{N}(\succsim)$ is always a partition of Ω. The above proposition, thus, says that an event is fully understood if and only if it is measurable with respect to the algebra generated by this partition.

Corollary 4. *Suppose that* $(\mathcal{S}^k, \Pi^k, (V_i^k)_{i\in\mathcal{S}^k}, (\sigma_i^k)_{i\in\mathcal{S}^k})$ *for* $k = 1, 2$ *are two representations with subjective contingencies, and* $\succsim^1$ *and* $\succsim^2$ *are the ex ante preference relations that they represent. Then, the first decision maker has a finer understanding of uncertainty if and only if* $\Pi^1 \cup \mathcal{N}(\succsim^1)$ *is a refinement of* $\Pi^2 \cup \mathcal{N}(\succsim^2)$.

A decision maker has the maximal understanding of uncertainty if her $\Pi \cup \mathcal{N}(\succsim)$ consists only of singleton sets. Full understanding of all events and maximal understanding of uncertainty is intrinsic to standard economic agents who have expected utility preferences and do Bayesian updating if they are viewed through the lenses of our model. Indeed, suppose that an expected utility maximizer's utility index $\Delta(X) \to \mathbb{R}$ is u and her belief is $p \in \Delta(\Omega)$, and Ω is enumerated as $\{\omega_1, \ldots, \omega_n, \ldots, \omega_{n+m}\}$ such that $p(\omega_l) > 0$ for all $l = 1, \ldots, n$ and $p(\{\omega_{n+1}, \ldots, \omega_{n+m}\}) = 0$. Then, we can let $\mathcal{S} = \{1, \ldots, n\}$, $\Pi = \{\{\omega_i\} \mid i \in \mathcal{S}\}$, $V_i(f) = u(f(\omega_i))\, p(\omega_i)$, and our representation (3)–(4) becomes identical to the standard procedure (1).

3.4.2 Updating errors

As we noted in Section 3.1 while discussing the intuition behind our axioms, the decision maker in our model makes inferences from arriving information and update her preferences so that individual states may be treated peculiarly. Indeed, conditional on an event E, some state $\omega \notin E$ may remain non-null or, conversely, some ex-ante-non-null state $\omega' \in E$ may become null after updating. Thus, from the analyst's point of view, the agent makes inference errors. We refer to the first type of spurious inference as an *inclusion error* and to the second one as an *exclusion error*. These errors emerge due to the fact that the agent has limited understanding of uncertainty — in a similar fashion, a statistician makes type-I or type-II errors in hypothesis testing because he is not omniscient and makes inferences on the basis of finite and imperfect data.

The notation below and Definition 7 provide a formal statement of a comparative notion that captures situations in which one agent is more prone to exclusion (resp. inclusion) errors than another one.

Notation 6. Let $(\succsim, \{\succsim_A\}_{A\in\mathscr{C}})$ be a system of binary relations and $E \in \mathscr{C}$. We define

$$M_-(E) := \{\omega \in E : P_\Omega(\omega) \text{ and } \neg P_E(\omega) \text{ hold}\},$$
$$M_+(E) := \{\omega \notin E : P_E(\omega) \text{ holds}\}.$$

Note that if sets $M_-(E)$ or $M_+(E)$ are nonempty for some event E, then the decision maker does not understand this event fully, and the two sets represent the states with respect to which she makes an exclusion and inclusion errors, respectively.

Definition 7. Suppose that two decision makers are described by $(\succsim^k, \{\succsim^k_A\}_{A\in\mathscr{C}^k}, \mathscr{C}^k)$ for $k = 1, 2$.

(i) We say that Decision maker 1 is *more prone to exclusion errors* than Decision maker 2 if, for any $E \in \mathscr{C}^1 \cap \mathscr{C}^2$, we have $M^2_-(E) \subseteq M^1_-(E)$.

(ii) We say that Decision maker 1 is *more prone to inclusion errors* than Decision maker 2 if, for any $E \in \mathscr{C}^2$ such that $M^2_+(E) \neq \varnothing$, we have $E \in \mathscr{C}^1$ and $M^2_+(E) \subseteq M^1_+(E)$.

The subsequent discussion relies heavily on the following notion.

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Notation 8. Suppose that $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, (V_i)_{i\in\mathcal{S}}, (\sigma_i)_{i\in\mathcal{S}})$ is a representation with subjective contingencies. Then, we refer to

$$\mathcal{S}_c := \{i \in \mathcal{S} : |C_i| \geq 2\}$$

as the decision maker's *coarse subjective contingencies.*

The term "coarse" refers to the fact that the corresponding cells of the partition representing contingencies are coarser than singletons. This fact has a direct connection to imperfections of the agent's understanding of uncertainty: Indeed, any set outside of $\bigcup_{i\in\mathcal{S}_c} C_i$ must be fully understood by her, and the sets $M_-(E)$ and $M_+(E)$ for any $E \in \mathscr{C}$ must be contained in $\bigcup_{i\in\mathcal{S}_c} C_i$.

For the purpose of linking the comparative notions introduced earlier to the objects of our representation — namely, the consistency maps $(\sigma_i)_{i\in\mathcal{S}}$ — we focus on the case in which two decision makers have equal understanding of uncertainty. Note that, as implied by Corollary 4, two such decision makers have the same (up to renumbering) sets of coarse subjective contingencies — that is, if $(\mathcal{S}^k, \Pi^k, (V_i^k)_{i\in\mathcal{S}^k}, (\sigma_i^k)_{i\in\mathcal{S}^k})$ for $k = 1, 2$ are the representations with subjective contingencies of the preferences of two agents with equal understanding of uncertainty, then there must exist a bijection $\pi : \mathcal{S}_c^1 \to \mathcal{S}_c^2$ such that $C_{\pi_i}^2 = C_i^1$ for all $i \in S_c^1$.

Now, we can state our comparative statics result.

Proposition 5. *Suppose that $(\mathcal{S}^k, (C_i^k)_{i\in\mathcal{S}^k}, (V_i^k)_{i\in\mathcal{S}^k}, (\sigma_i^k)_{i\in\mathcal{S}^k})$ for $k = 1, 2$ are two representations with subjective contingencies and the decision makers that they represent have equal understanding of uncertainty. Furthermore, suppose that the set of coarse subjective contingencies for both of them is $\{1, \dots, m\}$, and let $C_i := C_i^1 = C_i^2$ for $i = 1, \dots, m$. Then, the following conditions are equivalent:*

(i) Decision maker 1 is more prone to exclusion errors than Decision maker 2;

(ii) Decision maker 2 is more prone to inclusion errors than Decision maker 1;

(iii) For all $i = 1, \dots, m$, we have $\sigma_i^1(A) \leq \sigma_i^2(A)$ for all $A \subseteq C_i$.

This result claims, first, that the tendency to make exclusion errors versus inclusion ones are two faces of the same trait: If one agent is more inclined to make exclusion errors than the other, then, she must, at the same time, be less inclined to make inclusion errors. Second, in terms of the representation of their preferences, the fact that one agent is more prone to make inclusion errors (and less prone to make exclusion errors) than another is captured by the pointwise dominance relationship between their consistency maps that correspond to the agents' common coarse contingencies.

4 A More Structural Representation

While Theorem 1 captures the essential aspects of our theory, it leaves open the question about reasonable specifications of the evaluation functions V_i and consistency functions σ_i for contingencies $i \in \mathcal{S}$. In this section, we address this point by imposing a few additional axioms — Independence, Monotonicity, and one novel informational axiom — and deriving a more structural representation. It preserves the basic gist of Theorem 1 about subjective states and updating and, at the same time, provides a familiar expected utility-like structure for functions V_i and a related weight-based specification for σ_i.

4.1 Axioms

We start with recalling well-known properties that we impose on the ex ante preferences.

Axiom A13 (Monotonicity). *If $f, g \in \mathcal{F}$ and $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$.*

Axiom A14 (Independence). *If $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, then*

$$f \succsim g \quad \Leftrightarrow \quad \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h.$$

Axioms (A1), (A13), and (A14) are the classic assumptions of the subjective expected utility theory of Anscombe and Aumann (1963).

In addition to them, we introduce an axiom that provides a link between the decision maker's attitude towards betting on an event and her ability to evaluate acts conditional on that event.

Axiom A15 (Non-vacuousness of Valuable Bets). *Let $x, y \in \Delta(X)$ such that $x \succ y$, $A \in \mathscr{C}$, $B \subseteq \Omega$, and $P_A(\omega)$ holds for all $\omega \in B$. If $xBy \succsim xAy$, then $B \in \mathscr{C}$.*

This axiom has the following content: Suppose that events A and B are such that any state in B is revealed to be possible after event A has occurred. Intuitively, this suggests that A is a "larger" event, and, as an implication, A and B are comparable in their informational content. If, at the same time, a bet on B is weakly preferred to a bet on A, and A is a nondegenerate event (in the sense that $A \in \mathscr{C}$ and, hence, the decision maker can evaluate preferences conditional on A), then it must be that B is nondegenerate as well. As a side remark, note that, under the standard subjective expected utility theory with Bayesian updating, the above-postulated ranking of bets on events A and B, together with the assumption that A is nondegenerate (non-null), is sufficient to conclude that B is nondegenerate. In our model, this conclusion must hold only if A and B are comparable in their informational content.

4.2 The representation

The extended list of axioms leads to the following representation.

Theorem 6. *An ex ante preference relation $\succsim$, a collection $\{\succsim_A\}_{A\in\mathscr{C}}$, and $\mathscr{C}$ jointly satisfy Axioms (A1)–(A15) if and only if there exist*

- *a nonempty set of indices (subjective states) $\mathcal{S} = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ with $n \geq 3$;*
- *a collection $\Pi = \{C_1, \ldots, C_n\}$ of nonempty and mutually disjoint subsets of Ω;*
- *a nonconstant, continuous, and affine function $u : \Delta(X) \to \mathbb{R}$ with a compact range;*
- *a probability measure $p \in \Delta(\mathcal{S})$;*
- *a collection of functions $s_i : \Omega \to \mathbb{R}_+$ for $i \in \mathcal{S}$ such that C_i is the support of s_i and $\sum_{\omega\in\Omega} s_i(\omega) = 1$ for all $i \in \mathcal{S}$; and*
- *a collection of numbers $\alpha_i \in (0, 1]$ for $i \in \mathcal{S}$*

such that:

(i) The collection $\mathscr{C}$ can be computed as

$$\mathscr{C} = \left\{ E \subseteq \Omega : \sum_{\omega\in E} s_i(\omega) \geq \alpha_i \text{ for at least one } i \in \mathcal{S} \right\}; \tag{5}$$

(ii) $\succsim$ *is represented by*

$$V(f) = \sum_{i \in \mathcal{S}} \sum_{\omega \in \Omega} u(f(\omega))\, s_i(\omega)\, p_i \qquad \forall f \in \mathcal{F}; \tag{6}$$

(iii) For each $E \in \mathscr{C}$, $\succsim_E$ *is represented by*

$$V(f \mid E) = \frac{\sum_{i \in \mathcal{S}(E)} \sum_{\omega \in \Omega} u(f(\omega))\, s_i(\omega)\, p_i}{\sum_{i \in \mathcal{S}(E)} p_i} \qquad \forall f \in \mathcal{F}, \tag{7}$$

where $\mathcal{S}(E) := \{i \in \mathcal{S} : \sum_{\omega \in E} s_i(\omega) \geq \alpha_i\}$.

In comparison with the representation of Theorem 1, the decision maker here acts as if not only she thinks about contingencies that affect the resolution of uncertainty, but these contingencies also have attached *probabilities* that are captured by $p \in \Delta(S)$.

For the purpose of computing the decision maker's value that she derives from an act upon some contingency, the analyst can use a weighted average of her utilities from each of the states that he uses in his model. As an implication, the agent's ex ante preferences in this specification are compatible with the standard subjective expected utility model. Hence, the model of this section can be thought of as a minimal departure from the standard paradigm that can capture different views of uncertainty on behalf of the decision maker and the analyst, and the implications of that for processing information.

Finally, the important novelty in Theorem 6 is related to modeling the way the relevant contingencies are determined in updating. Instead of relying on arbitrary monotone set functions of Theorem 1, the procedure here consists of computing the total weight of the intersection of the event and the set of states representing a contingency, and then using a threshold strategy to determine whether or not this contingency is consistent with the event. Moreover, the weights used for these calculations are the same as the ones that are used for computing values of acts. As an interpretation, the decision maker captured by this procedure looks as if she is "overlooking" the low-weight states when incorporating arriving information in her decisions.

4.3 Uniqueness

Definition 9. We refer to a tuple $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, u, p, (s_i)_{i\in\mathcal{S}}, (\alpha_i)_{i\in\mathcal{S}})$ that satisfies the conditions of Theorem 6 as an *expected utility representation with subjective contingencies.*

The uniqueness properties of the objects of our expected utility representation with subjective contingencies are summarized by the following proposition.

Proposition 7. *Suppose that $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, u, p, (s_i)_{i\in\mathcal{S}}, (\alpha_i)_{i\in\mathcal{S}})$ is an expected utility representation with subjective contingencies of the system $(\succsim, \{\succsim_A\}_{A\in\mathscr{C}})$. Then, another tuple $(\mathcal{S}', (C'_i)_{i\in\mathcal{S}'}, u', p', (s'_i)_{i\in\mathcal{S}'}, (\alpha'_i)_{i\in\mathcal{S}'})$ represents the same system $(\succsim, \{\succsim_A\}_{A\in\mathscr{C}})$ as in (5)–(7) if and only if*

(i) there exist $k > 0$ and $b \in \mathbb{R}$ such that $u' = ku + b$;

(ii) there exists a bijection $\pi : \mathcal{S} \to \mathcal{S}'$ such that, for all $i \in \mathcal{S}$, $C'_{\pi_i} = C_i$, $s'_{\pi_i} = s_i$, $p'_{\pi_i} = p_i$;

(iii) $\alpha'_{\pi_i} \in (\alpha_i^{min}, \alpha_i^{max}]$, where

$$\begin{cases} \alpha_i^{min} := \max\left\{\sum_{\omega\in A} s_i(\omega) \mid A \subseteq C_i, A \notin \mathscr{C}\right\} \text{ and} \\ \alpha_i^{max} := \min\left\{\sum_{\omega\in A} s_i(\omega) \mid A \subseteq C_i, A \in \mathscr{C}\right\}. \end{cases} \tag{8}$$

In words, as usual, the utility function is unique up to a positive affine transformation, most objects related to subjective contingencies are unique up to renaming of the contingencies, and the only nontrivial part of the statement concerns the thresholds $(\alpha_i)_{i\in\mathcal{S}}$. These thresholds can be identified only up to an interval for a simple reason — our state space is finite, and, hence, the total weights $\sum_{\omega\in A} s_i(\omega)$ for different $A \subseteq \Omega$ form a discrete set. We also note that the thresholds α_i of the representation can be chosen within each interval $(\alpha_i^{min}, \alpha_i^{max}]$ arbitrarily and independently across different contingencies, and the boundaries of these intervals can be determined uniquely from the agent's choice behavior.[8]

As a final remark, we note that the limits on the identification of parameters $(\alpha_i)_{i\in\mathcal{S}}$ — up to an interval — does not preclude comparing these parameters across different agents

[8] To substantiate this claim, we give (without a proof) a behavioral definition of α^{min} and α^{max}. Suppose that $A \in \mathscr{C}$ is such that $\omega \in A \Leftrightarrow P_A(\omega)$, A does not have a proper subset with the same property, and $B \in \mathscr{C}$ is such that $A \cap B = \varnothing$. Then, A constitutes one of the agent's subjective contingencies,

(which will be given a behavioral meaning soon). Indeed, if two decision makers have the same ex ante preferences and equal understanding of uncertainty, then they may differ only in their thresholds $(\alpha_i)_{i\in\mathcal{S}}$, and the following will be true:

Observation 8. *Suppose that two decision makers are described by* $(\succsim, \{\succsim_A^k\}_{A\in\mathscr{C}^k}, \mathscr{C}^k)$ *for* $k = 1, 2$*, and their preferences admit expected utility representations with subjective contingencies* $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, u, p, (s_i)_{i\in\mathcal{S}}, (\alpha_i^k)_{i\in\mathcal{S}})$ *for* $k = 1, 2$*. Furthermore, suppose that* $(\alpha^{k\,min})_{i\in\mathcal{S}}$ *and* $(\alpha^{k\,max})_{i\in\mathcal{S}}$ *are defined by (8) for* $k = 1, 2$*. Then, for each* $i \in \mathcal{S}$*,*

$$(\alpha_i^{1\,min}, \alpha_i^{1\,max}] = (\alpha_i^{2\,min}, \alpha_i^{2\,max}] \text{ or } (\alpha_i^{1\,min}, \alpha_i^{1\,max}] \cap (\alpha_i^{2\,min}, \alpha_i^{2\,max}] = \varnothing.$$

The key implication of this observation is that if, for some $i \in \mathcal{S}$, the intervals for the thresholds α_i for these two agents are not identical, then one of the intervals must lie entirely to the left of the other one.

4.4 Comparative Statics

Since the expected utility representation with subjective contingencies is a special case of the general representation with subjective contingencies, the comparative notion of understanding of uncertainty and the corresponding characterization of Proposition 3 apply intactly to the model presented in Section 4.2. Therefore, we focus here on the behavioral properties of the new parameters introduced there — namely, $(\alpha_i)_{i\in\mathcal{S}}$.

Consider two decision makers described by $(\succsim^k, \{\succsim_A^k\}_{A\in\mathscr{C}^k}, \mathscr{C}^k)$ for $k = 1, 2$, with equal understanding of uncertainty and such that $\succsim^1 = \succsim^2$. In terms of the representation, this implies that $\mathcal{S}^1 = \mathcal{S}^2, \Pi^1 = \Pi^2, s_i^1 = s_i^2, p_i^1 = p_i^2$ for all i, and u^1 is a positive affine transformation of u^2. Hence, such decision makers differ only in the way they update their preferences. In the next proposition, we apply the comparative notion of proneness to

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and the corresponding threshold boundaries can be computed as

$$\begin{cases} \alpha_A^{min} := \max\left\{\frac{\text{CE}(x^*\,(E\cup B)\,x_*) - \text{CE}(x^*\,B\,x_*)}{\text{CE}(x^*\,A\,x_*)} \mid E \subseteq A, \succsim_{E\cup B} = \succsim_B\right\} \text{ and} \\ \alpha_A^{max} := \min\left\{\frac{\text{CE}(x^*\,(E\cup B)\,x_*) - \text{CE}(x^*\,B\,x_*)}{\text{CE}(x^*\,A\,x_*)} \mid E \subseteq A, \succsim_{E\cup B} \neq \succsim_B\right\}, \end{cases}$$

where $\text{CE} : \mathcal{F} \to [0,1]$ is the functional defined as $\text{CE}(f) = \gamma \Leftrightarrow f \sim \gamma x^* + (1-\gamma)x_*$.

inclusion and exclusion errors (Definition 7) to our expected utility model with subjective contingencies.

Proposition 9. *Suppose that two decision makers are described by* $(\succsim, \{\succsim_A^k\}_{A\in\mathscr{C}^k}, \mathscr{C}^k)$ *for* $k = 1, 2$*, and their preferences admit expected utility representations with subjective contingencies* $(\mathcal{S}, (C_i)_{i\in\mathcal{S}}, u, p, (s_i)_{i\in\mathcal{S}}, (\alpha_i^k)_{i\in\mathcal{S}})$ *for* $k = 1, 2$*. Then, the following conditions are equivalent:*

(i) Decision maker 1 is more prone to exclusion errors than Decision maker 2;

(ii) Decision maker 2 is more prone to inclusion errors than Decision maker 1;

(iii) For each $i \in \mathcal{S}$*,* $\alpha_i^{1\min} = \alpha_i^{2\min}$ *or* $\alpha_i^{1\min} \geq \alpha_i^{2\max}$*.*

As in Proposition 5, the tendency to make exclusion errors versus inclusion ones are two faces of the same trait. The novelty here lies in Condition (iii), in which the original characterization in terms of pointwise dominance of the consistency maps is replaced with a more structured form of an ordering of intervals on the real line: Two decision makers are comparable in terms of their predisposition to make errors in their conditional behavior whenever, for each contingency i, the interval of possible values of α_i for one agent weakly dominates the corresponding one of the other agent.

Appendix

Throughout the entire Appendix, we rely on the mapping $Q : \mathscr{C} \to 2^\Omega$ defined as $Q(E) := \{\omega \in \Omega : P_E(\omega) \text{ holds}\}$. This correspondence determines the states that are revealed to be possible after each admissible event.

A Properties of the possibility correspondence

Lemma 10. *Assume that axioms (A1)–(A12) hold. Then, for any* $A \in \mathscr{C}$*, we have* $Q(A) \in \mathscr{C}$*,* $\succsim_{Q(A)} = \succsim_A$*, and* $Q(Q(A)) = Q(A)$*.*

Proof. By using the Irrelevance of Impossible States axiom repeatedly, we obtain $A \cap Q(A) \in \mathscr{C}$ and $\succsim_{A \cup Q(A)} = \succsim_A$. Next, by Superficity of Possible States, $Q(A) \in \mathscr{C}$ and $\succsim_{Q(A)} = \succsim_A$. This, in turn, implies $Q(Q(A)) = Q(A)$ because the definition of $Q(E)$ depends only on $\succsim_E$ for any $E \in \mathscr{C}$. □

Lemma 11. *Assume that axioms (A1)–(A12) hold, and suppose that $A, B \subseteq \Omega$ are such that $Q(A) = A$ and $Q(B) = B$. Then, we have:*

(i) $Q(A \cup B) = A \cup B$;

(ii) If $B \backslash A \neq \varnothing$, *then* $B \backslash A \in \mathscr{C}$ *and* $Q(B \backslash A) = B \backslash A$;

(iii) If $A \cap B \neq \varnothing$, *then* $A \cap B \in \mathscr{C}$ *and* $Q(A \cap B) = A \cap B$.

Proof. Claim (i). By the Monotonicity of Possibility Correspondence axiom, $Q(A \cup B) \supseteq Q(A) \cup Q(B) = A \cup B$. To prove that $Q(A \cup B) \subseteq A \cup B$, take any $\omega \in \Omega$ such that $P_{A \cup B}(\omega)$ holds. Then, either $P_A(\omega)$ holds or, otherwise, $P_B(\omega)$ holds by the Understanding of Unions axiom. Overall, we have $\omega \in Q(A) \cup Q(B) = A \cup B$.

Claim (ii). Assume that $B \backslash A \neq \varnothing$. We note that $A \cup B = A \cup (B \backslash A)$ and apply the Understanding of Unions axiom to the sets A and $B \backslash A$. It gives $B \backslash A \in \mathscr{C}$ and $Q(A \cup B) \backslash Q(A) \subseteq Q(B \backslash A)$. Since $Q(A \cup B) = A \cup B$ by Claim (i), we have $B \backslash A \subseteq Q(B \backslash A)$. Next, we prove that $Q(B \backslash A) \subseteq B \backslash A$. By the Monotonicity of Possibility Correspondence axiom, $Q(B \backslash A) \subseteq B$. Assume, by contradiction, that there exists $\omega_0 \in \Omega$ such that $\omega_0 \in Q(B \backslash A) \cap Q(A)$. By the Intersection of Possibilities axiom applied to the sets A and $B \backslash A$, we obtain $\varnothing \in \mathscr{C}$, which contradicts to the Inadmissibility of Empty Events axiom. Therefore, we conclude that $Q(B \backslash A) \subseteq B \backslash Q(A) = B \backslash A$.

Claim (iii). Assume that $A \cap B \neq \varnothing$. If $A \cap B = A$ or $A \cap B = B$, then the claim holds trivially. Otherwise, let $C := B \backslash A$ and note that $C \neq \varnothing$. By Claim (ii), we have $C \in \mathscr{C}$ and $Q(C) = C$. Applying Claim (ii) to sets B and C, we have $B \backslash C \in \mathscr{C}$ and $Q(B \backslash C) = B \backslash C$. Since $B \backslash C = A \cap B$, Claim (iii) is proven. □

Lemma 12. *Assume that axioms (A1)–(A12) hold. For any $E \in \mathscr{C}$ and $f, g \in \mathcal{F}$, if $Q(E) \subseteq \{\omega \in \Omega : f(\omega) = g(\omega)\}$, then $f \sim_E g$.*

Proof. Fix an arbitrary $E \in \mathscr{C}$ and $f, g \in \mathcal{F}$ such that $Q(E) \subseteq \{\omega \in \Omega : f(\omega) = g(\omega)\}$. Let $\Omega = \{\omega_1, \dots, \omega_m\}$. For each $i = 0, \dots, m$, let $h^i \in \mathcal{F}$ be defined as $h^i = g\,\{\omega_{i+1}, \dots, \omega_m\}\,f$, and note that $h^0 = g$ and $h^m = f$. We claim that $h^{i-1} \sim_E h^i$ for all $i = 1, \dots, m$, which will prove the claim of the lemma by the transitivity of $\sim_E$.

Indeed $h^{i-1}(\omega_j) = h^i(\omega_j)$ for all $j \neq i$, while $h^{i-1}(\omega_i) = g(\omega_i)$ and $h^i(\omega_i) = f(\omega_i)$. If $g(\omega_i) = f(\omega_i)$, then $h^{i-1} = h^i$, and the claim is proven. Otherwise, we have $\neg P_E(\omega_i)$, and, therefore, $h^{i-1} \sim_E h^i$ by the definition of $\neg P_E(\omega_i)$. □

Lemma 13. *Assume that axioms (A1)–(A12) hold and suppose that $\{C_1, \dots, C_n\}$ for some $n \in \mathbb{N}$ is a partition of $Q(\Omega)$ such that $Q(C_i) = C_i$ for all $i = 1, \dots, n$. Then, for all $f, g \in \mathcal{F}$ such that $f \sim_{C_i} g$ for all $i = 1, \dots, n$, we have $f \sim g$.*

Proof. For each $i = 0, \dots, n$, let $h^i \in \mathcal{F}$ be defined as

$$h^i(\omega) := \begin{cases} g(\omega), & \text{if } \omega \in C_{i+1} \cup \dots C_n, \\ f(\omega), & \text{otherwise.} \end{cases}$$

We will prove by induction that $h^i \sim g$ for all $i = 0, \dots, n$, which will establish the claim of this lemma because $h^n = f$.

If $i = 0$, then $h^0 \sim g$ by Lemma 12 when Ω plays the role of E. Assume that $h^{i-1} \sim g$ for some $i = 1, \dots, n$. Then, observe that $h^{i-1} \sim_{C_i} g$ by Lemma 12, $g \sim_{C_i} f$ by the conditions of the lemma, and $f \sim_{C_i} h^i$ by Lemma 12. Therefore, by transitivity, we have $h^{i-1} \sim_{C_i} h^i$. Since that $h^{i-1}(\omega) = h^i(\omega)$ for all $\omega \notin C_i$, Dynamic Consistency implies that $h^{i-1} \sim h^i$, and, hence, $h^i \sim g$. □

B Proofs of the results of Section 3

Lemma 14. *Suppose that $\mathcal{S}$, Π, and collections $(V_i)_{i \in \mathcal{S}}$ and $(\sigma_i)_{i \in \mathcal{S}}$ are as described in Theorem 1. Furthermore, suppose that $\mathscr{C}$ is defined by (2) and binary relations $\succsim$ and $(\succsim_E)_{E \in \mathscr{C}}$ are defined by (3) and (4), respectively. Finally, fix an arbitrary $E \in \mathscr{C}$ and $\omega \in \Omega$. Then, the following conditions are equivalent:*

(i) *$P_E(\omega)$ holds;*

(ii) there exists $j \in \mathcal{S}$ such that $\omega \in C_j$ and $\sigma_j(E \cap C_j) = 1$, where C_j is a cell of the partition Π.

Proof. Suppose that $P_E(\omega)$ holds but there is no $j \in \mathcal{S}$ such that $\omega \in C_j$ and $\sigma_j(E \cap C_j) = 1$. Then, for any $f \in \mathcal{F}$ and $x, y \in \Delta(X)$,

$$\begin{aligned} V(x\{\omega\}f \mid E) &= \sum_{i \in \mathcal{S}:\sigma_i(E\cap C_i)=1} V_i(x\{\omega\}f) \\ &= \sum_{i \in \mathcal{S}:\sigma_i(E\cap C_i)=1} V_i(f) \qquad \text{(by the properties of } V_i\text{)} \\ &= V(y\{\omega\}f \mid E). \end{aligned}$$

Since $V(\cdot \mid E)$ represents $\succsim_E$, this contradicts to the fact that $P_E(\omega)$ holds.

Conversely, suppose that there exists $j \in \mathcal{S}$ such that $\omega \in C_j$ and $\sigma_j(E \cap C_j) = 1$. Next, due to the fact that C_j is the support of V_j, we can find $f \in \mathcal{F}$ and $x, y \in \Delta(X)$ such that $V_j(x\{\omega\}f) \neq V_j(y\{\omega\}f)$ and assume, without loss of generality, that $V_j(x\{\omega\}f) > V_j(y\{\omega\}f)$. Then,

$$\begin{aligned} V(x\{\omega\}f \mid E) &= \sum_{i \in \mathcal{S}:\sigma_i(E\cap C_i)=1} V_i(x\{\omega\}f) \\ &= V_j(x\{\omega\}f) + \sum_{i\in\mathcal{S}, i\neq j} V_i(x\{\omega\}f)\,\sigma_i(E\cap C_i) \\ &= V_j(x\{\omega\}f) + \sum_{i\in\mathcal{S}, i\neq j} V_i(f)\,\sigma_i(E\cap C_i) \\ &> V_j(y\{\omega\}f) + \sum_{i\in\mathcal{S}, i\neq j} V_i(f)\,\sigma_i(E\cap C_i) \\ &= V(y\{\omega\}f \mid E), \end{aligned}$$

and, therefore, $x\{\omega\}f \succ_E y\{\omega\}f$ because $V(\cdot \mid E)$ represents $\succsim_E$. □

Lemma 15 (Calculus)**.** *Suppose that $f : [0,1] \to \mathbb{R}$ is continuous, $f(0) \leq f(x) \leq f(1)$ for all $x \in [0,1]$, and a sequence $(y^m)_{m\in\mathbb{N}}$ in $f([0,1])$ converges to some y^0 as $m \to \infty$. Then, there exists a subsequence $(y^{m_k})_{k\in\mathbb{N}}$, a corresponding sequence $(x^k)_{k\in\mathbb{N}}$ in $[0,1]$ and $x^0 \in [0,1]$ such that $f(x^k) = y^{m_k}$ for all $k \in \mathbb{N}$, $f(x^0) = y^0$, and $x^k \to x^0$ as $k \to \infty$.*

Proof. Passing on a subsequence if necessary, we can assume that the sequence $(y^m)_{m\in\mathbb{N}}$ is monotone. Assume, without loss of generality, that it is decreasing. If $(y^m)_{m\in\mathbb{N}}$ is a constant sequence, then the claim holds trivially. Assume that it is not, and note that in this case $y^0 < f(1)$.

For each $k \in \mathbb{N}$, we define $a^k, b^k \in [0,1]$ inductively as follows: Let $a_1 := 0$, $b_1 := 1$. Then, for each $k \in \mathbb{N}$, let $c_k := \frac{1}{2}a^k + \frac{1}{2}b^k$, and let $a^{k+1} := a^k$ and $b^{k+1} := c^k$ if $f(c^k) > y^0$ and $a^{k+1} := c^k$ and $b^{k+1} := b^k$ if $f(c^k) \leq y^0$. Note that, for each $k \in \mathbb{N}$, we have $a^k < b^k$ and $f(a^k) \leq y^0 < f(b^k)$. Next, let $m_1 := 1$, and for each $k \in \mathbb{N}$, let $m_{k+1} > m_k$ be such that $y^{m_{k+1}} < f(b^k)$ (which can be done because $y^m \to y^0$ as $m \to \infty$). Finally, for each $k \in \mathbb{N}$, let x^k be such that $a^k \leq x^k < b^k$ and $f(x^k) = y^{m_k}$ (which can be done by the Intermediate Value Theorem because $f(a^k) \leq y^0 \leq y^{m_k} < f(b^k)$ for all $k \in \mathbb{N}$).

Since $[0,1]$ is compact and $([a^k, b^k])_{k\in\mathbb{N}}$ is a nested sequence of closed intervals of a length that goes to zero as $k \to \infty$, it has a nonempty intersection. Let $x^0 \in \bigcap_{k\in\mathbb{N}}[a^k, b^k]$. Clearly, $x^k \to x^0$ as $k \to \infty$. Since f is continuous, $f(x^0) = \lim_{k\to\infty} f(x^k) = \lim_{k\to\infty} y^{m_k} = y^0$, and the lemma is proven. □

Proof of Theorem 1. *Only if* part. *Step 1. The algebra of contingencies.*

Let $A, B \in \mathscr{C}$ be as given by the Nontriviality axiom. and let $A', B' \subset \Omega$ be defined as $A' := Q(A)$ and $B' := Q(B)$. As follows from Lemma 10, $A' \neq B'$, $A', B' \in \mathscr{C}$, $A' = Q(A')$, and $B' = Q(B')$.

Since $A' \neq B'$, it can be assumed without loss of generality that $B' \nsubseteq A'$. Next, we let $A_0, B_0 \in \mathscr{C}$ be defined as $A_0 = A'$ and $B_0 = B' \backslash A'$ and note that $A_0 \cap B_0 = \varnothing$. As follows from Lemma 11, $A_0 = Q(A_0)$, and $B_0 = Q(B_0)$.

Consider the collection

$$\mathcal{A} := \{Q(S) \cap Q(T) \mid S, T \subseteq \Omega, S \supseteq A_0, T \supseteq B_0\}.$$

We claim that this collection is a Boolean algebra — that is, for any $A_1, A_2 \in \mathcal{A}$, we have $A_1 \cup A_2 \in \mathcal{A}$ and $Q(\Omega) \backslash A_1 \in \mathcal{A}$. (These two properties also imply that $A_1 \cap A_2 \in \mathcal{A}$ for any $A_1, A_2 \in \mathcal{A}$.) To prove that, suppose that $S_1, S_2, T_1, T_2 \subseteq \Omega$ are such that $S_1 \supseteq A_0$, $S_2 \supseteq A_0$, $T_1 \supseteq B_0$, $T_2 \supseteq B_0$. First, let $\hat{S} := A_0 \cup \big(Q(S_1) \cap Q(T_1)\big) \cup \big(Q(S_2) \cap Q(T_2)\big)$ and $\hat{T} := B_0 \cup \big(Q(S_1) \cap Q(T_1)\big) \cup \big(Q(S_2) \cap Q(T_2)\big)$. As follows from Lemmas 11 and 10, $Q(\hat{S}) = \hat{S}$

and $Q(\hat{T}) = \hat{T}$ (and these equialities hold regardless of whether or not $Q(S_i) \cap Q(T_i) \neq \varnothing$ for $i = 1, 2$). Since $A_0 \cap B_0 = \varnothing$, we have $Q(\hat{S}) \cap Q(\hat{T}) = \big(Q(S_1) \cap Q(T_1)\big) \cup \big(Q(S_2) \cap Q(T_2)\big)$, which completes the proof that $\mathcal{A}$ is closed under taking unions. Second, let $\hat{S} := A_0 \cup \big(Q(\Omega) \backslash (Q(S_1) \cap Q(T_1))\big)$ and $\hat{T} := B_0 \cup \big(Q(\Omega) \backslash (Q(S_1) \cap Q(T_1))\big)$. By the same argument, $Q(\hat{S}) \cap Q(\hat{T}) = Q(\Omega) \backslash (Q(S_1) \cap Q(S_2))$, and the claim of this step is proven. Finally, note that, for any $C \in \mathcal{A}$ such that $C \neq \varnothing$, we have $C \in \mathscr{C}$ and $Q(C) = C$.

Step 2. The subjective state space. Since $\mathcal{A}$ is finite, we can let $\Pi = \{C_1, \ldots, C_n\}$ be the collection of the atoms of $\mathcal{A}$ — i.e., a collection of nonempty sets from $\mathcal{A}$ such that, for all $D \in \mathcal{A}$, we have either $C_i \cap D = \varnothing$ or $C_i \cap D = C_i$ for all $i = 1, \ldots, n$. Clearly, Π is a partition of $Q(\Omega)$, and, for any $D \in \mathcal{A}$, there exists $k \in \mathbb{N}$ and a collection of indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $D = \bigcup_{j=1}^{k} C_{i_j}$. Let $\mathcal{S} := \{1, \ldots, n\}$. Next we show that $n \geq 3$. The definition of $\mathcal{A}$ clearly ensures $A_0, B_0 \in \mathcal{A}$. Now, let $C \in \mathscr{C}$ be as given by the Nontriviality axiom and let $C' \subset \Omega$ be defined as $C' := Q(C)$. By Lemma 10, $C' \in \mathscr{C}$ and $C' = Q(C')$. Define $C_0 := C' \backslash A_0$ and note that $A_0 \cap C_0 = \varnothing$ and $B_0 \neq C_0$. As follows from Lemma 11, $C_0 \in \mathscr{C}$ and $C_0 = Q(C_0)$. Furthermore, by Lemma 11, again, $Q(A_0 \cup C_0) = A_0 \cup C_0$ and $Q(B_0 \cup C_0) = B_0 \cup C_0$. Hence, it follows that $A_0 \cup C_0, B_0 \cup C_0 \in \mathcal{A}$. Since $|\mathcal{A}| = 2^n$ and $\mathcal{A}$ is an algebra, we conclude that $n \geq 3$.

Step 3. Conditional utilities. The space $\mathcal{F}$ is connected and separable. Then, for each $i \in \mathcal{S}$, we can apply Debreu Theorem to get a utility function $U_i : \mathcal{F} \to \mathbb{R}$ that represents the preference relation $\succsim_{C_i}$. For each $i \in \mathcal{S}$, let $\Gamma_i := U_i(\mathcal{F})$ and note that Γ_i is a nondegenerate interval due to the Outcome Preference Consistency and Nontriviality axioms. Let $\Gamma := \prod_{i=1}^{n} \Gamma_i$.

Step 4. Additive separability. We define a binary relation $\trianglerighteq$ on Γ as $v \trianglerighteq w \Leftrightarrow f \succsim g$ for some $f, g \in \mathcal{F}$ such that $U_i(f) = v_i$ and $U_i(g) = w_i$ for all $i = 1, \ldots, n$. (Note that if $f \succsim g$ holds for some $f, g \in \mathcal{F}$ satisfying these conditions, then it holds for any such $f, g \in \mathcal{F}$ by Lemma 13.)

Now, we claim that $\trianglerighteq$ has the Coordinate Independence property (Wakker, 1989, Def. II.2.3): For any $v,w \in \Gamma$, $i \in \mathcal{S}$, $a,b \in \Gamma_i$,

$$(v_1,\ldots,v_{i-1},a,v_{i+1},\ldots,v_n) \trianglerighteq (w_1,\ldots,w_{i-1},a,w_{i+1},\ldots,w_n) \quad \Leftrightarrow \\ (v_1,\ldots,v_{i-1},b,v_{i+1},\ldots,v_n) \trianglerighteq (w_1,\ldots,w_{i-1},b,w_{i+1},\ldots,w_n).$$

Indeed, fix arbitrary $v,w \in \Gamma$, $i \in \mathcal{S}$, $a,b \in \Gamma_i$, and let $f,g \in \mathcal{F}$ be such that $U_j(f) = v_j$ and $U_j(g) = w_j$ for all $j = 1,\ldots,n$, $j \neq i$. For any $\zeta \in \Gamma_i$, let $z \in \Delta(X)$ be such that $U_i(z) = \zeta$, and observe that

$$\begin{aligned}
&(v_1,\ldots,v_{i-1},\zeta,v_{i+1},\ldots,v_n) \trianglerighteq (w_1,\ldots,w_{i-1},\zeta,w_{i+1},\ldots,w_n) \Leftrightarrow \\
&\quad z\,C_i\,f \succsim z\,C_i\,g \qquad \Leftrightarrow \quad \text{(by Dynamic Consistency)} \\
&\quad z\,C_i\,f \succsim_{Q(\Omega)\setminus C_i} z\,C_i\,g \quad \Leftrightarrow \quad \text{(by Lemma 12)} \\
&\qquad f \succsim_{Q(\Omega)\setminus C_i} g.
\end{aligned}$$

Note that the latter relationship does not depend on the value of ζ and, therefore, Coordinate Independence is proven.

Next, we prove that $\trianglerighteq$ is continuous — i.e., for any $w \in \Gamma$, the sets $\{v \in \Gamma : v \trianglerighteq w\}$ and $\{v \in \Gamma : w \trianglerighteq v\}$ are closed. Fix an arbitrary $w \in \Gamma$, and suppose that $v^0 \in \Gamma$ and a sequence $(v^m)_{m\in\mathbb{N}}$ in Γ are such that $v^m \trianglerighteq w$ for all $m \in \mathbb{N}$ and $v^m \to v^0$ as $m \to \infty$. Our objective is to find a subsequence $(v^{m_k})_{k\in\mathbb{N}}$, $f^0 \in \mathcal{F}$, and a sequence of acts $(f^k)_{k\in\mathbb{N}}$ such that, for all $i = 1,\ldots,n$, we have $U_i(f^k) = v_i^{m_k}$ for all $k \in \mathbb{N}$ and $U_i(f^0) = v_i^0$, and such that $f^k \to f^0$ as $k \to \infty$.

We do this by induction: Our claim is that, for all $j = 0,1,\ldots,n$, there exist a subsequence $(v^{m_k^j})_{k\in\mathbb{N}}$, $f_j^0 \in \mathcal{F}$, and a sequence of acts $(f_j^k)_{k\in\mathbb{N}}$ such that, for all $i = 1,\ldots,j$, we have $U_i(f_j^k) = v_i^{m_k^j}$ for all $k \in \mathbb{N}$ and $U_i(f_j^0) = v_i^0$, and such that $f_j^k \to f_j^0$ as $k \to \infty$. For $j = 0$, the claim holds trivially if we choose $m_k^0 = k$ and $f_j^k = f_j^0 = x$ for all $k \in \mathbb{N}$, where $x \in \Delta(X)$ is arbitrary. Assume that the claim holds for $j-1$. Then, we show that it also holds for j. Let $\varphi : [0,1] \to U_j(\mathcal{F})$ be defined as $\varphi(\alpha) = U_j(\alpha x^* + (1-\alpha)x_*)$, and note that it is continuous and surjective by the construction of U_j and the Best and Worst Outcomes axiom. Then, we apply Lemma 15 to the sequence $(v^{m_k^{j-1}})_{k\in\mathbb{N}}$ to obtain a subsequence $(v^{m_k^j})_{k\in\mathbb{N}}$, $\alpha_j^0 \in [0,1]$ and a sequence α_j^k in $[0,1]$ such that $U_j(\alpha_j^k x^* + (1-\alpha_j^k)x_*) = v_j^{m_k^j}$ for all $k \in \mathbb{N}$ and $U_j(\alpha_j^0 x^* + (1-\alpha_j^0)x_*) = v_j^0$, and such that

$\alpha_j^k \to \alpha_j^0$ as $k \to \infty$. Next, we let $f_j^k := (\alpha_j^k x^* + (1-\alpha_j^k)x_*)\, C_j\, f_{j-1}^k$ for all $k \in \mathbb{N} \cup \{0\}$. Observe that $U_j(f_j^k) = v_j^{m_k^j}$ by for all $k \in \mathbb{N}$ and $U_j(f_j^0) = v_j^0$ because U_j is a utility representation of $\succsim_{C_j}$, and $f_j^k \sim_{C_j} (\alpha_j^k x^* + (1-\alpha_j^k)x_*)$ for all $k \in \mathbb{N} \cup \{0\}$ by Lemma 12. Similarly, for all $i = 1, \ldots, j-1$, $U_i(f_j^k) = v_i^{m_k^j}$ for all $k \in \mathbb{N}$ and $U_j(f_i^0) = v_i^0$ by the assumption of the inductive step and Lemma 12. Also, $f_j^k \to f_j^0$ as $k \to \infty$ because the topology on $\mathcal{F}$ is the product one. The claim of the inductive step is proven.

Now, suppose that $g \in \mathcal{F}$ is such that $U_i(g) = w_i$ for all $i = 1, \ldots, n$. Then, $v^k \trianglerighteq w \Rightarrow f_n^k \succsim g$ by the construction of $\trianglerighteq$, which, by Continuity, implies that $f^0 \succsim g$, and, in turn, $v^0 \trianglerighteq w$. We conclude that the set $\{v \in \Gamma : v \trianglerighteq w\}$ is closed. Similarly, it can be shown that the set $\{v \in \Gamma : w \trianglerighteq v\}$ is closed, as well.

Finally, we claim that all coordinates in Γ are essential: Indeed, fix arbitrary $x, y \in \Delta(X)$ such that $x \succ y$. By Outcome Preference Consistency, we have $x \succ_{C_i} y$ and $U_i(x) > U_i(y)$ for all $i = 1, \ldots, n$, Furthermore, for all $i = 1, \ldots, n$, $x\, C_i\, y \sim_{C_i} x$ by Lemma 12, and, in turn, $x\, C_i\, y \succ_{C_i} y$ by transitivity, and, hence, $x\, C_i\, y \succ y$ by Dynamic Consistency. Then, $(U_1(y), \ldots, U_{i-1}(y), U_i(x), U_{i+1}(y), \ldots, U_n(y)) \triangleright (U_1(y), U_2(y), \ldots, U_n(y))$ and, therefore, coordinate i is essential for all $i = 1, \ldots, n$.

Having established all the above-listed properties of $\trianglerighteq$, we can conclude by Wakker (1989, Th. III.4.1) that there exist continuous functions $W_i : \Gamma_i \to \mathbb{R}$ for $i = 1, \ldots, n$ such that, for all $v, w \in \Gamma$, $v \trianglerighteq w \Leftrightarrow \sum_{i=1}^n W_i(v_i) \geq \sum_{i=1}^n W_i(w_i)$.

Step 5. The ex-ante representation. For any $f, g \in \mathcal{F}$, we let $v := (U_1(f), \ldots, U_n(f))$, $w := (U_1(g), \ldots, U_n(g))$, and observe that

$$f \succsim g \quad \Leftrightarrow \quad v \trianglerighteq w \quad \Leftrightarrow \quad \sum_{i=1}^n W_i(U_i(f)) \geq \sum_{i=1}^n W_i(U_i(g)).$$

Let $V_i := W_i \circ U_i$ for all $i \in \mathcal{S}$. For each i, the function V_i is continuous (as a composition of continuous functions), is nonconstant, and has compact range by the Best and Worst Outcomes axiom. Finally, we prove that C_i is the support of V_i for any $i = 1, \ldots, n$: Indeed, for any $f, g \in \mathcal{F}$ such that $f|_{C_i} = g|_{C_i}$, we have $f \sim_{C_i} g$ by Lemma 12; therefore, $U_i(f) = U_i(g)$ and, in turn, $V_i(f) = V_i(g)$. Moreover, for each $\omega \in C_i$, we have $P_{C_i}(\omega)$ by construction, and, therefore, there exist $f \in \mathcal{F}$ and $x, y \in \Delta(X)$ such that $x\,\{\omega\}\,f \succ_{C_i} y\,\{\omega\}f$, which implies that $V(x\,\{\omega\}\,f) > V(y\,\{\omega\}f)$.

Step 6. The conditional representation on $\mathcal{A}$. For all $E \in \mathcal{A}\backslash\{\varnothing\}$, define

$$V(f \mid E) \coloneqq \sum_{i\in\mathcal{S}:C_i\subseteq E} V_i(f) \quad \forall f \in \mathcal{F}.$$

We claim that, for each $E \in \mathcal{A}\backslash\{\varnothing\}$, $\succsim_E$ is represented by $V(\cdot \mid E)$. Indeed, fix an arbitrary $h \in \mathcal{F}$, and observe that, for any $f, g \in \mathcal{F}$ and $E \in \mathcal{A}\backslash\{\varnothing\}$, we have

$$\begin{aligned}
&f \succsim_E g && \Leftrightarrow \text{ (by Lemma 12)}\\
&\quad fEh \succsim_E gEh && \Leftrightarrow \text{ (by Dynamic Consistency)}\\
&\quad fEh \succsim gEh && \Leftrightarrow \text{ (by the ex-ante representation)}\\
&\quad \textstyle\sum_{i\in\mathcal{S}} V_i(fEh) \geq \sum_{i\in\mathcal{S}} V_i(gEh) && \Leftrightarrow \text{ (by properties of } \mathcal{A} \text{ and } V_i)\\
&\qquad \textstyle\sum_{i\in\mathcal{S}:C_i\subseteq E} V_i(f) \geq \sum_{i\in\mathcal{S}:C_i\subseteq E} V_i(g).
\end{aligned}$$

Step 7. Elements of the algebra $\mathcal{A}$. We claim that, for any $E \in \mathscr{C}$, $Q(E) \in \mathcal{A}$. Let $S, T \subseteq \Omega$ be defined as $S \coloneqq Q(E) \cup A_0$ and $T \coloneqq Q(E) \cup B_0$. Observe that $Q(S) = Q(Q(E)) \cup Q(A_0)$ by Lemma 14, and, hence, is equal to $Q(E) \cup A_0$ by Lemma 10; similarly, $Q(T) = Q(E) \cup B_0$. Since $A_0 \cap B_0 = \varnothing$, we have $Q(S) \cap Q(T) = Q(E)$, and, therefore, $Q(E) \in \mathcal{A}$ by definition.

Step 8. The functions σ_i. For each $i \in \mathcal{S}$, let $\sigma_i: 2^{C_i} \to \{0,1\}$ be a function defined by

$$\sigma_i(E) = 1 \quad \Leftrightarrow \quad C_i \subseteq Q(A_0 \cup E) \cap Q(B_0 \cup E),$$

where $A_0, B_0 \in \mathscr{C}$ are the sets defined in Step 1. We show that, for each $i \in \mathcal{S}$, the function σ_i satisfies all properties stated in the theorem.

First, if $E = \varnothing$, then $Q(A_0) \cap Q(B_0) = \varnothing \not\supseteq C_i$ and $\sigma_i(E) = 0$ for any $i \in \mathcal{S}$ Second, if $E = C_i$ for some $i \in \mathcal{S}$, then $Q(A_0 \cup C_i) \cap Q(B_0 \cup C_i) = (A_0 \cup C_i) \cap (B_0 \cup C_i) = C_i$ and $\sigma_i(E) = 1$. Third, the mapping $Q : \mathscr{C} \to 2^\Omega$ is monotone (inclusion-wise) by Monotonicity of Possibility Correspondence. Therefore, the mapping $\tilde{\sigma}_i : 2^{C_i} \to 2^\Omega$ defined as $\tilde{\sigma}_i(E) = Q(A_0 \cup E) \cap Q(B_0 \cup E)$, where $i \in \mathcal{S}$, is monotone for all $i \in \mathcal{S}$, which implies that σ_i is monotone for all $i \in \mathcal{S}$, as well.

Step 9. The relationship between σ_i and conditional preferences. We claim that, for any $i \in \mathcal{S}$ and any $E \in \mathscr{C}$, $C_i \subseteq Q(E)$ implies $\sigma_i(E \cap C_i) = 1$; and, conversely, for any $E \subseteq \Omega$, if there exists $i \in \mathcal{S}$ such that $\sigma_i(E \cap C_i) = 1$, then $E \in \mathscr{C}$ and, moreover, $C_i \subseteq Q(E)$.

Indeed, if, for some $i \in \mathcal{S}$ and $E \in \mathscr{C}$, we have $C_i \subseteq Q(E)$, then $E \cap C_i \in \mathscr{C}$ by Intersection of Possibilities. Therefore, $Q(E \cap C_i) \in \mathscr{C}$ by Lemma 10 and $Q(E \cap C_i) \neq \varnothing$ by Inadmissibility of Empty Events. At the same time, $Q(E \cap C_i) \subseteq Q(C_i) = C_i$ by the monotonicity of Q. Then, since $Q(E \cap C_i) \in \mathcal{A}$ by Step 7 and C_i is an atom of $\mathcal{A}$ (see Step 2), it must be that $Q(E \cap C_i) = C_i$. The monotonicity of Q, then, implies that $\sigma_i(E \cap C_i) = 1$.

Conversely, suppose that $E \subseteq \Omega$ and there exists $i \in \mathcal{S}$ such that $\sigma_i(E \cap C_i) = 1$. We have $A_0, B_0 \in \mathcal{A}$; C_i is an atom of $\mathcal{A}$; and $A_0 \cap B_0 = \varnothing$; therefore, it must be that $C_i \notin A_0$ or $C_i \notin B_0$ (or both). Assume without loss of generality that $C_i \notin A_0$. Since $\sigma_i(E \cap C_i) = 1$, then, for any $\omega \in C_i$, we have $\omega \in Q(A_0 \cup (E \cap C_i))$ and, in turn, $\omega \in Q(E \cap C_i)$ by Understanding of Unions. Therefore, $C_i \subseteq Q(E \cap C_i)$, and $C_i \subseteq Q(E)$ follows by the monotonicity of Q.

Step 10. The conditional representation. For any $f, g \in \mathcal{F}$ and $E \in \mathscr{C}$, we have

$$
\begin{aligned}
& f \succsim_E g && \Leftrightarrow \quad \text{(by Lemma 10)} \\
& \quad f \succsim_{Q(E)} g && \Leftrightarrow \quad \text{(by Steps 7 and 6)} \\
& \quad \textstyle\sum_{i \in \mathcal{S}: C_i \subseteq Q(E)} V_i(f) \geq \sum_{i \in \mathcal{S}: C_i \subseteq Q(E)} V_i(g) && \Leftrightarrow \quad \text{(by Step 9)} \\
& \qquad \textstyle\sum_{i \in \mathcal{S}: \sigma_i(E \cap C_i) = 1} V_i(f) \geq \sum_{i \in \mathcal{S}: \sigma_i(E \cap C_i) = 1} V_i(g),
\end{aligned}
$$

and Representation (4) is proven.

Step 11. Events in $\mathscr{C}$. We show that $\mathscr{C} = \{E \subseteq \Omega : \exists_{i \in \mathcal{S}} \, \sigma_i(E \cap C_i) = 1\}$. Indeed, as proven in Step 9, if $E \subseteq \Omega$ and $i \in \mathcal{S}$ are such that $\sigma_i(E \cap C_i) = 1$, then $E \in \mathscr{C}$. Conversely, suppose, by contradiction, that $E \in \mathscr{C}$ and $\sigma_i(E \cap C_i) = 0$ for all $i \in \mathcal{S}$. Then, by Step 10, the function $V(\cdot \mid E) \equiv 0$ is a utility representation of $\succsim_E$. As follows from this representation, $P_E(\omega)$ does not hold for any $\omega \in \Omega$. However, since $Q(E) \in \mathscr{C}$ by Lemma 10, we have $Q(E) \neq \varnothing$ by Inadmissibility of Empty Events, a contradiction. This completes the proof that our representation follows from the axioms.

If part. Assume that there exist a set $\mathcal{S} = \{1, \ldots, n\}$, a collection $\Pi = \{C_1, \ldots, C_n\}$ of subsets of Ω, a collection of functions $V_i : \mathcal{F} \to \mathbb{R}$, and a collection of functions $\sigma_i : 2^{C_i} \to \{0, 1\}$ for $i \in \mathcal{S}$ as described in Theorem 1 and such that statements (i)–(iv) hold.

Weak Order. This axiom follows from the existence of representations in (ii) and (iii).

Continuity. This follows easily from Representation (3) and the continuity of V_i for $i \in \mathcal{S}$.

Best and Worst Outcomes. Since the function V_1 has a compact range, there exist $x^*, x_* \in X$ such that $V_1(x^*) \geq V_1(f) \geq V_1(x_*)$ for all $f \in \mathcal{F}$. By Condition (iv), we have $V_i(x^*) \geq V_i(f) \geq V_i(x_*)$ for all $f \in \mathcal{F}$ and all $i \in \mathcal{S}$ and, therefore, $x^* \succsim f \succsim x_*$ for all $f \in \mathcal{F}$ by Representation (3).

Nontriviality. Since $n \geq 3$, we can consider the sets C_1, C_2, C_3 that belong to $\mathscr{C}$ by (2). It can also be easily verified that $\succsim, \succsim_{C_1}, \succsim_{C_2}, \succsim_{C_3}$ by using the representations (3)–(4) and checking the rankings of acts $x^* \, C_1 \, x_*$, $x^* \, C_2 \, x_*$, $x^* \, C_3 \, x_*$.

Outcome Preference Consistency. Fix an arbitrary $E \in \mathscr{C}$, and suppose that $x, y \in \Delta(X)$ are such that $x \succsim y$. Then, $\sum_{i \in \mathcal{S}} V_i(x) \geq \sum_{i \in \mathcal{S}} V_i(y)$ implies that there exists $j \in \mathcal{S}$ such that $V_j(x) \geq V_j(y)$. By (iv), we have that $V_i(x) \geq V_i(y)$ for all $i \in \mathcal{S}$. Therefore, $\sum_{i \in \mathcal{S}: \sigma_i(E \cap C_i)=1} V_i(x) \geq \sum_{i \in \mathcal{S}: \sigma_i(E \cap C_i)=1} V_i(y)$ and $x \succsim_E y$. Conversely, suppose that $x, y \in \Delta(X)$ are such that $x \succsim_A y$. Since $E \in \mathscr{C}$, there exists at least one $j \in \mathcal{S}$ such that $\sigma_j(E \cap C_j) = 1$. Then, there must be some $j \in \mathcal{S}$ such that $V_j(x) \geq V_j(y)$. By (iv), we have that $V_i(x) \geq V_i(y)$ for all $i \in \mathcal{S}$, and $\sum_{i \in \mathcal{S}} V_i(x) \geq \sum_{i \in \mathcal{S}} V_i(y)$, which means that $x \succsim y$.

Dynamic Consistency. Suppose that $E \in \mathscr{C}$ and $f, g \in \mathcal{F}$ such that $f(\omega) = g(\omega)$ for all $\omega \in \Omega$ such that $\neg P_E(\omega)$. We claim that $V_i(f) = V_i(g)$ for all $i \in \mathcal{S}$ such that $\sigma_i(E \cap C_i) = 0$: Indeed, fix any such $i \in \mathcal{S}$; for any $\omega \in C_i$, we have $\neg P_E(\omega)$ due to Lemma 14 and the fact that Π is a partition, and, hence, $f(\omega) = g(\omega)$; since C_i is the support of V_i, we obtain $V_i(f) = V_i(g)$. Therefore, $\sum_{i \in \mathcal{S}} V_i(f) \geq \sum_{i \in \mathcal{S}} V_i(g) \Leftrightarrow \sum_{i \in \mathcal{S}: \sigma_i(E \cap C_i)=1} V_i(f) \geq \sum_{i \in \mathcal{S}: \sigma_i(E \cap C_i)=1} V_i(g)$.

Monotonicity of the Possibility Correspondence. Suppose that $A \in \mathscr{C}$ and $B \in \Sigma$ are such that $B \supseteq A$, and $\omega \in \Omega$ such that $P_A(\omega)$ holds. By Lemma 14, there exists $j \in \mathcal{S}$ such that $\omega \in C_j$ and $\sigma_j(A \cap C_j) = 1$. By the monotonicity of σ_j, $B \cap C_j \supseteq A \cap C_j$ implies $\sigma_j(B \cap C_j) \geq \sigma_j(A \cap C_j)$. Thus, $B \in \mathscr{C}$ by (2) and $P_B(\omega)$ holds by Lemma 14.

Irrelevance of Impossible States. Suppose that $A \in \mathscr{C}$ and $\omega \in \Omega$ are such that $\neg P_A(\omega)$. Then, for any $i \in \mathcal{S}$, $\sigma_i(A \cap C_i) = 0$ implies $\sigma((A \backslash \{\omega\}) \cap C_i) = 0$ by the monotonicity of σ_i. If $\sigma_i(A \cap C_i) = 1$ for some $i \in \mathcal{S}$, then it must be that $\sigma_i((A \backslash \{\omega\}) \cap C_i) = 1$: Indeed,

$A \cap C_i \neq (A \backslash \{\omega\}) \cap C_i$ can occur only if $\omega \in A \cap C_i$, and in this case Lemma 14 implies $\sigma_i(A \cap C_i) = 0$ since $\neg P_A(\omega)$. We conclude that $\succsim_{A \backslash \{\omega\}} = \succsim_A$.

Superficity of Possible States. Suppose that $A \in \mathscr{C}$ and $\omega \in \Omega$ are such that $P_A(\omega)$. Then, by Lemma 14, there exists $j \in \mathcal{S}$ such that $\omega \in C_j$ and $\sigma_j(A \cap C_j) = 1$. Hence, by the monotonicity of σ_j, we have $\sigma_j((A \cup \{\omega\}) \cap C_j) = 1$. For all $i \neq j$, $(A \cup \{\omega\}) \cap C_i = A \cap C_i$, which implies $\sigma_i((A \cup \{\omega\}) \cap C_i) = \sigma_i(A \cap C_i)$. We conclude that $\succsim_{A \cup \{\omega\}} = \succsim_A$.

Intersection of Possibilities. Suppose that $A, B \in \mathscr{C}$, $P_A(\omega)$ implies $\omega \in A$ for any $\omega \in \Omega$, and there exists $\omega_0 \in \Omega$ such that $P_A(\omega_0)$ and $P_B(\omega_0)$ hold. Then, by Lemma 14, there exists $j \in \mathcal{S}$ such that $\omega_0 \in C_j$ and $\sigma_j(A \cap C_j) = 1$. Since Π is a partition, it follows from Lemma 14 that $\sigma_j(B \cap C_j) = 1$, as well. Moreover, as follows from the same Lemma, we have $P_A(\omega)$ for all $\omega \in C_j$, which, by the properties of A, means that $C_j \subseteq A$. Then, $A \cap B \cap C_j = B \cap C_j$, $A \cap B \in \mathscr{C}$ by (2), and $P_{A \cap B}(\omega_0)$ by Lemma 14.

Understanding of Unions. Suppose that $A \in \mathscr{C}$, $B \in \Sigma$, $P_A(\omega)$ holds for all $\omega \in A$, and suppose that there exists $\omega_0 \in \Omega$ such that $P_{A \cup B}(\omega_0)$ but not $P_A(\omega_0)$. Then, by Lemma 14, there exists $j \in \mathcal{S}$ such that $\omega_0 \in C_j$, $\sigma_j((A \cup B) \cap C_j) = 1$ but $\sigma_j(A \cap C_j) = 0$. We claim that $A \cap C_j = \varnothing$: Indeed, if there were some $\omega_1 \in A \cap C_j$, then we would have $P_A(\omega_1)$ by the property of the set A. Since Π is a partition and $\omega_1 \in C_j$, then Lemma 14 would imply $\sigma_j(A \cap C_j) = 1$, a contradiction. Therefore, $(A \cup B) \cap C_j = B \cap C_j$. Then, we have $B \in \mathscr{C}$ by (2) and $P_B(\omega_0)$ by Lemma 14.

Inadmissibility of Empty Events. The fact that $\varnothing \notin \mathscr{C}$ is guaranteed by Equation (2) and the property that $\sigma_i(\varnothing) = 0$ for all $i \in \mathcal{S}$. □

Lemma 16. *Suppose that two systems of binary relations — $(\succsim, \{\succsim_A\}_{A \in \mathscr{C}})$ and $(\succsim'$, $\{\succsim'_A\}_{A \in \mathscr{C}'})$ — satisfy Axioms (A1)–(A12) and are such that $\succsim = \succsim'$ and $\succsim_A = \succsim'_A$ for all $A \in \mathscr{C} \cap \mathscr{C}'$. Then, $\mathscr{C} = \mathscr{C}'$.*

Proof. Suppose, by contradiction, that there exists $A \in \mathscr{C}'$ such that $A \notin \mathscr{C}$.

For any $E \in \mathscr{C}'$, we denote by $P'_E(\omega)$ the statement that ω is not $\succsim'_E$-null and let $Q' : \mathscr{C}' \to 2^\Omega$ be defined as $Q'(E) := \{\omega \in \Omega : P_E(\omega) \text{ holds}\}$. Note that, due to the assumptions of the lemma, $Q(E) = Q'(E)$ for all $E \in \mathscr{C} \cap \mathscr{C}'$.

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Let $B \coloneqq Q'(A)$, and note that, by Inadmissibility of Empty Events, $B \neq \varnothing$. Applying Irrelevance of Impossible States and Superficity of Possible States repeatedly as needed, it follows that $B \in \mathscr{C}' \backslash \mathscr{C}$. Let $\{C_1, \ldots, C_n\}$ be the partition from the representation with subjective contingencies of $(\succsim, (\succsim_A)_{A \in \mathscr{C}})$. By Monotonicity of the Possibility Correspondence, we have $B \subseteq Q(\Omega)$, while $Q(\Omega) = C_1 \cup \cdots \cup C_n$. Therefore, $B \cap C_i \neq \varnothing$ for at least one $i = 1, \ldots, n$. Assume without loss of generality that $B \cap C_1 \neq \varnothing$, and let $E \coloneqq B \cup C_2$. By Monotonicity of the Possibility Correspondence, we have $E \in \mathscr{C} \cap \mathscr{C}'$ and $E \subseteq Q(E) = Q'(E)$. Since $Q(E) \cap C_1 \supseteq E \cap C_1 \neq \varnothing$, we have $C_1 \subseteq Q(E)$, as follows from Lemma 14. Then, by the Understanding of Unions axiom applied to sets C_2 and B, it must be that $B \in \mathscr{C}$, a contradiction. □

Proof of Proposition 2. The sufficiency of Conditions (i)–(iii) can be easily verified. We will prove the necessity. Let $(\mathcal{S}, (C_i)_{i \in \mathcal{S}}, (V_i)_{i \in \mathcal{S}}, (\sigma_i)_{i \in \mathcal{S}})$ and $(\mathcal{S}', (C'_i)_{i \in \mathcal{S}'}, (V'_i)_{i \in \mathcal{S}'}, (\sigma'_i)_{i \in \mathcal{S}'})$ be two representations with subjective contingencies of the same system $(\succsim, \{\succsim_A\}_{A \in \mathscr{C}})$ of binary relations.

Let $\mathcal{A} \coloneqq \{E \in \mathcal{S} : E = Q(E)\} \cup \{\varnothing\}$ (which is equivalent to the definition in Step 1 of the proof of Theorem 1), and note that $\mathcal{A}$ is an algebra by Lemma 11. As follows from Lemma 11, both $(C_i)_{i \in \mathcal{S}}$ and $(C'_i)_{i \in \mathcal{S}'}$ constitute the collections of atoms of $\mathcal{A}$. Therefore, there must exist a bijection $\pi : \mathcal{S} \to \mathcal{S}'$ such that $C'_{\pi_i} = C_i$ for all $i \in \mathcal{S}$.

Condition (ii) holds directly by Wakker (1989, Obs. III.6.6′).

Now, for any $i \in \mathcal{S}$ and any $A \subseteq \Omega$, we have that

$$\sigma_i(A \cap C_i) = 1 \Leftrightarrow C_i \subseteq Q(A) \Leftrightarrow C'_{\pi_i} \subseteq Q(A) \Leftrightarrow \sigma'_{\pi_i}(A \cap C_i) = 1$$

by Lemma 14. Thus, Condition (iii) holds, as well. □

Proof of Proposition 3. Suppose, first, that the decision maker fully understands an event $E \subseteq \Omega$. Consider an arbitrary $\omega \in E$. If ω is $\succsim$-null, then $\{\omega\} \in \mathcal{N}(\succsim)$. If ω is not $\succsim$-null, then E is not $\succsim$-null and, by the definition of full understanding, it implies that $E \in \mathscr{C}$. Moreover, by that definition, we have $P_E(\omega)$. By Lemma 14, there exists $i \in \mathcal{S}$ such that $\omega \in C_i$ and $\sigma_i(E \cap C_i) = 1$. By the same lemma, for any $\omega' \in C_i$, we have $P_E(\omega')$ and, in turn, $\omega' \in E$. Therefore, $C_i \subseteq E$. This establishes the *only if* part of the proposition.

Conversely, suppose that $E \subseteq \Omega$ is such that it is equal to a union of some elements of $\Pi \cup \mathcal{N}(\succsim)$, and assume that E is not $\succsim$-null. Our goal is to prove that $E \in \mathscr{C}$ and, for any $\omega \in \Omega$, $P_E(\omega)$ holds iff $P_\Omega(\omega)$ and $\omega \in E$.

Indeed, if E is not $\succsim$-null, then it must be that $C_i \subseteq E$ for some $i \in \mathcal{S}$. Then, $\sigma_i(E \cup C_i) = \sigma_i(C_i) = 1$, and, in turn, $E \in \mathscr{C}$ by (2).

Now, fix an arbitrary $\omega \in E$. If $P_E(\omega)$ holds, then $P_\Omega(\omega)$ holds by Dynamic Consistency and, by Lemma 14, it must be that $\omega \in C_i$ for some $i \in \mathcal{S}$ such that $\sigma_i(C_i \cap E) = 1$. Then, by the properties of σ, we have $C_i \subseteq E$ and, hence, $\omega \in E$. If $P_E(\omega)$ does not hold, then, as follows from Lemma 14, $\omega \notin C_i$ for any $i \in \mathcal{S}$ such that $C_i \subseteq E$. Therefore, either $\omega \notin E$ or $\omega \in E \cap \mathcal{N}(\succsim)$, in which case ω is $\succsim$-null and $P_\Omega(\omega)$ does not hold. □

Proof of Proposition 5. We prove the proposition by showing that (i) ⇔ (iii) and (ii) ⇔ (iii).

(iii) ⇒ *(i)* Suppose that $\sigma_i^1(A) \leq \sigma_i^2(A)$ for all $A \subseteq C_i$ and all $i = \{1, \ldots, m\}$, fix an arbitrary $E \in \mathscr{C}^1 \cap \mathscr{C}^2$, and let $A := M_-^2(E)$, noting that $A \subseteq E$. Let $\mathcal{I} := \{i \in \{1, \ldots, m\} : A \cap C_i \neq \varnothing\}$. As follows from Lemma 14, we have $\sigma^2(E \cap C_i) = 0$ for all $i \in \mathcal{I}$. Then, $\sigma^1(A \cap C_i) = 0$ and, hence, $A \cap C_i \subseteq M_-^1(E)$ for all $i \in \mathcal{I}$ by the same Lemma 14. Since $A = \bigcup_{i \in \mathcal{I}}(A \cap C_i)$, we obtain $A \subseteq M_-^1(E)$.

(iii) ⇒ *(ii)* Suppose that $\sigma_i^1(A) \leq \sigma_i^2(A)$ for all $A \subseteq C_i$ and all $i = \{1, \ldots, m\}$, fix an arbitrary $E \in \mathscr{C}^1$ such that $M_+^1(E) \neq \varnothing$, and let $A := M_+^1(E)$, noting that $A \cap E = \varnothing$. Let $\mathcal{I} := \{i \in \{1, \ldots, m\} : A \cap C_i \neq \varnothing\}$. As follows from Lemma 14, we have $\sigma^1(E \cap C_i) = 1$ and, hence, $\sigma^2(E \cap C_i) = 1$ for all $i \in \mathcal{I}$. Since $A \neq \varnothing$, we have $\mathcal{I} \neq \varnothing$ and, in turn, $E \in \mathscr{C}^2$ by (2). Moreover, by Lemma 14, $A \cap C_i \subseteq M_+^2(E)$ for all $i \in \mathcal{I}$, which implies that $A \subseteq M_+^2(E)$.

(i) ⇒ *(iii)* Suppose that $M_-^2(E) \subseteq M_-^1(E)$ for all $E \in \mathscr{C}^1 \cap \mathscr{C}^2$, and fix arbitrary $i \in \{1, \ldots, m\}$ and $A \subseteq C_i$. Our goal is to prove that $\sigma_i^2(A) = 0$ implies $\sigma_i^1(A) = 0$. Indeed, assume that $\sigma_i^2(A) = 0$, let $B := \Omega \backslash C_i$, and note that $B \in \mathscr{C}^1 \cap \mathscr{C}^2$ because representations with subjective contingencies must have at least three cells. Let $E := A \cup B$, note that $E \in \mathscr{C}^1 \cap \mathscr{C}^2$ as well by Monotonicity of Possibility Correspondence, and observe that $A \subseteq M_-^2(E)$ by Lemma 14. By assumption, we have $A \subseteq M_-^1(E)$ and, therefore, $\sigma_i^1(A) = 0$ by Lemma 14 again.

$(ii) \Rightarrow (iii)$ Suppose that, for all $E \in \mathscr{C}^1$ such that $M_+^1(E) \neq \varnothing$, we have $E \in \mathscr{C}^2$ and $M_+^1(E) \subseteq M_+^2(E)$, and fix arbitrary $i \in \{1, \dots, m\}$ and $A \subseteq C_i$. Our goal is to prove that $\sigma_i^1(A) = 1$ implies $\sigma_i^2(A) = 1$. Indeed, assume that $\sigma_i^1(A) = 1$. If $A = C_i$ then $A \in \mathscr{C}^2$ and $\sigma_i^2(A) = 1$ by normalization. Otherwise, by Lemma 14, $C_i \backslash A \subseteq M_+^1(A)$; in turn, $A \in \mathscr{C}^2$ by assumption and, hence, $\sigma_i^2(A) = 1$ by (2). □

C Proofs of the Results of Section 4

Proof of Theorem 6. *Only if* part. *Step 1.* Since $\succsim$ satisfies the Anscombe-Aumann axioms, it admits a SEU representation via the map $f \mapsto \sum_{\omega \in \Omega} u(f(\omega))\, \mu(\omega)$ for some nonconstant affine function $u : \Delta(X) \to \mathbb{R}$ and a probability measure $\mu \in \Delta(\Omega)$ (see, e.g., Fishburn, 1970, Theorem 13.3). Moreover, u is continuous and has a compact range by the Continuity and Best and Worst Outcomes axioms, respectively.

Step 2. By Theorem 1, there exist $\mathcal{S} := \{1, \dots, n\}$, a collection $\Pi = \{C_1, \dots, C_n\}$ of nonempty disjoint subsets of Ω, and functions $V_i : \mathcal{F} \to \mathbb{R}$ and $\sigma_i : 2^{C_i} \to \{0, 1\}$ satisfying the conditions listed in the theorem such that functions $V' : \mathcal{F} \to \mathbb{R}$ and $V'(\cdot \mid E) : \mathcal{F} \to \mathbb{R}$ defined as

$$V'(f) = \sum_{i \in \mathcal{S}} V_i(f)$$
$$V'(f \mid E) = \sum_{i \in \mathcal{S} : \sigma_i(E \cap C_i) = 1} V_i(f)$$

are utility representations of $\succsim$ and $\succsim_E$ for all $E \in \mathscr{C}$, respectively. By the uniqueness of additively separable representations (see, e.g., Wakker (1989, Obs. III.6.6′)), there exist $k > 0$ and $b_i \in \mathbb{R}$ for all $i \in \mathcal{S}$ such that $V_i(f) = k \sum_{\omega \in C_i} u(f(\omega))\, \mu(\omega) + b_i$, and we also have $\mu\left(\cup_{i \in \mathcal{S}} C_i\right) = 1$.

Step 3. Let $p \in \Delta(S)$ be defined as $p_i = \mu(C_i)$ for each $i \in \mathcal{S}$, and note that, for all $i \in \mathcal{S}$, $p_i > 0$ because V_i is nonconstant. For each $i \in \mathcal{S}$, let $s_i : \Omega \to \mathbb{R}_+$ be defined as

$$s_i(\omega) := \begin{cases} \mu(\omega)/p_i, & \text{if } \omega \in C_i, \\ 0, & \text{otherwise.} \end{cases}$$

As follows from the previous step, positive affine transformations $V(\cdot \mid E)$ of $V'(\cdot \mid E)$ that can be computed as

$$V(f \mid E) = \frac{\sum_{i\in\mathcal{S}(E)} \sum_{\omega\in\Omega} u(f(\omega))\, s_i(\omega)\, p_i}{\sum_{i\in\mathcal{S}(E)} p_i},$$

where $\mathcal{S}(E) \coloneqq \{i \in \mathcal{S} : \sigma_i(E \cap C_i) = 1\}$, are also utility representations $\succsim_E$ for all $E \in \mathscr{C}$. Note that, for all $i \in \mathcal{S}$, the support of s_i must be equal to the support of V_i (which is C_i), and s_i is normalized as required in the statement of the theorem.

Step 4. For each $i \in \mathcal{S}$, let $\alpha_i \coloneqq \min\{\mu(E)/p_i \mid E \subseteq C_i \text{ and } \sigma_i(E) = 1\}$. We claim that, for any $i \in \mathcal{S}$ and $E \subseteq C_i$, if $\mu(E) \geq \alpha_i p_i$ then $\sigma_i(E) = 1$. Indeed, fix arbitrary $i \in \mathcal{S}$ and $E \subseteq C_i$, and let $E_0 \subseteq C_i$ be such that $\sigma_i(E_0) = 1$ and $\alpha_i = \mu(E_0)/p_i$. Observe that $E_0 \in \mathscr{C}$ by (2) and that $P_{E_0}(\omega)$ holds for all $\omega \in C_i$ because C_i is the support of s_i. Therefore, we can apply the Non-vacuousness of Valuable Bets axiom to outcomes x^* and x_* and events E_0 and E to obtain $E \in \mathscr{C}$, which implies that $\sigma_i(E) = 1$.

Recalling the definition of s_i for $i \in \mathcal{S}$, we can conclude that $\sum_{\omega\in E} s_i(\omega) \geq \alpha_i \Leftrightarrow \sigma_i(E) = 1$ for all $E \subseteq C_i$ and all $i \in \mathcal{S}$. Therefore, $\mathscr{C} = \{E \subseteq \Omega : \sum_{\omega\in E} s_i(\omega) \geq \alpha_i$ for at least one $i \in \mathcal{S}\}$ and $\mathcal{S}(E) = \{i \in \mathcal{S} : \sum_{\omega\in E} s_i(\omega) \geq \alpha_i\}$, which completes the proof of the *only if* part of the theorem.

If part. Assume that there exist a set $\mathcal{S} = \{1, \dots, n\}$, a collection $\Pi = \{C_1, \dots, C_n\}$ of subsets of Ω, a utility index $u : X \to \mathbb{R}$, a probability measure $p \in \Delta(\mathcal{S})$, collections of functions $s_i : \Omega \to \mathbb{R}_+$ and numbers $\alpha_i \in (0, 1]$ for $i \in \mathcal{S}$ as described in Theorem 6 and such that statements (i)–(iii) hold.

Clearly, functions $V_i : \mathcal{F} \to \mathbb{R}$ and $\sigma_i : 2^{C_i} \to \{0, 1\}$ for $i \in \mathcal{S}$ defined as

$$V_i(f) = \sum_{\omega\in C_i} u(f(\omega))\, s(\omega)\, p_i \qquad \text{and} \qquad \sigma_i(E) = 1 \Leftrightarrow \sum_{\omega\in E} s_i(\omega) \geq \alpha_i$$

satisfy the conditions of Theorem 1, so Axioms (A1)–(A12) hold. Monotonicity and Independence axioms follow from the representation by the standard argument.

It remains to show that Non-vacuousness of Valuable Bets holds. Let $x, y \in \Delta(X)$ be such that $x \succ y$ and $A \in \mathscr{C}, B \subseteq \Omega$ such that $B \subseteq Q(A)$ and $xBy \succsim xAy$. Suppose, by contradiction, that $\sum_{\omega\in B} s_i(\omega) < \alpha_i$ for all $i \in \mathcal{S}$. Since $B \subseteq Q(A)$, we observe that if

$B \cap C_i \neq \varnothing$ for some $i \in \mathcal{S}$, then, as follows from Lemma 14 and the definition of σ_i, we have $\sum_{\omega \in A} s_i(\omega) \geq \alpha_i$ and, hence, $i \in \mathcal{S}(A)$. We also have $\mathcal{S}(A) \neq \varnothing$ because $A \in \mathscr{C}$. Then, we obtain:

$$\begin{aligned} V(xAy) &= u(y) + (u(x) - u(y)) \sum_{i \in \mathcal{S}} \sum_{\omega \in A} s_i(\omega)\, p_i \\ &\geq u(y) + (u(x) - u(y)) \sum_{i \in \mathcal{S}(A)} \sum_{\omega \in A} s_i(\omega)\, p_i \\ &\geq u(y) + (u(x) - u(y)) \sum_{i \in \mathcal{S}(A)} \alpha_i p_i \\ &> u(y) + (u(x) - u(y)) \sum_{i \in \mathcal{S}(A)} \sum_{\omega \in B} s_i(\omega)\, p_i \\ &\geq u(y) + (u(x) - u(y)) \sum_{i \in \mathcal{S} : B \cap C_i \neq \varnothing} \sum_{\omega \in B} s_i(\omega)\, p_i = V(xBy), \end{aligned}$$

a contradiction. □

Proof of Proposition 7. The sufficiency of the conditions can be easily verified. We will prove the necessity.

Suppose that $(\mathcal{S}, (C_i)_{i \in \mathcal{S}}, u, p, (s_i)_{i \in \mathcal{S}}, (\alpha_i)_{i \in \mathcal{S}})$ and $(\mathcal{S}', (C'_i)_{i \in \mathcal{S}'}, u', p', (s'_i)_{i \in \mathcal{S}'}, (\alpha'_i)_{i \in \mathcal{S}'})$ are expected utility representations with subjective contingencies of the same system $(\succsim, \{\succsim_A\}_{A \in \mathscr{C}})$. Since an expected utility representation with subjective contingencies is a special case of the general representation of Theorem 1, Proposition 2 implies that there exists a bijection $\pi : \mathcal{S} \to \mathcal{S}'$ such that $C'_{\pi_i} = C_i$ for all $i \in \mathcal{S}$. The claimed relationship between u' and u, and between p' and p and s' and s follows easily from the uniqueness of the subjective expected utility representation. It remains to prove the claim regarding the thresholds $(\alpha'_i)_{i \in \mathcal{S}'}$.

Let $V_i : \mathcal{F} \to \mathbb{R}$ and $\sigma_i : 2^{C_i} \to \{0, 1\}$ for $i \in \mathcal{S}$ be defined as

$$V_i(f) = \sum_{\omega \in C_i} u(f(\omega))\, s(\omega)\, p_i \qquad \text{and} \qquad \sigma_i(A) = 1 \Leftrightarrow \sum_{\omega \in A} s_i(\omega) \geq \alpha'_{\pi_i}.$$

Clearly, $(S, (C_i)_{i \in \mathcal{S}}, (V_i)_{i \in \mathcal{S}}, (\sigma_i)_{i \in \mathcal{S}})$ is a representation with subjective contingencies of $(\succsim, \{\succsim_A\}_{A \in \mathscr{C}})$. Therefore, it must be that, for any $i \in \mathcal{S}$ and $A \subseteq C_i$,

$$\sum_{\omega \in A} s_i(\omega) \geq \alpha'_{\pi_i} \Leftrightarrow \sigma_i(A) = 1 \Leftrightarrow A \in \mathscr{C},$$

and the situations $\alpha'_{\pi_i} > \sum_{\omega\in A} s_i(\omega)$ for some $A \subseteq C_i$, $A \in \mathscr{C}$, or $\alpha'_{\pi_i} \leq \sum_{\omega\in A} s_i(\omega)$ for some $A \subseteq C_i$, $A \notin \mathscr{C}$, would result in a contradiction. Hence, the fact that $\alpha'_{\pi_i} \leq \alpha_i^{max}$ and $\alpha'_{\pi_i} > \alpha_i^{min}$ for all $i \in \mathcal{S}$ is proven. □

Proof of Observation 8. Suppose that there exists $i \in \mathcal{S}$ such that $(\alpha_i^{1\,min}, \alpha_i^{1\,max}] \cap (\alpha_i^{2\,min}, \alpha_i^{2\,max}] \neq \varnothing$. Without loss of generality, assume that $\alpha_i^{1\,min} < \alpha_i^{2\,min} < \alpha_i^{1\,max}$. Let $A \subseteq C_i$ be such that $\sum_{\omega\in A} s_i(\omega) = \alpha_i^{2\,min}$. We have $\sum_{\omega\in A} s_i(\omega) > \alpha_i^{1\,min}$, so it must be that $A \in \mathscr{C}^1$ by the definition of α^{min}, and, therefore, $\sum_{\omega\in A} s_i(\omega) \geq \alpha_i^{1\,max}$ by the definition of α^{max}, a contradiction. □

Proof of Proposition 9. As follows from Proposition 5, Decision maker 1 is more prone to exclusion errors if and only if Decision maker 2 is more prone to inclusion errors if and only if

$$\sum_{\omega\in A} s_i(\omega) \geq \alpha_i^1 \Rightarrow \sum_{\omega\in A} s_i(\omega) \geq \alpha_i^2 \qquad \forall_{A\subseteq C_i} \forall_{i\in\mathcal{S}:|C_i|\geq 2}. \tag{9}$$

Clearly, if $\alpha_i^1 \geq \alpha_i^2$ for all $i \in \mathcal{S}$, then (9) holds.

Conversely, suppose that (9) holds, and assume, by contradiction, that $\alpha_i^{1\,min} < \alpha_i^{2\,min}$ for some $i \in \mathcal{S}$. Let $A \subseteq C_i$ be such that $\sum_{\omega\in A} s_i(\omega) = \alpha_i^{2\,min}$ and $A \notin \mathscr{C}^2$. As follows from Proposition 7 and Observation 8, it must be that $\alpha_i^{1\,min} < \alpha_i^1 \leq \alpha_i^{1\,max} \leq \alpha_i^{2\,min}$, and, therefore, $\sum_{\omega\in A} s_i(\omega) \geq \alpha_i^1$. By (9), we have $\sum_{\omega\in A} s_i(\omega) \geq \alpha_i^2$, which implies by (5) that $A \in \mathscr{C}^2$, a contradiction. We conclude that $\alpha_i^{1\,min} \geq \alpha_i^{2\,min}$ for all $i \in \mathcal{S}$, which, in the view of Observation 8, proves the claim of the proposition. □

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