Econometrica Supplementary Material

SUPPLEMENT TO "FISCAL RULES AND DISCRETION UNDER LIMITED ENFORCEMENT"

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APPENDIX B: OMITTED PROOFS

B.1. *Proof of Lemma 1*

WE PROCEED IN THREE STEPS.

STEP 1: Suppose $\theta^* \geq \underline{\theta}$. We show that (3) and (4) are satisfied for types $\theta \in [\underline{\theta}, \theta^*]$.

The claim follows immediately from the fact that all types $\theta \in [\underline{\theta}, \theta^*]$ are assigned their flexible debt levels with no penalty. Thus, given $\theta \in [\underline{\theta}, \theta^*]$, type θ's welfare cannot be increased, and (3) and (4) are trivially satisfied.

STEP 2: We show that (3) and (4) are satisfied for types $\theta \in (\theta^*, \theta^{**}]$.

Take first the enforcement constraint (4). We can rewrite it for $\theta \in (\theta^*, \theta^{**}]$ as

$$\theta U\big(\omega + b^r(\theta^*)\big) + \beta\delta V\big(b^r(\theta^*)\big) - \theta U\big(\omega + b^p(\theta)\big) - \beta\delta\big(V\big(b^p(\theta)\big) - \overline{P}\big(b^p(\theta)\big)\big) \geq 0. \quad \text{(B.1)}$$

Differentiating the left-hand side with respect to θ, given θ^* and the definition of $b^p(\theta)$, yields

$$U\big(\omega + b^r(\theta^*)\big) - U\big(\omega + b^p(\theta)\big),$$

which is weakly decreasing in θ, since $b^p(\theta)$ is nondecreasing. This means that the left-hand side of (B.1) is weakly concave. Since (B.1) holds as a strict inequality for $\theta = \theta^*$ and as an equality for $\theta = \theta^{**}$ (by (8)), this weak concavity implies that (B.1) holds as a strict inequality for all $\theta \in (\theta^*, \theta^{**})$. Thus, constraint (4) is satisfied for all $\theta \in (\theta^*, \theta^{**}]$.

Take next the truthtelling constraint (3). This constraint is trivially satisfied for all $\theta \in (\theta^*, \theta^{**}]$ given $\theta' \in [\theta^*, \theta^{**}]$, since all types $\theta \in [\theta^*, \theta^{**}]$ are assigned the same allocation. We next show that the constraint is also satisfied given $\theta' > \theta^{**}$ and $\theta' < \theta^*$:

Step 2a: We show that (3) is satisfied for all $\theta \in (\theta^*, \theta^{**}]$ given $\theta' > \theta^{**}$. Note that $(b(\theta'), P(\theta')) = (b^p(\theta'), \overline{P}(b^p(\theta')))$ for all $\theta' > \theta^{**}$, and by the definition of $b^p(\theta)$,

$$\begin{aligned} &\theta U\big(\omega + b^p(\theta)\big) + \beta\delta\big(V\big(b^p(\theta)\big) - \overline{P}\big(b^p(\theta)\big)\big) \\ &\quad \geq \theta U\big(\omega + b^p(\theta')\big) + \beta\delta\big(V\big(b^p(\theta')\big) - \overline{P}\big(b^p(\theta')\big)\big) \end{aligned}$$

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for all $\theta' \in \Theta$. Thus, the fact that the enforcement constraint (4) is satisfied for all $\theta \in (\theta^*, \theta^{**}]$ implies that (3) is satisfied for all such types given $\theta' > \theta^{**}$.

Step 2b: We show that (3) is satisfied for all $\theta \in (\theta^*, \theta^{**}]$ given $\theta' < \theta^*$. Suppose by contradiction that this is not the case, that is,

$$\theta\big(U\big(\omega + b^r(\theta^*)\big) - U\big(\omega + b^r(\theta')\big)\big) < \beta\delta\big(V\big(b^r(\theta')\big) - V\big(b^r(\theta^*)\big)\big) \tag{B.2}$$

for some $\theta \in (\theta^*, \theta^{**}]$ and $\theta' < \theta^*$. By Step 1, (3) holds for type θ^* given $\theta' < \theta^*$:

$$\theta^*\big(U\big(\omega + b^r(\theta^*)\big) - U\big(\omega + b^r(\theta')\big)\big) \geq \beta\delta\big(V\big(b^r(\theta')\big) - V\big(b^r(\theta^*)\big)\big). \tag{B.3}$$

Combining (B.2) and (B.3) yields

$$\big(\theta^* - \theta\big)\big(U\big(\omega + b^r(\theta^*)\big) - U\big(\omega + b^r(\theta')\big)\big) > 0,$$

which is a contradiction since $\theta > \theta^*$ and $b^r(\theta') \leq b^r(\theta^*)$. The claim follows.

STEP 3: Suppose $\theta^{**} < \overline{\theta}$. We show that (3) and (4) are satisfied for types $\theta \in (\theta^{**}, \overline{\theta}]$.

Constraint (4) is satisfied as an equality for all $\theta \in (\theta^{**}, \overline{\theta}]$. It is immediate that constraint (3) is satisfied for all $\theta \in (\theta^{**}, \overline{\theta}]$ given $\theta' \in (\theta^{**}, \overline{\theta}]$, since all such types are assigned their flexible debt level with maximum penalty. Consider next constraint (3) for $\theta \in (\theta^{**}, \overline{\theta}]$ given $\theta' \in [\theta^*, \theta^{**}]$. Note that $(b(\theta'), P(\theta')) = (b^r(\theta^*), 0)$ for all $\theta' \in [\theta^*, \theta^{**}]$. Thus, satisfaction of this constraint is ensured if (B.1) is violated for $\theta \in (\theta^{**}, \overline{\theta}]$. The latter is true since, as shown above, the left-hand side of (B.1) is weakly concave and (B.1) holds as an equality for $\theta = \theta^{**}$ and a strict inequality for $\theta \in (\theta^*, \theta^{**})$.

Finally, consider constraint (3) for $\theta \in (\theta^{**}, \overline{\theta}]$ given $\theta' < \theta^*$. Since (3) is satisfied given $\theta' \in [\theta^*, \theta^{**}]$, satisfaction of this constraint given $\theta' < \theta^*$ is ensured if

$$\theta\big(U\big(\omega + b^r(\theta^*)\big) - U\big(\omega + b^r(\theta')\big)\big) \geq \beta\delta\big(V\big(b^r(\theta')\big) - V\big(b^r(\theta^*)\big)\big)$$

for $\theta \in (\theta^{**}, \overline{\theta}]$. The latter follows from the same logic as in Step 2b above.

B.2. *Proof of Corollary 1*

Consider optimal rules with $b(\theta) \in (\underline{b}, \overline{b})$ for all $\theta \in \Theta$. We proceed in four steps.

STEP 1: We show that an optimal maximally enforced deficit limit solves

$$\max_{\theta^*, \theta^{**}} \left\{ \int_0^{\theta^*} U\big(\omega + b^r(\theta)\big)Q(\theta)\, d\theta + \int_{\theta^*}^{\theta^{**}} U\big(\omega + b^r(\theta^*)\big)Q(\theta)\, d\theta + \int_{\theta^{**}}^{\overline{\theta}} U\big(\omega + b^p(\theta)\big)Q(\theta)\, d\theta \right\}$$
$$\text{subject to (8),} \tag{B.4}$$

where $Q(\theta) = 1$ for $\theta < \underline{\theta}$ and, by convention, the last integral equals zero if $\theta^{**} \geq \overline{\theta}$.

By the arguments in the text, social welfare can be written as

$$\frac{1}{\beta}\underline{\theta}U\big(\omega+b(\underline{\theta})\big)+\delta\big(V\big(b(\underline{\theta})\big)-P(\underline{\theta})\big)+\frac{1}{\beta}\int_{\underline{\theta}}^{\overline{\theta}}U\big(\omega+b(\theta)\big)Q(\theta)\,d\theta,$$

which in turn can be rewritten as

$$\lim_{\underline{\theta}'\downarrow 0}\frac{1}{\beta}\underline{\theta}'U\big(\omega+b(\underline{\theta}')\big)+\delta\big(V\big(b(\underline{\theta}')\big)-P(\underline{\theta}')\big)+\frac{1}{\beta}\int_{0}^{\overline{\theta}}U\big(\omega+b(\theta)\big)Q(\theta)\,d\theta,$$

where $Q(\theta)=1$ for $\theta<\underline{\theta}$. Hence, social welfare under a maximally enforced deficit limit can be represented as

$$\begin{aligned}&\lim_{\underline{\theta}'\downarrow 0}\frac{1}{\beta}\underline{\theta}'U\big(\omega+b^r(\underline{\theta}')\big)+\delta\big(V\big(b^r(\underline{\theta}')\big)-P(\underline{\theta}')\big)\\&\quad+\frac{1}{\beta}\int_{0}^{\theta^*}U\big(\omega+b^r(\theta)\big)Q(\theta)\,d\theta+\frac{1}{\beta}\int_{\theta^*}^{\theta^{**}}U\big(\omega+b^r(\theta^*)\big)Q(\theta)\,d\theta\\&\quad+\frac{1}{\beta}\int_{\theta^{**}}^{\overline{\theta}}U\big(\omega+b^p(\theta)\big)Q(\theta)\,d\theta.\end{aligned}\tag{B.5}$$

Since the first term in (B.5) is independent of the choice of $\theta^*>0$ and $\theta^{**}>\theta^*$, and since the constant $\frac{1}{\beta}$ multiplies all other terms, the objective in (B.4) is equivalent to (B.5).

STEP 2: Consider the following relaxed program:

$$\max_{\theta^*}\bigg\{\int_{0}^{\theta^*}U\big(\omega+b^r(\theta)\big)Q(\theta)\,d\theta+\int_{\theta^*}^{\overline{\theta}}U\big(\omega+b^r(\theta^*)\big)Q(\theta)\,d\theta\bigg\}.$$

We show that any solution to this program yields strictly higher social welfare than any solution to program (B.4) with $\theta^{**}<\overline{\theta}$.

Take any solution $\{\theta^*,\theta^{**}\}$ to program (B.4) with $\theta^{**}<\overline{\theta}$. To prove the claim, it suffices to show that social welfare strictly increases if we change the allocation of types $\theta\in[\theta^{**},\overline{\theta}]$ from $(b(\theta),P(\theta))=(b^p(\theta),\overline{P}(b^p(\theta)))$ to $(b(\theta),P(\theta))=(b^r(\theta^*),0)$. To prove this, note first that by Step 1 in the proof of Proposition 2, the solution $\{\theta^*,\theta^{**}\}$ to program (B.4) has $\theta^{**}\geq\widehat{\theta}$. Hence, by Assumption 1, $Q(\theta)<0$ for all $\theta\in[\theta^{**},\overline{\theta}]$. Given the representation in (B.4), the claim then follows if $b^r(\theta^*)<b^p(\theta)$ for all $\theta\in[\theta^{**},\overline{\theta}]$. We show next that this inequality holds. Given the solution $\{\theta^*,\theta^{**}\}$, the following conditions hold for all $\theta\in[\theta^{**},\overline{\theta}]$:

$$\theta U\big(\omega+b^r(\theta^*)\big)+\beta\delta V\big(b^r(\theta^*)\big)\leq\theta U\big(\omega+b^p(\theta)\big)+\beta\delta\big(V\big(b^p(\theta)\big)-\overline{P}\big(b^p(\theta)\big)\big)$$

and

$$\theta^*U\big(\omega+b^r(\theta^*)\big)+\beta\delta V\big(b^r(\theta^*)\big)>\theta^*U\big(\omega+b^p(\theta)\big)+\beta\delta\big(V\big(b^p(\theta)\big)-\overline{P}\big(b^p(\theta)\big)\big).$$

Combining these two inequalities yields

$$(\theta-\theta^*)U\big(\omega+b^p(\theta)\big)>(\theta-\theta^*)U\big(\omega+b^r(\theta^*)\big),$$

which implies $b^p(\theta)>b^r(\theta^*)$ for all $\theta\in[\theta^{**},\overline{\theta}]$.

STEP 3: We show that the solution to the relaxed program in Step 2 is $\theta^* = \theta_e$, where $\theta_e \in [0, \overline{\theta})$ is uniquely defined by (11). Moreover, if $\theta^* = \theta_e$ satisfies constraint (8) for some $\theta^{**} \geq \overline{\theta}$, then these values correspond to the unique solution to program (B.4).

To prove the first claim, consider the first-order condition of the relaxed program in Step 2:

$$\frac{db^r(\theta^*)}{d\theta^*} U'(\omega + b^r(\theta^*)) \int_{\theta^*}^{\overline{\theta}} Q(\theta)\, d\theta = 0.$$

Since $\frac{db^r(\theta^*)}{d\theta^*} > 0$ and $U'(\omega + b^r(\theta^*)) > 0$, this condition requires that the integral be equal to 0. Hence, by the definition in (11), we obtain $\theta^* = \theta_e$. Note that this value is uniquely defined since, by Assumption 1, $\int_{\theta^*}^{\overline{\theta}} Q(\theta)\, d\theta = 0$ requires $\theta^* < \widehat{\theta}$ and $Q(\theta^*) > 0$, and hence $\int_{\theta^*}^{\overline{\theta}} Q(\theta)\, d\theta$ is strictly decreasing in θ^*. Since $\int_{\theta^*}^{\overline{\theta}} Q(\theta)\, d\theta$ is strictly positive for $\theta^* = \varepsilon$ and strictly negative for $\theta^* = \overline{\theta} - \varepsilon$ for sufficiently small $\varepsilon > 0$,[30] it follows that a unique interior $\theta_e \in (0, \overline{\theta})$ exists and is the unique optimum.

To prove the second claim, note that if constraint (8) holds under $\theta^* = \theta_e$ and some $\theta^{**} \geq \overline{\theta}$, then such a deficit limit $\{\theta_e, \theta^{**}\}$ is feasible in program (B.4). Moreover, since this deficit limit yields the same social welfare as the relaxed program, it follows from Step 2 and the above claim that it yields strictly higher social welfare than any other feasible deficit limit and is thus the unique solution to program (B.4).

STEP 4: We show that if (12) holds, then the solution to (B.4) has $\theta^* = \theta_e$ and $\theta^{**} \geq \overline{\theta}$.

The claim follows from Step 3 and the fact that if (12) holds, then constraint (8) is satisfied under $\theta^* = \theta_e$ and some $\theta^{**} \geq \overline{\theta}$.

B.3. *Proof of Proposition 4*

For any given threshold θ', denote by $\rho(\theta')$ the type exceeding θ' at which (8) holds:

$$\begin{aligned} &\rho(\theta')U(\omega + b^r(\theta')) + \beta\delta V(b^r(\theta')) \\ &\quad = \rho(\theta')U(\omega + b^p(\rho(\theta'))) + \beta\delta\big(V(b^p(\rho(\theta'))) - \overline{P}(b^p(\rho(\theta')))\big). \end{aligned} \tag{B.6}$$

Note that given θ', $\rho(\theta') > \theta'$ is uniquely defined. This follows from the same logic as in Step 2 in the proof of Lemma 1. We prove this proposition in five steps.

STEP 1: We show that $\frac{d\rho(\theta')}{d\theta'} > 0$.

Implicit differentiation of (B.6), taking into account the definition of $b^r(\theta')$, yields

$$\frac{d\rho(\theta')}{d\theta'} = \frac{(\rho(\theta') - \theta')U'(\omega + b^r(\theta'))\frac{db^r(\theta')}{d\theta'}}{U(\omega + b^p(\rho(\theta'))) - U(\omega + b^r(\theta'))}. \tag{B.7}$$

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[30]To see that $\int_{\varepsilon}^{\overline{\theta}} Q(\theta)\, d\theta > 0$ for ε sufficiently small, note that using integration by parts yields

$$\int_{\varepsilon}^{\overline{\theta}} Q(\theta)\, d\theta = -(1 - F(\varepsilon))\varepsilon + \int_{\varepsilon}^{\overline{\theta}} f(\theta)\theta\, d\theta - \int_{\varepsilon}^{\overline{\theta}} f(\theta)\theta(1-\beta)\, d\theta,$$

which approaches $\beta\mathbb{E}[\theta] > 0$ as ε goes to 0.

Note that since $\frac{db^r(\theta')}{d\theta'} > 0$ and $\rho(\theta') > \theta'$, the numerator in (B.7) is strictly positive. Moreover, by the arguments in Step 2 of the proof of Corollary 1, we have $b^p(\rho(\theta')) > b^r(\theta')$, which implies that the denominator is also strictly positive. Thus, we obtain $\frac{d\rho(\theta')}{d\theta'} > 0$.

STEP 2: We show that if $\theta_c \le \theta_e$, then condition (14) holds and the optimal maximally enforced deficit limit is unique and has $\theta^* = \theta_e$ and $\theta^{**} \ge \overline{\theta}$.

As noted in the text, if $\theta_c \le \theta_e$, Assumption 1 guarantees that $\int_{\theta_c}^{\overline{\theta}} Q(\theta)\,d\theta \ge \int_{\theta_e}^{\overline{\theta}} Q(\theta)\,d\theta = 0$, so condition (14) is satisfied. The claim then follows from Corollary 1.

STEP 3: We show that if $\theta_c > \theta_e$, then $\theta^* \le \theta_c$.

Assume $\theta_c > \theta_e$. Suppose by contradiction that an optimal maximally enforced deficit limit features $\theta^* > \theta_c$, which implies $\theta^{**} \ge \overline{\theta}$. Consider a perturbation that reduces θ^* by $\varepsilon > 0$ arbitrarily small. Since in the original rule the enforcement constraint of all types $\theta \in \Theta$ is slack, this perturbation is incentive feasible. The change in social welfare, using the representation in (B.4), is

$$-\int_{\theta^*}^{\overline{\theta}} \frac{db^r(\theta^*)}{d\theta^*} U'\big(\omega + b^r(\theta^*)\big) Q(\theta)\,d\theta. \tag{B.8}$$

Assumption 1 together with (11) imply $\theta_e < \widehat{\theta}$. It then follows from $\theta^* > \theta_c > \theta_e$ and Assumption 1 that $\int_{\theta^*}^{\overline{\theta}} Q(\theta)\,d\theta < 0$, and thus, since $\frac{db^r(\theta^*)}{d\theta^*} > 0$, (B.8) is strictly positive. Hence, the perturbation strictly increases social welfare, implying that $\theta^* > \theta_c$ cannot hold.

STEP 4: We show that if $\theta_c > \theta_e$ and condition (14) holds, then the optimal maximally enforced deficit limit is unique and has $\theta^* = \theta_c$ and $\theta^{**} = \overline{\theta}$.

Assume that $\theta_c > \theta_e$ and condition (14) holds. By Step 3, an optimal maximally enforced deficit limit has $\theta^* \le \theta_c$. Suppose by contradiction that $\theta^* < \theta_c$, which implies $\theta^{**} = \rho(\theta^*) < \overline{\theta}$ for $\rho(\cdot)$ as defined in (B.6). Consider a perturbation that changes θ^* by some $\varepsilon \gtrless 0$ for $|\varepsilon|$ arbitrarily small, where $\theta^{**} = \rho(\theta^*)$ is also changed to preserve (B.6). This perturbation is incentive feasible. Using the representation in (B.4), for this perturbation to not increase social welfare for any arbitrarily small $\varepsilon \gtrless 0$, we must have

$$\int_{\theta^*}^{\rho(\theta^*)} U'\big(\omega + b^r(\theta^*)\big)\frac{db^r(\theta^*)}{d\theta^*} Q(\theta)\,d\theta$$
$$+ \frac{d\rho(\theta^*)}{d\theta^*}\Big(U\big(\omega + b^r(\theta^*)\big) - U\big(\omega + b^p\big(\rho(\theta^*)\big)\big)\Big) Q\big(\rho(\theta^*)\big) = 0.$$

Using (B.7) to substitute for $\frac{d\rho(\theta^*)}{d\theta^*}$ and simplifying terms, we can rewrite this condition as

$$\int_{\theta^*}^{\rho(\theta^*)} \Big(Q(\theta) - Q\big(\rho(\theta^*)\big)\Big)\,d\theta = 0. \tag{B.9}$$

Given Assumption 1, (B.9) requires $\theta^* < \widehat{\theta} < \rho(\theta^*)$ with

$$Q(\theta^*) > Q\big(\rho(\theta^*)\big). \tag{B.10}$$

Now note that the derivative of the left-hand side of (B.9) with respect to θ^* is equal to

$$-\big(Q(\theta^*)-Q\big(\rho(\theta^*)\big)\big)-\int_{\theta^*}^{\rho(\theta^*)} Q'\big(\rho(\theta^*)\big)\frac{d\rho(\theta^*)}{d\theta^*}\,d\theta. \tag{B.11}$$

By (B.10), the first term is strictly negative. Moreover, since $\rho(\theta^*)>\widehat{\theta}$, Assumption 1 implies $Q'(\rho(\theta^*)))>0$. Given $\frac{d\rho(\theta')}{d\theta'}>0$ (as established in Step 1), it then follows that the second term in (B.11) is also strictly negative. Hence, the derivative of the left-hand side of (B.9) with respect to θ^* is strictly negative. However, using the contradiction assumption that $\theta^*<\theta_c$, condition (B.9) then requires that the left-hand side of (14) be strictly negative, contradicting the assumption that condition (14) holds. Therefore, there exists a perturbation that changes θ^* by some $\varepsilon \gtrless 0$ which strictly increases social welfare, implying that the unique optimal maximally enforced deficit limit has $\theta^*=\theta_c$ and $\theta^{**}=\overline{\theta}$.

STEP 5: We show that if $\theta_c>\theta_e$ and condition (14) does not hold, then the optimal maximally enforced deficit limit is unique and has $\theta^*\in(\theta_e,\theta_c)$ and $\theta^{**}<\overline{\theta}$.

Assume that $\theta_c>\theta_e$ and condition (14) is violated. By Step 3, an optimal maximally enforced deficit limit has $\theta^*\le\theta_c$. We begin by showing that $\theta^*=\theta_c$ cannot be optimal. Suppose by contradiction that an optimal maximally enforced deficit limit sets $\theta^*=\theta_c$ and thus $\theta^{**}=\rho(\theta_c)=\overline{\theta}$. Consider a perturbation that reduces θ^* by $\varepsilon>0$ arbitrarily small, where $\theta^{**}=\rho(\theta^*)$ is also changed to preserve (B.6). This perturbation is incentive feasible. Using the representation in (B.4), for this perturbation to not increase social welfare for any arbitrarily small $\varepsilon>0$, we must have

$$-\int_{\theta^*}^{\rho(\theta^*)} U'\big(\omega+b^r(\theta^*)\big)\frac{db^r(\theta^*)}{d\theta^*}Q(\theta)\,d\theta$$
$$-\frac{d\rho(\theta^*)}{d\theta^*}\big[U\big(\omega+b^r(\theta^*)\big)-U\big(\omega+b^p\big(\rho(\theta^*)\big)\big)\big]Q\big(\rho(\theta^*)\big)\le 0.$$

By analogous logic as in Step 4 above, we can rewrite this condition as

$$\int_{\theta_c}^{\overline{\theta}}\big(Q(\theta)-Q(\overline{\theta})\big)\,d\theta\ge 0,$$

where we have taken into account that $\theta^*=\theta_c$ and $\theta^{**}=\rho(\theta_c)=\overline{\theta}$. However, this inequality contradicts the assumption that condition (14) does not hold. Therefore, the perturbation strictly increases social welfare, implying that any optimal maximally enforced deficit limit has $\theta^*<\theta_c$ and $\theta^{**}=\rho(\theta^*)<\overline{\theta}$.

We next show that the optimal values of θ^* and $\theta^{**}=\rho(\theta^*)$ are unique with $\theta^*>\theta_e$. By analogous logic as in Step 4 above, the optimal value of θ^* must satisfy (B.9). As shown in Step 4, the left-hand side of (B.9) is strictly decreasing in θ^*. This has two implications. First, it implies that there is a unique value of θ^* and associated $\theta^{**}=\rho(\theta^*)$ which solve (B.9). Second, given (11), Assumption 1, and the fact that the left-hand side of (B.9) is strictly decreasing in $\rho(\theta^*)$, it implies that if $\theta^*\le\theta_e$, then the left-hand side of (B.9) must be strictly positive, a contradiction. Therefore, the unique value of θ^* that solves (B.9) must satisfy $\theta^*>\theta_e$.

B.4. *Proof of Proposition 5*

Let $\theta^L, \theta^H \in \Theta$ and $\Delta > 0$ be defined as in Definition 2. We prove the proposition by proving the following three claims.

CLAIM 1: *Suppose Assumption* 1 *is strictly violated. If a maximally enforced deficit limit* $\{\theta^*, \theta^{**}\}$ *is a solution to* (6) *for given functions* $V(b)$, $\overline{P}(b)$, *then* $\theta^* \leq \theta^L$ *and* $\theta^{**} \geq \theta^H$.

PROOF: Suppose Assumption 1 is strictly violated. Suppose by contradiction that a maximally enforced deficit limit with $\theta^* > \theta^L$ is a solution to (6). Then analogously to Step 2 (Case 2) in the proof of Proposition 1, consider a perturbation that drills a hole in the borrowing schedule in the range $[\theta^L, \theta^L + \varepsilon]$ for arbitrarily small $\varepsilon > 0$ satisfying $\theta^L + \varepsilon < \min\{\theta^*, \theta^L + \Delta\}$. This perturbation is incentive feasible. Moreover, since $Q(\theta)$ is strictly increasing in this range, the arguments in Step 2 in the proof of Proposition 1 imply that this perturbation strictly increases social welfare, yielding a contradiction.

Next, suppose by contradiction that a maximally enforced deficit limit with $\theta^{**} < \theta^H$ is a solution to (6). Then consider types $\theta \in [\theta^H - \varepsilon, \theta^H]$ for arbitrarily small $\varepsilon > 0$ satisfying $\theta^H - \varepsilon > \max\{\theta^{**}, \theta^H - \Delta\}$. For each such type θ, we have $(b(\theta), P(\theta)) = (b^p(\theta), \overline{P}(b^p(\theta)))$ and $Q'(\theta) < 0$. Thus, this is the same situation as in Step 1 in the proof of Proposition 2. Analogously to that step, we can show that there is an incentive feasible perturbation that strictly increases social welfare, yielding a contradiction. *Q.E.D.*

CLAIM 2: *Suppose Assumption* 1 *is strictly violated. For any function* $V(b)$, *there exists a function* $\overline{P}(b)$ *such that no solution to* (6) *is a maximally enforced deficit limit.*

PROOF: Suppose Assumption 1 is strictly violated. Given $V(b)$, define $\overline{P}(b) = P$ for $P > 0$. By Claim 1, if a maximally enforced deficit limit $\{\theta^*, \theta^{**}\}$ solves (6), then $\theta^* \leq \theta^L$ and $\theta^{**} \geq \theta^H$. Consider the indifference condition (8) which defines, for any given θ^*, a unique value of $\theta^{**} > \theta^*$. This condition shows that given $V(b)$ and $\overline{P}(b) = P$, the value of $(\theta^{**} - \theta^*)$ is continuous in P and approaches 0 as P goes to 0. Hence, if we take $P > 0$ small enough, then $\theta^* \leq \theta^L < \theta^H \leq \theta^{**}$ cannot hold. The claim follows. *Q.E.D.*

CLAIM 3: *Suppose Assumption* 1 *is weakly violated. For any function* $V(b)$, *there exists a function* $\overline{P}(b)$ *such that not every solution to* (6) *is a maximally enforced deficit limit.*

PROOF: Suppose Assumption 1 is weakly violated and a maximally enforced deficit limit $\{\theta^*, \theta^{**}\}$ is a solution to (6). Then $\{\theta^*, \theta^{**}\}$ satisfy condition (8) and analogous arguments as in the proof of Claim 2 above imply that, given $V(b)$, there exists a function $\overline{P}(b)$ such that $\theta^* \leq \theta^L < \theta^H \leq \theta^{**}$ cannot hold. This means that given such functions, any maximally enforced deficit limit $\{\theta^*, \theta^{**}\}$ solving (6) must have either $\theta^* > \theta^L$ or $\theta^{**} < \theta^H$ (or both). Suppose first that $\theta^* > \theta^L$. Then consider a perturbation as in the proof of Claim 1 above which drills a hole in the borrowing schedule in the range $[\theta^L, \theta^L + \varepsilon]$ for arbitrarily small $\varepsilon > 0$ satisfying $\theta^L + \varepsilon < \min\{\theta^*, \theta^L + \Delta\}$. The same arguments as in the proof of Claim 1, given $Q'(\theta) \geq 0$ for $\theta \in [\theta^L, \theta^L + \varepsilon]$, imply that this perturbation weakly increases social welfare. The resulting allocation is therefore a solution to (6), and it is not a maximally enforced deficit limit.

Suppose next that $\theta^{**} < \theta^H$. Then as in the proof of Claim 1 above, consider types $\theta \in [\theta^H - \varepsilon, \theta^H]$ for arbitrarily small $\varepsilon > 0$ satisfying $\theta^H - \varepsilon > \max\{\theta^{**}, \theta^H - \Delta\}$. For each such type θ, we have $(b(\theta), P(\theta)) = (b^p(\theta), \overline{P}(b^p(\theta)))$ and $Q'(\theta) \leq 0$. Thus, we can perturb the allocation of these types as in Step 1 in the proof of Proposition 2 and weakly increase social welfare. The resulting allocation is therefore a solution to (6), and it is not a maximally enforced deficit limit. *Q.E.D.*

B.5. *Proof of Proposition 6*

We prove each part of the proposition in order.

Part 1. Suppose the enforcement constraint binds under $\overline{P}(b)$. Then for $k = 0$, we have

$$\begin{aligned} &\overline{\theta}U\big(\omega + b^r(\theta_e)\big) + \beta\delta V\big(b^r(\theta_e)\big) \\ &\quad < \overline{\theta}U\big(\omega + b^p(\overline{\theta})\big) + \beta\delta\big(V\big(b^p(\overline{\theta})\big) - \overline{P}\big(b^p(\overline{\theta})\big) - k\big). \end{aligned} \tag{B.12}$$

Observe that there exists a finite value $k' > 0$ such that the right-hand side of (B.12) equals the left-hand side under $k = k'$. If $k \in [0, k')$, the inequality in (B.12) is preserved and the enforcement constraint continues to bind under $\overline{P}(b) + k$. If instead $k \geq k'$, this inequality no longer holds and the enforcement constraint does not bind under $\overline{P}(b) + k$.

Part 2. Suppose the enforcement constraint binds and on-path penalties are optimal under $\overline{P}(b)$. By analogous arguments as in the proof of Part 1 above, there exists a finite $k''' > 0$ such that the enforcement constraint under $\overline{P}(b) + k$ binds if $k \in [0, k''')$ and does not bind if $k \geq k'''$. To complete the proof, take $k \in [0, k''')$ and define $\theta_c(k)$ as the solution to

$$\begin{aligned} &\overline{\theta}U\big(\omega + b^r\big(\theta_c(k)\big)\big) + \beta\delta V\big(b^r\big(\theta_c(k)\big)\big) \\ &\quad = \overline{\theta}U\big(\omega + b^p(\overline{\theta})\big) + \beta\delta\big(V\big(b^p(\overline{\theta})\big) - \overline{P}\big(b^p(\overline{\theta})\big) - k\big). \end{aligned} \tag{B.13}$$

The value of $\theta_c(k)$ corresponds to the value of θ_c defined in (13) as a function of the additional penalty $k \in [0, k''')$. We show that $\theta_c(k)$ is strictly decreasing. Implicit differentiation of (B.13) yields

$$\frac{d\theta_c(k)}{dk} = -\frac{\beta\delta}{\big(\overline{\theta} - \theta_c(k)\big)\dfrac{db^r\big(\theta_c(k)\big)}{d\theta}U'\big(\omega + b^r\big(\theta_c(k)\big)\big)} < 0, \tag{B.14}$$

where we have used the fact that $\theta_c(k)U'(\omega + b^r(\theta_c(k))) = -\beta\delta V'(b^r(\theta_c(k)))$. Since on-path penalties are optimal under $k = 0$, Proposition 4 implies

$$\int_{\theta_c(0)}^{\overline{\theta}} \big(Q(\theta) - Q(\overline{\theta})\big)\, d\theta < 0. \tag{B.15}$$

By the definition of k''', the value of $\theta_c(k)$ approaches θ_e from above as k approaches k'''. Given the definition of θ_e in (11) and the fact that $Q(\overline{\theta}) < 0$, it follows that

$$\int_{\theta_c(k''')}^{\overline{\theta}} \big(Q(\theta) - Q(\overline{\theta})\big)\, d\theta > 0. \tag{B.16}$$

Equations (B.15) and (B.16) imply that there exists $k'' \in (0, k''')$ satisfying

$$\int_{\theta_c(k'')}^{\overline{\theta}} \big(Q(\theta) - Q(\overline{\theta})\big)\, d\theta = 0. \tag{B.17}$$

Note that k'' is unique: the derivative of the left-hand side of (B.17) with respect to k is

$$-\frac{d\theta_c(k'')}{dk}\big(Q\big(\theta_c(k'')\big) - Q(\overline{\theta})\big) > 0,$$

where the inequality follows from the fact that $\frac{d\theta_c(k'')}{dk} < 0$ (by (B.14)) and $Q(\theta_c(k'')) > Q(\overline{\theta})$ (by (B.17) and Assumption 1). Therefore, we obtain $\int_{\theta_c(k)}^{\overline{\theta}}(Q(\theta) - Q(\overline{\theta}))\,d\theta < 0$ if $k \in [0, k'')$ and $\int_{\theta_c(k)}^{\overline{\theta}}(Q(\theta) - Q(\overline{\theta}))\,d\theta > 0$ if $k \in (k'', k''')$. By Proposition 4, it follows that on-path penalties are optimal if $k \in [0, k'')$ and suboptimal if $k \in [k'', k''')$.

B.6. *Proof of Proposition 7*

We prove each part of the proposition in order.

Part 1. There are two cases to consider.

Case 1: Suppose that on-path penalties are suboptimal. By Proposition 4, the optimal rule sets $\theta^* = \theta_c(k)$ for $\theta_c(k)$ defined in (B.13) in the proof of Proposition 6. Since $\theta_c(k)$ is strictly decreasing in k by (B.14), it follows that θ^* strictly decreases (increases) when $\overline{P}(b)$ is shifted to $\overline{P}(b) + k$ for $k > 0$ ($k < 0$).

Case 2: Suppose that on-path penalties are optimal. We prove the result for the case of a positive penalty shift. The proof of the negative-shift case is analogous and thus omitted.

Given a penalty shift k, define $\rho^k(\theta)$ as the unique solution to

$$\begin{aligned}&\rho^k(\theta)U\big(\omega + b^r(\theta)\big) + \beta\delta V\big(b^r(\theta)\big)\\ &\quad = \rho^k(\theta)U\big(\omega + b^p\big(\rho^k(\theta)\big)\big) + \beta\delta\big(V\big(b^p\big(\rho^k(\theta)\big)\big) - \overline{P}\big(b^p\big(\rho^k(\theta)\big)\big) - k\big).\end{aligned}$$

Observe that $\rho^k(\theta)$ corresponds to the value of θ^{**} that satisfies the indifference condition (8) given $\theta = \theta^*$ and the penalty shift k, and for $k = 0$ it corresponds to $\rho(\theta^*)$ defined in the proof of Proposition 4. It follows from Step 1 in that proof that $\rho^k(\theta)$ is strictly increasing in θ. Moreover, by implicit differentiation,

$$\frac{d\rho^k(\theta)}{dk} = -\frac{\beta\delta}{U\big(\omega + b^r(\theta)\big) - U\big(\omega + b^p\big(\rho^k(\theta)\big)\big)} > 0,$$

where we have used the fact that $b^p(\rho^k(\theta)) > b^r(\theta)$, as implied by the arguments in Step 2 of the proof of Corollary 1.

Consider the optimal deficit limit $\{\theta^*, \theta^{**}\}$ under $\overline{P}(b)$ and denote by $\{\theta^{*k}, \theta^{**k}\}$ the optimal deficit limit under $\overline{P}(b) + k$. Since the enforcement constraint binds, we have $\theta^{**} = \rho(\theta^*)$ and $\theta^{**k} = \rho^k(\theta^{*k})$. By Step 4 in the proof of Proposition 4, the following first-order conditions uniquely define θ^* and θ^{*k}:

$$\int_{\theta^*}^{\rho(\theta^*)}\big(Q(\theta) - Q\big(\rho(\theta^*)\big)\big)\,d\theta = 0, \tag{B.18}$$

$$\int_{\theta^{*k}}^{\rho^k(\theta^{*k})}\big(Q(\theta) - Q\big(\rho^k(\theta^{*k})\big)\big)\,d\theta = 0. \tag{B.19}$$

By Assumption 1, these conditions require that $\theta^* < \widehat{\theta} < \rho(\theta^*)$ and $\theta^{*k} < \widehat{\theta} < \rho^k(\theta^{*k})$ and that $Q(\theta^*) > Q(\rho(\theta^*))$ and $Q(\theta^{*k}) > Q(\rho^k(\theta^{*k}))$.

Suppose by contradiction that $\theta^* \leq \theta^{*k}$ for some $k > 0$. Then, given Assumption 1, conditions (B.18) and (B.19), and the fact that $\rho^k(\theta)$ is strictly increasing in θ and k, we must have

$$\theta^* \leq \theta^{*k} < \widehat{\theta} < \rho(\theta^*) < \rho^k(\theta^{*k}) \tag{B.20}$$

and

$$Q(\theta^*) \geq Q(\theta^{*k}) > Q(\rho^k(\theta^{*k})) > Q(\rho(\theta^*)). \tag{B.21}$$

Note that by the arguments in Step 4 in the proof of Proposition 4, the function

$$\int_{\theta^L}^{\theta^H} (Q(\theta) - Q(\theta^H))\, d\theta$$

is strictly decreasing in θ^L and in θ^H for any θ^L and θ^H satisfying $Q(\theta^L) > Q(\theta^H)$ and $\theta^H > \widehat{\theta}$. However, combined with conditions (B.20) and (B.21), this implies

$$\begin{aligned}\int_{\theta^*}^{\rho(\theta^*)} (Q(\theta) - Q(\rho(\theta^*)))\, d\theta &\geq \int_{\theta^{*k}}^{\rho(\theta^*)} (Q(\theta) - Q(\rho(\theta^*)))\, d\theta \\ &> \int_{\theta^{*k}}^{\rho^k(\theta^{*k})} (Q(\theta) - Q(\rho^k(\theta^{*k})))\, d\theta,\end{aligned}$$

which cannot hold simultaneously with equations (B.18) and (B.19). Therefore, it follows that $\theta^* > \theta^{*k}$ for all $k > 0$.

Part 2. We prove the result for the case of a positive penalty shift. The proof of the negative-shift case is analogous and thus omitted.

Suppose by contradiction that $\theta^{**} = \rho(\theta^*) \geq \theta^{**k} = \rho^k(\theta^{*k})$ for some $k > 0$. Since $\theta^{*k} < \theta^*$ by Part 1, it follows by analogous reasoning as in the proof of Part 1 that

$$\begin{aligned}\int_{\theta^*}^{\rho(\theta^*)} (Q(\theta) - Q(\rho(\theta^*)))\, d\theta &< \int_{\theta^{*k}}^{\rho(\theta^*)} (Q(\theta) - Q(\rho(\theta^*)))\, d\theta \\ &\leq \int_{\theta^{*k}}^{\rho^k(\theta^{*k})} (Q(\theta) - Q(\rho^k(\theta^{*k})))\, d\theta.\end{aligned}$$

However, this cannot hold simultaneously with equations (B.18) and (B.19). Therefore, it follows that $\theta^{**} < \theta^{**k}$ for all $k > 0$.

B.7. *Proof of Proposition 8*

We prove each part of the proposition in order.

Part 1. Suppose that on-path penalties are suboptimal under $f(\theta)$. By Proposition 4, the following condition holds:

$$\int_{\theta_c}^{\overline{\theta}} (Q(\theta) - Q(\overline{\theta}))\, d\theta \geq 0. \tag{B.22}$$

Consider a Q-decreasing perturbation that yields $\widetilde{f}(\theta)$ over $\widetilde{\Theta} = \Theta$. Observe that the value of θ_c defined in (13) does not vary with the perturbation since $\overline{\theta} = \widetilde{\overline{\theta}}$. Suppose by contradiction that on-path penalties are optimal under $\widetilde{f}(\theta)$. By Proposition 4, this implies

$$\int_{\theta_c}^{\overline{\theta}} \big(\widetilde{Q}(\theta) - \widetilde{Q}(\overline{\theta})\big)\, d\theta < 0. \tag{B.23}$$

Combining (B.22) and (B.23) yields

$$\int_{\theta_c}^{\overline{\theta}} \big(\widetilde{Q}(\overline{\theta}) - Q(\overline{\theta})\big)\, d\theta > \int_{\theta_c}^{\overline{\theta}} \big(\widetilde{Q}(\theta) - Q(\theta)\big)\, d\theta. \tag{B.24}$$

However, since the perturbation is Q-decreasing and support-preserving, it necessarily admits

$$\widetilde{Q}(\overline{\theta}) - Q(\overline{\theta}) < \widetilde{Q}(\theta) - Q(\theta)$$

for all $\theta \leq \overline{\theta}$. For $\theta \in [\underline{\theta}, \overline{\theta}]$, this inequality follows by the definition of Q-decreasing. For $\theta < \underline{\theta}$, the inequality follows from the fact that $\widetilde{Q}(\theta) = Q(\theta) = 1$ for all $\theta < \underline{\theta}$ and $Q(\overline{\theta}) \geq \widetilde{Q}(\overline{\theta})$, where the latter follows from the fact that $\widetilde{f}(\overline{\theta}) \geq f(\overline{\theta})$ in a support-preserving Q-decreasing perturbation.[31] Hence, we obtain that (B.24) cannot hold, which yields a contradiction and proves that on-path penalties are suboptimal under $\widetilde{f}(\theta)$.

Part 2. Suppose that on-path penalties are optimal under $f(\theta)$. By Proposition 4, the following condition holds:

$$\int_{\theta_c}^{\overline{\theta}} \big(Q(\theta) - Q(\overline{\theta})\big)\, d\theta < 0.$$

Consider a Q-increasing perturbation that yields $\widetilde{f}(\theta)$ over $\widetilde{\Theta} = \Theta$. Suppose by contradiction that on-path penalties are suboptimal under $\widetilde{f}(\theta)$. By Proposition 4, this implies

$$\int_{\theta_c}^{\overline{\theta}} \big(\widetilde{Q}(\theta) - \widetilde{Q}(\overline{\theta})\big)\, d\theta \geq 0.$$

Analogous arguments as in the proof of Part 1 imply that these two inequalities cannot simultaneously hold under a support-preserving, Q-increasing perturbation. We thus obtain a contradiction, which proves that on-path penalties are optimal under $\widetilde{f}(\theta)$.

B.8. *Proof of Proposition 9*

Denote by $\{\widetilde{\theta}^*, \widetilde{\theta}^{**}\}$ the optimal deficit limit under $\widetilde{f}(\theta)$. Observe that given the binding enforcement constraint, $\widetilde{\theta}^{**} = \rho(\widetilde{\theta}^*)$ for $\rho(\cdot)$ defined in Step 1 of the proof of Proposition 4. We prove each part of the proposition in order.

Part 1. Suppose that on-path penalties are suboptimal. By Proposition 4, the optimal deficit limits under $f(\theta)$ and $\widetilde{f}(\theta)$ set $\theta^* = \theta_c$ and $\widetilde{\theta}^* = \widetilde{\theta}_c$, respectively, where $\widetilde{\theta}_c = \theta_c$ if $\overline{\theta} = \widetilde{\overline{\theta}}$ (since θ_c and $\widetilde{\theta}_c$ are defined by (13)). To complete the proof, it is thus sufficient to prove that $\widetilde{\theta}_c$ strictly increases in $\widetilde{\overline{\theta}}$. Note that $\widetilde{\overline{\theta}} = \rho(\widetilde{\theta}_c)$, where $\rho(\cdot)$ (defined in Step 1 of the proof of Proposition 4) is strictly increasing. It thus follows that $\widetilde{\theta}_c = \rho^{-1}(\widetilde{\overline{\theta}})$ is strictly increasing in $\widetilde{\overline{\theta}}$.

[31]See footnote 25.

Part 2. We prove the result for the case of a Q-increasing perturbation. The proof for the case of a Q-decreasing perturbation is analogous and thus omitted.

Suppose that on-path penalties are optimal. By Step 4 in the proof of Proposition 4, the following two first-order conditions uniquely define θ^* and $\widetilde{\theta}^*$:

$$\int_{\theta^*}^{\rho(\theta^*)} \big(Q(\theta) - Q\big(\rho(\theta^*)\big)\big)\, d\theta = 0, \tag{B.25}$$

$$\int_{\widetilde{\theta}^*}^{\rho(\widetilde{\theta}^*)} \big(\widetilde{Q}(\theta) - \widetilde{Q}\big(\rho(\widetilde{\theta}^*)\big)\big)\, d\theta = 0. \tag{B.26}$$

By Assumption 1, these conditions require that $\theta^* < \widehat{\theta} < \rho(\theta^*)$ and $\widetilde{\theta}^* < \widetilde{\widehat{\theta}} < \rho(\widetilde{\theta}^*)$, where $\widetilde{\widehat{\theta}}$ corresponds to the analog of $\widehat{\theta}$ under the perturbed distribution. Moreover, we must have that $Q(\theta^*) > Q(\rho(\theta^*))$ and $\widetilde{Q}(\widetilde{\theta}^*) > \widetilde{Q}(\rho(\widetilde{\theta}^*))$.

Suppose that $\widetilde{f}(\theta)$ is the result of a Q-increasing perturbation satisfying the conditions in the proposition. Suppose by contradiction that $\widetilde{\theta}^* \geq \theta^*$. It then follows that

$$\theta^* \leq \widetilde{\theta}^* < \widetilde{\widehat{\theta}} < \rho\big(\widetilde{\theta}^*\big) \quad \text{and} \quad \widehat{\theta} < \rho\big(\theta^*\big) \leq \rho\big(\widetilde{\theta}^*\big) \tag{B.27}$$

and

$$\widetilde{Q}\big(\theta^*\big) \geq \widetilde{Q}\big(\widetilde{\theta}^*\big) > \widetilde{Q}\big(\rho\big(\widetilde{\theta}^*\big)\big), \tag{B.28}$$

where we observe that $\widetilde{Q}(\theta)$ is well defined at all $\theta \leq \widetilde{\overline{\theta}}$ and thus at θ^* and $\rho(\theta^*)$. Since the perturbation is Q-increasing, we can show that

$$\int_{\theta^*}^{\rho(\theta^*)} \big(Q(\theta) - Q\big(\rho(\theta^*)\big)\big)\, d\theta > \int_{\theta^*}^{\rho(\theta^*)} \big(\widetilde{Q}(\theta) - \widetilde{Q}\big(\rho(\theta^*)\big)\big)\, d\theta. \tag{B.29}$$

The inequality follows from the fact that $\widetilde{Q}(\theta) - Q(\theta) < \widetilde{Q}(\rho(\theta^*)) - Q(\rho(\theta^*))$ for all $\theta \in (\max\{\underline{\theta}, \underline{\widetilde{\theta}}\}, \rho(\theta^*))$ with $\theta^* \geq \max\{\underline{\theta}, \underline{\widetilde{\theta}}\}$. Moreover, by arguments analogous to those in the proof of Part 1 of Proposition 7, and appealing to (B.27) and (B.28), we obtain

$$\begin{aligned}\int_{\theta^*}^{\rho(\theta^*)} \big(\widetilde{Q}(\theta) - \widetilde{Q}\big(\rho(\theta^*)\big)\big)\, d\theta &\geq \int_{\theta^*}^{\rho(\widetilde{\theta}^*)} \big(\widetilde{Q}(\theta) - \widetilde{Q}\big(\rho\big(\widetilde{\theta}^*\big)\big)\big)\, d\theta \\ &\geq \int_{\widetilde{\theta}^*}^{\rho(\widetilde{\theta}^*)} \big(\widetilde{Q}(\theta) - \widetilde{Q}\big(\rho\big(\widetilde{\theta}^*\big)\big)\big)\, d\theta.\end{aligned} \tag{B.30}$$

However, combining (B.29) and (B.30) yields

$$\int_{\theta^*}^{\rho(\theta^*)} \big(Q(\theta) - Q\big(\rho(\theta^*)\big)\big)\, d\theta > \int_{\widetilde{\theta}^*}^{\rho(\widetilde{\theta}^*)} \big(\widetilde{Q}(\theta) - \widetilde{Q}\big(\rho\big(\widetilde{\theta}^*\big)\big)\big)\, d\theta,$$

which cannot hold simultaneously with equations (B.25) and (B.26). Therefore, it follows that $\widetilde{\theta}^* < \theta^*$.

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