Dynamic Opinion Aggregation: Long-run Stability and Disagreement*

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February 2021

Abstract

This paper proposes a functional-form free model of non-Bayesian social learning in networks that accounts for heuristics and biases in opinion aggregation, as well as the coexistence of layers of networks corresponding to different interaction levels. We provide conditions on the layers of networks that guarantee long-run stability of opinions, consensus formation, and efficient or biased information aggregation. Under the descriptive phenomena that we capture, at times agents ignore some of their neighbors' opinions, reducing the number of effective connections. This generates new channels toward the formation of disagreement and polarization of opinions in networks. Besides, we show that our class of updating procedures precisely characterizes agents' optimal behavior in response to a concern of disagreeing with others. Our framework bridges several scattered models and phenomena in the non-Bayesian social learning literature, thereby providing a unifying approach to the field.

JEL: *D81, D83, D85*

Keywords: *Social networks, non-Bayesian learning, polarization, wisdom of the crowd.*

*We are particularly indebted to Drew Fudenberg, Benjamin Golub, and Stephen Morris, for several suggestions that substantially improved the paper. We also wish to thank Abhijit Banerjee, Pierpaolo Battigalli, Alessandro Citanna, Krishna Dasaratha, Glenn Ellison, Joel Flynn, Mira Frick, Nicola Gennaioli, Sanjeev Goyal, Ryota Iijima, Matthew Jackson, Fabio Maccheroni, Massimo Marinacci, Pooya Molavi, Elchanan Mossel, Manuel Mueller-Frank, Paolo Pin, Drazen Prelec, Giulio Principi, Evan Sadler, Fernando Vega-Redondo, Alexander Wolitzky, and Muhamet Yildiz for useful comments. Simone Cerreia-Vioglio gratefully acknowledges the financial support of ERC (grant SDDM-TEA). Roberto Corrao gratefully acknowledges the financial support of the Marco Fanno scholarship. Giacomo Lanzani gratefully acknowledges the financial support of the Guido Cazzavillan scholarship.

1 Introduction

The rise in social media use and the parallel formation of global social networks have given more importance to studying how people change and influence their opinions over time. Economists' classical approach considers agents as statisticians whose opinions (or beliefs) are influenced by the information received and evolve according to *Bayesian* rationality. At the other extreme, we might have naive agents that repeatedly take weighted averages of the opinions they observe, i.e., *DeGroot's learning.* In both cases, the social network of links determining neighbors' identities is *fixed* and independent of the current distribution of opinions.

However, in many real-life economic interest situations, individuals fail to adjust their opinions according to either of the procedures described, for example, because of uncertainty concerns or behavioral biases.[1] First, the complex information structures in a social network induce high uncertainty about the data generating process, suggesting that even entirely rational but cautious agents should rely on more *robust estimation procedures.* Second, as we consider lower degrees of sophistication and expertise about a given topic, we observe that people rely on simpler *heuristics* to aggregate information. In these cases, the process of opinion aggregation is often influenced by documented *biases* such as *attraction to extreme or intermediate* opinions and *confirmatory bias,* which cannot be captured by naive repeated averaging. Furthermore, all these phenomena imply that the network structure revealed by the opinion dynamics is *different* from the one revealed by the totality of social interactions. For example, individuals who disregard extreme stances (robustness concerns) or are attracted by positions closer to their own (confirmatory bias) might sometimes ignore a subset of their friends' opinions. Overall, these aspects also induce a degree of long-run *disagreement* and opinions' *polarization* that would be inconsistent with either Bayesian or DeGroot's learning.

This paper proposes a unifying and functional-form-free social learning model based on intuitive properties that account for robustness concerns for the uncertainty in social networks and heuristics and biases in opinion aggregation. These features alter from period to period the links of the underlying social network depending on society's current opinions. Therefore, rather than fixing a unique exogenous network structure, we derive, directly from the updating rules, two extreme layers of networks that capture the maximum and the minimum possible degrees of influence among them. We provide conditions on the extreme layers of networks that guarantee long-run stability of opinions, consensus formation, and efficient or biased information aggregation. Besides, we show that our class of updating procedures precisely characterizes agents' optimal behavior in response to a concern of disagreeing with others (i.e., distance-based loss functions).

Along the way, we illustrate the broad reach of our model in terms of old and new documented phenomena captured. In particular, to more standard properties such as *long-run agreement* and the *"wisdom of the crowd"*, we contrast novel effects such as *opinion segregation* and the *"bias of the crowd"*. The former obtains when two possibly connected groups of agents have intrinsic preferences for median opinions but receive very different initial information: each agent of one group would ignore the other group's opinions because they were considered too extreme for their tastes. The

[1] See the empirical evidence in Breza et al. [12], Chandrasekhar et al. [20], and the references therein. In addition, when modeling Bayesian updating in a network, tractability is easily lost, see Breza et al. [11]. Notable exceptions are Mossel et al. [52] and Mueller-Frank [53].

latter obtains when in large networks, despite consensus being reached and the noise of the original sources of information being washed away, the common long-run opinion is bounded away from the truth due to society's systematic biases toward extreme opinions. In all these cases, agents tend to ignore some of their neighbors' opinions, reducing the number of effective social network connections. Importantly, these effects are obtained without relying on the presence of unrealistic stubborn agents. More in general, our framework bridges several scattered models and phenomena in the social learning literature, thereby providing a unifying approach to answer the main questions arising in this growing field.

Robust opinion aggregators Concretely, we consider a discrete-time model of opinion dynamics in a social network. Agents observe the opinions of their neighbors and repeatedly incorporate these opinions to update their own. The initial opinions are equal to a common fundamental parameter plus some agent-specific noise. Motivated by the considerations above, we model agents that, either for robustness concerns about the uncertainty on the data-generating process or intrinsic biases, pool their neighbors' opinions through *robust opinion aggregators*. These aggregators map the last-period opinions of each agent' neighbors to her current stance and satisfy the following properties:

1. **Normalization**: If the agents have reached a consensus, then none of them further updates her opinion;

2. **Monotonicity**: If two opinion profiles are such that the first one coordinatewise dominates the second, then this relation is preserved after aggregation;

3. **Translation invariance**: If *each* agent's opinion is shifted by the *same constant*, then the updates are shifted accordingly.

The first two properties have a straightforward interpretation as a minimal trust in the neighbors' opinions. Translation invariance is equivalent to assume that agents only care about the opinions' differences rather than their intrinsic levels and rules out explosive dynamics. Besides, this property is a natural consequence of a distance-minimization procedure consistent with the interpretations of the updating rule proposed in our foundation.[2]

A key feature of our model implied by these properties is that the influence among agents depends on their original opinions. Hence, it is particularly suited to capture economic phenomena such as dislike for (or attraction to) relatively extreme positions, confirmatory bias, disregard for redundant information, and assortativeness. For example, if there is an attraction to extreme opinions, then each agent's influence depends on her centrality and how extreme her original belief is relative to the entire population. This novel effect has immediate and relevant implications for designing intervention policies in networks. The intrinsic level of extremism among the agents might prevent establishing a moderate consensus even when central agents propose the latter.[3] To the best of our knowledge, the current work is the first to propose a unifying framework that allows for all the aforementioned effects.

[2] We defer a proper comparison with existing models to the discussion of the related literature.

[3] These policy interventions can assume different forms such as incentive distortions (Galeotti et al. [27]) or information design (Galperti and Perego [29]).

On the empirical side, the recent field studies that compare Bayesian to non-Bayesian social learning models have obtained evidence consistent with our properties. For instance, Chandrasekhar et al. [20] find that if the sampled subjects come to a consensus most of the time, they remain stuck on their beliefs even when such behavior is objectively suboptimal: this is consistent with normalization. Similarly, they also find that the overwhelming majority of subjects respond monotonically to changes in their neighbors' opinions.

The dynamics of robust opinion aggregation A few fundamental questions arise. Are the new dynamics induced by robust opinion aggregators completely undisciplined? Is it still possible to obtain convergence of opinions? Also, if the answer is yes, can we say anything about the formation of consensus? Do large crowds learn the true parameter?

In Theorem 1, we show that the *time-average* opinions induced by any robust opinion aggregators *uniformly* converge, so that a profile of long-run opinions always exists. This first benchmark result implies that an external agent can test the long-run learning properties of the updating procedure by computing time averages, a feature that we exploit in our results on large networks. Moreover, it is the stepping stone to derive proper convergence and consensus formation from the opinion aggregators' network properties. Indeed, we already pointed out that robust aggregators may induce several layers of networks, each capturing an increasing influence level. We next focus on the two extremes of such layers and show their relevance for the long-run evolution of opinions.

We say that an agent is *strongly influenced* by another if the former *always* reacts to variations in the latter's opinion, regardless of the current opinion profile in the society. This is a sharp form of connection among agents and may capture relationships such as parenthood, mentorship, or partnership with esteemed colleagues. In Theorem 2, we show that if each agent has at least one strong link and each closed class of the induced *strong network* is aperiodic, opinion convergence obtains. This result is powerful for two reasons. First, it guarantees that, in a comprehensive class of models, the aggregation process's sole iteration always conduces to a stable distribution of opinions in the population (i.e., Nash equilibria under best-response dynamics interpretation). Second, it highlights the critical role of the strong ties in the society to stabilize opinions in the long run.[4]

Instead, we say that an agent is *weakly influenced* by another if the former reacts to variations in the latter's opinion for *at least one opinion profile.* This is a minimal form of connection that captures any relationship between two agents that can hear their reciprocal opinions (e.g., some agents might be listened to only when stating very extreme opinions). In Theorem 3, we link the formation of consensus in the long run to the two layers of networks defined. On the one hand, if the strong network forms a unique, cohesive class, convergence to consensus always obtains. On the other hand, a necessary condition for forming consensus, regardless of the initial opinions, is that the weak network is composed of a unique, cohesive class. This last result points out that if, even by looking at the weakest links among agents, it is impossible to construct a global connection in the network, then

[4] We follow one of the two interpretations that Granovetter [36] assigned to the adjectives "strong" and "weak" for social ties. Indeed, as also argued by Centola and Macy [16], there is a dual meaning behind the "strong-weak" classification of ties: one is relational, and the other is structural. Even though only the latter one has been traditionally associated with Granovetter's theory, we stick to the former interpretation, whereby "strong" ties "connect close friends or kin whose interactions are frequent, affectively charged, and highly salient to each other" , [16, pp. 703].

there can be disagreement in the limit.

Motivated by a more detailed study of long-run disagreement and opinion segregation, we next focus on *rank-dependent robust aggregators*, whereby agents assign to each other fixed weights, which are then distorted depending on the ranking among the agents' current opinions.[5] Relevant examples of this class are: weighted means (no distortion), quantiles (extreme distortion), and trimmed means (flat distortion at the extremes). For this class, we first illustrate that opinion segregation naturally arises even when the weights induce a strongly connected network, and we perturb an established consensus. The intuition is that flat regions of the distortion function endogenously remove links induced by having positive weights. With this, original connections are broken, and opinions may become more and more segregated. In Proposition 3 and the following corollary, we characterize the structure of long-run disagreement for rank-dependent aggregators formalizing the previous intuition.

Vox populi, vox Dei? We next study the information-aggregation properties of the consensus emerging from robust aggregators in large networks.[6] This question is critical to understand to what extent the phenomenon of the *wisdom of the crowd*, whereby agents' consensus coincide with the fundamental parameter (cf. Golub and Jackson [32]), is robust to a broader class of opinion aggregators. It would seem plausible that, similarly to DeGroot's model, if a generalized measure of the influence of every agent in the revealed network is vanishing, then efficient information aggregation obtains. However, Proposition 4 shows that this condition is sufficient only for making the variance of the consensus disappear in the limit, regardless of the data generating process. In this case, if we also have symmetry of the distribution of agents' signals and the opinion aggregator, then we obtain the wisdom of the crowd. Otherwise, we obtain a phenomenon that we call the *bias of the crowd*: the consensus opinion in large populations converges to a constant that can be bounded away from the fundamental parameter. Interestingly, as we formally show for a parametric example, the bias's magnitude depends on the original information sources' noisiness. This unveils a critical link between more dispersed sources of information and the polarization between the consensus of two biased and disconnected populations.

Foundation and discrete opinions The properties of robust opinion aggregators arise when we generalize the two foundations for non-Bayesian opinion dynamics: repeated estimation of the underlying parameter with naive agents (cf. DeMarzo et al. [23]) and best response dynamics in coordination games (cf. Golub and Jackson [33]). In both cases, we allow for more general distance-based loss functions that nest the one proposed by Huber [38] for robust estimation of a location parameter. In particular, an opinion aggregator is robust if and only if there is a distance-based loss function with positive complementaries whose unique solution map coincides with the aggregator itself. Given

[5] The adjective "rank-dependent" has been extensively used in decision theory to describe decision criteria under risk or uncertainty where a probability over states is distorted depending on the ranking over the possible outcomes. Given the analogy with the class of opinion aggregators we study, we have used the same adjective. However, the analogy is purely mathematical, and we interpret our weights as degrees of confidence rather than likelihoods.

[6] Vox populi, vox Dei is Latin for: The voice of the people is the voice of God. It is often shortened to just "Vox populi" as in the original paper of Galton [30] on the wisdom of the crowd. In that paper, Galton "aggregated" opinions using the empirical median, a robust opinion aggregator.

this characterization, it is possible to reinterpret the previous results in terms of convergence to and stability of Nash equilibria and consistency of robust estimators induced by opinion aggregation.

Finally, our foundation highlights the common structure of two network phenomena that are usually modeled with very different methods: aggregation of continuous opinions and diffusion/contagion of a binary behavior such as adopting a new technology. Indeed, when we look at a subclass of robust opinion aggregators that we call *discrete*, we obtain a generalization of the threshold models of Morris [51], Kempe et al. [42], and Centola and Macy [16]. This implies that the new tools and results that we develop can help bridge the two different approaches to opinion aggregation and diffusion in networks.

Related literature This paper belongs to the literature on non-Bayesian opinion aggregation. In particular, we nest the benchmark of this class: DeGroot's model [22].[7] In this simple model, there is a clear link between the underlying network structure's properties and the long-run evolution of opinions. These features are exploited in Golub and Jackson [32] to fully characterize convergence and convergence to consensus in terms of the network structure and the wisdom of the crowd. For the former, we substantially extend the conditions of [32, Theorem 2] and we show that they are still sufficient for convergence and convergence to consensus when imposed on the strong network. Nevertheless, given the different layers of networks in our model, these conditions fail to be necessary. For the latter, we derive a general law of large numbers for robust aggregators specializing to the one of [32] for the linear case. Here the two main novelties are that: i) the maximal influence in the network, which generalizes the notion of maximal eigenvector centrality, has to vanish *sufficiently fast*; ii) Both the noise distribution and the aggregators must satisfy a symmetry property without which we obtain the bias of the crowd.

DeMarzo et al. [23] provide a microfoundation of the DeGroot's model as a repeated naive *maximum likelihood estimation* of an underlying parameter that captures a form of persuasion bias.[8] In their model, aggregation's linearity crucially relies on the assumption that the error terms are normal and independent. Our approach can also be seen as a generalization of [23] that does not impose any parametric specification and independence assumption.

Among the recent papers, the one closest to ours is Molavi et al. [50]. However, both the questions and the methodology are rather different from ours. First, they follow Jadbabaie et al. [41] in considering social learning when agents both repeatedly receive external signals about an underlying state of the world and naively combine the beliefs of their neighbors. Instead, we follow the wisdom of the crowd approach of [32], and we study the long-run opinions as the size of the society grows to infinity. Therefore, we single out the role of the network structure and the opinion aggregator in efficiently combining the agents' *initial* information as the network's size increases. In the Online Appendix, we show that for the questions we explore, log-linear aggregators a la [50] can be studied in an equivalent linear system, thus making use of the results developed for DeGroot's model and its time-varying versions. So, our results cover their aggregators too.

[7] For a comprehensive treatment of this literature see, for example, Acemoglu and Ozdaglar [1] and Golub and Sadler [35].

[8] They also allow agents to vary over time the weight they give to their own past beliefs relative to the others. Banerjee et al. [8] consider a different asynchronous departure from the DeGroot's model.

Our work is related to a multidisciplinary literature on repeated averaging procedures mostly focused on convergence to consensus. In this setting, Krause [43, Theorem 8.3.4] characterizes convergence to *consensus* with respect to a form of strict internality of the averaging procedure. Importantly, a similar internality condition is linked to the underlying network structure by Mueller-Frank [54]. Our opinion aggregators do not satisfy in general these forms of internality and have an additional normative appeal given by our characterization. Both Mueller-Frank [54] and Arieli et al. [4] address different robustness concerns in a social learning setting: in [54] it is with respect to external manipulation of the initial opinions, while in [4] it is with respect to the initial information structure of the agents. In the Appendix (see Remarks 2–5), we discuss the relation between our contribution and the mathematical literature.

Our results also make use of techniques coming from decision theory, and in particular Ghirardato et al [31], Maccheroni et al. [48], and Schmeidler [58]. The first two papers were the first to study functionals that satisfy normalization, monotonicity, and translation invariance, using nonstandard differential techniques. These techniques turn out to be useful when we discuss the wisdom of the crowd. The third paper introduced the class of comonotonic additive functionals that include the rank-dependent aggregators.

2 The model

In this section, we introduce a model of opinion aggregation in social networks that captures either an heuristic process of information acquisition or an intrinsic preference to conform. Let $N = \{1, ..., n\}$, $n \in \mathbb{N}$, denote a finite set of agents and let I be a closed interval of $\mathbb{R}$ with nonempty interior denoting the set of possible opinions. For example, if $I = [0, 1]$, then we interpret $c \in I$ as a measure of agreement on a particular instance. Let $B = I^n \subseteq \mathbb{R}^n$ denote the set of opinion profiles $x = (x_i)_{i=1}^n$. Agents are vertices of a directed *observation network* (N, A), where A is an adjacency matrix, that is, $a_{ij} = 1$ if there is a directed link from agent i to agent j, and $a_{ij} = 0$ otherwise. With this, let $N_i = \{j \in N : a_{ij} = 1\}$ denote the neighborhood of agent i. The interpretation is that agent i can only observe the opinions of her neighbors $j \in N_i$.

Time is discrete, $t \in \mathbb{N}$, and the initial opinion of agent $i \in N$ at period 0 is given by a signal

$$X_i^0 = \mu + \varepsilon_i \tag{1}$$

where $\mu \in \mathbb{R}$ is an underlying fundamental parameter and each $\varepsilon_i : \Omega \to \mathbb{R}$ is a random variable defined over a common probability space $(\Omega, \mathcal{F}, P)$.[9] Let x_i^0 denote the realization of the period-0 opinion of agent i. We model the evolution of opinions in the following periods through an *opinion aggregator*, that is, a selfmap $T : B \to B$ that for each profile of period-t opinions $x^t \in B$ returns the profile of period-$(t+1)$ updates $x^{t+1} = T(x^t)$. In particular, we let $T_i : B \to I$ denote the i-th component of T, the updating rule of agent i. Conditional on the initial opinions x^0, the deterministic sequence of updates that we study is $\{T^t(x^0)\}_{t\in\mathbb{N}}$.[10] We are particularly interested in the long-run properties of

[9] For completeness, we present the stochastic structure of initial opinions here. However, this does not have a relevant role in the analysis until Section 5 on the wisdom of the crowd.

[10] The network structure (N, A) can be reflected in the opinion aggregator T by assuming that for each $i \in N$ and for

these dynamics with respect to stability, convergence, and consensus formation.

Next, we propose three descriptively appealing properties of opinion aggregators. Let $e \in \mathbb{R}^n$ denote the vector whose components are all 1s.

Definition 1 *Let T be an opinion aggregator. We say that:*

1. *T is* normalized *if and only if $T(ke) = ke$ for all $k \in I$.*
2. *T is* monotone *if and only if for each $x, y \in B$*

$$x \geq y \implies T(x) \geq T(y).$$

3. *T is* translation invariant *if and only if*

$$T(x + ke) = T(x) + ke \qquad \forall x \in B, \forall k \in \mathbb{R} \text{ s.t. } x + ke \in B.$$

We say that T is robust *if and only if T is normalized, monotone, and translation invariant.*

Normalization requires that, whenever all the agents share the same opinion, each of the next-period updates coincides with that opinion. Monotonicity embodies a form of trust of the agents in the opinions observed from others.[11] Finally, translation invariance naturally arises when agents only care about their differences of opinions. In our related work [18], we provide a game-theoretic foundation that relaxes this property to translation subinvariance, that is, agents react less than proportionally to uniform shifts. It turns out that our main convergence results would continue to hold.

Robust opinion aggregators nest several opinion aggregation models and, at the same time, are rich enough to describe new behavioral phenomena such as aversion/attraction for extreme opinions, rank-dependent social influence, confirmatory bias, and optimism/pessimism (for illustrations, see Section 3). For example, in the widely studied DeGroot's model T is linear, hence robust. Even though this simple model has been often generalized to capture some of the aforementioned behavioral phenomena, each of these extensions rely on ad hoc tools to analyze convergence and the consensus properties. Differently, our three functional properties define a unifying but tractable framework to analyze the dynamics of opinion aggregation as well as diffusion on networks (for the latter, see Section 6.3).[12]

In Section 6.1, we characterize the dynamics induced by robust opinion aggregators in terms of repeated minimization of loss functions by agents that dislike to disagree with each other. This provides

each $x, x' \in B$

$$x_j = x'_j \qquad \forall j \in N_i \implies T_i(x) = T_i(x').$$

It is a natural assumption satisfied by all the illustrations in Section 3, but it can be dispensed with for the general analysis.

[11] Although natural, monotonicity may exclude some patterns of behavior such as when agents listen to each other only when their opinions are closer than some threshold (cf. Krause [43]). However, we can capture a similar effect in a continuous monotone model of confirmation bias (see Section 3).

[12] Note that the defining properties of our framework are well defined even for discrete opinions. In particular, we can always consider discrete opinions by assuming that the support of the initial opinions is finite. In Section 6.3, we show how our robust opinion aggregators encompass models of simple and complex contagion as Morris [51], Kempe et al [42], and Centola and Macy [16].

a foundation for our class of aggregators that naturally generalizes the one of the linear model, without committing to any specific functional form and gives a rationale for their names.[13] Here, we briefly mention two particular economic interpretations for these dynamics.

Best response dynamics Consider n agents playing a pure coordination game where I is the set of feasible actions of each player and the payoff function of agent i is

$$u_i(x) = -\phi_i(x - x_i e) \qquad \forall x \in B$$

for some loss function $\phi_i : \mathbb{R}^n \to \mathbb{R}$. In particular, the payoff of each agent only depends on the differences between her actions and the ones of her coplayers capturing, together with additional complementarity properties considered in Section 6.1, the willingness of the agents to adapt to each other's actions. Under these properties, the updating procedure T corresponds to myopic best-response dynamics of this game whenever each T_i is a selection from the best-response correspondence of agent i.[14] As a concrete example, consider the quadratic loss function

$$\phi_i(x - x_i e) = \sum_{j=1}^{n} w_{ij}(x_j - x_i)^2 \tag{2}$$

for some vector of weights $w_i \in \Delta$ such that $w_{ij} = 0$ for all $j \in N \backslash N_i$. This is the simplest payoff function considered in coordination games on networks (Ballester et al. [7]), beauty contests (Golub and Morris [34]), and team problems (Calvó-Armengol et al. [15]). In this case, the best response map is $T(x) = Wx$ for all $x \in B$, where $W \in \mathcal{W}$ is the matrix collecting the vectors of weights, and $\mathcal{W}$ denotes the collection of stochastic matrices. Finally, according to this interpretation, the fixed points of T coincide with the Nash equilibria of the underlying coordination game.

Repeated robust estimation Alternatively, consider agents on a network that try to estimate μ by repeatedly pooling the last-period estimates of their neighbors via the aggregator T, that is, T_i is the time-independent estimator used by i. DeMarzo et al. [23] considered agents updating their opinions by maximum likelihood and proposed a "persuasion bias" justification of the DeGroot's iteration procedure where agents ignore the information redundancies in their neighbors' estimates. However, even in the first period, this estimation approach is optimal only when the agents know the specific (Gaussian) parametric form of the errors. Instead, in many real-world situations (see Breza et al. [12]), the complexity of the environment does not allow the agents to attach probabilistic beliefs to the data generating process, including the network structure. Under this uncertainty, the standard approach taken in robust statistics is to minimize a loss function (see for example the seminal contribution by Huber [38]) that usually induces an aggregator satisfying the properties we studied above: e.g., the absolute loss, the p-loss where the quadratic function in (2) is replaced by a general power $p \geq 1$

[13] In particular, we point out a tight link between our aggregation procedure and the theory of robust statistics (see the dedicated paragraph below).

[14] Specifically, our updating procedure is strictly more general as it allows each agent to best respond to her own last-period opinion.

function, and the Huber loss. Therefore, one foundation for the behavior of the agents we study is that, in face of uncertainty, they repeatedly perform a generalized robust estimation procedure.

Turning to the analysis of opinion dynamics, we will be dealing with two kinds of limit of $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$, the standard one induced by the supnorm $\| \|_\infty$ as well as the one of Cesaro (i.e., time averages limit):

$$\text{C-}\lim_t T^t(x) = \lim_\tau \frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x)$$

where the limit on the right-hand side of the definition is the standard one.

Definition 2 *Let T be an opinion aggregator. We say that T is* Cesaro convergent *if and only if* C-$\lim_t T^t\left(x^0\right)$ *exists for each $x^0 \in B$. We say that T is* convergent *if and only if* $\lim_t T^t\left(x^0\right)$ *exists for each $x^0 \in B$.*

Given the initial opinions x^0, if the updates converge, then Cesaro convergence obtains and the Cesaro and the standard limit coincide. When the time average converges we denote it as

$$\bar{T}(x) = \text{C-}\lim_t T^t(x) \qquad \forall x \in B, \tag{3}$$

and we refer to $\bar{T}$ as the *long-run opinion aggregator* $\bar{T}$. Once we have stability and convergence, we study whether the profile of long-run opinions is represented by a unique consensus across all agents or by several coexisting conventions, i.e., long-run disagreement. We denote by $D \subseteq B$ the consensus subset, that is, $x \in D$ if and only if $x_i = x_j$ for all $i, j \in N$.

Definition 3 *Let T be an opinion aggregator. We say that* convergence to consensus always obtains under T *if T is convergent and $\bar{T}(x) \in D$ for all $x \in B$.*

When the opinion aggregator T is clear from the context, we drop the qualification "under T" in the previous definition.

3 Illustrations

In this section, we provide some examples of robust opinion aggregators and illustrate the long-run dynamics of opinions induced.

Median Assume that the agents best reply to the previous opinions of the opponents, but instead of minimizing a weighted quadratic loss function (2), they minimize the weighted absolute deviations:

$$\phi_i(x - ce) = \sum_{j=1}^{n} w_{ij}|x_j - c| \qquad \forall x \in B, \forall c \in I \tag{4}$$

where the values w_{ij} are the entries of a stochastic matrix W. It is well known that the solution correspondence admits as a selection the robust opinion aggregator T,

$$T_i(x) = \min \left\{ c \in \mathbb{R} : \sum_{j: x_j \leq c} w_{ij} \geq 0.5 \right\} \qquad \forall x \in B, \forall i \in N, \tag{5}$$

that is, $T_i(x)$ is the (weighted) median of x. We next illustrate via a simple example the dynamics induced by this particular robust opinion aggregator.

Example 1 A group of agents $N = \{1, 2, 3, 4\}$ share their opinions $x^0 \in B = [0, 1]^4$. The weights assigned to the other agents are represented by the matrix

$$W = \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.1 & 0.2 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}.$$

Aggregation through weighted *averages* would achieve consensus in the limit (see, e.g., [32]). However, the dynamics induced by using the median are qualitatively different.

If $x^0 = \left(x_1^0, 1, 1, 1\right)$, then the block of agents agreeing on the higher opinion is sufficiently large to attract agent 1 to the same opinion, and the limit (consensus) opinion of $(1, 1, 1, 1)$ is reached in one round of updating. Note that the initial opinion x_1^0 of agent 1 is irrelevant given the agreement of the other agents. Similarly, the same limit consensus obtains if agent 2 disagrees with the initial consensus, that is, if $x^0 = \left(1, x_2^0, 1, 1\right)$.

Instead, convergence to consensus fails if the initial opinions of *both* agents 1 and 2 fall. If $x^0 = (0, 1/2, 1, 1)$, then their first round of updating gives $x^1 = (1/2, 1/2, 1, 1)$, and this opinion segregation will be the limit outcome: a strongly connected society fails to reach consensus without a sufficiently large block of initial agreement. This highlights how, contrarily to the DeGroot aggregator, with median aggregation a *joint* deviation from consensus by a group of agents might be necessary to destabilize an initial consensus.[15] In Proposition 3 and Corollary 3, focusing on a class of aggregators that we introduce next, we characterize the coalitions of agents that, with a joint deviation, can destabilize consensus.

If $x^0 = (0, 1/2, 0, 1)$, then the agents' first update is $x^1 = (0, 0, 1, 0)$ and agents 1 and 2 never change their opinions again, whereas agents 3 and 4 keep reciprocally switching their opinions. This shows that even convergence may not be guaranteed. On the one hand, in Proposition 7 we give necessary and sufficient conditions for convergence of a class of robust opinion aggregators including the median. On the other hand, in spite of the failure of global convergence, the current example suggests to focus on the convergence of the time averages of the updates given their cyclical dynamics.

[15] Note that in the corresponding DeGroot model with matrix W, both an individual and a joint deviation would still lead to a consensus but on a different opinion.

In this case, it is possible to show that

$$\text{C-}\lim_t T^t\left(x^0\right) = (0,0,1/2,1/2)\,.$$

In general, Theorem 1 shows that each robust opinion aggregator is convergent in this weaker sense, no matter what is the initial condition. ▲

Rank-dependent influence The median aggregator features at the same time rank-dependent influence across agents and, possibly, polarizing dynamics. The former property characterizes a more general class of robust opinion aggregators that we now define. Consider a stochastic matrix W, whose positive entries implicitly define the observation network. But differently from the DeGroot's model, we allow for agents who use a distorted collection of weights. Formally, we say that T^f is a rank-dependent aggregator if, for all $i \in N$,

$$T_i^f(x) = \sum_{j=1}^n x_j \left[f_i\left(\sum_{l:x_l \leq x_j} w_{il}\right) - f_i\left(\sum_{l:x_l < x_j} w_{il}\right)\right] \qquad \forall x \in B, \tag{6}$$

where $f = (f_i : [0,1] \to [0,1])_{i=1}^n$ is a profile of weakly increasing *distortion functions* such that $f_i(0) = 0$ and $f_i(1) = 1$ for all $i \in N$.[16]

In Proposition 2 we show that it is easy to link the matrix of weights W to the convergence of the corresponding updating dynamics whenever the distortion functions are *strictly increasing*. In particular, Theorem 2 guarantees (via Corollary 1) that T^f is convergent whenever the diagonal of W has strictly positive entries. Moreover, by Theorem 3, convergence to consensus always obtains under T^f if and only if there is a unique strongly connected and closed group of agents in the matrix of connections derived from W.

An example of a strictly increasing distortion function with a clear psychological interpretation is given by

$$f_i(s) = q_i^{(-\ln(s))^{\alpha_i}} \qquad \forall s \in (0,1] \tag{7}$$

where $q_i \in (0,1)$ and $\alpha_i \in \mathbb{R}_{++}$.[17] The parameter α_i captures the attitudes of agent i with respect to extreme opinions: attraction ($\alpha_i \in (0,1)$) or aversion ($\alpha_i \in (1,\infty)$). The parameter q_i captures the relative concern of agent i for stating an opinion that is higher ($q_i \in (0,1/2)$) or lower ($q_i \in (1/2,1)$) than the opinions of her neighbors. To see why the parameter q_i captures the asymmetric concern for disagreement of agent i, note that, as aversion to extreme opinions increases ($\alpha_i \to \infty$), the

[16] For each $i \in N$, the rank-dependent aggregator $T_i^f : B \to I$ of agent i is a Choquet integral against the capacity obtained by distorting the probability distribution $w_i \in \Delta$ with respect to f_i (see [49, Example 4.6]), hence, by [58], it is robust. Note in particular that the functional form of the aggregator in (6) is analogous to the decision criterion in cumulative prospect theory.

[17] Clearly, f_i is defined only on $(0,1]$, but it also admits a unique continuous extension to $[0,1]$. The extension takes value 0 in 0. In particular, we obtain Prelec's probability weighting function [56] when $q = 1\backslash e$. More generally, using an f_i different from the identity map is a way to introduce a *perception bias* a la Banerjee and Fudenberg [9] in a model of naive and nonequilibrium learning.

corresponding rank-dependent aggregator of agent i converges pointwise to

$$T_i^{q_i}(x) = \min\left\{c \in \mathbb{R} : \sum_{j:x_j \le c} w_{ij} \ge q_i\right\} \qquad \forall x \in B,$$

that is, the weighted q_i-quantile. In particular, we get back to the weighted median in (5) when $q_i = 0.5$. The q-quantile aggregators capture the idea of a (extreme) truncation of the sample of opinions effectively taken into account. Indeed, the essential feature of these particular rank-dependent aggregators is the extreme flatness of the corresponding weight distortion function $f_i(s) = \mathbf{1}[s \ge q_i]$, $s \in [0,1]$. With this, for each opinion profile $x \in B$, agent i is only influenced by the neighbor with the opinion corresponding to q_i-quantile of the distribution of opinions induced by the profile x and the weights $w_i \in \Delta$. In the case of continuous opinions, a less extreme form of truncation might be desirable. For example, agent i aggregates opinions with a trimmed mean with thresholds $\underline{q}_i, \bar{q}_i \in [0,1]$, $\underline{q}_i < \bar{q}_i$, if her distortion function is

$$f_i(s) = \begin{cases} 0 & if \quad s < \underline{q}_i \\ \frac{s-\underline{q}_i}{\bar{q}_i - \underline{q}_i} & if \quad s \in [\underline{q}, \bar{q}_i] \\ 1 & if \quad s > \bar{q}_i. \end{cases} \qquad \forall s \in [0,1] \tag{8}$$

The q_i-quantile is the limit case in which both $\underline{q}_i$ and $\bar{q}_i$ converges to q.

In Figure 1 we illustrate some natural distortions. The first graph shows two distortion functions as in (7), where $\alpha_i < 1$ for the red agent and $\alpha_i > 1$ for the blue agent. The second graph shows two distortions that truncate part of the observed sample. The third graph shows pure directional biases: indeed, concave (resp. convex) distortion functions capture overweighting of higher (lower) opinions.

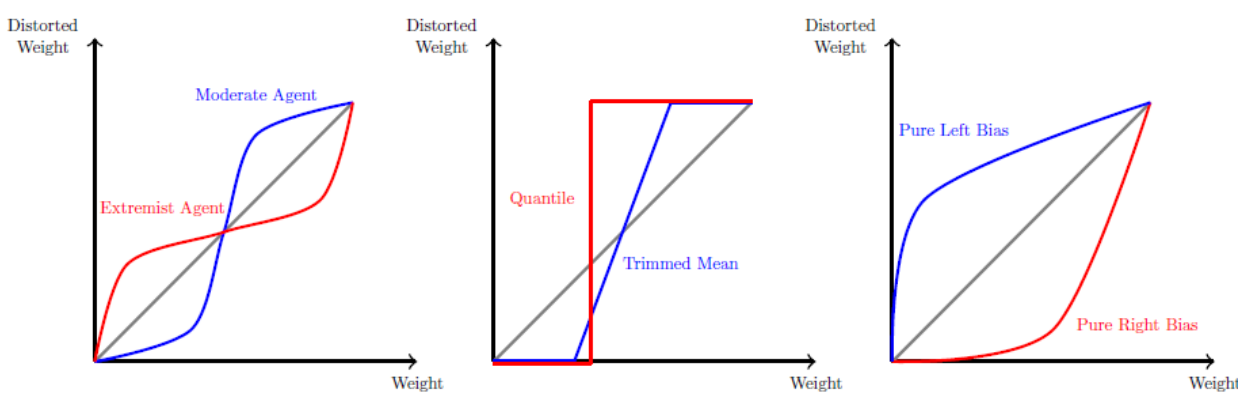

Figure 1

Notice that flat regions of f_i imply that agent i disregards the opinions of some of her neighbors depending on the current ranking of opinions. For example, suppose that the opinion of j is currently the lowest among the opinions of the neighbors of agent i. If the weight that agent i puts on j's opinion is not too high, that is $w_{ij} < \underline{q}_i$, then i completely ignores j's opinion. Differently, whenever the weight on the opinion of j is high enough, that is $w_{ij} > \max\left\{\underline{q}_i, \bar{q}_i\right\}$, agent i will be always influenced by j regardless of the current opinion profile. In this second case, we say that j *strongly influences* i. It turns out that this relation can be defined for an arbitrary robust opinion aggregator T and it allows us to derive (see Definition 4) an endogenous network structure $\underline{A}(T)$. This *strong network* is at the heart of all our sufficient conditions for convergence and convergence to consensus in Section 4. Importantly, for rank-dependent aggregators, it is clear that flat regions of the distortion functions f generate a wedge between the observation network implicit in W and the strong network $\underline{A}\left(T^f\right)$. This is a crucial feature that we exploit in Section 4.4 to show how, with our opinion aggregators, long-run disagreement may or may not arise in the same strongly connected network. We next illustrate this point in a particular example.

Example 2 (The islands model) Suppose that the network of agents (N, A) is partitioned in m groups $\{N_p\}_{p=1}^m$, that is,

$$N = \bigcup_{p \in G} N_p,$$

where $N_p \cap N_{p'} = \emptyset$ for all $p, p' \in G = \{1, ..., m\}$ such that $p \neq p'$. For example, these groups might capture similar cultural or social background of the agents. Also, consider a strongly connected observation network A with $a_{ii} = 1$ for each $i \in N$. So far, there is no relation between the neighborhood N_i of an agent i and the group she belongs to, denoted N_{p_i}. In order to relate these two objects, let us define the *internal linking fraction* of $i \in N$ as

$$\ell_i = \frac{|\{j \in N_{p_i} : a_{ij} = 1\}| - 1}{|N_i|}.$$

According to our interpretation of the groups, the $(\ell_i)_{i \in N}$ capture the degree of homophily in the given network structure: agents with an high ℓ_i are connected with many neighbors belonging to their own group N_{p_i}. A stylized picture of real-world networks that has been fruitfully used in the literature (cf. Golub and Jackson [33]) is the island structure with a large internal linking fraction ℓ_i of every agent i.

Consider the uniform stochastic matrix W such that $w_{ij} = \frac{1}{|N_i|}$ if $j \in N_i$ and $w_{ij} = 0$ otherwise. Suppose that each agent $i \in N$ aggregates the opinions she observes in her neighborhood N_i using a trimmed mean T_i with weights given by W and $\underline{q}_i = 1 - \bar{q}_i = \alpha/2$ where $\alpha \in [0, 1)$. In words, every agent computes the arithmetic average of the opinions she observes discarding both the $\alpha/2$ highest and lowest opinions. The DeGroot's model, obtained as a particular case by setting $\alpha = 0$, would still predict convergence to consensus in the long run. However, if $\ell_i \geq 1 - \alpha/2$ for each $i \in N$ (i.e., high homophily), then disagreement is a typical outcome for the long-run opinion dynamics.

We next illustrate this point by studying the evolution of opinions in the society when, starting from a consensus $ke \in B$, the opinions of a subset $M \subseteq N$ of agents are shifted upwards, that is,

$$x_i^0 = \begin{cases} k + \delta_i & if \quad i \in M \\ k & \text{otherwise,} \end{cases}$$

with $\delta_i > 0$ and $k + \delta_i \in I$ for all $i \in M$. For example, we can interpret this shock as follows:

a subset of agents M is targeted by a marketing campaign and persuaded to increase the use of a certain technology (as in Sadler [57]). Crucially, the extent of opinion segregation in the new long-run dynamics will depend on the identities of the agents in the subgroup in relation to the islands' structure. If the shock is local, that is, $M = N_p$ for some $p \in G$, then the long-run limit will be such that $\lim_t T_i^t\left(x^0\right) > k$ if $i \in M$, and $\lim_t T_i^t\left(x^0\right) = k$ if instead $i \notin M$. Differently, if the shock is dispersed, that is $|M \cap N_p| \leq 1$ for every $p \in \{1, ..., m\}$, then the long-run limit will be such that $\lim_t T_i^t\left(x^0\right) = k$ for every $i \in N$.

If the number of islands is greater than the size of each island, that is, $m \gg |N_p|$ for all $p \in G$, then the second shock involves a much larger subgroup of agents. Nevertheless, the deviation of each agent in the subgroup is washed out within each island and the original consensus is restored. Instead, if the targeted set of agents M is smaller but more cohesive, as in the first case, then the original consensus is broken. This phenomenon resembles the so called "complex contagion" theory of Centola and Macy [16], whereby a few long ties are not sufficient to spread an increased opinion globally. In contrast, in the DeGroot model, both shocks would have lead to the formation of a new higher consensus. ▲

The failure of global convergence to consensus in spite of a strongly connected observation network is due to the wedge between the latter and the strong network. It is easy to see that, whenever $\ell_i \geq 1 - \alpha/2$ for each $i \in N$, no agent strongly influences any other agent, that is, the strong network is empty. Examples like this one suggest to analyze other layers of networks that are weaker than the one of strong ties. The weakest possible layer can be defined as follows: agent i weakly influences j if there exists at least one opinion profile such that the aggregator of i is responsive to marginal changes in the opinion of j (see Definition 7). This *weak network* $\bar{A}(T)$ is at the heart of our necessary conditions for convergence to consensus in Section 4. In the island model just analyzed, the weak network coincides with the observation network.

Confirmatory bias In some societies, individuals tend to trust more those sources of information whose opinion confirms their original stance.[18] This phenomenon can be captured by a modification of DeGroot's linear model which is a generalization of the one proposed by Jackson [39]. Let $I = [0, 1]$ and assume that the observation network is represented by an adjacency matrix A with strictly positive diagonal. As in the linear model, society is represented by a stochastic matrix $W(x)$ where $w_{ij}(x)$ is the weight assigned by individual i to agent j. Differently from the linear model, the weight is allowed to depend on the vector x. Moreover, it is assumed that each individual downweights the agents who disagree the most with her:

$$T(x) = W(x)\, x \qquad \forall x \in B$$

where

$$w_{ij}(x) = \frac{a_{ij} e^{-\gamma_{ij}|x_i - x_j|}}{\sum_{l=1}^{n} a_{il} e^{-\gamma_{il}|x_i - x_l|}} \qquad \forall x \in B$$

with $\gamma_{ij} \in (0, 1]$ for all $i, j \in N$ and $\sum_{j=1}^{n} \gamma_{ij} \leq 1$ for all $i \in N$. Here, $1/\gamma_{ij}$ captures the relative strength of the weight assigned by individual i to agent j net of the difference in their opinions. It is easy to see that the aggregator T is robust. In Section 3, we obtain two general results that covers the

[18] See, e.g., Golub and Jackson [33].

dynamics induced by confirmatory bias. In particular, here we exploit the fact that both the strong and the weak network induced by T coincide with the observation network. First, by Theorem 2, T is convergent. Second, by Theorem 3, convergence to consensus always obtains under T if and only if the network (N, A) features a unique strongly connected and closed group of agents. Finally, even if we have considered the aggregator T as a primitive object, our characterization theorem in Section 6.1 provides a formal microfoundation of these dynamics.

Biased aggregation and opinions' dispersion Consider again agents that best reply to the previous opinions of the opponents at each period. Within this interpretation of our dynamics, a restriction imposed by the quadratic loss in (2) is that upward and downward discrepancies are felt as equally harming by every agent. It might be the case that (some) agents dislike more one or the other. A smooth and tractable robust opinion aggregator that takes in account these asymmetries is obtained by minimizing

$$\phi_i^\lambda (x - ce) = \sum_{j=1}^{n} w_{ij} \left[\exp\left(\lambda \left(x_j - c\right)\right) - \lambda \left(x_j - c\right)\right] \qquad \forall x \in B, \forall c \in I \tag{9}$$

where $\lambda \neq 0$ and the values w_{ij} are the entries of a stochastic matrix W. In particular, whenever $\lambda > 0$, upward deviations from i's current opinion are more penalized than downward deviations and vice versa whenever $\lambda < 0$.

We next show that there exists a unique solution function T_i^λ for each minimization problem induced by ϕ_i^λ. Furthermore, the specific properties of the induced robust opinion aggregator offer a glimpse into our more general findings about long-run opinions and how they differ from the linear case. In particular, for this parametric class, we derive an explicit formula for the induced long-run opinion aggregator.

Proposition 1 *Let ϕ be the profile of loss functions $\left(\phi_i^\lambda : \mathbb{R}^n \to \mathbb{R}\right)_{i=1}^n$ as in (9) with $W \in \mathcal{W}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. The following statements are true:*

1. *For each $i \in N$ we have that*

$$T_i^\lambda (x) = \operatorname{argmin}_{c \in \mathbb{R}} \phi_i^\lambda (x - ce) = \frac{1}{\lambda} \ln \left(\sum_{j=1}^{n} w_{ij} \exp\left(\lambda x_j\right) \right) \qquad \forall x \in B \tag{10}$$

and T^λ is a robust opinion aggregator.

2. *For each $i \in N$ we have that*

$$\lim_{\lambda \to \hat{\lambda}} T_i^\lambda (x) = \begin{cases} \max_{j : w_{ij} > 0} x_j & \text{if } \hat{\lambda} = \infty \\ \sum_{j=1}^{n} w_{ij} x_j & \text{if } \hat{\lambda} = 0 \\ \min_{j : w_{ij} > 0} x_j & \text{if } \hat{\lambda} = -\infty \end{cases} \qquad \forall x \in B.$$

3. If there exists a vector $s \in \Delta$ such that

$$\lim_t W^t x = \left(\sum_{i=1}^n s_i x_i \right) e \qquad \forall x \in \mathbb{R}^n, \tag{11}$$

then T^λ is convergent and

$$\bar{T}^\lambda (x) = \frac{1}{\lambda} \ln \left(\sum_{i=1}^n s_i \exp (\lambda x_i) \right) e \qquad \forall x \in B.$$

Point 1 gives an explicit functional form for the opinion aggregator. Point 2 shows that this functional form encompasses the linear case as a limit, but also allows for behavior which is nonneutral toward the direction of disagreement. In point 3, we see how it is not just the network structure that determines the limit influence of each agent, but the initial opinion also plays a key role. Indeed, the marginal contribution to the limit of agent i's opinion x_i is proportional to $s_i \exp(\lambda x_i)$. Therefore, when $\lambda > 0$, the higher the initial signal realization of an individual is, the higher is her marginal contribution to the limit. This fact has extremely relevant consequences. For example, consider one of the classical applications of non-Bayesian learning, technology adoption in a village of a developing country, with an opinion vector representing how much the agents have invested in the new technology (e.g., the share of land cultivated with the new technology). There, $\lambda > 0$ captures the idea that the most innovative members of the society have a disproportionate influence on the others, maybe because their performance attracts relatively more attention. If resources are limited, i.e., if the external actor can only increase adoption for an agent directly, relying on the network aggregation for the rest, the policy prescription is qualitatively different. Indeed, she should choose the agent j for which $s_j \exp(\lambda x_j)$ is maximized, combining the standard eigenvector centrality s_j with a distortion increasing in the initial opinion x_j of agent j.

Finally, we observe that the functional form obtained in Proposition 1 links the dispersion of the initial opinions of the agents to the level of their limit consensus. More precisely, consider two agents i, j sharing the same influence under the linear model, i.e., such that $s_i = s_j > 0$. If their initial opinions are strictly more dispersed (resp. more concentrated) in the second-order dominance sense, then the limit consensus is strictly higher (lower) if and only if $\lambda > 0$. Therefore, there is scope for manipulating the informativeness of the agents' initial opinion to increase (or decrease) the limit consensus. More in general, in Section 5, we show that, for large populations, the long-run consensus converges in probability to a constant that may differ from the fundamental parameter μ, and if the error terms $\{\varepsilon_i\}_{i \in N}$ are uniformly bounded and independent, then the probability limit is

$$\mu + \frac{1}{\lambda} \ln (\mathbb{E} (\exp (\lambda \varepsilon))) .$$

Here, the bias of the crowd depends on the distribution of ε_i and increases with the noisiness of the initial signals. In example 4, we leverage this observation to match some stylized facts that link the noisiness of information to opinions' bias and polarization.

We close this section with an observation. Even though in each illustration presented all the agents aggregate opinions according to aggregators of the same subclass, almost all the results we provide in the next sections apply to arbitrary combinations of robust opinion aggregators.

4 The dynamics of robust opinion aggregation

In the rest of the paper, we analyze the dynamics induced by iterated robust opinion aggregators. In this section, we focus on the opinions' limit for a given population size. In Section 5, we let the size of the network grow to analyze the asymptotic properties of the consensus opinion emerging from aggregation.

4.1 Convergence of time averages

Example 1 suggests that, even if the sequence of updates of a robust opinion aggregator might not converge, their time averages do.

Theorem 1 *If T is a robust opinion aggregator, then T is Cesaro convergent. Moreover, the long-run opinion aggregator $\bar{T} : B \to B$ is a robust opinion aggregator such that $\bar{T} \circ T = \bar{T}$, and if $\hat{B}$ is a bounded subset of B, then*

$$\lim_{\tau} \left(\sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^{t}(x) - \bar{T}(x) \right\|_{\infty} \right) = 0. \tag{12}$$

The previous theorem shows that the time-average opinions stabilize in the long run. The *Cesaro limit* is formally described by the long-run opinion aggregator $\bar{T}$ that, for every initial profile of stances $x \in B$, returns the long-run average opinion of each agent (see equation (3)). In particular, $\bar{T}$ is robust and satisfies the fixed point equation $\bar{T} \circ T = \bar{T}$, hence generalizing the well known notion of *eigenvector centrality* of the DeGroot model (cf. influence vector in [32]). In addition, whenever the initial opinions of the agents are known to belong to a bounded set, the initial realizations of their signals do not affect the *rate of convergence* of time averages. Therefore, the time needed for the information to stabilize on average does not depend on the objective data generating process, but only on the updating procedures used by the agents.

In Section 5, we give conditions under which the *Cesaro limit* of the updates converges in probability to the true underlying parameter μ, provided the size of society goes to infinity. If the robust opinion aggregator T happens to be convergent, then this implies the *wisdom of the crowd*: agents are going to learn the true parameter. Instead, if T is not convergent, still there is *wisdom from the crowd*: an external observer that has access to the time averages of at least a subset of the society can extract enough information to learn the truth.

4.2 Strong ties and stable long-run opinions

In the standard DeGroot's linear model, convergence is implied by the properties of an underlying network structure. This can either be implicit and given by the matrix of weights W (e.g., Golub and Jackson [32]) or be explicit and given by a primitive observation network (e.g., DeMarzo et al. [23]).

Here, we explore both paths and show they are also linked when T is robust. We extend the notion of indicator matrix to the nonlinear case by defining when an agent is strongly connected to another one.[19] A piece of notation: $e^j \in \mathbb{R}^n$ denotes the j-th vector of the canonical basis.

Definition 4 *Let T be an opinion aggregator. We say that j* strongly influences *i if and only if there exists $\varepsilon_{ij} \in (0,1)$ such that for each $x \in B$ and for each $h > 0$ such that $x + he^j \in B$*

$$T_i\left(x + he^j\right) - T_i\left(x\right) \geq \varepsilon_{ij} h. \tag{13}$$

We say that $\underline{A}(T)$ is the matrix of strong ties *of T if and only if for each $i, j \in N$ the ij-th entry is such that*

$$\underline{a}_{ij} = \begin{cases} 1 & \text{if } j \text{ strongly influences } i \\ 0 & \text{otherwise} \end{cases}.$$

The interpretation of (13) is simple: the update of i increases at least linearly in the opinion of j no matter what is the current opinion profile. The strong directed network given by $\underline{A}(T)$ is the *minimal* network underlying the opinion aggregator T. This is because the condition for $\underline{a}_{ij}$ to be equal to 1 requires that the updates of i are responsive to changes in the opinion of j, starting from every vector of opinions (cf. Example 2). When T is linear, the adjacency matrix $\underline{A}(T)$ coincides with the indicator matrix $A(W)$ of W, that is the matrix collecting the unweighted links.

The strong network $(N, \underline{A}(T))$ is instrumental in providing sufficient conditions for convergence that have an immediate interpretation in terms of connections among agents. We recall some terminology from the network literature first. Consider an arbitrary network (N, A') and $M \subseteq N$. A path in M is a finite sequence of agents $i_1, i_2, ..., i_K \in M$ with $K \geq 2$, not necessarily distinct, such that $a'_{i_k i_{k+1}} = 1$ for all $k \in \{1, ..., K-1\}$. In this case, the length of the path is $K - 1$. A cycle in M is a path such that $i_1 = i_K$. A cycle is simple if and only if the only agent appearing twice in the sequence is the starting (and ending) one. We say that M is strongly connected if and only if, for each $i, j \in M$, there exists a path in N such that $i_1 = i$ and $i_K = j$. We say that M is closed if and only if for each $i \in M$, $a'_{ij} = 1$ implies $j \in M$. Finally, M is aperiodic if and only if the greatest common divisor of the lengths of its simple cycles is 1.[20]

Definition 5 *Let T be an opinion aggregator. We say that T is* strongly aperiodic *if and only if each closed group M is aperiodic in $\underline{A}(T)$.*

This definition coincides with the definition of strongly aperiodic proposed by Golub and Jackson [32, Definition 7] for the linear case. One key difference with the linear case is that it might be emptily satisfied: for example, if an agent is not strongly influenced by anyone. This is always ruled out in the linear case since W is assumed to have a nonzero element in each row. With the next property, we rule out this possibility.

[19] Formally, the indicator matrix $A(W)$ of an arbitrary $W \in \mathcal{W}$ is such that its ijth entry is equal to 1 if w_{ij} is strictly positive and 0 otherwise.

[20] With a slight abuse of notation, we often mention only the adjacency matrix A' instead of the explicit network (N, A') when referring to properties of a class M of agents in that network.

Definition 6 *Let T be an opinion aggregator. We say that T has a* nontrivial network *if and only if, for each $i \in N$, there exists $j \in N$ such that $\underline{a}_{ij} = 1$.*

The next result links the properties of the network of strong ties with the stability of the long-run opinions.

Theorem 2 *Let T be a robust opinion aggregator. If T is strongly aperiodic and has a nontrivial network, then T is convergent.*

Our convergence theorem significantly generalizes in one direction Golub and Jackson [32, Theorem 2]. Similarly to the latter, it obtains convergence even under the existence of multiple closed groups that do not strongly influence each other. In both cases, the necessary requirement is that the aggregator is aperiodic within each of these groups. Besides its generality, there are two main differences between Theorem 2 and the result in [32, Theorem 2]. First, our sufficient condition for convergence does not completely prevent any form of communication among the closed groups (see Section 4.3). Second, these conditions are not necessary for convergence (see Example 6).

As an immediate corollary of our convergence theorem, if each agent strongly influences herself then convergence is always guaranteed.

Corollary 1 *Let T be a robust opinion aggregator. If T is* self-influential*, that is $\underline{a}_{ii} = 1$ for all $i \in N$, then T is convergent.*

Self-influentiality characterizes a situation where the opinion of each agent has a form of own-history dependence: information gathered in the past is not entirely dismissed in light of new evidence. Relatedly, Chandrasekhar et al. [20] find that the behavior of most of the subjects is consistent with self-influentiality, even when this is objectively suboptimal.

4.3 Layers of network and consensus

The existence of a strong influence of j over i requires that i listens to j no matter what are the current stances of the society. In real social networks, these strong links characterize only a subset of all the connections: close friends, own past opinions (anchoring effect), or an extremely reliable source (more generally, "strong ties" a la Granovetter [36]). However, there might be additional links (i.e., "weak ties") not in $\underline{A}(T)$ that are active only under exceptional circumstances. For example, a person may completely discard the opinion of a non-close friend of her when this is too extreme compared to the ones of the rest of her neighbors. Conversely, there are topics involving potential high stakes risks (e.g., vaccinations) for which a person may well be influenced by the opinion of someone outside her personal network, whenever the latter reports an extremely negative stance (e.g., isolated serious adverse reactions to vaccines). These observations suggest the presence of several *layers of influence networks* associated with the same observation network.[21]

[21] These layers can be mapped into the data collected on the field. For example, in their analysis of the network structure of the Indonesian villages Alatas et al. [2] identify both the strong familial ties and the links due to the extreme relative wealth of some agents.

In our model, it is easy to differentiate these facets of a social networks. On one extreme, $\underline{A}(T)$ captures the strong connections in the society as already observed. On the other extreme, it is possible to capture the weak connections as follows.

Definition 7 *Let T be an opinion aggregator. We say that j* weakly influences *i if and only if there exist $x \in B$ and $h > 0$ such that $x + he^j \in B$ and*

$$T_i\left(x + he^j\right) - T_i(x) > 0.$$

We say that $\bar{A}(T)$ is the matrix of weak ties *of T if and only if for each $i, j \in N$ the ij-th entry is such that*

$$\bar{a}_{ij} = \begin{cases} 1 & \text{if } j \text{ weakly influences } i \\ 0 & \text{otherwise} \end{cases}.$$

Intuitively, i is weakly influenced by j if there are circumstances under which a change in the opinion of j affects her update. It is plain to see that, in general, $\underline{A}(T) \leq \bar{A}(T)$ and, if T is linear with matrix W, then $A(W) = \underline{A}(T) = \bar{A}(T)$. Therefore, in the DeGroot model, it is impossible to separate these two extreme layers of networks. The additional richness of our model has several implications for the opinion dynamics. In particular, our next result shows that the same network property, capturing cohesiveness, is sufficient for convergence to consensus when satisfied by the strong network, whereas it is necessary when satisfied by the weak network.

Theorem 3 *Let T be a robust opinion aggregator. The following statements are true:*

1. *Convergence to consensus always obtains if T has a nontrivial network, $\underline{A}(T)$ has a unique strongly connected and closed group M, and M is aperiodic under $\underline{A}(T)$.*

2. *Convergence to consensus always obtains only if $\bar{A}(T)$ has a unique strongly connected and closed group M and M is aperiodic under $\bar{A}(T)$.*

The network condition considered above characterizes convergence to consensus whenever $\underline{A}(T) = \bar{A}(T)$, and, in particular, whenever T is linear (cf. Jackson [39, Corollary 8.1]). Furthermore, there are simpler properties of the strong network structure $\underline{A}(T)$ that imply this condition. For example, whenever $\underline{A}(T)$ as a whole is strongly connected and aperiodic, then Theorem 3 implies that convergence to consensus always obtains.

Another condition implying always convergence to consensus via Theorem 3 is that T has the *pairwise common influencer property*: for all agents $i, j \in N$, there is a common influencer $l \in N$ such that l strongly influences both i and j. This is a minimal requirement about the presence of a direct source of information relied upon by both agents. A typical situation where we expect the pairwise common influencer property to hold is one of asymmetric networks with a bunch of media followed by the agents. If there is a minimal overlapping in the media trusted by the agents, the property holds. Moreover, Jackson et al. [40] shows that strategic considerations naturally lead to networks that satisfy this property.

Corollary 2 *Let T be a robust opinion aggregator. If T has the pairwise common influencer property, then convergence to consensus always obtains.*

Note that a particular case in which T has the pairwise common influencer property is that each agent is strongly influenced by a common influencer $k \in N$. Therefore, Corollary 2 implies that, if agents share (pairwise or uniformly) a strong first-hand source of information, then consensus is always achieved in the limit.

Remark 1 The results in this section leveraged on our main convergence result, Theorem 2. This proof strategy has the drawback of not elaborating on the rate of convergence. Alternatively, we could obtain these results via a nonlinear version of a well-known fact: in DeGroot's model, convergence to consensus happens if and only if there exists $\hat{t} \in \mathbb{N}$ such that some column k of $W^{\hat{t}}$ is strictly positive (see, e.g., Jackson [39, Corollary 8.2]). In our model, this condition is generalized as follows: there exist $\hat{t} \in \mathbb{N}$ and $k \in N$ such that, for every $t \geq \hat{t}$, agent k strongly influences every other agent in the population under the endogenous network $\left(N, \underline{A}\left(T^t\right)\right)$. This is equivalent to (i) in Theorem 3 from which, not only we would have that convergence to consensus always obtains, but we could also obtain bounds on the rate of convergence: there exists $\varepsilon \in (0,1)$ such that[22]

$$\left\|\bar{T}(x) - T^t(x)\right\|_\infty \leq 2(1-\varepsilon)^{\left\lfloor \frac{t}{\hat{t}} \right\rfloor} \|x\|_\infty \qquad \forall t \in \mathbb{N}, \forall x \in B.$$

In particular, if T satisfies (i) in Theorem 3 with $M = N$, then $\hat{t}$ can be chosen to be the smallest integer such that each entry of $\underline{A}(T)^{\hat{t}}$ is strictly positive, i.e., $\hat{t}$ is the smallest integer such that for each $i, j \in N$ there exists a path of length $\hat{t}$ from i to j. This allows us to provide several bounds, for example, it is known that $\hat{t} \leq d^2 + 1$ where d is the diameter of the network $(N, \underline{A}(T))$ (see, e.g., Neufeld [55]) or $\hat{t} \leq n + s(n-2)$, provided the shortest (simple) cycle has length $s \geq 1$ (see, e.g., Horn and Johnson [37, Theorem 8.5.7]). ▲

Finally, note that an implication of point 2 in Theorem 3 is that, if the network of weak ties $\bar{A}(T)$ is not too cohesive, then there exists at least one profile of initial opinions such that *long-run disagreement* obtains, that is, $\bar{T}(x) \in B \backslash D$, for some $x \in B$.

4.4 Rank-dependence and disagreement

We close this section by focusing on rank-dependent aggregators introduced in Section 3 and their opinion segregation properties. We first show that both the network layers of a rank-dependent aggregator are equivalent to the observation network, provided that each distortion function is strictly increasing.

Proposition 2 *Let T^f be a rank-dependent aggregator with matrix of weights $W \in \mathcal{W}$. If f_i is strictly increasing for all $i \in N$, then $\underline{A}\left(T^f\right) = \bar{A}\left(T^f\right) = A(W)$.*

When the distortion functions of all the agents are strictly increasing, as in equation (7), the effective weights used by the agents depend on the current opinion profile but they are strictly positive

[22] Recall that, given $s \in (0, \infty)$, $\lfloor s \rfloor$ is the integer part of s, that is, the greatest integer $l \in \mathbb{N}_0$ such that $s \geq l$.

if and only if the corresponding agents are connected in W. We then immediately have that T^f is convergent if W is strongly aperiodic. Moreover, convergence to consensus always obtains under T^f if and only if W has a unique strongly connected and closed group M, and M is aperiodic under $A(W)$.

At the same time, Proposition 2 and Example 2 point out the importance of flat regions of the distortion functions for differentiating the layers of network under rank-dependence (e.g., trimmed means). In particular, regardless of the matrix of weights, locally flat distortion functions can induce richer dynamics where disagreement obtains in the limit. This point is formalized in our next proposition. For all $W \in \mathcal{W}$, $i \in N$, and $N' \subseteq N$, define $w_i(N') = \sum_{j \in N'} w_{ij}$.

Proposition 3 *Let T^f be a rank-dependent aggregator with matrix of weights $W \in \mathcal{W}$. We have (i) $\implies$ (ii), where*

(i) There exist two nonempty disjoint subsets $\overline{N}, \underline{N} \subseteq N$ such that, for all $i \in \overline{N}$ and for all $l \in \underline{N}$

$$f_i\left(w_i\left(N \backslash \overline{N}\right)\right) = 0 \quad \text{and} \quad f_l\left(w_l\left(\underline{N}\right)\right) = 1; \tag{14}$$

(ii) There exists $x \in B$, such that $\bar{T}^f(x) \in B \backslash D$.

Moreover, if T^f is convergent, then (i) and (ii) are equivalent and, for all $\overline{N}, \underline{N} \subseteq N$ as in (i) and $x \in B$,

$$\min_{j \in \overline{N}} x_j > \max_{j \in \underline{N}} x_j \implies \min_{j \in \overline{N}} \bar{T}_j^f(x) > \max_{j \in \underline{N}} \bar{T}_j^f(x).$$

Proposition 3 establishes that if there are two groups of agents that distort sufficiently toward zero the total weights of the outsiders, then long-run disagreement might obtain. This can happen, without distortions (i.e., DeGroot model), when the two groups are completely isolated from the rest of the society. More interestingly, this can happen even in a completely connected society provided that the distortions are sufficiently large compared to the weights among groups (e.g., society with very cohesive groups as in Example 2). If T^f is convergent, then our condition fully characterizes long-run disagreement and, in addition, whenever the two groups start with non-overlapping opinions, these remain separated in the long run as well, that is, the opinions remain segregated.

Proposition 3 is mute with respect to the residual group of agents outside the segregated sets. This suggests that the exact composition of these sets is flexible and might change depending on their initial stances. This point is illustrated in the following example.

Example 3 Consider a society segmented along two different dimensions: left vs. right and younger vs. elder. For simplicity, restrict attention to four "representative" agents $N = \{1, 2, 3, 4\}$ such that agents 1 and 2 are left-leaning and agents 3 and 4 are right-leaning, whereas agents 1 and 3 are younger and agents 2 and 4 are elder. For every agent $i \in N$, the neighborhood N_i is the set of agents that share their type with i, at least on one dimension. Each agent i assigns a weight of 0.25 to j for every dimension with a shared trait:

$$W = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0.25 & 0.5 & 0 & 0.25 \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

Observe that the linear opinion aggregator induced by W is always convergent to consensus. Instead, we assume that every agent $i \in N$ aggregates opinions via a trimmed mean over the observations in her neighborhood with $\underline{q}_i = 1 - \bar{q}_i = 0.3$ (see equation (8)). One can easily show that the induced opinion aggregator T is self-influential, hence convergent by Corollary 1. Also, suppose that each opinion profile $x \in B = [0,1]^4$ collects the measures of agreement of the agents on a particular instance.

Consider first degrees of agreement with a policy that reduces restrictions on work migration. In this case, it is realistic to assume that the distribution of initial opinions is supported on profiles $x \in B$ such that $x_1 > x_2 > x_3 > x_4$. For each of such initial opinion profile x, the long-run limit is $\bar{T}(x) = (x_2, x_2, x_3, x_3)$. On the one hand, the long-run polarization is obtained along the *political dimension.* On the other hand, agents 1 and 4 are the ones adjusting their original stances toward the ones of agents 2 and 3 respectively.

Second, consider degrees of agreement with a policy that incentives the creation of an individual pension system and cut on expenses for a national pension system. In this case, it is realistic to assume that the distribution of initial opinions is supported on profiles $x \in B$ such that $x_3 > x_1 > x_4 > x_2$. For this initial opinion profile x, the long-run limit is $\bar{T}(x) = (x_1, x_4, x_1, x_4)$. In this case, the long-run polarization is obtained along the *age dimension.* Moreover, agents 2 and 3 are the ones adjusting their original stances toward the ones of agents 4 and 1 respectively. ▲

Note that, in the previous example, we obtained polarization even if the agents were symmetric in the way they aggregate the observations of their neighbors opinions. Moreover, we reached different segmentations of the population of agents with respect to their long-run opinions depending on the initial ranking of stances. Indeed, both the pairs of groups $\overline{N} = \{1,2\}$ and $\underline{N} = \{3,4\}$ as well as $\overline{N}' = \{1,3\}$ and $\underline{N}' = \{2,4\}$ satisfy condition (i) in Proposition 3. Next corollary, instead, provides necessary and sufficient conditions such that the society is segregated along a *given binary segmentation.*

Corollary 3 *Let T^f be a convergent rank-dependent aggregator with matrix of weights $W \in \mathcal{W}$ and fix a nonempty subset $\underline{N} \subset N$. The following are equivalent:*

(i) There exists $x^* \in B$ such that $\min_{j \in N \setminus \underline{N}} x_j^* > \max_{j \in \underline{N}} x_j^*$ and $T^f(x^*) = x^*$;

(ii) For all $i \in N \setminus \underline{N}$, $f_i(w_i(\underline{N})) = 0$, and for all $l \in \underline{N}$, $f_l(w_l(\underline{N})) = 1$;

(iii) For all $x \in B$, if $\min_{j \in N \setminus \underline{N}} x_j > \max_{j \in \underline{N}} x_j$, then $\min_{j \in N \setminus \underline{N}} \bar{T}_j^f(x) > \max_{j \in \underline{N}} \bar{T}_j^f(x)$.

Consider a candidate binary segmentation of the society $\underline{N}$ and $N \setminus \underline{N}$. Corollary 3 gives two alternative characterizations of disagreement persistence across this segmentation (i.e., point (iii)).

Condition (ii) is the simpler binary version of the distortion-weight structure in Proposition 3. Condition (i) instead requires the existence of a stable opinion profile where the two groups are completely separated. In particular, under our best-response interpretation of T^f, the previous corollary gives a characterization for the existence of a Nash equilibrium with *coexisting conventions* as in Morris [51].

5 Vox populi, vox Dei?

In the previous sections, we always considered a given deterministic profile of initial opinions and studied the corresponding evolution of opinions. In particular, for every given population size $n \in \mathbb{N}$, the stochastic nature of the vector of initial opinions $X = \mu + \varepsilon$ implies that the long-run outcome $\bar{T}(X)$ will be stochastic as well. In this section, we consider large networks (i.e., $n \to \infty$) to study the aggregate variability of opinions under robust opinion aggregation. This limit variability disappears provided that no single agent is excessively influential. However, without additional symmetry properties of the error terms and the opinion aggregators, the long-run opinions will concentrate around a biased estimate of the fundamental parameter μ. This *bias of the crowd* implies that completely disconnected but still large subgroups may have persistent disagreements even if their initial signals follow the same data generating process. More interestingly, the size of this disagreement depends on the noisiness of the initial signals, thereby establishing a connection between noisiness of the sources of information and polarization in large networks.

Formally, we consider the same setup of Section 2, with the caveat that here everything is parametrized by the size n of the population.

Assumptions In this section, we maintain the following assumptions:

1. $I = \mathbb{R}$.

2. For each $n \in \mathbb{N}$ we assume that $X_i(n) = \mu + \varepsilon_i(n)$ for all $i \in N$, where $\{\varepsilon_i(n)\}_{i \in N, n \in \mathbb{N}}$ is an array of uniformly bounded and independent random variables.

3. $\bar{T}_i(n) = \bar{T}_j(n)$ for all $i, j \in N$ and for all $n \in \mathbb{N}$.

Assumption 3 holds whenever $T(n)$ is a robust opinion aggregator that satisfies (i) of Theorem 3. Even though Assumption 3 is not strictly necessary in order for the entire set of agents to learn μ, it seems natural to consider a situation in which the result of the updating procedure is common across agents. Some additional notation is useful for the following analysis.

Notation With $\hat{I}$, we denote a bounded open interval such that $X_i(n)(\omega) \in \hat{I}$ for all $\omega \in \Omega$, $i \in N$, and $n \in \mathbb{N}$. We denote by $\ell \overset{\text{def}}{=} \sup \hat{I} - \inf \hat{I}$ the length of $\hat{I}$. Moreover, we denote the collection of probability vectors in $\mathbb{R}^n$ by Δ_n.

We are interested in whether the society becomes wise in the limit (cf. Golub and Jackson [32]).

Definition 8 *Let $\{T(n)\}_{n \in \mathbb{N}}$ be a sequence of robust opinion aggregators. The sequence $\{T(n)\}_{n \in \mathbb{N}}$ is* wise *if and only if*

$$\max_{i \in N} \left| \bar{T}_i(n)(X_1(n), ..., X_n(n)) - \mu \right| \overset{P}{\to} 0. \tag{15}$$

If $T(n)$ is linear with matrix $W(n)$, so is $\bar{T}(n)$ and the latter is represented by a matrix $\bar{W}(n)$. In this case, Assumption 3 yields that all the rows of $\bar{W}(n)$ coincide with the left Perron-Frobenius eigenvector $s(n) \in \Delta_n$ associated with the eigenvalue 1 of $W(n)$. DeMarzo et al. [23] as well as Golub and Jackson [32] call $s(n)$ the *influence vector* and the latter show that if $\lim_{n\to\infty} \max_{k\in N} s_k(n) \to 0$, then $\{T(n)\}_{n\in\mathbb{N}}$ is wise. In this case, the vector $s(n)$ coincides with the gradient of $\bar{T}_i(n)$, thereby capturing the idea of "marginal contribution" of the agents to the common limit opinion.

As suggested by Theorem 1, in our general model the marginal contributions to the limit opinion are captured by the partial derivatives of $\bar{T}_i(n)$. Unfortunately, we face the technical complication that our opinion aggregators might be nondifferentiable. Nevertheless, being Lipschitz continuous, by Rademacher's Theorem, they are almost everywhere differentiable. Let $\mathcal{D}\left(\bar{T}(n)\right) \subseteq \hat{I}^n$ be the subset of $\hat{I}^n$ where $\bar{T}(n)$ is differentiable.

Definition 9 *Let $T(n) : \mathbb{R}^n \to \mathbb{R}^n$ be a robust opinion aggregator. We say that $s(T(n)) \in \mathbb{R}^n$ is the* influence vector *of $T(n)$ if and only if*

$$s_i(T(n)) = \sup_{x\in\mathcal{D}\left(\bar{T}(n)\right)} \frac{\partial \bar{T}_1(n)}{\partial x_i}(x) \qquad \forall i \in N.$$

As we mentioned, the above definition of influence vector coincides with the one of Golub and Jackson whenever $T(n)$ is linear since $s(T(n)) = s(n)$. In the general case, $s_i(T(n))$ captures the maximal marginal contribution of a change of the opinion i on the final consensus estimate.

We say that the array $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ is symmetric if, for each $i \in N$ and for each $n \in \mathbb{N}$,

$$P(\{\omega \in \Omega : \varepsilon_i(n)(\omega) \in B\}) = P(\{\omega \in \Omega : -\varepsilon_i(n)(\omega) \in B\})$$

for all Borel sets $B \subseteq \mathbb{R}$. Moreover, we say that the sequence $\{T(n)\}_{n\in\mathbb{N}}$ is *odd*, if $T(n)(-x) = -T(n)(x)$ for all $x \in B$ such that $-x \in B$ and for all $n \in \mathbb{N}$, that is, $T(n)$ is an odd operator for all $n \in \mathbb{N}$.[23]

Proposition 4 *Let $\{T(n)\}_{n\in\mathbb{N}}$ be a sequence of robust opinion aggregators. If there exist two sequences $\{c(n)\}_{n\in\mathbb{N}}$ and $\{w(n)\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$: $c(n) \in \mathbb{R}$, $w(n) \in \Delta_n$, and*

$$s(T(n)) \leq c(n)\, w(n) \text{ as well as } c(n)^2 \max_{k\in N} w_k(n) \to 0 \text{ as } n \to \infty, \tag{16}$$

then

$$\operatorname{Var}\left(\bar{T}_i(n)(X_1(n), ..., X_n(n))\right) \to 0. \tag{17}$$

If in addition $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ is symmetric and $\{T(n)\}_{n\in\mathbb{N}}$ is odd, then $\{T(n)\}_{n\in\mathbb{N}}$ is wise.

As long as the maximal influence in the society is vanishing fast enough, the variance of the limit consensus is going to 0, thereby establishing the absence of aggregate variability of the long-run opinion

[23] In the foundation of robust opinion aggregators that we propose in Section 6.1, loss functions that are symmetric with respect to opinions' deviations naturally induce odd aggregators.

in large crowds.[24] In particular, this result exploits McDiarmid's concentration inequality to link the variance of the long-run consensus to the maximal influence in the society. Moreover, if both the errors and the opinion aggregator are symmetric, then the long-run opinion of large crowds is also correct.[25] Relatedly, the next corollary shows that a convergence rate of $1/\sqrt{n}$ is sufficient to guarantee wisdom.

Corollary 4 *Let $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ be symmetric and let $\{T(n)\}_{n\in\mathbb{N}}$ be odd. The sequence $\{T(n)\}_{n\in\mathbb{N}}$ is wise provided that*

$$\max_{k\in N} s_k(T(n)) = o\left(\frac{1}{\sqrt{n}}\right). \tag{18}$$

In general, there are two main differences between the previous results (e.g., Golub and Jackson [32] and Gilat and Razin [45]) about the wisdom of the crowd and ours. First, we neither impose any parametric form of the opinion aggregators nor assume that agents use the same class of robust aggregators. Second, our results encompass the case of a non-convergent aggregator $T(n)$. In such a case, $\bar{T}(n)$ is the limit of the time averages of the updates. This extra layer of generality is interesting if we think about the following question: can an external observer learn μ by observing only part of the updating dynamics of a subset of the agents? Theorem 1 and Proposition 4 together yield a positive answer: the external observer can use $\bar{T}(n)$ to extract information about the underlying parameter, even if the opinions of the agents in the network are not stabilizing.

One positive message of the wisdom of the crowd result in Golub and Jackson is that, even if the society is partitioned in almost disconnected, yet large, echo chambers, the existence of an "objective truth μ" still leads to an agreement between them. However, if each component features some homophily in their behavioral biases, then this may not happen and differences in beliefs may persist in the limit. Even more importantly, our model predicts that the extent of this polarization increases in the noisiness of the agents' initial information. This suggests a formal channel to link the increased polarization of political opinions to the diffusion of noisier sources of information such as social rather than professional media (see, e.g., Bail et al. [5], Lelkes et al. [44], and Gilat and Razin [46]).

Example 4 Consider a sequence of stochastic matrices $\{W(n)\}_{n\in\mathbb{N}}$ such that, for every $n\in\mathbb{N}$, $N=\{1,...,n\}$ is *partitioned* in two *closed, strongly connected,* and *aperiodic* classes $\{H(n), L(n)\}$, that is

$$W(n) = \begin{pmatrix} W_H(n) & \mathbf{0} \\ \mathbf{0} & W_L(n) \end{pmatrix}$$

where the sizes n_H and n_L of the matrices in both sequences $\{W_H(n)\}_{n\in\mathbb{N}}$ and $\{W_L(n)\}_{n\in\mathbb{N}}$ are unboundedly increasing with n.[26] For every $n\in\mathbb{N}$, the evolution of opinions in class $H(n)$ is governed by a quasi-arithmetic opinion aggregator $T^{\lambda_H}(n)$ as defined in Section 3 with respect to the stochastic

[24] The proof of Proposition 4 also gives a bound on the speed of convergence of the variance of $\bar{T}_i$ to 0 as a function of the maximum influence $\max_{k\in N} w_k(n)$ and the range of initial realizations ℓ.

[25] Note that linear opinion aggregators are always odd and that our condition in (16) is equivalent to the one of Golub and Jackson. Therefore, compared to the linear case, Proposition 4 differs only in one central aspect: our result relies on the signals being symmetric around μ.

[26] For notational simplicity, we avoid repeating the index n for the classes H and L when it is already clear, e.g., we write $W_H(n)$ instead of $W_{H(n)}(n)$.

matrix $W_H(n)$ and $\lambda_H > 0$. Similarly, for each class $L(n)$ we consider the aggregator $T^{\lambda_L}(n)$ with matrix $W_L(n)$ and $\lambda_L < 0$.[27] The interpretation here is that agents are partitioned in two groups characterized by their intrinsic preference for one extreme or the other of the spectrum of opinions. We assume that errors are identically distributed with $\varepsilon_i(n) \overset{d}{\sim} \varepsilon$ for some bounded random variable ε.

By Proposition 1, our assumptions guarantee that, for each $n \in \mathbb{N}$,

$$\bar{T}_i^{\lambda_H}(n)(x) = \frac{1}{\lambda_H} \ln \left(\sum_{j=1}^{n_H} s_{j,H}(n) \exp(\lambda_H x_j) \right) e \qquad \forall x \in \mathbb{R}^n, \forall i \in H(n)$$

where $s_H(n)$ is the left Perron-Frobenius eigenvector of $W_H(n)$. Since each $\bar{T}^{\lambda_H}(n)$ is translation invariant, we immediately obtain that, for each $n \in \mathbb{N}$,

$$\mathbb{E}\left(\bar{T}_i^{\lambda_H}(n)(X_1(n), ..., X_n(n))\right) = \mu + \mathbb{E}\left(\frac{1}{\lambda_H} \ln \left(\sum_{j=1}^{n_H} s_{j,H}(n) \exp(\lambda_H \varepsilon_j(n)) \right)\right) \qquad \forall i \in H(n). \tag{19}$$

Clearly, the second term on the right hand side will be typically different from 0 and, in particular, $\bar{T}_i^{\lambda_H}(n)$ will be biased. At the same time, by point 3.b of Proposition 1, it can be seen immediately that there exists $c \in \mathbb{R}$ such that $s\left(\bar{T}^{\lambda_H}(n)\right) \leq c s_H(n)$ for all $n \in \mathbb{N}$. Thus, by (17) and (19), if $\lim_{n\to\infty} \max_{j\in H(n)} s_{j,H}(n) \to 0$, then, for every sequence of agents $\{i_n\}_{n\in\mathbb{N}}$ such that $i_n \in H(n)$ for all $n \in \mathbb{N}$, we have

$$\bar{T}_{i_n}^{\lambda_H}(n)(X_1(n), ..., X_n(n)) \overset{P}{\to} \mu + \frac{1}{\lambda_H} \ln(\mathbb{E}(\exp(\lambda_H \varepsilon))). \tag{20}$$

The right-hand side of the previous equation gives an explicit representation for the *"bias" of the crowd* as a function of λ_H as well as of all the moments of the distribution of ε. A completely analogous formula of the limit bias for a sequence of agents in classes $L(n)$ is obtained by replacing λ_H with λ_L. In both cases, it is easy to see that the magnitude of the limit bias within each class is increasing in the absolute value the corresponding λ.

Next, define the *polarization* in the society as the difference between the limit consensus in the two classes obtained as in (20):

$$POL(\lambda_H, \lambda_L, \varepsilon) = \frac{1}{\lambda_H} \ln(\mathbb{E}(\exp(\lambda_H \varepsilon))) - \frac{1}{\lambda_L} \ln(\mathbb{E}(\exp(\lambda_L \varepsilon))) \geq 0$$

It is immediate to see that whenever ε is a mean-preserving spread of ε', then

$$POL(\lambda_H, \lambda_L, \varepsilon) \geq POL\left(\lambda_H, \lambda_L, \varepsilon'\right)$$

that is, the polarization is larger when the information is more noisy. Notice that in the linear case, i.e., $\lambda_H, \lambda_L \to 0$, the quality of information would be irrelevant with respect to the extent of polarization.

[27] Notice that $T^{\lambda_H}(n)$ and $T^{\lambda_L}(n)$ are respectively selfmaps on $I^{H(n)}$ and $I^{L(n)}$.

In particular, we have $\lim_{\lambda_H,\lambda_L\to 0} POL(\lambda_H,\lambda_L,\varepsilon)=0$ regardless of the distribution of ε. ▲

Proposition 4 and Corollary 4 give us easy-to-interpret sufficient conditions on the sequence of long-run aggregators for both absence of aggregate variability and wisdom. However, it is important to have properties of the primitive sequences of robust opinion aggregators that induce long-run wisdom. We close this section with a result addressing this point.

Proposition 5 *Let $\{\varepsilon_i(n)\}_{i\in N,n\in\mathbb{N}}$ be symmetric and let $\{T(n)\}_{n\in\mathbb{N}}$ be odd. The sequence $\{T(n)\}_{n\in\mathbb{N}}$ is wise provided that,*

$$\max_{i,j\in N}\sup_{x\in\mathcal{D}(T(n))}\frac{\partial T_j(n)}{\partial x_i}(x)=o\left(\frac{1}{\sqrt{n}}\right). \tag{21}$$

This last condition is exactly the same as the one in Corollary 4 but imposed on the original robust aggregator rather than directly on its long-run counterpart. Intuitively, if the maximum one-period influence across agents is converging to 0 fast enough then no group of agents is able to subvert the efficient aggregation of information in the social network. In Example 5 in the next section, we illustrate how to use the sufficient conditions of Proposition 5 to obtain the wisdom of the crowds in a model where agents repeatedly solve a robust estimation problem to estimate the fundamental parameter.

6 Discussion: foundation and discrete opinions

In this closing section, we discuss two important points that were left out in the main analysis: a microfoundation of robust opinion aggregators and the relation with models of diffusion/contagion in networks.

6.1 A characterization of robust opinion aggregators

Here, we characterize robust opinion aggregators as the solution of a distance minimization problem. Formally, we endow each agent i with a loss function $\phi_i:\mathbb{R}^n\to\mathbb{R}_+$ and we assume that at every period she solves

$$\min_{c\in\mathbb{R}}\phi_i(x-ce) \tag{22}$$

where $x\in B$ is the opinion profile in the previous period. Intuitively, in choosing her current opinion c, agent i minimizes a loss function that penalizes the disagreement (i.e., differences of opinions) with the last-period opinions of her neighbors.[28]

We next impose two minimal restrictions on the profile of loss functions $\phi=(\phi_i)_{i=1}^n$.

Definition 10 *The profile of loss functions ϕ is* sensitive *if and only if $\phi_i(he)>\phi_i(0)$ for each $i\in N$ and $h\in\mathbb{R}\setminus\{0\}$.*

[28] The network structure (N,A) can be reflected in the profile of loss functions $(\phi_i)_{i=1}^n$ by assuming that for each $i\in N$ and for each $z,z'\in\mathbb{R}^n$

$$z_j=z'_j\qquad\forall j\in N_i\implies\phi_i(z)=\phi_i(z').$$

It is a natural assumption but it can be dispensed with.

If agent i observes a unanimous opinion (including herself), then her loss is minimized by declaring that same opinion. In particular, under our best-response dynamics interpretation of Section 2, sensitivity of ϕ implies that all the constant profiles of actions ke, $k \in I$, are Nash equilibria of the induced game.

Definition 11 *The profile of loss functions ϕ has* increasing shifts *if and only if for each $i \in N$, $z, v \in \mathbb{R}^n$, and $h \in \mathbb{R}_{++}$*

$$z \geq v \implies \phi_i(z + he) - \phi_i(z) \geq \phi_i(v + he) - \phi_i(v).$$

It has strictly increasing shifts *if and only if the above inequality is strict whenever $z \gg v$.*

The property of increasing shifts is a form of complementarity in disagreeing with two or more agents from the same side. It is implied by stronger properties usually required on games played on networks, such as degree complementarity (see, e.g., Galeotti et al. [28]). In particular, the game induced by ϕ is supermodular if and only if ϕ has increasing shifts.

We call *robust* a profile of loss functions which is sensitive and has increasing shifts. The collection of all these profiles is denoted by Φ_R. Given a robust profile of loss functions ϕ, we denote with $T^\phi : B \to B$ an arbitrary function satisfying

$$T^\phi(x) \in \prod_{i=1}^{n} \operatorname{argmin}_{c \in \mathbb{R}} \phi_i(x - ce) \qquad \forall x \in B. \tag{23}$$

The selfmap T^ϕ is an opinion aggregator and describes one possible updating rule induced by ϕ. In particular, note how updating procedure is consistent with both best-response dynamics and repeated robust estimation as pointed out in Section 2. Next, we first characterize robust opinion aggregators as solution functions derived from robust loss functions. Then, we will come back to the two interpretations of the opinions dynamics induced.

The next theorem shows that our loss-function-based updating procedure naturally generalizes the one of the DeGroot model (cf. Golub and Sadler [35]) without committing to any specific functional form (e.g., quadratic) of the loss function.[29]

Theorem 4 *Let T be an opinion aggregator. The following statements are equivalent:*

(i) There exists $\phi \in \Phi_R$ which has strictly increasing shifts and is such that $T = T^\phi$, that is, for each $i \in N$

$$T_i(x) = \operatorname{argmin}_{c \in \mathbb{R}} \phi_i(x - ce) \qquad \forall x \in B; \tag{24}$$

(ii) T is a robust opinion aggregator.

In showing (ii) implies (i), given a robust opinion aggregator T, we construct a robust loss function which generalizes (2). For each opinion profile $x \in B$, every agent i considers a set of weights $\Gamma_i(x) \subseteq \Delta$ to be assigned to all the agents in the population, and minimizes the quadratic loss (2) with

[29] In particular, it is always possible to derive a DeGroot aggregator via the loss function (2).

the *worst* weights $w_i(x)$ in $\Gamma_i(x)$. Therefore, this construction rationalizes our opinion aggregators as the solutions of *robust* distance-minimization problems. Moreover, this point highlights the most important aspect of our theory: with robust opinion aggregators the current stances x in the population affect both if and how agents influence each other.[30] We have seen this feature already in Sections 3 and 4, with a particular focus on rank-dependent aggregators.

6.2 Loss functions and long-run dynamics

Next, we illustrate how our foundation is linked to the convergence and wisdom results for robust opinion aggregators. We focus on the familiar and particularly tractable class of loss-functions given by

$$\phi_i(z) = \sum_{j=1}^{n} w_{ij}\rho_i(z_j) \qquad \forall z \in \mathbb{R}^n, \forall i \in N$$

where $W \in \mathcal{W}$ is a stochastic matrix whose positive entries implicitly define the observation network, and $\rho = (\rho_i : \mathbb{R} \to \mathbb{R}_+)_{i=1}^n$ is a profile of positive functions. The weight w_{ij} captures the relative importance of the opinion of j as perceived by i. We call such a profile *additively separable* and write $\phi = (W, \rho)$. We denote the set of *robust* and *additively separable* profile of loss functions with Φ_A. Easy computations yield that $(W, \rho) \in \Phi_A$ if and only if each ρ_i is convex, strictly decreasing on $\mathbb{R}_-$, and strictly increasing on $\mathbb{R}_+$. If on top, each ρ_i is strictly convex, then there exists a unique robust opinion aggregator T^ϕ that satisfies (23).

Three relevant examples of robust opinion aggregators stemming from additively separable loss functions are the DeGroot aggregator (2), the quantile aggregators (see equation (4) for the loss function inducing the median aggregator), and the biased opinion aggregator of Proposition 1 (with loss function in equation (9)). More in general, the functional properties characterizing additively separable robust loss functions correspond to the ones considered by Huber [38], thereby highlighting the link between our foundation and robust statistics (see also the related discussion in Section 2).

Natural conditions on the profile of loss functions $\phi = (W, \rho)$ yield that the both the strong network $\left(N, \underline{A}\left(T^\phi\right)\right)$ and the weak network $\left(N, \bar{A}\left(T^\phi\right)\right)$ coincide with the observation network given by W.[31]

Proposition 6 *Let $\phi = (W, \rho) \in \Phi_A$. If I is compact and ρ_i is twice continuously differentiable and strongly convex for all $i \in N$, then there exists a unique T^ϕ that satisfies (23) and $\underline{A}\left(T^\phi\right) = \bar{A}\left(T^\phi\right) = A(W)$.*

Note that Proposition 6 together with Theorem 3 characterize convergence to consensus in terms of the observation network $A(W)$, provided that each ρ_i is sufficiently smooth and convex.

Finally, we illustrate how Proposition 5 can be applied to check wisdom within the setting of this section. As a by-product, we will obtain that, under assumptions 1-3 of Section 5, the wisdom of the

[30] In Theorem 4, the property of strictly increasing shifts guarantees that $\operatorname{argmin}_{c\in\mathbb{R}} \phi_i(x - ce)$ is a singleton. However, it is violated for few interesting specifications of ϕ (see, e.g., (4)). In Appendix C (see Proposition 11), we show that the solution correspondence of problem (22) always admits a selection which is a robust opinion aggregator.

[31] In general, we can prove a similar result for profiles of loss functions which are not additively separable. In this case, the assumptions of differentiability and strong convexity can also be weakened and replaced with a coercivity condition and a Lipschitz property of the difference quotients.

crowd can be achieved as long as the minimum degree of connections gets larger as the population size increases.

Example 5 Consider a sequence $\{T(n)\}_{n\in\mathbb{N}}$ of robust opinion aggregators as in Section 5 such that:

$$T_i(n)(x) \in \operatorname{argmin}_{c\in\mathbb{R}} \sum_{j\in N_i(n)} \frac{\rho_i(n)(x_j - c)}{|N_i(n)|} \qquad \forall i \in N, \forall x \in \mathbb{R}^n, \forall n \in \mathbb{N}.$$

where, for all $n \in \mathbb{N}$, the profile of loss functions $\phi(n) = (W(n), \rho(n)) \in \Phi_A$ used by the agents satisfy the assumptions in Proposition 6. In particular, the weights $w_{ij}(n)$ of each $W(n)$ are uniform over their neighborhoods $N_i(n)$. Moreover, assume that there exists $\bar{c} \in \mathbb{R}$ such that

$$\frac{\rho_i''(n)(z)}{\rho_i''(n)(z')} \leq \bar{c} \quad \forall i \in N, \forall n \in \mathbb{N}, \forall z, z' \in [-2\ell, 2\ell].$$

The previous condition is requiring that, as the size of the population increases, new agents are not becoming too sensible to the opinions of others. In particular, this condition is satisfied if $\rho_i(n) = \bar{\rho}$ for all $i \in N$ and $n \in N$. By the Implicit Function Theorem, we have that $T(n)$ is Frechet differentiable and

$$\frac{\partial T_h(n)}{\partial x_i}(x) \leq \bar{c}\frac{1}{\min_{k\in N}|N_k(n)|} \qquad \forall i, h \in N, \forall x \in \hat{B}, \forall n \in \mathbb{N}.$$

In words, the uniform bound on the sensibility of the loss functions implies that the reciprocal influence among the agents can be bounded using the size of the minimal neighborhood in the growing network. By Proposition 5, wisdom is reached at the limit (i.e., (15) holds) if the minimal degree in the society is growing sufficiently fast, that is,

$$\frac{1}{\min_{k\in N}|N_k(n)|} = o\left(\frac{1}{\sqrt{n}}\right). \tag{25}$$

In particular, note that condition (25) allows each agent to be connected to a vanishing fraction of the society. ▲

6.3 Discrete robust opinion aggregators and contagion

We next show how our framework can easily deal with discrete opinions. Even if we considered continuous opinions that belong to a connected set, the properties defining robust aggregators do not strictly rely on these assumptions and allow us to consider diffusion models with binary opinions. A set function $\nu : 2^N \to \{0,1\}$ is a $\{0,1\}$-*valued capacity* if $\nu(\emptyset) = 0$, $\nu(N) = 1$, and $\nu(M) \geq \nu(M')$ for all $M, M' \in 2^N$ such that $M' \subseteq M$. We say that T is a *discrete robust opinion aggregator* if there exists a profile $(\nu_i)_{i\in N}$ of $\{0,1\}$-valued capacities such that

$$T_i(x) = \min\{c \in \mathbb{R} : \nu_i(\{j \in N : x_j \leq c\})\} \qquad \forall x \in B, \forall i \in N. \tag{26}$$

it is immediate to see that these aggregators satisfy the properties in Definition 1, thereby falling within the class of robust aggregators. We call them "discrete" because they satisfy:

$$T_i(x) \in \{x_1, ..., x_n\} \qquad \forall x \in B, \forall i \in N.$$

Next, let $B = [0,1]^N$ and consider a distribution of initial opinions such that $x^0 \in \{0,1\}^N$ almost surely. The interpretation is that opinion 1 describes the adoption of a certain technology/behavior or the contagion of an idea, and all the agents i with $x_i^0 = 1$ are the initial seeds. In particular, it is easy to show that for discrete robust opinion aggregators one has $T^t(x^0) \in \{0,1\}^N$ for all $t \in \mathbb{N}$. With this, we can keep track of the evolution of adopters/infected in the society just by considering the set of agents whose opinions at a given period are all equal to 1.

Note that this is a generalization of the q-threshold contagion models in Morris [51], Kempe et al [42], and Centola and Macy [16]. In particular, we obtain the aforementioned models whenever each T_i is a q-quantile aggregator. In this case, given a stochastic matrix of weights $W \in \mathcal{W}$, the defining $\{0,1\}$-valued capacity ν_i of agent i is given by $\nu_i(M) = 1$ if and only if $\sum_{j \in M} w_{ij} \geq q_i$. Indeed, quantile aggregators are at the intersection of rank-dependent aggregators and discrete ones. In general, this discussion shows how robust opinion aggregators, and in particular rank-dependent aggregators, provide a bridge between standard models of continuous opinion aggregation such as the DeGroot model and standard models of simple and complex diffusion in networks.

Although weak, the sufficient conditions of Theorem 2 rule out these discrete robust opinion aggregators. Our final result shows that, for these aggregators, if convergence happens, then it will happen in a finite number of periods.

Proposition 7 *Let T be a discrete robust opinion aggregator and fix $x \in B$. Then, either $\{T^t(x)\}_{t \in \mathbb{N}}$ converges or it is eventually periodic, that is, there exists $\bar{t}, p \leq n^n$ such that*

$$T^{t+p}(x) = T^t(x) \qquad \forall t \geq \bar{t}.$$

Moreover, $\{T^t(x)\}_{t \in \mathbb{N}}$ converges if and only if $T^{m^n}(x) = T^{m^n+1}(x)$, where m is the number of distinct opinions in x.

For discrete robust opinion aggregators, there is a finite number of opinion profiles that can be reached, each corresponding to an assignment of the agents to the stances that were present in the initial vector x_0. Therefore, either one of these configurations is a fixed point of the operator and it is reached in a finite time, or the system alternates forever between unstable opinion profiles.

We conclude this section by arguing that other relevant social phenomena can be easily captured by rank-dependent and discrete aggregators. For example, in DeGroot's model, one can interpret the sum of weights $w_{ij} + w_{il}$ given by $i \in N$ to two agents $j, l \in N$ as a measure of the joint informativeness of the opinions of j and l, as perceived by i. However, in social networks, it might well be the case that i perceives the information sources of j and l as strongly *correlated*. In this case, given the observation from j, the additional information obtained from l is (perceived as) much lower than if observed alone (see, e.g., Liang and Mu [47]). Therefore, i might assign a total weight to j and l that is less than the sum $w_{ij} + w_{il}$. Clearly, this behavior is ruled out by the linear model, but can be described by discrete aggregators with a capacity that is subadditive with respect to j and l, i.e., such that $\nu_i(\{j,l\}) < \nu_i(\{j\}) + \nu_i(\{l\})$. Finally, another important social phenomenon is *assortativeness*: individuals with higher opinions assign a higher weight to individuals with high opinions (see, e.g., Frick et al. [26]). Given that under rank-dependent aggregation the updating matrix depends on the ranking of opinions, these aggregators can capture this social behavior as well. In the working paper version of this manuscript [17], we formalize this idea and show how it simplifies the analysis of the updating dynamics and the limit opinions.

A Appendix: Convergence

All the missing proofs are in the Online Appendix. The next ancillary lemmas highlight the properties of T and the limiting operator $\bar{T}$, whenever it exists.

Lemma 1 *Let T be an opinion aggregator. The following statements are true:*

1. *If T is robust, then it admits an extension $S : \mathbb{R}^n \to \mathbb{R}^n$ which is also robust.*
2. *If T is normalized and monotone, then $\left\|T^t(x)\right\|_\infty \leq \|x\|_\infty$ for all $x \in B$ and for all $t \in \mathbb{N}$.*
3. *If $x \in B$, then there exists $\tilde{I} \subseteq I$ which is a compact subinterval of I with nonempty interior and $x \in \tilde{I}^n \stackrel{\text{def}}{=} \tilde{B}$. Moreover, if T is robust, the restriction $\tilde{T} = T_{|\tilde{B}}$ is a robust opinion aggregator and $\tilde{T}^t(x) = T^t(x)$ as well as $\bar{\tilde{T}}(x) = \bar{T}(x)$ for all $t \in \mathbb{N}$ and for all $x \in \tilde{B}$ where $\bar{T}$ (resp., $\bar{\tilde{T}}$) is defined as in (3).*

Lemma 2 *If T is a robust opinion aggregator, then T^t is nonexpansive (i.e., Lipschitz continuous of order 1) for all $t \in \mathbb{N}$. In particular, T is nonexpansive.*

Despite being easy to derive, the property of nonexpansivity plays an important role in what follows and it also rules out the presence of chaotic behavior. The proof of next lemma instead relies on the property of "being a limit". It thus shows that the properties of T are often inherited by $\bar{T}$, provided the latter exists.

Lemma 3 *Let T be an opinion aggregator. If T is such that*

$$\text{C-}\lim_t T^t(x) \ \textit{exists} \qquad \forall x \in B,$$

then $\bar{T} : B \to B$, defined by $\bar{T}(x) = \text{C-}\lim_t T^t(x)$ for all $x \in B$, is well defined and $\bar{T} \circ T = \bar{T}$. Moreover,

1. *If T is nonexpansive, so is $\bar{T}$. In particular, $\bar{T}$ is continuous.*
2. *If T is normalized and monotone, so is $\bar{T}$.*
3. *If T is robust, so is $\bar{T}$.*

4. If T is odd, so is $\bar{T}$, provided I is a symmetric interval, that is, $k \in I$ if and only if $-k \in I$.

We can now prove that any sequence of updates of a robust opinion aggregator converges a la Cesaro and this convergence is uniform on bounded subsets of B.

Proof of Theorem 1. Consider $x \in B$. By point 2 of Lemma 1, we have that $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ is a bounded sequence and, in particular, relatively compact. By Lemma 2, T is nonexpansive. By Baillon et al. [6, Theorem 3.2 and Corollary 3.1], we can conclude that C-$\lim_t T^t(x)$ exists for all $x \in B$. By Lemma 3, $\bar{T}$ is a robust opinion aggregator such that $\bar{T} \circ T = \bar{T}$. Next, consider a bounded subset $\hat{B}$ of B. Define by $\tilde{B}$ the closed convex hull of $\hat{B}$. Since $\hat{B}$ is bounded and B is closed and convex, $\tilde{B}$ is a closed and bounded subset of B and, in particular, compact. For each $\tau \in \mathbb{N}$ define $S_\tau : \tilde{B} \to \mathbb{R}^n$ by

$$S_\tau(x) = \frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) \qquad \forall x \in \tilde{B}.$$

By Lemma 2, S_τ is well defined and nonexpansive for all $\tau \in \mathbb{N}$. The collection $\{S_\tau\}_{\tau\in\mathbb{N}}$ belongs to the space $C\left(\tilde{B},\mathbb{R}^n\right)$ of continuous functions from $\tilde{B}$ to $\mathbb{R}^n$. This space is a Banach space once endowed with the supnorm: $\|f\|_* = \sup_{x\in\tilde{B}} \|f(x)\|_\infty$ for all $f \in C\left(\tilde{B},\mathbb{R}^n\right)$. By [24, pp. 135–136] and since $\{S_\tau\}_{\tau\in\mathbb{N}}$ is a collection of nonexpansive maps, this implies that the sequence $\{S_\tau\}_{\tau\in\mathbb{N}} \subseteq C\left(\tilde{B},\mathbb{R}^n\right)$ is equicontinuous. By contradiction, assume that $S_\tau \overset{\|\ \|_*}{\not\to} \bar{T}_{|\tilde{B}}$. This would imply that there exists $\varepsilon > 0$ and a subsequence $\{S_{\tau_m}\}_{m\in\mathbb{N}} \subseteq \{S_\tau\}_{\tau\in\mathbb{N}}$ such that $\left\|S_{\tau_m} - \bar{T}_{|\tilde{B}}\right\|_* \geq \varepsilon$ for all $m \in \mathbb{N}$. By the Arzela-Ascoli Theorem (see, e.g., [24, Theorem 7.5.7]) and since $\{S_{\tau_m}\}_{m\in\mathbb{N}}$ is equicontinuous and $\{S_{\tau_m}(x)\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^n$ is bounded for all $x \in \tilde{B}$, this would imply that there exists a subsequence $\left\{S_{\tau_{m(l)}}\right\}_{l\in\mathbb{N}}$ and a function $\hat{S} \in C\left(\tilde{B},\mathbb{R}^n\right)$ such that $\lim_l \left\|S_{\tau_{m(l)}} - \hat{S}\right\|_* = 0$. By the previous part of the proof, recall that $\lim_\tau S_\tau(x) = \bar{T}(x)$ for all $x \in \tilde{B}$. By definition of $\|\ \|_*$, it would follow that $\bar{T}(x) = \lim_l S_{\tau_{m(l)}}(x) = \hat{S}(x)$ for all $x \in \tilde{B}$, that is, $\bar{T} = \hat{S}$ on $\tilde{B}$. This would imply that $0 < \varepsilon \leq \lim_l \left\|S_{\tau_{m(l)}} - \bar{T}_{|\tilde{B}}\right\|_* = 0$, a contradiction. We can conclude that

$$0 \leq \lim_\tau \sup_{x\in\hat{B}} \left\|\frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) - \bar{T}(x)\right\|_\infty \leq \lim_\tau \sup_{x\in\tilde{B}} \left\|\frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) - \bar{T}(x)\right\|_\infty = \lim_\tau \left\|S_\tau - \bar{T}_{|\tilde{B}}\right\|_* = 0,$$

proving the last part of the statement. ■

Remark 2 Theorem 1 could be seen as a version of the classic nonlinear ergodic theorem of Baillon. The generalization we are relying upon is the one contained in Baillon et al. [6, Theorem 3.2 and Corollary 3.1]. Compared to our version, the part that would be missing is the one contained in (12). Observe that (12), not only guarantees uniform Cesaro convergence of $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$, but also the independence from the initial condition of the rate of such convergence. This latter property might play an important role in applications and is missing in the aforementioned works. Finally, in the working paper version of this manuscript, exploiting the finite dimensionality of our framework, we provide a self-contained proof. ▲

We next prove our main result on standard convergence: Theorem 2. First, we identify a technical property, termed asymptotic regularity, which characterizes convergence. Second, we show how T having a nontrivial network is equivalent to T having a useful decomposition. Finally, via this decomposition, we show that strong aperiodicity yields asymptotic regularity, hence convergence.

Lemma 4 *Let T be a robust opinion aggregator. The following statements are equivalent:*

(i) T is asymptotically regular, that is, $\lim_t \left\|T^{t+1}(x) - T^t(x)\right\|_\infty = 0$ for all $x \in B$;

(ii) T is convergent.

Remark 3 Asymptotic regularity is weaker than the Cauchy property, yet paired with Cesaro convergence, is enough to grant convergence. Our techniques, use Lorentz's Theorem to transform Cesaro convergence of the orbits of T into standard convergence. This technique seems to have first appeared in Bruck [14]. Proving that asymptotic regularity is equivalent to convergence can also be obtained using the techniques of Browder and Petryshyn [13, Theorem 2]. ▲

Proposition 8 *Let T be a robust opinion aggregator. The following statements are equivalent:*

(i) T has a nontrivial network;

(ii) There exist $W \in \mathcal{W}$ and $\varepsilon \in (0,1)$ such that

$$T(x) = \varepsilon W x + (1-\varepsilon) S(x) \qquad \forall x \in B \tag{27}$$

where S is a robust opinion aggregator.

Moreover, we have that W in (ii) can be chosen to be such that $A(W) = \underline{A}(T)$.

Proof. (i) implies (ii). For each $i, j \in N$ if j strongly influences i, consider $\varepsilon_{ij} \in (0,1)$ as in (13) otherwise let $\varepsilon_{ij} = 1/2$. Define $\tilde{W}$ to be such that $\tilde{w}_{ij} = \underline{a}_{ij}\varepsilon_{ij}$ for all $i, j \in N$ where $\underline{a}_{ij}$ is the ij-th entry of $\underline{A}(T)$. Since each row of $\underline{A}(T)$ is not null, for each $i \in N$ there exists $j \in N$ such that $\underline{a}_{ij} = 1$ and, in particular, $\tilde{w}_{ij} > 0$. This implies that $\sum_{l=1}^n \tilde{w}_{il} > 0$ for all $i \in N$. Define also $\varepsilon = \min\{\min_{i\in N} \sum_{l=1}^n \tilde{w}_{il}, 1/2\} \in (0,1)$. Define $W \in \mathcal{W}$ to be such that $w_{ij} = \tilde{w}_{ij} / \sum_{l=1}^n \tilde{w}_{il}$ for all $i, j \in N$. Clearly, we have that for each $i, j \in N$

$$w_{ij} > 0 \iff \tilde{w}_{ij} > 0 \iff \underline{a}_{ij} = 1. \tag{28}$$

This yields that $A(W) = \underline{A}(T)$. Next, consider $x, y \in B$ such that $x \geq y$. Define $y^0 = y$. For each $t \in \{1, ..., n-1\}$ define $y^t \in B$ to be such that $y_i^t = x_i$ for all $i \leq t$ and $y_i^t = y_i$ for all $i \geq t+1$. Define $y^n = x$. Note that $x = y^n \geq ... \geq y^1 \geq y^0 = y$. It follows that for each $i \in N$

$$T_i(x) - T_i(y) = \sum_{j=1}^n \left[T_i\left(y^j\right) - T_i\left(y^{j-1}\right)\right] \geq \sum_{j=1}^n \underline{a}_{ij}\varepsilon_{ij}\left(y_j^j - y_j^{j-1}\right) = \sum_{j=1}^n \tilde{w}_{ij}(x_j - y_j)$$
$$= \left(\sum_{l=1}^n \tilde{w}_{il}\right)\left(\sum_{j=1}^n \frac{\tilde{w}_{ij}}{\sum_{l=1}^n \tilde{w}_{il}}(x_j - y_j)\right) = \left(\sum_{l=1}^n \tilde{w}_{il}\right)\left(\sum_{j=1}^n w_{ij}(x_j - y_j)\right) \geq \varepsilon \sum_{j=1}^n w_{ij}(x_j - y_j).$$

It follows that

$$x \geq y \implies T(x) - T(y) \geq \varepsilon W(x-y) = \varepsilon(Wx - Wy). \tag{29}$$

Define $S : B \to \mathbb{R}^n$ by

$$S(x) = \frac{T(x) - \varepsilon Wx}{1-\varepsilon} \qquad \forall x \in B. \tag{30}$$

By definition of S and since $W \in \mathcal{W}$ and T is normalized and translation invariant, it is immediate to see that $S(ke) = ke$ for all $k \in I$ and that S is translation invariant. Since (29) holds and $\varepsilon \in (0,1)$, routine computations yield that S is monotone. Since S is normalized and monotone, then $S(B) \subseteq B$, that is, S is a selfmap and, in particular, S is a robust opinion aggregator. By rearranging (30), (27) follows.

(ii) implies (i). Consider $i \in N$. Since W is a stochastic matrix, there exists $j \in N$ such that $w_{ij} > 0$. Let $x \in B$ and $h > 0$ be such that $x + he^j \in B$. By (27) and since S is monotone, we have that

$$T_i\left(x + he^j\right) - T_i(x) = \varepsilon w_{ij} h + (1-\varepsilon) S_i\left(x + he^j\right) - (1-\varepsilon) S_i(x) \geq \varepsilon w_{ij} h,$$

proving that j strongly influences i and $\underline{a}_{ij} = 1$. It follows that the i-th row of $\underline{A}(T)$ is not null. Since i was arbitrarily chosen, the statement follows.

Finally, by (28), note that W in (ii) can be chosen to be such that $A(W) = \underline{A}(T)$. ■

Theorem 2 builds on two assumptions: a) the matrix of strong ties $\underline{A}(T)$ has no null row and b) each closed group of $(N, \underline{A}(T))$ is aperiodic. The first assumption allows for a decomposition of T into a convex linear combination of a linear opinion aggregator with matrix W and a robust opinion aggregator S (cf. Proposition 8). We next show that if W takes a very particular form, which we dub partition matrix, then T is asymptotically regular and, in particular, convergent (see Lemma 5 and Proposition 9). The second assumption yields that W can be always chosen such that W^t is eventually a partition matrix. This will prove Theorem 2.

Definition 12 *Let $J : B \to B$ be an opinion aggregator. We say that J is a* partition operator/matrix *if and only if there exists a family of disjoint nonempty subsets $\{N_l\}_{l=1}^m$ of N such that $\cup_{l=1}^m N_l = N$ and for each $l \in \{1, ..., m\}$ there exists $k_l \in N_l$ such that $J_i(x) = x_{k_l}$ for all $i \in N_l$.*

Note that a partition operator is linear. With a small abuse of notation, we will denote the matrix and the operator by the same symbol.

Lemma 5 *Let T be a robust opinion aggregator such that $T = \varepsilon J + (1-\varepsilon) S$ where $\varepsilon \in (0,1)$, J is a partition operator, and $S : B \to B$ is a robust opinion aggregator. Let A be a nonempty subset of B such that there exists $k > 0$ satisfying*

$$\|T(x) - x\|_\infty < k \qquad \forall x \in A. \tag{31}$$

If there exists $\delta > 0$ such that for each $t \in \mathbb{N}_0$ there exists $x \in A$ satisfying

$$\left\|T^{t+1}(x) - T^t(x)\right\|_\infty \geq \delta, \tag{32}$$

then $\left\{T^t(x) : x \in A \text{ and } t \in \mathbb{N}_0\right\}$ is unbounded.

Remark 4 The proof of Lemma 5, which is contained in the Online Appendix, is a tedious adaptation of the techniques contained in the proof of Edelstein and O'Brien [25, Lemma 1]. Their case is more general in terms of domain of T in that B can be any convex subset of a normed vector space. Their generality comes at the cost of having J equal to the identity operator which in our case would only cover Corollary 1. A similar observation applies to Proposition 9. ▲

Proposition 9 *Let T be a robust opinion aggregator. If T is such that $T = \varepsilon J + (1-\varepsilon) S$ where $\varepsilon \in (0,1)$, J is a partition operator, and S is a robust opinion aggregator, then T is asymptotically regular and, in particular, convergent.*

Proof. Fix $x \in B$. In Lemma 5, set $A = \{x\}$. Clearly, there exists $k > 0$ that satisfies $\|T(x) - x\|_\infty < k$. By point 2 of Lemma 1 and since T is a robust opinion aggregator, it follows that $\{T^t(x)\}_{t\in\mathbb{N}_0}$ is bounded. By Lemma 5, we have that for each $\delta > 0$ there exists $\bar{t} \in \mathbb{N}$ such that

$$\left\|T^{\bar{t}+1}(x) - T^{\bar{t}}(x)\right\|_\infty < \delta. \tag{33}$$

Since T is nonexpansive, $\{\|T^{t+1}(x) - T^t(x)\|_\infty\}_{t\in\mathbb{N}_0}$ is a decreasing sequence. By (33) and since $\{\|T^{t+1}(x) - T^t(x)\|_\infty\}_{t\in\mathbb{N}_0}$ is a decreasing sequence, we have that for each $\delta > 0$ there exists $\bar{t} \in \mathbb{N}$ such that $\|T^{t+1}(x) - T^t(x)\|_\infty < \delta$ for all $t \geq \bar{t}$, that is, $\lim_t \|T^{t+1}(x) - T^t(x)\|_\infty = 0$. Since x was arbitrarily chosen, it follows that T is asymptotically regular. By Lemma 4, this implies that T is convergent. ■

Lemma 6 shows that if T is strongly aperiodic and has a nontrivial network, then there exists $\bar{t} \in \mathbb{N}$ such that $T^{\bar{t}} = \gamma J + (1-\gamma) S$ (resp., $T^{\bar{t}+1} = \gamma J + (1-\gamma) S$) where J is a partition operator, $\gamma \in (0,1)$, and S is a robust opinion aggregator. The operator J only depends on $\underline{A}(T)$ while γ and S both depend on $\bar{t}$ (resp., $\bar{t}+1$). In turn, Proposition 9 yields that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent. This will be sufficient to imply the convergence of T.

Lemma 6 *Let T be a robust opinion aggregator. If T is strongly aperiodic and has a nontrivial network, then there exists $\bar{t} \in \mathbb{N}$ such that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent.*

Proof. By Proposition 8 and since T has a nontrivial network, we have that there exists $W \in \mathcal{W}$, $\varepsilon \in (0,1)$, and a robust opinion aggregator $S : B \to B$ such that

$$T(x) = \varepsilon W x + (1-\varepsilon) S(x) \quad \forall x \in B. \tag{34}$$

Moreover, W can be chosen to be such that $A(W) = \underline{A}(T)$. By [32, Theorems 2 and 3] and since T is strongly aperiodic, this implies that there exist $\bar{t} \in \mathbb{N}$ and a partition $\{N_l\}_{l=1}^m$ of N such that for each $l \in \{1, ..., m\}$ there exists $k_l \in N_l$ satisfying $w_{ik_l}^{(\bar{t})}, w_{ik_l}^{(\bar{t}+1)} > 0$ for all $i \in N_l$.[32] It follows that

$$W^{\bar{t}} = \delta_{\bar{t}} J + (1-\delta_{\bar{t}}) \tilde{W}_{\bar{t}} \text{ and } W^{\bar{t}+1} = \delta_{\bar{t}+1} J + (1-\delta_{\bar{t}+1}) \tilde{W}_{\bar{t}+1} \tag{35}$$

[32] As usual, we denote by $w_{ik_l}^{(\bar{t})}$ (resp., $w_{ik_l}^{(\bar{t}+1)}$) the entry in the i-th row and k_l-th column of the matrix $W^{\bar{t}}$ (resp., $W^{\bar{t}+1}$).

where $\delta_{\bar{t}}, \delta_{\bar{t}+1} \in (0,1)$, J is a partition operator/matrix,[33] and $\tilde{W}_{\bar{t}}$ as well as $\tilde{W}_{\bar{t}+1}$ are stochastic matrices. By (34) and induction, we also have that

$$T^{\bar{t}}(x) = \varepsilon^{\bar{t}} W^{\bar{t}} x + \left(1 - \varepsilon^{\bar{t}}\right) \tilde{S}_{\bar{t}}(x) \qquad \forall x \in B$$

and

$$T^{\bar{t}+1}(x) = \varepsilon^{\bar{t}+1} W^{\bar{t}+1} x + \left(1 - \varepsilon^{\bar{t}+1}\right) \tilde{S}_{\bar{t}+1}(x) \qquad \forall x \in B$$

where $\tilde{S}_{\bar{t}}$ and $\tilde{S}_{\bar{t}+1}$ are robust opinion aggregators. By (35), it follows that

$$T^{\bar{t}} = \gamma_{\bar{t}} J + (1 - \gamma_{\bar{t}}) \hat{S}_{\bar{t}} \text{ and } T^{\bar{t}+1} = \gamma_{\bar{t}+1} J + (1 - \gamma_{\bar{t}+1}) \hat{S}_{\bar{t}+1}$$

where $\gamma_{\bar{t}} = \varepsilon^{\bar{t}} \delta_{\bar{t}}$ (resp., $\gamma_{\bar{t}+1} = \varepsilon^{\bar{t}+1} \delta_{\bar{t}+1}$) and $\hat{S}_{\bar{t}}(x) = \frac{\varepsilon^{\bar{t}}(1-\delta_{\bar{t}})}{1-\varepsilon^{\bar{t}}\delta_{\bar{t}}} \tilde{W}_{\bar{t}} x + \frac{1-\varepsilon^{\bar{t}}}{1-\varepsilon^{\bar{t}}\delta_{\bar{t}}} \tilde{S}_{\bar{t}}(x)$ (resp., $\hat{S}_{\bar{t}+1}(x) = \frac{\varepsilon^{\bar{t}+1}(1-\delta_{\bar{t}+1})}{1-\varepsilon^{\bar{t}+1}\delta_{\bar{t}+1}} \tilde{W}_{\bar{t}+1} x + \frac{1-\varepsilon^{\bar{t}+1}}{1-\varepsilon^{\bar{t}+1}\delta_{\bar{t}+1}} \tilde{S}_{\bar{t}+1}(x)$) for all $x \in B$. It follows that $\gamma_{\bar{t}}, \gamma_{\bar{t}+1} \in (0,1)$ and $\hat{S}_{\bar{t}}$ as well as $\hat{S}_{\bar{t}+1}$ are robust opinion aggregators. By Proposition 9, this implies that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent. ■

Proof of Theorem 2. We adopt the usual convention $T^0(x) = x$ for all $x \in B$. By Lemma 6 and since T is strongly aperiodic and has a nontrivial network, there exists $\bar{t} \in \mathbb{N}$ such that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent. We next show that this implies that T is convergent. Fix $x \in B$. Since $T^{\bar{t}}$ is convergent, we can conclude that $\lim_k T^{k\bar{t}}(x)$ exists. Denote $\bar{x} = \lim_k T^{k\bar{t}}(x)$. Since T is continuous and so is $T^{\bar{t}}$, it is plain that $T^{\bar{t}}(\bar{x}) = \bar{x}$. This implies that

$$T^{\bar{t}}(T^s(\bar{x})) = T^{\bar{t}+s}(\bar{x}) = T^{s+\bar{t}}(\bar{x}) = T^s\left(T^{\bar{t}}(\bar{x})\right) = T^s(\bar{x}) \qquad \forall s \in \mathbb{N}_0.$$

By induction on k, this yields that for each $s \in \mathbb{N}_0$

$$T^{(k+1)\bar{t}}(T^s(\bar{x})) = T^{k\bar{t}}\left(T^{\bar{t}}(T^s(\bar{x}))\right) = T^{k\bar{t}}(T^s(\bar{x})) = T^s(\bar{x}) \quad \forall k \in \mathbb{N}.$$

In particular, by setting $k = s$, we obtain that for each $s \in \mathbb{N}$

$$T^{s(\bar{t}+1)}(\bar{x}) = T^{s\bar{t}}(T^s(\bar{x})) = T^s(\bar{x}). \tag{36}$$

Since $T^{\bar{t}+1}$ is convergent, we have that $\lim_s T^{s(\bar{t}+1)}(\bar{x})$ exists. By (36), this implies that $\lim_s T^s(\bar{x})$ exists. Denote $\hat{x} = \lim_s T^s(\bar{x})$. Since T is continuous, it is plain that $T(\hat{x}) = \hat{x}$. Since $\left\{T^{k\bar{t}}(\bar{x})\right\}_{k\in\mathbb{N}} \subseteq \{T^s(\bar{x})\}_{s\in\mathbb{N}}$ and $T^{k\bar{t}}(\bar{x}) = \bar{x}$ for all $k \in \mathbb{N}$, we have that

$$\bar{x} = \lim_k T^{k\bar{t}}(\bar{x}) = \lim_s T^s(\bar{x}) = \hat{x} \text{ and } T(\hat{x}) = \hat{x}. \tag{37}$$

We can now prove that $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ converges too. By (37) and since T is nonexpansive, we have that

$$\left\|\bar{x} - T^{t+1}(x)\right\|_\infty = \left\|T(\bar{x}) - T\left(T^t(x)\right)\right\|_\infty \leq \left\|\bar{x} - T^t(x)\right\|_\infty \qquad \forall t \in \mathbb{N},$$

[33] That is, $J_i(x) = x_{k_l}$ for all $i \in N_l$ and for all $l \in \{1, ..., m\}$ where $\{N_l\}_{l=1}^m$ and $\{k_l\}_{l=1}^m$ have been defined above.

yielding that $\left\{\left\|\bar{x}-T^{t}(x)\right\|_{\infty}\right\}_{t\in\mathbb{N}}$ is a decreasing sequence. Moreover, since $\bar{x}=\lim_{k}T^{k\bar{t}}(x)$, we have that the subsequence $\left\{\left\|\bar{x}-T^{k\bar{t}}(x)\right\|_{\infty}\right\}_{k\in\mathbb{N}}\subseteq\left\{\left\|\bar{x}-T^{t}(x)\right\|_{\infty}\right\}_{t\in\mathbb{N}}$ converges to 0. This implies that $\lim_{t}T^{t}(x)=\bar{x}$. Since x was arbitrarily chosen, the statement follows. ■

Remark 5 Our Theorem 3 belongs to the literature on discrete dynamical systems/repeated averaging, where agents aggregate opinions at each point in time with a potential time-varying averaging procedure. These works are typically concerned in providing the more general conditions possible on the sequence of averaging procedures which guarantee convergence to consensus. In this setting, Krause [43, Theorem 8.3.4] shows that convergence to consensus is achieved if and only if a form of strict internality is satisfied, that is, the range of opinions of the agents eventually shrinks no matter what is the initial vector of opinions. Our results differ from the ones above in two dimensions. In our Theorems 1 and 2, we tackle the issue of convergence in general and we do not restrict ourselves just to convergence to consensus. This significantly complicates the analysis and we need to resort to completely different techniques coming from functional analysis which we discussed in the previous remarks of this appendix. The overlap with Krause's result is therefore restricted to the first part of Theorem 3 which we obtain from our more general Theorem 2. Finally, since our opinion aggregators are microfounded, under mild conditions, they inherit the primitive observation network structure of the foundation (see Proposition 6). In turn, this imposes a strong discipline on the averaging process that has never been exploited before and allows us to provide bounds on the rate of convergence which are function of the underlying network (see Remark 1). ▲

In order to prove Theorem 3, we begin by making two simple observations about convergence and fixed points of the opinion aggregator T: 1) convergence is always toward a fixed point of T, 2) simple properties on the network $\underline{A}(T)$ yield that those fixed points are constant vectors. We denote by $E(T)$ the set of fixed points/equilibria of T. Recall that D is the consensus subset, that is, $x\in D\subseteq B$ if and only if $x_i=x_j$ for all $i,j\in N$.

Proposition 10 *Let T be a robust opinion aggregator. We have $E(T)=D$ provided one of the following holds:*

a. T has the pairwise common influencer property;

b. T has a nontrivial network, $\underline{A}(T)$ has a unique strongly connected and closed group M, and M is aperiodic under $\underline{A}(T)$.

Proof of Theorem 3. 1. Suppose that T has a nontrivial network, $\underline{A}(T)$ has a a unique strongly connected and closed group M, and M is aperiodic under $\underline{A}(T)$. By Theorem 2 and Lemma 7, it follows that $\bar{T}(x)=\lim_{t}T^{t}(x)\in E(T)$ for all $x\in B$. Moreover, by Proposition 10, it follows that $E(T)=D$, proving the implication.

2. By assumption, T is convergent and $\bar{T}(x)\in D$ for all $x\in B$. Define $\bar{W}(T)\in\mathcal{W}$ by

$$\bar{w}_{ij}(T)=\frac{\bar{a}_{ij}(T)}{\left|\left\{l\in N:\bar{a}_{il}(T)=1\right\}\right|}\qquad\forall i,j\in N.$$

Observe that T is normalized, hence that, for each $i \in N$, there exists $j \in N$ such that $\bar{a}_{ij}(T) = 1$, showing that $\bar{W}(T)$ is well defined. Moreover, it is immediate to see that, for all $t \in \mathbb{N}$, the ij-th element of $\left(\bar{W}(T)\right)^t$ is equal to 0 if and only if the ij-th element of $\left(\bar{A}(T)\right)^t$ is equal to 0. Assume by contradiction that it is not true that $\bar{A}(T)$ has a unique strongly connected and closed group M and the latter is aperiodic. By [39, Corollaries 8.1 and 8.2], for all $t \in \mathbb{N}$ and for all $j \in N$, there exists $i \in N$ such that the ij-th element of $\left(\bar{W}(T)\right)^t$, hence the ij-th element of $\left(\bar{A}(T)\right)^t$, is equal to 0. By Lemma 8, for all $j \in N$, there exists $i \in N$ such that $\bar{a}_{ij}\left(\bar{T}\right) = 0$. As argued above, observe that there exists $j \in N$ such that $\bar{a}_{1j}\left(\bar{T}\right) = 1$, in particular, there exist $x \in B$ and $h \in \mathbb{R}_+$ such that $\bar{T}_1\left(x + he^j\right) - \bar{T}_1(x) > 0$. Moreover, there exists $l \in N$ such that $\bar{a}_{lj}\left(\bar{T}\right) = 0$. There are two cases:

a. If $\bar{T}(x) \notin D$, then we have reached a contradiction.

b. If $\bar{T}(x) \in D$, then

$$\bar{T}_l\left(x + he^j\right) = \bar{T}_l(x) = \bar{T}_1(x) < \bar{T}_1\left(x + he^j\right),$$

that is, $\bar{T}\left(x + he^j\right) \notin D$, hence reaching a contradiction.

This proves the statement. ■

B Appendix: Vox populi, vox Dei?

All the missing proofs are in the Online Appendix.

Proof of Proposition 4. Define $\hat{B} = \hat{I}^n$. We first show that (16) yields that $\bar{T}_i(n)$ is not extremely sensitive to changes coming from a single observation. Then, by applying McDiarmid's inequality, we obtain (17). Before starting, we make a few observations. By Assumption 3, we have that $\bar{T}_i(n) = \bar{T}_j(n)$ for all $i, j \in N$ and for all $n \in \mathbb{N}$. Since the random variables $\{X_i(n)\}_{i \in N, n \in \mathbb{N}}$ are uniformly bounded and $\bar{T}_i(n)$ is continuous for all $i \in N$ and for all $n \in \mathbb{N}$, it follows that $\omega \mapsto \bar{T}_i(n)(X_1(n)(\omega), ..., X_n(n)(\omega))$ is integrable for all $i \in N$ and for all $n \in \mathbb{N}$.

Claim. For each $i, j \in N$ and for each $n \in \mathbb{N}$

$$\sup_{\left\{(x,t) \in \hat{B} \times \mathbb{R} : x + te^j \in \hat{B}\right\}} \left|\bar{T}_i(n)\left(x + te^j\right) - \bar{T}_i(n)(x)\right| \leq \ell c(n) w_j(n).$$

Proof of the Claim. Fix $i \in N$ and $n \in \mathbb{N}$. By Rademacher's Theorem and since $\bar{T}(n)$ is nonexpansive, this implies that $\bar{T}(n)$ is almost everywhere Frechet differentiable. Let $\mathcal{D}\left(\bar{T}(n)\right) \subseteq \hat{I}^n = \hat{B}$ be the subset of $\hat{B}$ where $\bar{T}(n)$ is Frechet differentiable. Clearly, $\bar{T}_i(n)$ is Frechet differentiable on $\mathcal{D}\left(\bar{T}(n)\right)$ and, in particular, Clarke differentiable. Since $\bar{T}_i(n)$ is monotone and translation invariant, note that $\nabla \bar{T}_i(n)(x) \in \Delta_n$ for all $x \in \mathcal{D}\left(\bar{T}(n)\right)$. Consider $\bar{x} \in \hat{B}$. Recall that Clarke's differential is the set (see, e.g., [21, Theorem 2.5.1]):

$$\partial \bar{T}_i(n)(\bar{x}) = \operatorname{co}\left\{p \in \Delta_n : p = \lim_k \nabla \bar{T}_i(n)\left(x^k\right) \text{ s.t. } x^k \to \bar{x} \text{ and } x^k \in \mathcal{D}\left(\bar{T}(n)\right)\right\}. \qquad (38)$$

By Definition 9 and (38) and since $\bar{T}_1(n) = \bar{T}_i(n)$, note that

$$0 \leq p_j \leq s_j(T(n)) \quad \forall p \in \partial \bar{T}_i(n)(x), \forall x \in \hat{B}, \forall j \in N. \tag{39}$$

Consider $j \in N$, $x \in \hat{B}$, and $t \in \mathbb{R}$ such that $x + te^j \in \hat{B}$. Define $y = x + te^j$. By Lebourg's Mean Value Theorem, we have that there exist $\lambda \in (0,1)$ and $\bar{p} \in \partial \bar{T}_i(n)(z)$ where $z = \lambda y + (1-\lambda)x \in \hat{B}$ such that

$$\bar{T}_i(n)\left(x + te^j\right) - \bar{T}_i(n)(x) = \bar{T}_i(n)(y) - \bar{T}_i(n)(x) = \sum_{l=1}^{n} \bar{p}_l(y_l - x_l).$$

It follows that $\left|\bar{T}_i(n)\left(x + te^j\right) - \bar{T}_i(n)(x)\right| = |\bar{p}_j(y_j - x_j)| = \bar{p}_j|y_j - x_j| \leq \ell \bar{p}_j$. By (39), this implies that

$$\left|\bar{T}_i(n)\left(x + te^j\right) - \bar{T}_i(n)(x)\right| \leq \ell \bar{p}_j \leq \ell s_j(T(n)) \leq \ell c(n) w_j(n).$$

Since x and t were arbitrarily chosen, it follows that

$$\sup_{\left\{(x,t) \in \hat{B} \times \mathbb{R} : x + te^j \in \hat{B}\right\}} \left|\bar{T}_i(n)\left(x + te^j\right) - \bar{T}_i(n)(x)\right| \leq \ell c(n) w_j(n).$$

Since i, n, and j were also arbitrarily chosen, the statement follows. □

Let $n \in \mathbb{N}$ and $i \in N$. By McDiarmid's inequality and Assumption 3 as well as the previous claim, we can conclude that, for each $\delta > 0$,

$$\begin{aligned}
&P\left(\left\{\omega \in \Omega : \left|\bar{T}_i(n)(X_1(n)(\omega), ..., X_n(n)(\omega)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n), ..., X_n(n))\right)\right|^2 \geq \delta\right\}\right) \\
&= P\left(\left\{\omega \in \Omega : \left|\bar{T}_i(n)(X_1(n)(\omega), ..., X_n(n)(\omega)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n), ..., X_n(n))\right)\right| \geq \sqrt{\delta}\right\}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\exp\left(-\frac{2\delta}{\sum_{j=1}^{n}(\ell c(n) w_j(n))^2}\right) \leq 2\exp\left(-\frac{2\delta}{\ell^2 c(n)^2 \max_{k \in N} w_k(n) \sum_{j=1}^{n} w_j(n)}\right) \\
&= 2\exp\left(-\frac{2\delta}{\ell^2 c(n)^2 \max_{k \in N} w_k(n)}\right).
\end{aligned}$$

Next, by [10, Equation 21.9], observe that

$$\begin{aligned}
&\operatorname{Var}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\\
&=\mathbb{E}\left(\left(\bar{T}_i(n)(X_1(n),...,X_n(n))-\mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right)^2\right)\\
&=\int_0^\infty P\left(\left\{\omega\in\Omega:\left(\bar{T}_i(n)(X_1(n)(\omega),...,X_n(n)(\omega))-\mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right)^2\geq t\right\}\right)dt\\
&=\int_0^{\ell^2} P\left(\left\{\omega\in\Omega:\left|\bar{T}_i(n)(X_1(n)(\omega),...,X_n(n)(\omega))-\mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right|^2\geq t\right\}\right)dt\\
&\leq\int_0^{\ell^2} 2\exp\left(-\frac{2t}{\ell^2 c(n)^2\max_{k\in N} w_k(n)}\right)dt\\
&=\ell^2 c(n)^2\max_{k\in N} w_k(n)\left[1-\exp\left(-\frac{2}{c(n)^2\max_{k\in N} w_k(n)}\right)\right]\to 0\quad\text{as } n\to\infty,
\end{aligned}$$

proving (17).

For the second part, assume that $\{\varepsilon_i(n)\}_{i\in N,n\in\mathbb{N}}$ is symmetric and that $T(n)$ is odd for each $n\in\mathbb{N}$. It is enough to show that $\bar{T}_i(n)$ is an unbiased estimator of μ for all $i\in N$ and for all $n\in\mathbb{N}$. By points 3 and 4 of Lemma 3 and since $I=\mathbb{R}$, we have that $\bar{T}(n)$ is a well defined odd robust opinion aggregator for all $n\in\mathbb{N}$. Next, recall that $X_i(n)=\mu+\varepsilon_i(n)$ for all $i\in N$ and for all $n\in\mathbb{N}$ where $\{\varepsilon_i(n)\}_{i\in N,n\in\mathbb{N}}$ is a collection of uniformly bounded and independent random variables. Since $\bar{T}_i(n)$ is continuous for all $i\in N$ and for all $n\in\mathbb{N}$, it follows that $\omega\mapsto\bar{T}_i(n)(\varepsilon_1(n)(\omega),...,\varepsilon_n(n)(\omega))$ is integrable for all $i\in N$ and for all $n\in\mathbb{N}$. Since $\bar{T}(n)$ is odd for all $n\in\mathbb{N}$, this implies that for each $i\in N$ and for each $n\in\mathbb{N}$

$$\begin{aligned}
\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP&=\int_\Omega \bar{T}_i(n)(-\varepsilon_1(n),...,-\varepsilon_n(n))\,dP\\
&=-\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP.
\end{aligned}$$

It follows that for each $i\in N$ and for each $n\in\mathbb{N}$

$$2\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP=0.$$

Since $\bar{T}(n)$ is translation invariant, we can conclude that for each $i\in N$ and for each $n\in\mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)=\int_\Omega \bar{T}_i(n)(X_1(n),...,X_n(n))\,dP\\
&=\int_\Omega \bar{T}_i(n)(\mu+\varepsilon_1(n),...,\mu+\varepsilon_n(n))\,dP=\mu+\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP=\mu,
\end{aligned}$$

proving that $\bar{T}_i(n)$ is an unbiased estimator of μ. Given that i and n were arbitrarily chosen, this concludes the proof. ■

C Appendix: Discussion

All the missing proofs are in the Online Appendix. The next result generalizes (i) implies (ii) of Theorem 4 and it is based on routine optimization and monotone comparative statics arguments. Given the profile of loss functions $\phi = (\phi_i)_{i=1}^n$, define $\mathbf{T}^\phi : B \rightrightarrows B$ as

$$\mathbf{T}^\phi (x) = \prod_{i=1}^{n} \text{argmin}_{c\in\mathbb{R}}\, \phi_i (x - ce) \qquad \forall x \in B.$$

Proposition 11 *Let ϕ be a profile of loss functions. If $\phi \in \Phi_R$, then the correspondence $\mathbf{T}^\phi$ admits a selection T^ϕ which is a robust opinion aggregator. Moreover, if ϕ has strictly increasing shifts, then $\mathbf{T}^\phi = T^\phi$ is single-valued and, in particular, is a robust opinion aggregator.*

Proof of Theorem 4. (i) implies (ii). By Proposition 11 and since $\phi \in \Phi_R$ and has strictly increasing shifts, the implication follows.

(ii) implies (i). Let $T : B \to B$ be a robust opinion aggregator. By Lemma 1, there exists an extension from $\mathbb{R}^n$ to $\mathbb{R}^n$. With a small abuse of notation, we denote it by the same symbol T. By Lemma 2, we have that T is nonexpansive and, in particular, continuous and so are the maps T_i. Fix $i \in N$. By [19, Corollary 3], there exists a closed and convex set of probability vectors, $C_i \subseteq \Delta$, and a function $\alpha_i : \mathbb{R}^n \to [0, 1]$ such that

$$T_i (x) = \alpha_i (x) \min_{w\in C_i} w \cdot x + [1 - \alpha_i (x)] \max_{w\in C_i} w \cdot x \qquad \forall x \in \mathbb{R}^n. \tag{40}$$

Fix $x \in \mathbb{R}^n$. Since C_i is compact, define $w^{i,1} (x), w^{i,2} (x) \in C_i$ to be such that $w^{i,1} (x)\cdot x = \min_{w\in C_i} w\cdot x$ and $w^{i,2} (x) \cdot x = \max_{w\in C_i} w \cdot x$. By (40) and since C_i is convex, we have that $w^i (x) = \alpha_i (x) w^{i,1} (x) + [1 - \alpha_i (x)] w^{i,2} (x) \in C_i$ and $T_i (x) = w^i (x) \cdot x$. Since x and i were arbitrarily chosen, it follows that for each $i \in N$ and for each $x \in \mathbb{R}^n$ there exists $w \in C_i$ such that $T_i (x) = w \cdot x$. For each $i \in N$ and for each $x \in \mathbb{R}^n$ define the correspondence $\Gamma_i : \mathbb{R}^n \rightrightarrows \Delta$ by

$$\Gamma_i (x) = \{w \in C_i : T_i (x) = w \cdot x\} \qquad \forall x \in \mathbb{R}^n.$$

We first prove the following ancillary claim.
Claim: *For each $i \in N$, the correspondence Γ_i is nonempty-, convex-, and compact-valued, upper hemicontinuous, and such that*

$$\Gamma_i (x) = \Gamma_i (x + he) \qquad \forall x \in \mathbb{R}^n, \forall h \in \mathbb{R}. \tag{41}$$

Proof of Claim. Fix $i \in N$. Consider $x \in \mathbb{R}^n$. By the previous part of the proof, we have that $w^i (x) \in \Gamma_i (x)$, proving that $\Gamma_i (x)$ is nonempty. Since C_i is convex and compact, it is immediate to check that $\Gamma_i (x)$ is convex and compact. Since x was arbitrarily chosen, it follows that Γ_i is nonempty-, convex- and compact-valued. We next show that Γ_i is upper hemicontinuous. Consider a sequence $\{(x^n, w^n)\}_{n\in\mathbb{N}}$ such that $x^n \to x$ and $w^n \in \Gamma_i (x^n)$ for all $n \in \mathbb{N}$. By definition, we have that $w^n \in C_i$ as well as $T_i (x^n) = w^n \cdot x^n$ for all $n \in \mathbb{N}$. Since C_i is compact, there exists a subsequence $\{w^{n_k}\}_{k\in\mathbb{N}}$ such that $w^{n_k} \to w \in C_i$. Since T_i is continuous, we have that $T_i(x) = \lim_k T_i(x^{n_k}) = \lim_k w^{n_k} \cdot x^{n_k} = w \cdot x$, proving that $\{w^n\}_{n \in \mathbb{N}}$ has a limit point in $\Gamma_i(x)$. By [3, Theorem 17.20], we can conclude that Γ_i is upper hemicontinuous. Finally, consider $x \in \mathbb{R}^n$ and $h \in \mathbb{R}$. Since T is translation invariant and $C_i \subseteq \Delta$, note that

$$\begin{aligned}
w \in \Gamma_i(x) &\iff w \in C_i \text{ and } T_i(x) = w \cdot x \iff w \in C_i \text{ and } T_i(x) + h = w \cdot x + w \cdot he \\
&\iff w \in C_i \text{ and } T_i(x + he) = w \cdot (x + he) \iff w \in \Gamma_i(x + he),
\end{aligned}$$

proving (41). □

Fix $i \in N$. Define $f : \mathbb{R}^n \times \Delta \to \mathbb{R}_+$ by $f(x, w) = \sum_{j=1}^n w_j x_j^2$ for all $(x, w) \in \mathbb{R}^n \times \Delta$. It is immediate to see that f is continuous in the product topology. Define $\phi_i^T : \mathbb{R}^n \to \mathbb{R}_+$ by

$$\phi_i^T(x) = \max_{w \in \Gamma_i(x)} f(x, w) \qquad \forall x \in \mathbb{R}^n.$$

Next, consider $h \in \mathbb{R} \setminus \{0\}$. It follows that $f(he, w) = h^2$ for all $w \in \Delta$. We can conclude that

$$\phi_i^T(he) = h^2 > 0 = \phi_i^T(0).$$

Since i and h were arbitrarily chosen, this implies that $\phi = \left(\phi_i^T\right)_{i=1}^n$ is sensitive. Next, we move to the property of strictly increasing shifts. By (41), we have that

$$\phi_i^T(x + he) = \max_{w \in \Gamma_i(x+he)} \sum_{j=1}^n w_j (x_j + h)^2 = \max_{w \in \Gamma_i(x)} \left[\sum_{j=1}^n w_j x_j^2 + 2h \sum_{j=1}^n w_j x_j + h^2\right] \tag{42}$$

$$= \max_{w \in \Gamma_i(x)} \sum_{j=1}^n w_j x_j^2 + 2h T_i(x) + h^2 \qquad \forall x \in \mathbb{R}^n, \forall h \in \mathbb{R}. \tag{43}$$

Consider $z, v \in \mathbb{R}^n$ and $h \in \mathbb{R}_{++}$. By (42) and (43) and since T is monotone, we can conclude that

$$z \geq v \implies \phi_i^T(z + he) - \phi_i^T(z) = 2h T_i(z) + h^2 \geq 2h T_i(v) + h^2 = \phi_i^T(v + he) - \phi_i^T(v).$$

Since i was arbitrarily chosen, it follows that $\phi = \left(\phi_i^T\right)_{i=1}^n$ has increasing shifts and, in particular, $\phi \in \Phi_R$. Next, consider $z, v \in \mathbb{R}^n$ such that $z \gg v$. Set $k = \min_{j \in N}(z_j - v_j)$. It follows that $k > 0$ and $z \geq v + ke$. Since T is monotone and translation invariant and $k > 0$, we can conclude that $T(z) \geq T(v + ke) = T(v) + ke$. Since $z, v \in \mathbb{R}^n$ were arbitrarily chosen, it follows that

$$z \gg v \implies T(z) \gg T(v).$$

By (42) and (43), this implies that if $z, v \in \mathbb{R}^n$ and $h \in \mathbb{R}_{++}$, then

$$z \gg v \implies \phi_i^T(z + he) - \phi_i^T(z) = 2h T_i(z) + h^2 > 2h T_i(v) + h^2 = \phi_i^T(v + he) - \phi_i^T(v).$$

Since i was arbitrarily chosen, it follows that $\phi = \left(\phi_i^T\right)_{i=1}^n$ has strictly increasing shifts. We are left to prove (24). By Proposition 11 and since $\phi = \left(\phi_i^T\right)_{i=1}^n \in \Phi_R$ has strictly increasing shifts, we have that $\mathbf{T}_i^\phi(x) = \operatorname{argmin}_{c\in\mathbb{R}} \phi_i^T(x - ce)$ is well defined and single-valued for all $x \in B$ and for all $i \in N$. We are left to prove that it coincides with $T_i(x)$ for all $x \in B$ and for all $i \in N$. Fix $i \in N$ and $x \in B$. By (42) and (43), we have that

$$\phi_i^T(x - ce) = \max_{w\in\Gamma_i(x)} \sum_{j=1}^n w_j x_j^2 - 2cT_i(x) + c^2 \quad \forall c \in \mathbb{R}$$

which, as a function of c, is quadratic and minimized at $c = T_i(x)$, proving the statement. ■

Proof of Proposition 7. Let $x \in B$. Call V the set of values the components of x take: $V = \{x_1, ..., x_n\}$. Define U to be the subset of vectors y in B such that each component of y coincides in value to the value of some component of x, formally,

$$U = \{y \in B : y_i \in V \qquad \forall i \in \{1, ..., n\}\}.$$

Note that the cardinality of U is at most m^n. Since T is a discrete opinion aggregator, note that $T_i(y) \in V$ for all $y \in U$ and for all $i \in \{1, ..., n\}$. This implies that $T(x) \in U$. By induction, it follows that $T^t(x) \in U$ for all $t \in \mathbb{N}$. This implies that the sequence $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ can take at most a finite number of values. We have two cases:

1. $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ converges. If $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ converges, then the previous part implies that $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ becomes constant, that is, there exists $\tilde{t} \in \mathbb{N}$ such that

$$T^t(x) = T^{\tilde{t}}(x) \in U \qquad \forall t \geq \tilde{t}. \tag{44}$$

Call $\bar{x}$ the limit of $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$. Note that $\bar{x} = T^{\tilde{t}}(x)$ and $T^t(\bar{x}) = \bar{x}$ for all $t \in \mathbb{N}$. In particular, we have that

$$T(\bar{x}) = \bar{x}. \tag{45}$$

Define now $\bar{t} \in \mathbb{N}$ to be such that $\bar{t} = \min\left\{t \in \mathbb{N} : T^t(x) = \bar{x}\right\}$. By (44), $\bar{t}$ is well defined. By (45), we have that $T^t(x) = \bar{x}$ for all $t \geq \bar{t}$. If $\bar{t} = 1$, then $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ is constant to begin with and so it becomes constant after at most n^n periods. Assume $\bar{t} > 1$. We next show that $T^t(x) \neq T^m(x)$ for all $m, t < \bar{t}$ such that $m \neq t$. By contradiction, assume that there exist $m, t < \bar{t}$ such that $m \neq t$ and $T^t(x) = T^m(x)$. Without loss of generality, we assume that $m > t$. This would imply that $T^{t+n}(x) = T^n\left(T^t(x)\right) = T^n(T^m(x)) = T^{m+n}(x)$ for all $n \in \mathbb{N}$. In particular, by setting $n = \bar{t} - m > 0$, we would have that $T^{t+n}(x) = T^{m+n}(x) = T^{\bar{t}}(x) = \bar{x}$. Note that $\hat{t} = t + n < m + n = \bar{t}$. Thus, this would imply that

$$T^{\hat{t}}(x) = \bar{x} \text{ and } \hat{t} < \bar{t},$$

a contradiction with the minimality of $\bar{t}$. By definition of $\bar{t}$, we can also conclude that $T^t(x) \neq \bar{x}$ for all $t < \bar{t}$. This implies that $\left\{T^t(x)\right\}_{t=1}^{\bar{t}-1}$ is contained in $U\backslash\{\bar{x}\}$. Since U contains at most n^n elements and the elements of $\left\{T^t(x)\right\}_{t=1}^{\bar{t}-1}$ are pairwise distinct, it follows that $\bar{t} - 1 \leq n^n - 1$, proving that $\{T^t(x)\}_{t\in\mathbb{N}}$ converges only if it becomes constant after at most n^n periods.

2. $\{T^t(x)\}_{t\in\mathbb{N}}$ does not converge. Define $\tilde{n} = n^n$. Recall that $\{T^t(x)\}_{t=1}^{\tilde{n}+1} \subseteq U$ where the latter set has cardinality at most $\tilde{n}$. This implies that there exist $\hat{m}, \hat{t} \leq \tilde{n}+1$ such that $T^{\hat{m}}(x) = T^{\hat{t}}(x)$ and $\hat{m} \neq \hat{t}$. Without loss of generality, we assume that $\hat{m} > \hat{t}$. It follows that

$$T^{\hat{t}+n}(x) = T^n\left(T^{\hat{t}}(x)\right) = T^n\left(T^{\hat{m}}(x)\right) = T^{\hat{m}+n}(x) \qquad \forall n \in \mathbb{N}_0.$$

Define $p = \hat{m} - \hat{t} > 0$. Since $\hat{t} \geq 1$ and $\hat{m} \leq \tilde{n}+1$, note that $\hat{m} - \hat{t} \leq \tilde{n}$ and $\hat{t} \leq \tilde{n}$. We have that $T^{\hat{t}+n}(x) = T^{\hat{t}+n+p}(x)$ for all $n \in \mathbb{N}_0$, proving that $T^t(x) = T^{t+p}(x)$ for all $t \geq \hat{t}$.

Points 1 and 2 prove the first part of the statement as well as the "only if" of the second part. The "if" part is trivial. ■

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D Online Appendix

D.1 Missing proofs

In this section, we confine all the missing proofs. They appear in the order in which the corresponding statements appear in the text, unless they are new ancillary results.

Proof of Proposition 1. We omit the proof of point 2 which follows from well-known facts.[34] Recall that for each $i \in N$

$$\phi_i^\lambda(z) = \sum_{j=1}^n w_{ij}\rho_i(z_j) \qquad \forall z \in \mathbb{R}^n$$

where $\lambda \in \mathbb{R}\setminus\{0\}$ and $\rho_i : \mathbb{R} \to \mathbb{R}_+$ is defined by $\rho_i(\tilde{s}) = e^{\lambda\tilde{s}} - \lambda\tilde{s}$ for all $\tilde{s} \in \mathbb{R}$. It is easy to see that ρ_i is convex, strictly decreasing on $\mathbb{R}_-$, and strictly increasing on $\mathbb{R}_+$ for all $i \in N$. As shown in Section 6.2, this implies that $\phi \in \Phi_A \subseteq \Phi_R$. Since $\rho_i'' > 0$ for all $i \in N$, ρ_i is strictly convex for all $i \in N$. Standard computations yield that ϕ has strictly increasing shifts.

1. By Proposition 11, it follows that $\mathbf{T}^\phi = T^\phi = T^\lambda$ is single-valued and is a robust opinion aggregator. We are only left to compute it. Fix $i \in N$. Consider $x \in B$. Note that the function $c \mapsto \phi_i^\lambda(x - ce)$ is strictly convex and differentiable. We compute the first order conditions where $c^\star$ is the optimal value:

$$-\sum_{j=1}^n w_{ij}\left[\lambda\exp\left(\lambda\left(x_j - c^\star\right)\right) - \lambda\right] = 0 \implies \sum_{j=1}^n w_{ij}\exp\left(\lambda x_j\right) = \exp\left(\lambda c^\star\right) \implies$$
$$\implies c^\star = \frac{1}{\lambda}\ln\left(\sum_{j=1}^n w_{ij}\exp\left(\lambda x_j\right)\right),$$

proving the rest of the statement.

3. Let $S : \mathbb{R}^n \to \mathbb{R}^n_{++}$ be defined by $S_i(x) = \exp(\lambda x_i)$ for all $i \in N$ and for all $x \in \mathbb{R}^n$. Define $\hat{T} : \mathbb{R}^n \to \mathbb{R}^n$ by $\hat{T}(x) = Wx$ for all $x \in \mathbb{R}^n$. We next show that

$$\left(T^\lambda\right)^t = S^{-1}\hat{T}^t S \qquad \forall t \in \mathbb{N}. \tag{46}$$

By definition of T^λ, if $t = 1$, then $T^\lambda(x) = S^{-1}(WS(x))$ for all $x \in \mathbb{R}^n$, yielding (46). Next, assume that (46) holds for t. We have that

$$\left(T^\lambda\right)^{t+1} = T^\lambda\left(T^\lambda\right)^t = S^{-1}\hat{T}SS^{-1}\hat{T}^tS = S^{-1}\hat{T}^{t+1}S,$$

proving that (46) holds for $t+1$. By induction, (46) follows. Consider $x \in B$. By (11), it follows that

$$\lim_t \hat{T}^t(S(x)) = \lim_t W^tS(x) = \left(\sum_{i=1}^n s_i\exp\left(\lambda x_i\right)\right)e \in \mathbb{R}^n_{++}.$$

[34] The result for $\hat{\lambda} = \infty$ is also known as Laplace's method (see, e.g., [3, Theorem 4.1]). The case for $\hat{\lambda} = -\infty$ is instead obtained from the previous one and by observing that $\lambda x_j = -\lambda(-x_j)$ and that $\lambda \to -\infty$ yields $-\lambda \to \infty$. The case of $\hat{\lambda} = 0$ is a standard result in risk theory.

By (46) and since S^{-1} is continuous, we have that

$$\lim_t \left(T^{\lambda}\right)^t(x) = \left(\frac{1}{\lambda}\ln\left(\sum_{i=1}^n s_i \exp\left(\lambda x_i\right)\right)\right) e = \bar{T}^{\lambda}(x).$$

Since x was arbitrarily chosen, the statement follows. ■

Proof of Lemma 1. 1. Since T is robust, we have that $T_i : B \to \mathbb{R}$ is monotone and translation invariant for all $i \in N$.[35] By [4, Theorem 4], T_i is a niveloid for all $i \in N$. By [4, Theorem 1], T_i admits an extension $S_i : \mathbb{R}^n \to \mathbb{R}$ which is a niveloid for all $i \in N$. By [4, Theorem 4], S_i is monotone and translation invariant for all $i \in N$. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ to be such that the i-th component of $S(x)$ is $S_i(x)$ for all $i \in N$ and for all $x \in B$. It is immediate to see that S is monotone and translation invariant. Fix $k' \in I$. Since S is translation invariant and T is normalized, it follows that for all $k \in \mathbb{R}$

$$S(ke) = S\left(k'e + \left(k - k'\right)e\right) = S\left(k'e\right) + \left(k - k'\right)e = T\left(k'e\right) + \left(k - k'\right)e = k'e + \left(k - k'\right)e = ke,$$

proving that S is normalized and, in particular, that S is robust.

2. By induction, if T is normalized and monotone, then T^t is normalized and monotone for all $t \in \mathbb{N}$. Consider $x \in B$ and $t \in \mathbb{N}$. Define $k_\star = \min_{i\in N} x_i$ and $k^\star = \max_{i\in N} x_i$. Note that $\|x\|_\infty = \max\{|k_\star|, |k^\star|\}$, $k_\star, k^\star \in I$, and $k_\star e \le x \le k^\star e$. Since T^t is normalized and monotone, we have that

$$k_\star e = T^t(k_\star e) \le T^t(x) \le T^t(k^\star e) = k^\star e, \tag{47}$$

yielding that $\left|T^t(x)\right| \le \max\{|k_\star|, |k^\star|\} e$ and $\left\|T^t(x)\right\|_\infty \le \|x\|_\infty$. Since t and x were arbitrarily chosen, the statement follows.

3. Let $x \in B$. Define $k_\star = \min_{i\in N} x_i$ and $k^\star = \max_{i\in N} x_i$. We have two cases:

a. $k_\star < k^\star$. Clearly, we have that $k_\star, k^\star \in I$. Note that $\tilde{I} = [k_\star, k^\star] \subseteq I$ is compact and with nonempty interior. Moreover, $x \in \tilde{I}^n = \tilde{B}$.

b. $k_\star = k^\star$. Since I has nonempty interior, there exists $\varepsilon > 0$ such that either $\tilde{I} = [k_\star, k_\star + \varepsilon] \subseteq I$ or $\tilde{I} = [k_\star - \varepsilon, k_\star] \subseteq I$. In all these cases, $\tilde{I}$ is compact and with nonempty interior. Moreover, $x \in \tilde{I}^n = \tilde{B}$.

Consider the restriction $\tilde{T} = T_{|\tilde{B}}$. By (47), note that $T\left(\tilde{B}\right) \subseteq \tilde{B}$, yielding that $\tilde{T}$ is a robust opinion aggregator. By induction, we have that $\tilde{T}^t(x) = T^t(x)$ for all $t \in \mathbb{N}$ and for all $x \in \tilde{B}$. It follows that

$$\bar{T}(x) = \lim_\tau \frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) = \lim_\tau \frac{1}{\tau}\sum_{t=1}^{\tau} \tilde{T}^t(x) = \bar{\tilde{T}}(x) \qquad \forall x \in \tilde{B},$$

proving the point. ■

Proof of Lemma 2. Since T is a robust opinion aggregator, T_i is normalized, monotone, and translation invariant for all $i \in N$. By [4, Theorem 4], it follows that T_i is a niveloid for all $i \in N$. By [4, p. 346], it follows that $|T_i(x) - T_i(y)| \le \|x - y\|_\infty$ for all $x, y \in B$ and for all $i \in N$. This implies that

$$\|T(x) - T(y)\|_\infty = \max_{i\in N} |T_i(x) - T_i(y)| \le \|x - y\|_\infty \qquad \forall x, y \in B,$$

proving that T is nonexpansive.

By induction, we next show that T^t is nonexpansive for all $t \in \mathbb{N}$. Since we have shown that T is nonexpansive, T^t is nonexpansive for $t = 1$, proving the initial step. By the induction hypothesis,

[35] With a small abuse of terminology, we use the same name for similar properties that pertain to, respectively, functionals and operators.

assume that T^t is nonexpansive, we have that for each $x, y \in B$

$$\left\|T^{t+1}(x) - T^{t+1}(y)\right\|_\infty = \left\|T\left(T^t(x)\right) - T\left(T^t(y)\right)\right\|_\infty \leq \left\|T^t(x) - T^t(y)\right\|_\infty \leq \|x - y\|,$$

proving the inductive step. The statement follows by induction. ■

Proof of Lemma 3. Let $x \in B$. Since T is a selfmap, we have that $\left\{T^t(x)\right\}_{t\in\mathbb{N}} \subseteq B$. Since B is convex, we have that

$$\frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) \in B \qquad \forall \tau \in \mathbb{N}.$$

Since x was arbitrarily chosen, this implies that $A_\tau : B \to B$, defined by $A_\tau(x) = \sum_{t=1}^{\tau} T^t(x)/\tau$ for all $x \in B$, is well defined for all $\tau \in \mathbb{N}$. Since B is closed, we have that $\bar{T}(x) = \lim_\tau A_\tau(x) = \lim_\tau \frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) \in B$ for all $x \in B$, proving that $\bar{T}$ is well defined. By the same computations contained in [1, Lemma 20.12], despite T being nonlinear, one can prove that

$$A_\tau(T(x)) = \frac{\tau+1}{\tau} A_{\tau+1}(x) - \frac{1}{\tau} T(x) \qquad \forall x \in B, \forall \tau \in \mathbb{N}.$$

This implies that

$$\bar{T}(T(x)) = \lim_\tau A_\tau(T(x)) = \lim_\tau \frac{\tau+1}{\tau} \lim_\tau A_{\tau+1}(x) - \lim_\tau \frac{1}{\tau} T(x) = \bar{T}(x) \qquad \forall x \in B,$$

proving that $\bar{T} \circ T = \bar{T}$.

1. By the same inductive argument contained in the proof of Lemma 2, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is nonexpansive. Since the convex linear combination of nonexpansive maps is nonexpansive, the map $A_\tau : B \to B$ is nonexpansive for all $\tau \in \mathbb{N}$. We can conclude that for each $x, y \in B$

$$\left\|\bar{T}(x) - \bar{T}(y)\right\|_\infty = \left\|\lim_\tau A_\tau(x) - \lim_\tau A_\tau(y)\right\|_\infty = \lim_\tau \|A_\tau(x) - A_\tau(y)\|_\infty \leq \|x - y\|_\infty,$$

proving that $\bar{T}$ is nonexpansive. Continuity of $\bar{T}$ trivially follows.

2. By induction, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is normalized and monotone. Since the convex linear combination of normalized and monotone operators is normalized and monotone, the map $A_\tau : B \to B$ is normalized and monotone for all $\tau \in \mathbb{N}$. We can conclude that $\bar{T}(ke) = \lim_\tau A_\tau(ke) = ke$ for all $k \in I$ as well as

$$x \geq y \implies \bar{T}(x) = \lim_\tau A_\tau(x) \geq \lim_\tau A_\tau(y) = \bar{T}(y),$$

proving that $\bar{T}$ is normalized and monotone.

3. Since T is robust, T is normalized, monotone, and translation invariant. By the previous point, $\bar{T}$ is normalized and monotone. By induction, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is translation invariant. Since the convex linear combination of translation invariant operators is translation invariant, the map $A_\tau : B \to B$ is translation invariant for all $\tau \in \mathbb{N}$. We can conclude that for each $x \in B$ and for each $k \in \mathbb{R}$ such that $x + ke \in B$

$$\bar{T}(x + ke) = \lim_\tau A_\tau(x + ke) = \lim_\tau [A_\tau(x) + ke] = \bar{T}(x) + ke,$$

proving that $\bar{T}$ is translation invariant and, in particular, robust.

4. By induction, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is odd. Since the convex linear combination of odd maps is odd, the map $A_\tau : B \to B$ is odd for all $\tau \in \mathbb{N}$. We can conclude that

$$\bar{T}(-x) = \lim_\tau A_\tau(-x) = \lim_\tau [-A_\tau(x)] = -\bar{T}(x) \quad \forall x \in B \text{ s.t. } -x \in B,$$

proving that $\bar{T}$ is odd. ■

In order to prove Lemma 4, we are going to rely upon Lorentz's Theorem.

Theorem 5 (Lorentz) *Let $\{x^t\}_{t\in\mathbb{N}} \subseteq \mathbb{R}^n$ be a bounded sequence. The following statements are equivalent:*

(i) There exists $\bar{x} \in \mathbb{R}^n$ such that

$$\forall \varepsilon > 0 \ \exists \bar{\tau} \in \mathbb{N} \ \forall m \in \mathbb{N} \ s.t. \ \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_\infty < \varepsilon \qquad \forall \tau \geq \bar{\tau} \tag{48}$$

and $\lim_t \left\|x^{t+1} - x^t\right\|_\infty = 0$*;*

(ii) $\lim_t x^t = \bar{x}$*.*

Proof of Lemma 4. By Theorem 1 and since T is robust, we have that if $\hat{B}$ is a bounded subset of B, then

$$\lim_\tau \left(\sup_{x\in\hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_\infty \right) = 0 \tag{49}$$

where $\bar{T} : B \to B$ is a robust opinion aggregator such that $\bar{T} \circ T = \bar{T}$. Since $\bar{T}(T(x)) = \bar{T}(x)$ for all $x \in B$, by induction, we have that $\bar{T}(T^m(x)) = \bar{T}(x)$ for all $m \in \mathbb{N}$ and for all $x \in B$.

(i) implies (ii). Fix $x \in B$. Define the sequence $x^t = T^t(x)$ for all $t \in \mathbb{N}$. By point 2 of Lemma 1, we have that $\{x^t\}_{t\in\mathbb{N}}$ is bounded. Set $\hat{B} = \{x^t\}_{t\in\mathbb{N}}$. Note that for each $\tau \in \mathbb{N}$ and for each $m \in \mathbb{N}$

$$\frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} = \frac{1}{\tau} \sum_{t=1}^{\tau} T^{m+t}(x) = \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x)).$$

Since (49) holds, if we define $\bar{x} = \bar{T}(x)$, then we have that for each $m \in \mathbb{N}$

$$\lim_\tau \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} = \lim_\tau \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x)) = \bar{T}(T^m(x)) = \bar{T}(x) = \bar{x}.$$

It follows that

$$\sup_{m\in\mathbb{N}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_\infty = \sup_{m\in\mathbb{N}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x)) - \bar{T}(T^m(x)) \right\|_\infty \leq \sup_{x\in\hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_\infty.$$

Since (49) holds and T is asymptotically regular, we have that $\{x^t\}_{t\in\mathbb{N}}$ satisfies point (i) of Theorem 5. By Theorem 5, we have that $\lim_t T^t(x) = \lim_t x^t$ exists. Since x was arbitrarily chosen, the implication follows.

(ii) implies (i). Fix $x \in B$. Define $x^t = T^t(x)$ for all $t \in \mathbb{N}$. Since T is convergent, we have that $\{x^t\}_{t\in\mathbb{N}}$ converges and, in particular, is bounded. By Theorem 5, we have that $\lim_t \left\|T^{t+1}(x) - T^t(x)\right\|_\infty = \lim_t \left\|x^{t+1} - x^t\right\|_\infty = 0$. Since x was arbitrarily chosen, the implication follows. ■

Proof of Lemma 5. We first offer two definitions and make two observations. Define the diameter of $\left\{T^t(x) : x \in A \text{ and } t \in \mathbb{N}_0\right\}$ by D.[36] Given $x \in B$, define $x^t = T^t(x)$ as well as $y^t = S\left(x^t\right)$ for all $t \in \mathbb{N}_0$. Since T is nonexpansive, recall that $\left\{\left\|x^t - x^{t-1}\right\|_\infty\right\}_{t\in\mathbb{N}}$ is a decreasing sequence for all $x \in B$. Note that this implies that $\|T(x) - x\|_\infty \geq \left\|T^{t+1}(x) - T^t(x)\right\|_\infty$ for all $t \in \mathbb{N}_0$ and for all $x \in B$, yielding that $k > \delta$.

By contradiction, assume that $\left\{T^t(x) : x \in A \text{ and } t \in \mathbb{N}_0\right\}$ is bounded. This implies that $D < \infty$. Consider $M \in \mathbb{N}\backslash\{1\}$ and $P \in \mathbb{N}$ to be such that

$$M\delta > D + \delta + 1 \text{ and } \left\lfloor \frac{P}{M} \right\rfloor > \max\left\{1, \frac{k}{(1-\varepsilon)\varepsilon^M}\right\}.$$

By (32) and since $P \in \mathbb{N}$, there exists $x \in A$ such that

$$\left\|x^{P+1} - x^P\right\|_\infty = \left\|T^{P+1}(x) - T^P(x)\right\|_\infty \geq \delta.$$

Now, we enlist seven useful facts:

1. By (31) and since $\left\{\left\|x^t - x^{t-1}\right\|_\infty\right\}_{t\in\mathbb{N}}$ is a decreasing sequence, it follows that

$$k \geq \left\|x^{i+1} - x^i\right\|_\infty \geq \delta \qquad \forall i \in \{1, ..., P\}.$$

2. By definition of $\left\{y^t\right\}_{t\in\mathbb{N}_0}$ and since S is nonexpansive, we have that

$$\left\|y^t - y^{t-1}\right\|_\infty = \left\|S\left(x^t\right) - S\left(x^{t-1}\right)\right\|_\infty \leq \left\|x^t - x^{t-1}\right\|_\infty \qquad \forall t \in \mathbb{N}.$$

3. By definition of $\left\{x^t\right\}_{t\in\mathbb{N}_0}$ and since $T = \varepsilon J + (1-\varepsilon)S$, we have that $x^t = T\left(x^{t-1}\right) = \varepsilon J\left(x^{t-1}\right) + (1-\varepsilon)y^{t-1}$ for all $t \in \mathbb{N}$, that is,

$$y^{t-1} = \frac{1}{1-\varepsilon}x^t - \frac{\varepsilon}{1-\varepsilon}J\left(x^{t-1}\right) \qquad \forall t \in \mathbb{N}.$$

   By point 2, this yields that for each $t \in \mathbb{N}$

$$\left\|\frac{1}{1-\varepsilon}\left(x^{t+1} - x^t\right) - \frac{\varepsilon}{1-\varepsilon}\left(J\left(x^t\right) - J\left(x^{t-1}\right)\right)\right\|_\infty = \left\|y^t - y^{t-1}\right\|_\infty \leq \left\|x^t - x^{t-1}\right\|_\infty.$$

4. Let L be an integer in $\mathbb{N}$ such that

$$L > \frac{k}{(1-\varepsilon)\varepsilon^M}. \tag{50}$$

   Define $b_m = \delta + m(1-\varepsilon)\varepsilon^M$ for all $m \in \{0, ..., L\}$. It follows that the collection of intervals $\{[b_m, b_{m+1}]\}_{m=0}^{L-1}$ contains L elements whose union is a superset of $[\delta, k]$.

5. Note that $\varepsilon^{M-1}\frac{1-\varepsilon^i}{\varepsilon^i} = \varepsilon^{M-i-1} - \varepsilon^{M-1} \leq \varepsilon^{M-i-1}$ for all $i \in \{1, M-1\}$. Since $\varepsilon \in (0,1)$, this

[36] Recall that the diameter of a subset $\hat{A}$ of B is the quantity

$$\sup\left\{\|x - y\|_\infty : x, y \in \hat{A}\right\}.$$

implies that

$$(1-\varepsilon)\,\varepsilon^{M}\sum_{i=1}^{M-1}\frac{1-\varepsilon^{i}}{\varepsilon^{i}}\leq(1-\varepsilon)\,\varepsilon\sum_{i=1}^{M-1}\varepsilon^{M-i-1}=(1-\varepsilon)\,\varepsilon\sum_{i=0}^{M-2}\varepsilon^{i}$$
$$\leq(1-\varepsilon)\,\varepsilon\frac{1}{1-\varepsilon}\leq\varepsilon<1.$$

6. Let $t\in\mathbb{N}$, $j\in N$, and $b,\kappa,c\geq 0$. If $x_j^{t+1}-x_j^t\geq b-c$ and $\left\|x^t-x^{t-1}\right\|_\infty\leq b+\kappa$, then (by point 3):
$$\frac{b-c}{1-\varepsilon}-\frac{\varepsilon}{1-\varepsilon}\left(x_{k_l}^{t}-x_{k_l}^{t-1}\right)=\frac{b-c}{1-\varepsilon}-\frac{\varepsilon}{1-\varepsilon}\left(J_j\left(x^t\right)-J_j\left(x^{t-1}\right)\right)\leq b+\kappa$$
where l is such that $j\in N_l$. This yields that
$$x_{k_l}^{t}-x_{k_l}^{t-1}\geq b-\frac{c}{\varepsilon}-\frac{1-\varepsilon}{\varepsilon}\kappa. \tag{51}$$

7. Let $t\in\mathbb{N}$, $j\in N$, and $b,\kappa,c\geq 0$. If $x_j^{t}-x_j^{t+1}\geq b-c$ and $\left\|x^t-x^{t-1}\right\|_\infty\leq b+\kappa$, then (by point 3):
$$\frac{b-c}{1-\varepsilon}-\frac{\varepsilon}{1-\varepsilon}\left(x_{k_l}^{t-1}-x_{k_l}^{t}\right)=\frac{b-c}{1-\varepsilon}-\frac{\varepsilon}{1-\varepsilon}\left(J_j\left(x^{t-1}\right)-J_j\left(x^{t}\right)\right)\leq b+\kappa$$
where l is such that $j\in N_l$. This yields that
$$x_{k_l}^{t-1}-x_{k_l}^{t}\geq b-\frac{c}{\varepsilon}-\frac{1-\varepsilon}{\varepsilon}\kappa. \tag{52}$$

By definition of P, we have that $\lfloor P/M\rfloor$ satisfies (50). By point 4, there exists a collection of intervals $\{[b_m,b_{m+1}]\}_{m=0}^{\lfloor P/M\rfloor-1}$ which covers $[\delta,k]$. By point 1, $[\delta,k]$ contains $\left\{\left\|x^{i+1}-x^i\right\|_\infty\right\}_{i=1}^P$. Since we have $\lfloor P/M\rfloor$ intervals and the first P elements (of the sequence $\left\{\left\|x^{t+1}-x^t\right\|_\infty\right\}_{t\in\mathbb{N}}$) belong to these intervals, we have that there exists one of them, $\hat{I}=[b_{\bar{m}},b_{\bar{m}+1}]$, which contains at least M elements of $\left\{\left\|x^{i+1}-x^i\right\|_\infty\right\}_{i=1}^P$. Since $\left\{\left\|x^t-x^{t-1}\right\|_\infty\right\}_{t\in\mathbb{N}}$ is decreasing, we have that there exists $K\in\mathbb{N}_0$ such that
$$\left\|x^{K+i+1}-x^{K+i}\right\|_\infty\in\hat{I}\qquad\forall i\in\{1,...,M\}\,.$$
This implies that there exists $j\in\{1,...,n\}$ such that $\left|x_j^{K+M+1}-x_j^{K+M}\right|\geq b_{\bar{m}}$ and $||x^{K+M}-x^{K+M-1}||_\infty\leq b_{\bar{m}+1}=b_{\bar{m}}+(1-\varepsilon)\,\varepsilon^M$. We have two cases:

a. $x_j^{K+M+1}-x_j^{K+M}\geq b_{\bar{m}}$. Set $b=b_{\bar{m}}$, $c=0$, and $\kappa=(1-\varepsilon)\,\varepsilon^M$. By (51), we can conclude that
$$x_{k_l}^{K+M}-x_{k_l}^{K+M-1}\geq b_{\bar{m}}-(1-\varepsilon)\,\varepsilon^M\frac{(1-\varepsilon)}{\varepsilon}. \tag{53}$$

By (finite) induction, we next prove that
$$x_{k_l}^{K+M+1-i}-x_{k_l}^{K+M-i}\geq b_{\bar{m}}-(1-\varepsilon)\,\varepsilon^M\frac{\left(1-\varepsilon^i\right)}{\varepsilon^i}\qquad\forall i\in\{1,M-1\}\,, \tag{54}$$
that is,
$$x_{k_l}^{K+M+1-i}\geq x_{k_l}^{K+M-i}+b_{\bar{m}}-(1-\varepsilon)\,\varepsilon^M\frac{\left(1-\varepsilon^i\right)}{\varepsilon^i}\qquad\forall i\in\{1,M-1\}\,. \tag{55}$$

By (53), the statement is true for $i = 1$. Next, we assume it is true for $i \in \{1, ..., M-1\}$ and prove it is still true for $i+1$ when $i+1 \in \{1, ..., M-1\}$. This implies that $i \leq M-2$. Define $t = K + M - i$. By the induction hypothesis, we have that

$$x_{k_l}^{t+1} - x_{k_l}^{t} = x_{k_l}^{K+M+1-i} - x_{k_l}^{K+M-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i}.$$

Moreover, we also have that $\left\|x^t - x^{t-1}\right\|_\infty = \left\|x^{K+M-i} - x^{K+M-i-1}\right\|_\infty \leq b_{\bar{m}} + (1-\varepsilon)\,\varepsilon^M$. Set $b = b_{\bar{m}}$, $c = (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i}$, and $\kappa = (1-\varepsilon)\,\varepsilon^M$. By (51), we can conclude that

$$\begin{aligned} x_{k_l}^{K+M+1-(i+1)} - x_{k_l}^{K+M-(i+1)} &= x_{k_l}^{K+M-i} - x_{k_l}^{K+M-i-1} = x_{k_l}^{t} - x_{k_l}^{t-1} \\ &\geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i}\frac{1}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon}(1-\varepsilon)\,\varepsilon^M \\ &= b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^{i+1})}{\varepsilon^{i+1}}, \end{aligned}$$

proving (54). By (55), repeated substitution, and point 5, this implies that

$$x_{k_l}^{K+M} \geq x_{k_l}^{K+1} + (M-1)\,b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \sum_{i=1}^{M-1} \frac{1-\varepsilon^i}{\varepsilon^i} \geq x_{k_l}^{K+1} + (M-1)\,b_{\bar{m}} - 1,$$

that is,

$$\left\|x^{K+M} - x^{K+1}\right\|_\infty \geq x_{k_l}^{K+M} - x_{k_l}^{K+1} \geq (M-1)\,b_{\bar{m}} - 1.$$

Since $b_{\bar{m}} \geq \delta > 0$, we have that $(M-1)\,b_{\bar{m}} \geq (M-1)\,\delta > D+1$. We can conclude that

$$D \geq \left\|x^{K+M} - x^{K+1}\right\|_\infty \geq (M-1)\,b_{\bar{m}} - 1 > D,$$

a contradiction.

b. $x_j^{K+M} - x_j^{K+M+1} \geq b_{\bar{m}}$. Set $b = b_{\bar{m}}$, $c = 0$, and $\kappa = (1-\varepsilon)\,\varepsilon^M$. By (52), we can conclude that

$$x_{k_l}^{K+M-1} - x_{k_l}^{K+M} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{1-\varepsilon}{\varepsilon}. \tag{56}$$

By (finite) induction, we next prove that

$$x_{k_l}^{K+M-i} - x_{k_l}^{K+M+1-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i} \qquad \forall i \in \{1, M-1\}, \tag{57}$$

that is,

$$x_{k_l}^{K+M-i} \geq x_{k_l}^{K+M+1-i} + b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i} \qquad \forall i \in \{1, M-1\}. \tag{58}$$

By (56), the statement is true for $i = 1$. Next, we assume it is true for $i \in \{1, ..., M-1\}$ and prove it is still true for $i+1$ when $i+1 \in \{1, ..., M-1\}$. This implies that $i \leq M-2$. Define $t = K + M - i$. By the induction hypothesis, we have that

$$x_{k_l}^{t} - x_{k_l}^{t+1} = x_{k_l}^{K+M-i} - x_{k_l}^{K+M+1-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i}.$$

Moreover, we also have that $\left\|x^t - x^{t-1}\right\|_\infty = \left\|x^{K+M-i} - x^{K+M-i-1}\right\|_\infty \leq b_{\bar{m}} + (1-\varepsilon)\varepsilon^M$. Set $b = b_{\bar{m}}$, $c = (1-\varepsilon)\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i}$, and $\kappa = (1-\varepsilon)\varepsilon^M$. By (52), we can conclude that

$$\begin{aligned} x_{k_l}^{K+M-(i+1)} - x_{k_l}^{K+M+1-(i+1)} &= x_{k_l}^{K+M-i-1} - x_{k_l}^{K+M-i} = x_{k_l}^{t-1} - x_{k_l}^{t} \\ &\geq b_{\bar{m}} - (1-\varepsilon)\varepsilon^M \frac{\left(1-\varepsilon^i\right)}{\varepsilon^i}\frac{1}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon}(1-\varepsilon)\varepsilon^M \\ &= b_{\bar{m}} - (1-\varepsilon)\varepsilon^M \frac{\left(1-\varepsilon^{i+1}\right)}{\varepsilon^{i+1}}, \end{aligned}$$

proving (57). By (58), repeated substitution, and point 5, this implies that

$$x_{k_l}^{K+1} \geq x_{k_l}^{K+M} + (M-1) b_{\bar{m}} - (1-\varepsilon)\varepsilon^M \sum_{i=1}^{M-1} \frac{1-\varepsilon^i}{\varepsilon^i} \geq x_{k_l}^{K+M} + (M-1) b_{\bar{m}} - 1,$$

that is,

$$\left\|x^{K+1} - x^{K+M}\right\|_\infty \geq x_{k_l}^{K+1} - x_{k_l}^{K+M} \geq (M-1) b_{\bar{m}} - 1.$$

Since $b_{\bar{m}} \geq \delta > 0$, we have that $(M-1) b_{\bar{m}} \geq (M-1)\delta > D+1$. We can conclude that

$$D \geq \left\|x^{K+1} - x^{K+M}\right\|_\infty \geq (M-1) b_{\bar{m}} - 1 > D,$$

a contradiction.

Points a and b prove the statement. ■

Proof of Corollary 1. Since T is self-influential, it follows that each row of $\underline{A}(T)$ is not null, yielding that T has a nontrivial network. Moreover, since there is a simple cycle of length 1 from i to i for all $i \in N$, each closed group is trivially aperiodic. By Theorem 2, the statement follows. ■

Lemma 7 *Let T be a robust opinion aggregator. If T is convergent and $\bar{T}$ is defined as in (3), then $\bar{T}(x) = \lim_t T^t(x) \in E(T)$ for all $x \in B$.*

Proof. By Lemma 2 and since T is robust, T is nonexpansive and, in particular, continuous. By Theorems 1 and 5 and since T is robust and convergent, we have that

$$\bar{T}(x) = \text{C-}\lim_t T^t(x) = \lim_t T^t(x) \qquad \forall x \in B.$$

For ease of notation, for each $x \in B$ define $\bar{T}(x) = \bar{x}$. Since T is continuous, we have that $T(\bar{x}) = T\left(\lim_t T^t(x)\right) = \lim_t T\left(T^t(x)\right) = \lim_t T^{t+1}(x) = \bar{x}$, proving the statement. ■

Proof of Proposition 10. By Proposition 8, if T has a nontrivial network, then there exist $W \in \mathcal{W}$ and $\varepsilon \in (0,1)$ such that

$$T(x) = \varepsilon W x + (1-\varepsilon) S(x) \qquad \forall x \in B \tag{59}$$

where $S : B \to B$ is a robust opinion aggregator. Moreover, W can be chosen to be such that $A(W) = \underline{A}(T)$. Finally, by induction and (59), we have that if $t \in \mathbb{N}$, then there exist $\gamma \in (0,1)$ and a robust opinion aggregator $\tilde{S} : B \to B$ (which both depend on t) such that

$$T^t(x) = \gamma W^t x + (1-\gamma)\tilde{S}(x) \qquad \forall x \in B. \tag{60}$$

a. Before starting, observe that if T has the pairwise common influencer property, then T has a nontrivial network. Assume that T has the pairwise common influencer property. By contradiction, assume that there exists $x \in B \backslash D$ such that $T(x) = x$. Define $x_i = \min_{l \in N} x_l$ and $x_j = \max_{l \in N} x_l$. It follows that $x_j > x_i$ and $i \neq j$. Since W is scrambling, there exists $k = k(i,j) \in N$ such that $w_{ik} > 0$ and $w_{jk} > 0$. We have two cases:

1. $x_k < x_j$. It follows that

$$0 = \|T(x) - x\|_\infty \geq |T_j(x) - x_j| = \left| \varepsilon \sum_{l=1}^{n} w_{jl} x_l + (1-\varepsilon) S_j(x) - x_j \right|$$
$$= \varepsilon \sum_{l=1}^{n} w_{jl}(x_j - x_l) + (1-\varepsilon)(x_j - S_j(x)) \geq \varepsilon w_{jk}(x_j - x_k) > 0,$$

   a contradiction.

2. $x_k > x_i$. It follows that

$$0 = \|T(x) - x\|_\infty \geq |T_i(x) - x_i| = \left| \varepsilon \sum_{l=1}^{n} w_{il} x_l + (1-\varepsilon) S_i(x) - x_i \right|$$
$$= \varepsilon \sum_{l=1}^{n} w_{il}(x_l - x_i) + (1-\varepsilon)(S_i(x) - x_i) \geq \varepsilon w_{ik}(x_k - x_i) > 0,$$

   a contradiction.

Case 1 and 2 prove that the only equilibria of T are the constant vectors in B, proving the statement.

b. By induction, if $t \in \mathbb{N}$, then the equilibria of T are a subset of the ones of $T^t : B \to B$. Since $A(W) = \underline{A}(T)$, it follows that $A(W)$ has a a unique strongly connected and closed group M, and M is aperiodic under $A(W)$. Fix $x \in E(T)$.

We first show that $x_i = x_j$ for all $i, j \in M$. Define $x_i = \min_{l \in M} x_l$ and $x_j = \max_{l \in M} x_l$, and assume by contradiction that $x_j > x_i$. Given the properties of W and M, there exists $\bar{t} \in \mathbb{N}$ such that $w_{ij}^{(\bar{t})} > 0$. Since $x \in E(T)$, we have that $T^{\bar{t}}(x) = x$. By (60), this implies that

$$0 = \left\| T^{\bar{t}}(x) - x \right\|_\infty \geq \left| T_i^{\bar{t}}(x) - x_i \right| = \left| \gamma \sum_{l \in M} w_{il}^{(\bar{t})} x_l + (1-\gamma) \tilde{S}_i(x) - x_i \right|$$
$$= \gamma \sum_{l \in M} w_{il}^{(\bar{t})}(x_l - x_i) + (1-\gamma)\left(\tilde{S}_i(x) - x_i\right) \geq \gamma w_{ij}^{(\bar{t})}(x_j - x_i) > 0,$$

a contradiction.

Next, assume by contradiction that there exist $j \in M$ and $i \in N \backslash M$ such that $x_j < \max_{l \in N} x_l = x_i$. Given the properties of W and M, there exists $\bar{t} \in \mathbb{N}$ such that $w_{ij}^{(\bar{t})} > 0$ (cf. [8, Corollaries 8.1 and 8.2]). Since $x \in E(T)$, we have that $T^{\bar{t}}(x) = x$. By (60), this implies that

$$0 = \left\| T^{\bar{t}}(x) - x \right\|_\infty \geq \left| T_i^{\bar{t}}(x) - x_i \right| = \left| \gamma \sum_{l \in N} w_{il}^{(\bar{t})} x_l + (1-\gamma) \tilde{S}_i(x) - x_i \right|$$
$$= \gamma \sum_{l \in M} w_{il}^{(\bar{t})}(x_i - x_l) + (1-\gamma)\left(x_i - \tilde{S}_i(x)\right) \geq \gamma w_{ij}^{(\bar{t})}(x_i - x_j) > 0,$$

a contradiction. Finally, assuming by contradiction that there exist $j \in M$ and $i \in N \backslash M$ such that $x_j > \min_{l \in N} x_l = x_i$ and following similar steps, we would also get a contradiction. This proves the statement. ■

Lemma 8 *Let T be a convergent robust opinion aggregator. There exists $\tau \in \mathbb{N}$ such that, for each $t \geq \tau$, $\left(\bar{A}(T)\right)^t \geq \bar{A}(\bar{T})$.*

Proof. We first observe that, for all robust opinion aggregators S and $\hat{S}$, $\bar{A}(S)\bar{A}\left(\hat{S}\right) \geq \bar{A}\left(S\hat{S}\right)$. To see this, define the matrices $A' = \bar{A}(S)\bar{A}\left(\hat{S}\right)$ and $A'' = \bar{A}\left(S\hat{S}\right)$ and fix $i, j \in N$ such that $a'_{ij} = 0$. This implies that, for all $l \in N$, either $\bar{a}_{il}(S) = 0$ or $\bar{a}_{lj}\left(\hat{S}\right) = 0$. Then, for all $l \in N$ such that $\bar{a}_{il}(S) = 1$, it follows that $\hat{S}_l\left(x + he^j\right) = \hat{S}_l(x)$, for every $x \in B$ and $h \in \mathbb{R}_+$ such that $x + he^j \in B$. Finally, define $\hat{N}_i = \{l \in N : \bar{a}_{il}(S) = 1\}$ and observe that, for every $x \in B$ and $h \in \mathbb{R}_+$ such that $x + he^j \in B$,

$$S_i\left(\hat{S}\left(x + he^j\right)\right) = S_i\left(\hat{S}(x) + \sum_{l \in N}\left[\hat{S}\left(x + he^j\right) - \hat{S}(x)\right]e^l\right)$$

$$= S_i\left(\hat{S}(x) + \sum_{l \in N \backslash \hat{N}_i}\left[\hat{S}\left(x + he^j\right) - \hat{S}(x)\right]e^l\right) = S_i\left(\hat{S}(x)\right),$$

proving that $\bar{a}_{ij}\left(S\hat{S}\right) = 0$.

Next, given the previous claim, by induction we have that, for all $t \in \mathbb{N}$, $\left(\bar{A}(T)\right)^t \geq \bar{A}\left(T^t\right)$. Moreover, since T is convergent, for all $i, j \in N$, $x \in B$ and $h \in \mathbb{R}_+$, such that $x + he^j \in B$, if $\bar{T}_i\left(x + he^j\right) - \bar{T}_i(x) > 0$, then there exists $\tau \in \mathbb{N}$, such that for all $t \geq \tau$, $T_i^t\left(x + he^j\right) - T_i^t(x) > 0$. Fix $i, j \in N$. Suppose that $\bar{a}_{ij}\left(\bar{T}\right) = 1$. Therefore, there exist $x \in B$ and $h \in \mathbb{R}_+$, such that $x + he^j \in B$, if $\bar{T}_i\left(x + he^j\right) - \bar{T}_i(x) > 0$. Since T is convergent, there exists $\tau_{ij} \in \mathbb{N}$, such that, for all $t \geq \tau_{ij}$, $T_i^t\left(x + he^j\right) - T_i^t(x) > 0$ implying that, for all $t \geq \tau_{ij}$, $\bar{a}_{ij}\left(T^t\right) = 1$. With this, it follows that, for all $t \geq \max_{i,j \in N} \tau_{ij}$, $\bar{A}\left(T^t\right) \geq \bar{A}\left(\bar{T}\right)$, which on turn implies that, for all $t \geq \max_{i,j \in N} \tau_{ij}$, $\left(\bar{A}(T)\right)^t \geq \bar{A}\left(\bar{T}\right)$, proving the statement. ■

Proof of Corollary 2. Suppose that T has the pairwise common influencer property, it follows that T has a nontrivial network. Moreover, we have that there exists $\hat{t} \in \mathbb{N}$ such that $\underline{A}(T)^t$ has a column, say k, which is strictly positive for all $t \geq \hat{t}$.[37] If M is a closed group, then this implies that for each $i \in M$ there exists a path to k, proving that $k \in M$. Recall that $\underline{a}_{kk}^{(\hat{t})} = 1$ yields the existence of a cycle, be it simple or not, starting and ending at k. It is easy to show that the greatest common divisor of the lengths of the simple cycles of M, call it d, divides all the $t \in \mathbb{N}$ such that $\underline{a}_{kk}^{(t)} = 1$. Since $\underline{a}_{kk}^{(\hat{t})}, \underline{a}_{kk}^{(\hat{t}+1)} = 1$, we can conclude that $d = 1$, proving aperiodicity. By Theorem 2, Lemma 7, and Proposition 10, it follows that $\bar{T}(x) = \lim_t T^t(x) \in E(T) = D$. ■

Proof of Proposition 2. Before proving the statement, we introduce some terminology and some useful facts. Call Π the collection of all permutations of N, that is, the collection of all bijections $\pi : \{1, ..., n\} \to \{1, ..., n\}$. Fix $i \in N$. Given $\pi \in \Pi$, and $i \in N$, let $p^{\pi,i}$

$$p_{\pi(l)}^{\pi,i} = f_i\left(\sum_{j=1}^{l} w_{i\pi(j)}\right) - f_i\left(\sum_{j=1}^{l-1} w_{i\pi(j)}\right) \qquad \forall l \in N. \tag{61}$$

[37] Consider a stochastic matrix W such that $A(W) = \underline{A}(T)$. By [13, Theorem 4.11 and Exercise 4.13], we have that there exists $\hat{t} \in \mathbb{N}$ such that W^t has a column, say k, which is strictly positive for all $t \geq \hat{t}$. Finally, it is enough to observe that $\underline{a}_{ij}^{(t)} > 0$ if and only if $w_{ij}^{(t)} > 0$.

with the convention that $f_i\left(\sum_{j=1}^{0} w_{i\pi(j)}\right) = 0$. Define by D_i the set $\left\{p^{\pi,i} \in \Delta : \pi \in \Pi\right\}$. By (61) and the fact that f_i is strictly increasing, for each $\pi \in \Pi$ and for each $j \in N$

$$p_{\pi(j)}^{\pi,i} > 0 \iff w_{i\pi(j)} > 0. \tag{62}$$

Finally, since each f_i induces a functional from $\mathbb{R}^n$ to $\mathbb{R}$, in this case, we can assume without loss of generality that $B = \mathbb{R}^n$. By [5, Theorem 14 and Example 17] and (62), we have that the Clarke's differential of T_i^f at 0, $\partial T_i^f(0)$, coincides with the convex hull of D_i. This yields that

$$p_j > 0 \quad \forall p \in \partial T_i^f(0) = \operatorname{co}(D_i) \iff w_{ij} > 0. \tag{63}$$

Next, fix $i, j \in N$, define $\varepsilon_{ij} = \min_{\pi\in\Pi} p_j^{\pi,i}/2$, and assume $w_{ij} > 0$. By (62) and since $w_{ij} > 0$, we have that $\varepsilon_{ij} \in (0,1)$. Consider $x \in B$ and $h \in \mathbb{R}_+$ such that $x + he^j \in B$. Define $y = x + he^j$. By Lebourg's Mean Value Theorem and (63) and since T_i^f is a Choquet integral, there exist $\gamma \in (0,1)$ and $p \in \partial T_i^f(\gamma y + (1-\gamma)x) \subseteq \partial T_i^f(0)$ such that

$$T_i^f\left(x + he^j\right) - T_i^f(x) = T_i^f(y) - T_i^f(x) = \sum_{l=1}^{n} p_l (y_l - x_l) = p_j h \geq \varepsilon_{ij} h,$$

proving that j strongly influences i. Given that $i, j \in N$, were arbitrarily chosen, this proves that $A(W) \leq \underline{A}\left(T^f\right) \leq \bar{A}\left(T^f\right)$.

Next fix $i, j \in N$, assume that $w_{ij} = 0$. and fix arbitrarily $x \in B$ and $h \in \mathbb{R}_+$ such that $x + he^j \in B$. Define $y = x + he^j$, we have

$$\begin{aligned}
T_i^f(y) &= \sum_{k=1}^{n} y_k \left[f_i\left(\sum_{l:y_l \leq y_k} w_{il}\right) - f_i\left(\sum_{l:y_l < y_k} w_{il}\right)\right] = \sum_{k\in N\setminus\{j\}} y_k \left[f_i\left(\sum_{l:y_l \leq y_k, l\neq j} w_{il}\right) - f_i\left(\sum_{l:y_l < y_k, l\neq j} w_{il}\right)\right] \\
&= \sum_{k\in N\setminus\{j\}} x_k \left[f_i\left(\sum_{l:x_l \leq x_k, l\neq j} w_{il}\right) - f_i\left(\sum_{l:x_l < x_k, l\neq j} w_{il}\right)\right] \\
&= \sum_{k=1}^{n} x_k \left[f_i\left(\sum_{l:x_l \leq x_k} w_{il}\right) - f_i\left(\sum_{l:x_l < x_k} w_{il}\right)\right] = T_i^f(x)
\end{aligned}$$

this proves that $A(W) \geq \bar{A}\left(T^f\right)$ and the statement. ■

Given two nonempty disjoint subsets $\overline{N}, \underline{N} \subseteq N$, define

$$B\left(\overline{N}, \underline{N}\right) = \left\{x \in B : \min_{j\in\overline{N}} x_j > \max_{j\in\underline{N}} x_j\right\}.$$

Proof of Proposition 3. (i) implies (ii). Consider $x \in B$ and $a, b \in I$ such that $b > a$, $x_i = b$ for all $i \in \overline{N}$, $x_i \in [a,b]$ for all $i \in \left(\overline{N} \cup \underline{N}\right)^c$, $x_i = a$ for all $i \in \underline{N}$. By induction and since T^f is a rank-dependent aggregator and (14) holds, then $\left(T^f\right)_i^t(x) = b$ for all $i \in \overline{N}$, $\left(T^f\right)_i^t(x) \in [a,b]$ for all $i \in \left(\overline{N} \cup \underline{N}\right)^c$, and $\left(T^f\right)_i^t(x) = a$ for all $i \in \underline{N}$. Therefore, we can conclude that $\bar{T}^f(x) = \lim_t \left(T^f\right)^t(x)$ and $\bar{x} = b$ for all $i \in \overline{N}$ and $\bar{x}_i = a$ for all $i \in \underline{N}$, yielding that $\bar{x} \in \bar{T}^f(x) \in B\setminus D$.

Now, suppose T is convergent.

(ii) implies (i) It immediately follows from Lemma 7 that there exists $x \in B \backslash D$, such that $T^f(x) = x$. Define $\overline{N} = \{i \in N : x_i = \max_{j\in N} x_j\}$ and $\underline{N} = \{i \in N : x_i = \min_{j\in N} x_j\}$. Clearly, we have that $\overline{N}$ and $\underline{N}$ are two nonempty disjoint subsets of N. Since $T^f(x) = x$, by (6) we have that

$$T_i^f(x) = x_i = \max_{j\in N} x_j \quad \forall i \in \overline{N} \implies f_i\left(1 - \sum_{j\in\overline{N}} w_{ij}\right) = 0 \quad \forall i \in \overline{N}$$

and

$$T_i^f(x) = x_i = \min_{j\in N} x_j \quad \forall i \in \underline{N} \implies f_i\left(\sum_{j\in\underline{N}} w_{ij}\right) = 1 \quad \forall i \in \underline{N},$$

proving the implication.

For the last part of the statement, consider $x \in B\left(\overline{N}, \underline{N}\right)$. We will prove by induction that for all $t \in \mathbb{N}$

$$\min_{j\in\overline{N}} \left(T^f\right)_j^t(x) \geq \min_{j\in\overline{N}} x_j > \max_{j\in\underline{N}} x_j \geq \max_{j\in\underline{N}} \left(T^f\right)_j^t(x).$$

Suppose the statement holds for every $\tau \leq t$. Let $\bar{N}(t+1) = \left\{j \in N : \left(T^f\right)_j^t \geq \min_{l\in\overline{N}} \left(T^f\right)_l^t\right\}$. For every $i \in \overline{N}$, we have

$$\begin{aligned}
\left(T^f\right)_i^{t+1}(x) &= \sum_{j=1}^n \left(T^f\right)_j^t(x)\left[f_i\left(\sum_{l=1:(T^f)_l^t(x)\leq(T^f)_j^t(x)} w_{il}\right) - f_i\left(\sum_{l=1:(T^f)_l^t(x)<(T^f)_j^t(x)} w_{il}\right)\right] \\
&= \sum_{j\notin\bar{N}(t+1)}^n \left(T^f\right)_j^t(x)\left[f_i\left(\sum_{l=1:(T^f)_l^t(x)\leq(T^f)_j^t(x)} w_{il}\right) - f_i\left(\sum_{l=1:(T^f)_l^t(x)<(T^f)_j^t(x)} w_{il}\right)\right] \\
&\sum_{j\in\bar{N}(t+1)}^n \left(T^f\right)_j^t(x)\left[f_i\left(\sum_{l=1:(T^f)_l^t(x)\leq(T^f)_j^t(x)} w_{il}\right) - f_i\left(\sum_{l=1:(T^f)_l^t(x)<(T^f)_j^t(x)} w_{il}\right)\right] \\
&\geq \sum_{j\notin\bar{N}(t+1)}^n \left(T^f\right)_j^t(x)\, 0 \\
&+ \sum_{j\in\bar{N}(t+1)}^n \max_{j\in\underline{N}} \left(T^f\right)_j^t(x)\left[f_i\left(\sum_{l=1:(T^f)_l^t(x)\leq(T^f)_j^t(x)} w_{il}\right) - f_i\left(\sum_{l=1:(T^f)_l^t(x)<(T^f)_j^t(x)} w_{il}\right)\right] \\
&= \max_{j\in\underline{N}} \left(T^f\right)_j^t(x)\left[1 - f_i\left(\sum_{l=1:(T^f)_l^t(x)<\min_{l\in\overline{N}}(T^f)_l^t(x)} w_{il}\right)\right] \\
&\geq \min_{l\in\overline{N}} \left(T^f\right)_l^t(x)\left[1 - f_i\left(\sum_{l\notin\overline{N}} w_{il}\right)\right] = \min_{l\in\overline{N}} \left(T^f\right)_l^t(x) \geq \min_{j\in\overline{N}} x_j.
\end{aligned}$$

Let $\underline{N}(t+1) = \left\{j \in N : \left(T^f\right)_j^t \leq \max_{j\in\underline{N}} \left(T^f\right)_j^t(x)\right\}$. For every $i \in \underline{N}$, we have

$$
\begin{aligned}
\left(T^f\right)_i^{t+1}(x) &= \sum_{j=1}^{n} \left(T^f\right)_j^{t}(x) \left[f_i \left(\sum_{l=1:(T^f)_l^t(x) \leq (T^f)_j^t v} w_{il} \right) - f_i \left(\sum_{l=1:(T^f)_l^t(x) < (T^f)_j^t(x)} w_{il} \right) \right] \\
&= \sum_{j \in \underline{N}(t+1)}^{n} \left(T^f\right)_j^{t}(x) \left[f_i \left(\sum_{l=1:(T^f)_l^t(x) \leq (T^f)_j^t(x)} w_{il} \right) - f_i \left(\sum_{l=1:(T^f)_l^t(x) < (T^f)_j^t(x)} w_{il} \right) \right] \\
&+ \sum_{j \notin \underline{N}(t+1)}^{n} \left(T^f\right)_j^{t}(x) \left[f_i \left(\sum_{l=1:(T^f)_l^t(x) \leq (T^f)_j^t(x)} w_{il} \right) - f_i \left(\sum_{l=1:(T^f)_l^t(x) < (T^f)_j^t(x)} w_{il} \right) \right] \\
&\leq \sum_{j \in \underline{N}(t+1)}^{n} \max_{j \in \underline{N}} \left(T^f\right)_j^{t}(x) \left[f_i \left(\sum_{l=1:(T^f)_l^t(x) \leq (T^f)_j^t(x)} w_{il} \right) - f_i \left(\sum_{l=1:(T^f)_l^t(x) < (T^f)_j^t(x)} w_{il} \right) \right] \\
&= \max_{j \in \underline{N}} \left(T^f\right)_j^{t}(x) \left[1 - f_i \left(\sum_{l=1:(T^f)_l^t(x) \geq \max_{j \in \underline{N}} (T^f)_j^t(x)} w_{il} \right) \right] = \max_{j \in \underline{N}} \left(T^f\right)_j^{t}(x) \leq \max_{j \in \underline{N}} x_j .
\end{aligned}
$$

■

Proof of Corollary 3. (i) implies (ii) Fix $x \in B\left(N \backslash \underline{N}, \underline{N}\right)$ such that $T^f(x) = x$. By (6) we have that, for all $i \in N \backslash \underline{N}$,

$$T_i^f(x) = x_i = \max_{j \in N} x_j \implies f_i\left(w_i\left(\underline{N}\right)\right) = 0$$

and, for all $l \in \underline{N}$,

$$T_l^f(x) = x_l = \min_{j \in N} x_j \implies f_l\left(w_l\left(\underline{N}\right)\right) = 1,$$

proving the implication.

(ii) implies (iii) It follows from the last part of Proposition 3.

(iii) implies (i) It immediately follows from Lemma 7. ■

Proof of Corollary 4. For each $n \in \mathbb{N}$ and for each $i \in N$ set

$$w_i(n) = \frac{1}{n} \text{ and } c(n) = n \max_{k \in N} s_k\left(T(n)\right).$$

It is immediate to see that $c(n) \in \mathbb{R}$ and $w(n) \in \Delta_n$ for all $n \in \mathbb{N}$. Moreover, we have that for each $n \in \mathbb{N}$

$$s_i\left(T(n)\right) \leq \max_{k \in N} s_k\left(T(n)\right) = c(n)\, w_i(n) \qquad \forall i \in N. \tag{64}$$

Since $\max_{k \in N} s_k\left(T(n)\right) = o\left(\frac{1}{\sqrt{n}}\right)$, we have that

$$c(n)^2 \max_{k \in N} w_k(n) = n\left(\max_{k \in N} s_k\left(T(n)\right)\right)^2 \to 0 \text{ as } n \to \infty. \tag{65}$$

By (64) and (65), it follows that (16) holds. By Proposition 4 and since $\{T(n)\}_{n \in \mathbb{N}}$ is a sequence of odd robust opinion aggregators, this implies the statement. ■

Lemma 9 *Let $\{T(n)\}_{n \in \mathbb{N}}$ be a sequence of odd robust opinion aggregators. If there exist two sequences $\{c(n)\}_{n \in \mathbb{N}}$ and $\{w(n)\}_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$: $c(n) \in \mathbb{R}$, $w(n) \in \Delta_n$, and for each $x \in \mathcal{D}\left(T(n)\right)$*

and $i, j \in N$

$$\frac{\partial T_j(n)}{\partial x_i}(x) \leq c(n) w_i(n) \text{ and } c(n)^2 \max_{k \in N} w_k(n) \to 0 \text{ as } n \to \infty, \tag{66}$$

then (16) holds and $\{T(n)\}_{n \in \mathbb{N}}$ *is wise.*

Proof of Lemma 9. Fix $n \in \mathbb{N}$. Since $T(n)$ is a robust opinion aggregator, we have that $T(n)$ is Lipschitz continuous. By Rademacher's Theorem, this implies that $T(n)$ is almost everywhere Frechet differentiable and, in particular, Clarke differentiable. Clearly, $T_j(n)$ is Frechet differentiable on $\mathcal{D}(T(n))$ for all $j \in N$. Since $T_j(n)$ is monotone and translation invariant, note that $\nabla T_j(n)(x) \in \Delta_n$ for all $x \in \mathcal{D}(T(n))$. Consider $\bar{x} \in \hat{B}$. Recall that Clarke's differential is the set (see, e.g., [2, Theorem 2.5.1]):

$$\partial T_j(n)(\bar{x}) = \text{co}\left\{p \in \Delta_n : p = \lim_t \nabla T_j(n)\left(x^t\right) \text{ s.t. } x^t \to \bar{x} \text{ and } x^t \in \mathcal{D}(T(n))\right\}. \tag{67}$$

Similarly, we have that

$$\partial \bar{T}_1(n)(\bar{x}) = \text{co}\left\{p \in \Delta_n : p = \lim_t \nabla \bar{T}_1(n)\left(x^t\right) \text{ s.t. } x^t \to \bar{x} \text{ and } x^t \in \mathcal{D}\left(\bar{T}(n)\right)\right\}.$$

By Theorem 1, recall that $\bar{T}(n) \circ T(n) = \bar{T}(n)$, yielding that $\bar{T}_1(n) \circ T(n) = \bar{T}_1(n)$. Fix $\bar{x} \in \hat{B}$. Define by $\Pi_{j=1}^n \partial T_j(n)(\bar{x})$ the collection of all $n \times n$ square matrices whose j-th row is an element of $\partial T_j(n)(\bar{x})$. From the previous part of the proof, we have that $\Pi_{j=1}^n \partial T_j(n)(\bar{x}) \subseteq \mathcal{W}$. Define

$$\begin{aligned}&\partial \bar{T}_1(n)(T(n)(\bar{x}))\, \Pi_{j=1}^n \partial T_j(n)(\bar{x}) \\ &= \left\{\tilde{w} \in \Delta_n : \exists p \in \partial \bar{T}_1(n)(T(n)(\bar{x})), \exists W \in \Pi_{j=1}^n \partial T_j(n)(\bar{x}) \text{ s.t. } p^{\mathrm{T}} W = \tilde{w}^{\mathrm{T}}\right\}.\end{aligned}$$

In words, $\tilde{w} \in \partial \bar{T}_1(n)(T(n)(\bar{x}))\, \Pi_{j=1}^n \partial T_j(n)(\bar{x})$ only if it is a stochastic vector which is a convex linear combination of the rows of some matrix $W \in \Pi_{j=1}^n \partial T_j(n)(\bar{x})$. By the Chain Rule (see, e.g., [2, Theorem 2.6.6 and point e of Proposition 2.6.2]) and since $\bar{T}_1(n) = \bar{T}_1(n) \circ T(n)$, we have that

$$\partial \bar{T}_1(n)(\bar{x}) \subseteq \text{co}\left\{\partial \bar{T}_1(n)(T(n)(\bar{x}))\, \Pi_{j=1}^n \partial T_j(n)(\bar{x})\right\}. \tag{68}$$

By assumption, we have that for each $j \in N$

$$\sup_{x \in \mathcal{D}(T(n))} \frac{\partial T_j(n)}{\partial x_i}(x) \leq c(n) w_i(n) \text{ and } c(n)^2 \max_{k \in N} w_k(n) \to 0 \text{ as } n \to \infty. \tag{69}$$

By (67) and (69), we have that

$$0 \leq p_i \leq c(n) w_i(n) \quad \forall p \in \partial T_j(n)(\bar{x}), \forall i, j \in N.$$

By (68), we can conclude that

$$0 \leq p_i \leq c(n) w_i(n) \quad \forall p \in \partial \bar{T}_1(n)(\bar{x}), \forall i \in N.$$

Since $\bar{x}$ was arbitrarily chosen, this implies that

$$0 \leq p_i \leq c(n) w_i(n) \quad \forall p \in \partial \bar{T}_1(n)(x), \forall x \in \hat{B}, \forall i \in N.$$

Finally, observe that if $x \in \mathcal{D}\left(\bar{T}(n)\right)$, we have that $\nabla \bar{T}_1(n)(x) \in \partial \bar{T}_1(n)(x)$ and, in particular, $\frac{\partial \bar{T}_1(n)}{\partial x_i}(x) \leq c(n) w_i(n)$ for all $i \in N$. This yields that

$$s_i(T(n)) = \sup_{x \in \mathcal{D}(\bar{T}(n))} \frac{\partial \bar{T}_1(n)}{\partial x_i}(x) \leq c(n) w_i(n),$$

proving the statement. ■

Proof of Proposition 5. Similarly to the proof of Corollary 4, for each $n \in \mathbb{N}$ and for each $i \in N$ set

$$w_i(n) = \frac{1}{n} \text{ and } c(n) = n \max_{i,j \in N} \sup_{x \in \mathcal{D}(T(n))} \frac{\partial T_j(n)}{\partial x_i}(x).$$

The result then follows from Lemma 9. ■

Next, we state and prove two ancillary lemmas that will be instrumental in the proof Proposition 11.

Lemma 10 *Let ϕ be a profile of loss functions. If $\phi \in \Phi_R$, then for each $i \in N$ and $\tilde{z} \in \mathbb{R}^n$*

$$\tilde{z} \gg 0 \implies \phi_i(\tilde{z}) > \phi_i\left(\tilde{z} - \min_{j \in N} \tilde{z}_j e\right),$$

and

$$0 \gg \tilde{z} \implies \phi_i(\tilde{z}) > \phi_i\left(\tilde{z} - \max_{j \in N} \tilde{z}_j e\right).$$

Proof. Fix $i \in N$. Consider $\tilde{z} \in \mathbb{R}^n$ such that $\tilde{z} \gg 0$. Define $z = \tilde{z} - \min_{j \in N} \tilde{z}_j e$, $v = 0$, and $h = \min_{j \in N} \tilde{z}_j$. Note that $z \geq v$ as well as $h \in \mathbb{R}_{++}$. Since ϕ has increasing shifts and is sensitive, we obtain that

$$\begin{aligned}\phi_i(\tilde{z}) - \phi_i\left(\tilde{z} - \min_{j \in N} \tilde{z}_j e\right) &= \phi_i(z + he) - \phi_i(z) \\ &\geq \phi_i(v + he) - \phi_i(v) = \phi_i\left(\min_{j \in N} \tilde{z}_j e\right) - \phi_i(0) > 0,\end{aligned}$$

proving the first inequality. A symmetric argument yields the second inequality. ■

Lemma 11 *Let ϕ be a profile of loss functions. If $\phi \in \Phi_R$, then for each $i \in N$ and for each $x \in \mathbb{R}^n$ the function $f_{i,x} : \mathbb{R} \to \mathbb{R}_+$, defined by $f_{i,x}(c) = \phi_i(x - ce)$ for all $c \in \mathbb{R}$, is continuous and convex. Moreover, if ϕ has strictly increasing shifts, then $f_{i,x}$ is strictly convex for all $i \in N$ and for all $x \in \mathbb{R}^n$.*

Proof. Fix $i \in N$ and $x \in \mathbb{R}^n$. Define $g_{i,x} : \mathbb{R} \to \mathbb{R}_+$ by $g_{i,x}(c) = \phi_i(x + ce)$ for all $c \in \mathbb{R}$. Consider $c_1, c_2 \in \mathbb{R}$ such that $c_1 > c_2$ and $h > 0$. Since $\phi \in \Phi_R$ and $x + c_1 e \geq x + c_2 e$, it follows that

$$\begin{aligned}g_{i,x}(c_1 + h) - g_{i,x}(c_1) &= \phi_i((x + c_1 e) + he) - \phi_i(x + c_1 e) \\ &\geq \phi_i((x + c_2 e) + he) - \phi_i(x + c_2 e) = g_{i,x}(c_2 + h) - g_{i,x}(c_2).\end{aligned}$$

By [12, Problem N, pp. 223–224], it follows that $g_{i,x}$ is midconvex. Next, fix $c \in \mathbb{R}$ and note that the step above and $\phi_i \geq 0$ together imply that for all $c' \in (c - 1, c + 1)$

$$g_{i,x}(c') - g_{i,x}(c - 1) \leq g_{i,x}(c + 1) - g_{i,x}(2c - c') \implies g^c_{i,x}(c') \leq g_{i,x}(c - 1) + g_{i,x}(c + 1),$$

proving that $g_{i,x}$ is bounded on $(c-1, c+1)$. By [12, Theorem C, p. 215], it follows that $g_{i,x}$ is continuous and convex. Finally, observe that $f_{i,x} = g_{i,x} \circ h$ where $h(c) = -c$ for all $c \in \mathbb{R}$, yielding that $f_{i,x}$ is convex and continuous being the composition of a convex and continuous function with an affine and continuous function. Next, assume that ϕ has strictly increasing shifts and, in particular, has increasing shifts. By the previous part of the proof, $g_{i,x}$ is convex. By contradiction, assume that $g_{i,x}$ is not strictly convex. This implies that there exists an interval $[d_2, d_1]$, with $d_2 < d_1$, where $g_{i,x}$ is affine. Define $c_1 = \frac{1}{2}d_1 + \frac{1}{2}d_2$, $c_2 = d_2$, and $h = (d_1 - d_2)/2$. Note that $c_1 > c_2$ and $h > 0$. Since ϕ has strictly increasing shifts, by the same computations of the previous part of the proof, we have that

$$g_{i,x}(d_1) - g_{i,x}\left(\frac{1}{2}d_1 + \frac{1}{2}d_2\right) = g_{i,x}(c_1 + h) - g_{i,x}(c_1)$$
$$> g_{i,x}(c_2 + h) - g_{i,x}(c_2) = g_{i,x}\left(\frac{1}{2}d_1 + \frac{1}{2}d_2\right) - g_{i,x}(d_2),$$

yielding that $g_{i,x}\left(\frac{1}{2}d_1 + \frac{1}{2}d_2\right) < \frac{1}{2}g_{i,x}(d_1) + \frac{1}{2}g_{i,x}(d_2)$, a contradiction with affinity. Since $g_{i,x}$ is strictly convex, so is $f_{i,x} = g_{i,x} \circ h$. ■

Proof of Proposition 11. Fix $i \in N$. We begin by considering the correspondence $\mathbf{T}_i^{\phi} : B \rightrightarrows I$ defined by

$$\mathbf{T}_i^{\phi}(x) = \operatorname{argmin}_{c \in \mathbb{R}} \phi_i(x - ce) \qquad \forall x \in B.$$

We next show that $\mathbf{T}_i^{\phi}$ is well defined, nonempty-, convex-, and compact-valued, and such that for each $x, y \in B$

$$x \geq y \implies \mathbf{T}_i^{\phi}(x) \geq_{\text{SSO}} \mathbf{T}_i^{\phi}(y) \tag{70}$$

where $\geq_{\text{SSO}}$ is the strong set order. Fix $x \in B$. We next show that

$$\forall d \notin \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right], \exists c \in \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right] \text{ s.t. } \phi_i(x - ce) < \phi_i(x - de). \tag{71}$$

Consider d as above. We have two cases either $d < \min_{j \in N} x_j$ or $d > \max_{j \in N} x_j$. In the first case, we have that $x - de \gg 0$, in the second case, we have that $0 \gg x - de$. By Lemma 10 and since $\phi \in \Phi_R$, if we set $\tilde{c} = \min_{j \in N} x_j - d$ (resp., $\max_{j \in N} x_j - d$), we obtain that

$$\phi_i(x - de) > \phi_i(x - de - \tilde{c}e) = \phi_i(x - ce)$$

where $c = \min_{j \in N} x_j \in [\min_{j \in N} x_j, \max_{j \in N} x_j]$ (resp., $c = \max_{j \in N} x_j \in [\min_{j \in N} x_j, \max_{j \in N} x_j]$), proving (71). By (71), we can conclude that

$$\min_{c \in \mathbb{R}} \phi_i(x - ce) = \min_{c \in I} \phi_i(x - ce) = \min_{c \in [\min_{j \in N} x_j, \max_{j \in N} x_j]} \phi_i(x - ce) \tag{72}$$

as well as

$$\operatorname{argmin}_{c \in \mathbb{R}} \phi_i(x - ce) = \operatorname{argmin}_{c \in I} \phi_i(x - ce) = \operatorname{argmin}_{c \in [\min_{j \in N} x_j, \max_{j \in N} x_j]} \phi_i(x - ce).$$

By Weierstrass' Theorem (see, e.g., [1, Theorem 2.43]) and since, by Lemma 11, the map $c \mapsto \phi_i(x - ce)$ is continuous and convex, it follows that the above minimization problems admit solution and each argmin is a compact and convex set. Since x was arbitrarily chosen, this implies that $\mathbf{T}_i^{\phi}$ is well defined, nonempty-, convex-, and compact-valued and, in particular,

$$\emptyset \neq \mathbf{T}_i^{\phi}(x) \subseteq \left[\min_{j\in N} x_j, \max_{j\in N} x_j\right] \subseteq I \quad \forall x \in B. \tag{73}$$

We next prove (70). In order to do so, we rewrite explicitly (72) as a problem of parametric optimization/monotone comparative statics. Next, define $f : I \times B \to \mathbb{R}$ by

$$f(c,x) = -\phi_i(x - ce) \qquad \forall (c,x) \in I \times B.$$

It is immediate to see that

$$\mathbf{T}_i^{\phi}(x) = \operatorname{argmax}_{c\in I} f(c,x) \qquad \forall x \in B.$$

We next show that f has increasing differences in (c,x). Consider $x, y \in B$ as well as $c, d \in I$ such that $c \geq d$ and $x \geq y$. Define $z = x - ce$, $v = y - ce$, and $h = c - d$. Note that $z \geq v$ and $h \in \mathbb{R}_+$. Since $\phi \in \Phi_R$, it follows that

$$\begin{aligned} f(c,x) - f(d,x) &= \phi_i(x - de) - \phi_i(x - ce) = \phi_i(z + he) - \phi_i(z) \\ &\geq \phi_i(v + he) - \phi_i(v) = \phi_i(y - de) - \phi_i(y - ce) = f(c,y) - f(d,y). \end{aligned}$$

This shows that f satisfies the property of increasing differences in (c,x). By [10, Theorem 5], $\mathbf{T}_i^{\phi}$ satisfies (70). We finally show that $\mathbf{T}_i^{\phi}$ is such that for each $x \in B$ and for each $k \in \mathbb{R}$ such that $x + ke \in B$

$$c^\star \in \mathbf{T}_i^{\phi}(x) \iff c^\star + k \in \mathbf{T}_i^{\phi}(x + ke). \tag{74}$$

Fix $x \in B$. Consider $k \in \mathbb{R}$ such that $x + ke \in B$. Consider $c^\star \in \mathbf{T}_i^{\phi}(x)$. By definition, it follows that $\phi_i(x - c^\star e) \leq \phi_i(x - ce)$ for all $c \in \mathbb{R}$. This implies that

$$\phi_i(x + ke - (c^\star + k)e) = \phi_i(x - c^\star e) \leq \phi_i(x - (d - k)e) = \phi_i(x + ke - de) \quad \forall d \in \mathbb{R}.$$

By definition of $\mathbf{T}_i^{\phi}$, this implies that $c^\star + k \in \mathbf{T}_i^{\phi}(x + ke)$. Vice versa, if $c^\star + k \in \mathbf{T}_i^{\phi}(x + ke)$, then

$$\phi_i(x + ke - (c^\star + k)e) \leq \phi_i(x + ke - de) \qquad \forall d \in \mathbb{R},$$

yielding that

$$\phi_i(x - c^\star e) = \phi_i(x + ke - (c^\star + k)e) \leq \phi_i(x - ce) \qquad \forall c \in \mathbb{R},$$

proving that $c^\star \in \mathbf{T}_i^{\phi}(x)$.

To sum up, since $i \in N$ was arbitrarily chosen, we proved that, for each $i \in N$, $\mathbf{T}_i^{\phi}$ is well defined, nonempty-, convex-, and compact-valued, and satisfies (70) as well as (74). Observe also that $\mathbf{T}^{\phi} : B \rightrightarrows B$ is the product correspondence $\mathbf{T}^{\phi} = \prod_{i=1}^{n} \mathbf{T}_i^{\phi}$. We are ready to show that $\mathbf{T}^{\phi}$ admits a selection T^{ϕ} which is a robust opinion aggregator. Define $T^{\phi} : B \to B$ to be such that

$$T_i^{\phi}(x) = \min \mathbf{T}_i^{\phi}(x) \qquad \forall x \in B, \forall i \in N.$$

Since $\mathbf{T}_i^{\phi}(x)$ is compact for all $x \in B$ and for all $i \in N$, it follows that $T_i^{\phi}(x)$ is well defined and, in particular, $T_i^{\phi}(x) \in \mathbf{T}_i^{\phi}(x)$ for all $x \in B$ and for all $i \in N$, proving that T^{ϕ} is a selection of $\mathbf{T}^{\phi}$. By (73), it follows that $\mathbf{T}_i^{\phi}(ke) = \{k\}$ for all $k \in I$ and for all $i \in N$, proving that $T_i^{\phi}(ke) = k$ for all $k \in I$ and for all $i \in N$, that is, that T^{ϕ} is normalized. Next, consider $x, y \in B$ such that $x \geq y$. By (70), we have that $T_i^{\phi}(x) \geq T_i^{\phi}(y)$ for all $i \in N$, proving monotonicity of T_i^{ϕ} for all $i \in N$ and so of T^{ϕ}. Finally, consider $x \in B$ and $k \in \mathbb{R}$ such that $x+ke \in B$. By (74) and definition of $T_i^\phi(x)$ as well as $T_i^\phi(x+ke)$, we have that $T_i^\phi(x) \in \mathbf{T}_i^\phi(x)$ for all $i \in N$, yielding that $T_i^\phi(x)+k \in \mathbf{T}_i^\phi(x+ke)$ for all $i \in N$ and, in particular, $T_i^\phi(x)+k \geq T_i^\phi(x+ke)$ for all $i \in N$. This implies that $T_i^\phi(x+ke) = T_i^\phi(x)+k$ for all $i \in N$, proving translation invariance.[38]

Finally, by Lemma 11, if ϕ has strictly increasing shifts, then the map $c \mapsto \phi_i(x-ce)$ is strictly convex, yielding that each $\mathbf{T}_i^\phi$ is single-valued and so is $\mathbf{T}^\phi$. ■

Proof of Proposition 6. Before starting, we make few observations about strong convexity (see, e.g., [12, p. 268]). Since each ρ_i is strongly convex and twice continuously differentiable, we have that for each $i \in N$ there exists $\alpha_i > 0$ such that $\rho_i''(s) \geq \alpha_i$ for all $s \in \mathbb{R}$. Moreover, we have that for each $i \in N$

$$\left(\rho_i'(s_1) - \rho_i'(s_2)\right)(s_1 - s_2) \geq \alpha_i (s_1 - s_2)^2 \quad \forall s_1, s_2 \in \mathbb{R}. \tag{75}$$

Finally, since each ρ_i is twice continuously differentiable and I is compact, for each $i \in N$ we have that there exists $L_i > 0$ such that

$$\left|\rho_i'(s_1) - \rho_i'(s_2)\right| \leq L_i |s_1 - s_2| \quad \forall s_1, s_2 \in [\min I - \max I, \max I - \min I]. \tag{76}$$

Recall that $\phi_i : \mathbb{R}^n \to \mathbb{R}_+$ is defined by $\phi_i(z) = \sum_{j=1}^n w_{ij}\rho_i(z_j)$ for all $z \in \mathbb{R}^n$ and for all $i \in N$. By assumption, $\phi \in \Phi_A \subseteq \Phi_R$. Since $\rho_i'' \geq \alpha_i > 0$, this implies that ρ_i is strictly convex for all $i \in N$. Standard computations yield that ϕ has strictly increasing shifts. By Proposition 11, we have that $\mathbf{T}^\phi = T^\phi$ is single-valued and a robust opinion aggregator from B to B. Moreover, $T_i^\phi(x)$ is the unique solution of

$$\min_{c \in \mathbb{R}} \phi_i(x-ce) = \min_{c \in I} \phi_i(x-ce) \qquad \forall i \in N, \forall x \in B. \tag{77}$$

Fix $i \in N$. Since ρ_i is differentiable and convex, so is the map $c \mapsto \phi_i(x-ce)$ for all $x \in B$. The solution of (77) is then given by the first order condition

$$\sum_{j=1}^n w_{ij}\rho_i'\left(x_j - T_i^\phi(x)\right) = 0 \qquad \forall x \in B.$$

Consider $x \in B$, $h > 0$, and $l \in N$ such that $x + he^l \in B$. We have that

$$\sum_{j=1}^n w_{ij}\rho_i'\left(x_j - T_i^\phi(x)\right) = 0 \text{ and } \sum_{j=1}^n w_{ij}\rho_i'\left(x_j + he_j^l - T_i^\phi\left(x+he^l\right)\right) = 0. \tag{78}$$

Note that if $w_{il} = 0$, then $\sum_{j=1}^n w_{ij}\rho_i'\left(x_j + he_j^l - c\right) = \sum_{j=1}^n w_{ij}\rho_i'(x_j - c)$ for all $c \in \mathbb{R}$, proving that $T_i^\phi\left(x+he^l\right) = T_i^\phi(x)$ and that l does not strongly influence i. Next, assume that $w_{il} > 0$. By (76),

[38] Fix $i \in N$. By the previous part of the proof, for each $x \in B$ and for each $k \in \mathbb{R}$ such that $x+ke \in B$, we have that $T_i^\phi(x+ke) \leq T_i^\phi(x)+k$. Next, note that if $x \in B$ and $x+ke \in B$, then $(x+ke)-ke = x \in B$. It follows that

$$T_i^\phi(x) = T_i^\phi((x+ke)-ke) \leq T_i^\phi(x+ke) - k,$$

proving the opposite inequality.

(78), and (75) and since T^{ϕ} is monotone and $h > 0$, we can conclude that

$$
\begin{aligned}
L_i\left(T_i^{\phi}\left(x+he^l\right)-T_i^{\phi}(x)\right) &\geq \sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j+he_j^l-T_i^{\phi}(x)\right)-\sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j+he_j^l-T_i^{\phi}\left(x+he^l\right)\right) \\
&= \sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j+he_j^l-T_i^{\phi}(x)\right)-\sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j-T_i^{\phi}(x)\right) \\
&= w_{il}\left[\rho_i'\left(x_l+h-T_i^{\phi}(x)\right)-\rho_i'\left(x_l-T_i^{\phi}(x)\right)\right] \geq w_{il}\alpha_i h,
\end{aligned}
$$

proving that $T_i^{\phi}\left(x+he^l\right)-T_i^{\phi}(x) \geq \varepsilon_{il}h$ where $\varepsilon_{il} = L_i^{-1}w_{il}\alpha_i/2 \in (0,1)$. Since x and h were arbitrarily chosen, we have that l strongly influences i. With this, we have $\bar{A}\left(T^{\phi}\right) \geq \underline{A}\left(T^{\phi}\right) = A(W)$. Next, fix $i,j \in N$ such that $a_{ij}(W) = 0$, that is, $w_{ij} = 0$. This clearly implies that, for all $x \in B$ and $h \in \mathbb{R}_+$ such that $x + he^j$, and for all $c \in I$, $\phi_i\left(\left(x+he^j\right)-ce\right) = \phi_i(x-ce)$, implying that $T_i^{\phi}\left(x+he^j\right) = T_i^{\phi}(x)$, hence that $\bar{a}_{ij}\left(T^{\phi}\right) = 0$. Given that i,j were arbitrarily chosen, it follows that $A(W) \geq \bar{A}\left(T^{\phi}\right) \geq \underline{A}\left(T^{\phi}\right) = A(W)$, proving the statement. ■

D.2 Additional Examples

Example 6 Let $B = \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be such that

$$
T_1(x) = \begin{cases} x_2 & \text{if } x_1 > x_2 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 & \text{if } x_2 \geq x_1 \end{cases} \quad \text{and } T_2(x) = x_1 \quad \forall x \in B.
$$

Agent 2 always discards her own opinion. Agent 1 discards her own opinion if and only if it is the highest of the two. Easy computations yield that T is a robust opinion aggregator and

$$
\underline{A}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Thus, T is strongly connected, but it fails to be strongly aperiodic. Moreover, it is easy to show that $\lim_t T^t(x) = \min\{x_1, x_2\} e$ for all $x \in B$, showing that, under strong connectedness, strong aperiodicity is sufficient for convergence to consensus, but not necessary. ▲

D.3 Comparison with Molavi, Tahbaz-Salehi, and Jadbabaie

In this section we show that for the issues analyzed in this paper, namely convergence and the wisdom of the crowd, when each agent observes a unique *initial* signal, the log-linear learning rule axiomatized in Molavi, Tahbaz-Salehi, and Jadbabaie [10] can be equivalently analyzed by means of a linear system. However, the equivalence with a linear system may be lost for a problem of learning with repeated signals like the one in [10].

Formally, Molavi, Tahbaz-Salehi, and Jadbabaie consider the case of agents having a full support belief $\mu \in \Delta(\Theta)$ where Θ is a finite set of possible states of the world. In particular, there is a bijection between beliefs and the profile of likelihood ratios $\left(x\left(\theta,\hat{\theta}\right)\right)_{(\theta,\hat{\theta}) \in \Theta\times\Theta}$ with

$$
x\left(\theta,\hat{\theta}\right) = \frac{\mu(\theta)}{\mu\left(\hat{\theta}\right)} \qquad \forall\left(\theta,\hat{\theta}\right) \in \Theta \times \Theta.
$$

Their assumption of IIA allows to study the evolution of $x\left(\theta,\hat{\theta}\right) \in \mathbb{R}_{++}$ independently from the value of the other likelihood ratios. Therefore, we fix a particular $\left(\theta,\hat{\theta}\right)$ and denote the likelihood ratio obtained from the belief of agent i at time t as

$$x_i^t = \frac{\mu_i^t(\theta)}{\mu_i^t\left(\hat{\theta}\right)} \qquad \forall i \in N, \forall t \in \mathbb{N}.$$

When the agents do not observe any additional signal at period t equation (3) in [10] reads as

$$\ln x_i^t = \sum_{j \in N_i} a_{ij,t} \ln x_j^{t-1}$$

where $a_{ij,t} > 0$ for all $i, j \in N$ and for all $t \in \mathbb{N}$. We can explicitly write the law of motion of $\left\{x^t\right\}_{t\in\mathbb{N}}$ as

$$x_i^t = \exp\left(\sum_{j\in N_i} a_{ij,t} \ln x_j^{t-1}\right) \qquad \forall i \in N, \forall t \in \mathbb{N}. \tag{79}$$

For each $t \in \mathbb{N}$ define the operator $S_t : \mathbb{R}_{++}^n \to \mathbb{R}_{++}^n$ to be such that the i-th component is $x \mapsto \exp\left(\sum_{j\in N_i} a_{ij,t} \ln x_j\right)$. Thus, we can rewrite (79) as $x^t = S_t\left(x^{t-1}\right)$ for all $t \in \mathbb{N}$. Each operator S_t is *not* a robust opinion aggregator, since it does not satisfy translation invariance.

At the same time, we can analyze an equivalent system whose law of motion is described by time-varying linear aggregation. Toward this end, fix x^0 and let $\left\{x^t\right\}_{t\in\mathbb{N}}$ be recursively defined as in equation (79). Define A_t to be the matrix whose ij-th entry is $a_{ij,t}$. Next, define $\left\{y^t\right\}_{t\in\mathbb{N}_0}$ by

$$y_i^t = \ln x_i^t \qquad \forall i \in N, \forall t \in \mathbb{N}_0 \tag{80}$$

and note that

$$y^t = A_t y^{t-1} \qquad \forall t \in \mathbb{N}.$$

Note that, whenever each A_t is equal to the same stochastic matrix $W \in \mathcal{W}$, the iterates $\left\{y^t\right\}_{t\in\mathbb{N}}$ are described by the standard DeGroot's model with matrix W. Therefore, in this case, it is possible to appeal to the results in [6] to study the limit behavior of $\left\{y^t\right\}_{t\in\mathbb{N}}$. In general, whenever each $A_t \in \mathcal{W}$, one can rely on the more general results for time-varying matrices (see, e.g., Seneta [13] and Krause [7]). Given the continuity of the transformation in (80), we have that

$$\lim_t x^t \text{ exists in } \mathbb{R}_{++}^n \iff \lim_t y^t \text{ exists in } \mathbb{R}^n.$$

The previous simple equivalence shows that $\lim_t y^t$ uniquely pins down $\lim_t x^t$.

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