

A POSSIBILITY THEOREM ON INFORMATION AGGREGATION IN ELECTIONS

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ABSTRACT. We provide a simple condition that is both necessary and sufficient for aggregation of private information in large elections where all voters have the same preference. In some states of the world, all voters prefer alternative A ; and in other states, all voters prefer alternative B , and A wins if the corresponding vote share is higher than a threshold $q \in (0, 1)$. Each voter draws a private signal independently from a distribution conditional on the state. According to our condition, there should be a hyperplane in the simplex over signals that separates the conditional distributions in states where A is preferred from those in states where B is preferred. If this condition is satisfied, information is aggregated in an equilibrium sequence: even under incomplete information, the preferred outcome obtains with a very high ex-ante likelihood. If the hyperplane condition is violated, there exists no feasible strategy profile that aggregates information. While the hyperplane condition is satisfied only in special environments, it holds generically if the state space is discrete and the number of available signals is more than or equal to the number of states.

1. INTRODUCTION

In large elections, the decision relevant information is often dispersed throughout the electorate. This poses the classic problem of information aggregation in voting: even if voters potentially agree on who the right candidate is, each individual's vote contains only his own *private* information. It is therefore not guaranteed whether the election outcome obtained by aggregating everyone's vote leads to the right outcome, i.e., the outcome that any voter would have preferred if he knew all the information dispersed within the electorate. In this paper, we are interested in the question of information aggregation in large elections: when does voting lead to the same outcome that would have prevailed if all the private information were publicly known? We provide a simple condition on the diversity of information in the electorate that is both necessary and sufficient for information aggregation in equilibrium.

In our model, there are two alternatives (A and B), and preferences are represented by states: in some states of the world, all voters prefer A while in others, all voters prefer B . Alternative A wins if it obtains more than a threshold $q \in (0, 1)$ share of votes. The dispersion of information is captured by the probability distribution over signals conditional on the state. Our central result (Theorem 1) is that there exists some strategy profile that aggregates

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information efficiently in large voting populations if and only if there is a hyperplane that separates the probability distributions arising from states where A is preferred from those arising from states where B is preferred. This result suggests that information aggregation happens only in special environments. If the hyperplane condition is not satisfied, then there is no *feasible* voting strategy that aggregates information. In such an environment, even if voters could commit to playing any strategy, information would not be aggregated. On the other hand, if the hyperplane condition is satisfied, then full information aggregation obtains as an equilibrium phenomenon (Theorem 2). The set of information aggregating strategies is identified by the hyperplane condition.

To see what our result implies in a very simple setting, suppose that voters' private information is categorized two signals a and b , and voter rankings depend only on the proportion of a -signals in the population. To capture this more formally, suppose the state θ is a number in $[0, 1]$, and in state θ , each voter obtains signal a with probability θ . The state then is the same as the expected proportion of a -signals in the electorate, which is also, almost surely, the actual proportion if the population is large. According to our result, information is aggregated for the following type of voter preference: alternative A is preferred for high states (say, larger than θ^*) and alternative B is preferred for low states (i.e., lower than θ^*). On the other hand, information aggregation fails if the voters prefer to elect A for moderate states (say, between θ_1 and θ_2) and B for extreme states (i.e., lower than θ_1 or higher than θ_2). Since a strategy maps individual signals into votes, any strategy that leads lower vote shares for A at low proportions of a -signals and higher vote shares for A at moderate proportions of a -signals must also produce high vote shares for A when the proportion of a -signals is high. Thus, it is impossible to find a strategy that guarantees a win for B both in the very high and very low states, but a win for A in the moderate states. The above example suggests that the possibility of aggregation depends on what information the signals convey: if a higher proportion of signal a (resp. b) indicates higher quality of candidate A (resp. B), then information aggregation holds. On the other hand, if a very high or very low proportion of a -signals conveys that candidate A has a very extreme position on a policy issue, then information aggregation fails.

There is a large literature going back to Condorcet (1786) that argues that information is aggregated in common value environments. In the canonical Condorcet Jury model, there are two states (A and B) and two signals (a and b). State A (resp. B) is simply interpreted as all situations where candidate A (resp. B) is the commonly preferred candidate. In each state, voters draw their signals independently from a distribution: $\Pr(a|A) = p_A > \frac{1}{2}$ and $\Pr(b|B) = p_B > \frac{1}{2}$. Thus, the signal a (resp. b) can be interpreted as an assessment that A (resp. B) is the right candidate: however, the assessment may be mistaken. In this setting, if all voters vote according to their private signal, the majority votes for the correct alternative almost surely by the Law of Large Numbers. In this sense, individual uncertainty does not matter for the aggregate outcome in a large electorate, and information is always aggregated. This result is popularly known as the Condorcet Jury Theorem (CJT).

The earlier statistical work on the theorem have always equated (implicitly or explicitly) the state of the world with a ranking over the two alternatives (see Ladha (1992) and Berg (1993) among others). This strand has also assumed “sincere voting”, i.e., that voters vote their signals. The game theoretic literature started with the insight in Austen-Smith and Banks (1996) that the sincere voting profile may not be a Nash equilibrium. Since then, there have been other proofs of CJT showing that information can be aggregated in Nash equilibrium for majority and supermajority rules and for more varied information structure (e.g. Wit (1998), Feddersen and Pesendorfer (1997), Myerson (1998), Duggan and Martinelli (2005)). However, all the papers dealing with common value environments (except Feddersen and Pesendorfer (1997)) have retained the two-state structure, effectively assuming that all situations where one alternative is better for all voters can be lumped into a single state.

Our starting point is that electorates often have a far richer informational diversity than is supposed by the canonical two-state model. Electability of a candidate depends on myriad factors like his policy positions on different issues, his past history, party affiliation, the state of the economy, the geopolitical situation and so forth. We use the state variable to capture all the different factors that affect the preference of the voters. It is also likely that different individuals hold information on different factors, and the overall preference of the electorate is based on the distribution of this dispersed information. Thus, reducing every individual’s private signal to a probability assessment of which candidate is better seems to be too narrow a way to describe the dispersion of private information in the society. Our main contribution is to demonstrate how the property of information aggregation depends on the relationship between (common) preference and distribution of information in the electorate.

In the current paper, there is a compact state space Θ , and a compact set of signals X . We allow for both discrete and continuous signals. In each state θ , each voter receives a signal $x \in X$ that is an independent and random draw from the distribution $\eta(\cdot|\theta)$. The state space is partitioned into two sets \mathcal{A} and \mathcal{B} : in states lying in the set \mathcal{A} , all voters prefer alternative A and in the states lying in the set \mathcal{B} , all voters prefer alternative B . Notice that, in a formal sense, voter rankings are simply defined over the space of probability distributions over signals, henceforth denoted by $\Delta(X)$. In large electorates, given a state, the frequency distribution over signals approximates the probability distribution. Therefore, our setup is approximately equivalent to one where there are a large number of voters whose ranking depends on the entire profile of private signals in the electorate (with the added restriction that identity of individuals does not matter for preference).

Formally stated, we have two sets of results. First, we show that there exists some feasible strategy profile that aggregates information if and only if the conditional probability distributions arising from states in \mathcal{A} can be separated from those arising from states in \mathcal{B} by a hyperplane on $\Delta(X)$ (Theorem 1). Our result extends to both symmetric and asymmetric strategies (Corollary 1). The proof of this result simply follows from the fact that the vote share for A is a linear functional of the vectors in $\Delta(X)$. This condition also allows us to

identify the class of strategies that do aggregate information for a given voting rule. Moreover, the particular voting threshold is not important - if information aggregation is feasible for a given threshold, it is feasible for every other non-unanimous threshold rule (Corollary 2).

The second result says that if the hyperplane condition is satisfied, then there exists a profile of information-aggregating strategy that is also a Nash equilibrium (Theorem 2). This result borrows the insight from McLennan (1998) that in any voting game, a symmetric strategy profile that maximizes the ex-ante payoff must also be a Nash equilibrium. Combining these two results, the hyperplane condition is both necessary and sufficient for information aggregation in the limit.

At this stage, it is important to point out the relationship of our work with McLennan (1998). McLennan points out that if there is some *feasible* symmetric strategy that fully aggregates information, then there exists an equilibrium that aggregates information too. The implication of this result is that aggregation failure is not an equilibrium phenomenon at all. We identify necessary and sufficient conditions for existence of a feasible symmetric strategy that aggregates information, and then use the insight from McLennan to show that in environments where information aggregation is feasible, it is also an equilibrium property.

As implications to our main theorem, we provide certain sufficient conditions for information aggregation in elections which are new to the literature. In a setting where there are r states and k signals, we show that information is always aggregated if (i) there are at least as many signals as states, i.e., $r \leq k$, and (ii) the conditional probability distribution over signals in any given state cannot be obtained as a convex combination of the conditional distributions in the other states (Corollary 4). This result implies that information is aggregated if the signals are rich enough for the electorate to distinguish between all states. A simple corollary of this result is that whenever there are just two states, information is aggregated as long as the probability distribution over signals is different in the two states. When both states and signals have a natural order, a commonly studied informativeness condition on signals is Monotone Likelihood Ratio Property (MLRP) (Milgrom 1981). We show that the sufficient condition for information aggregation boils down to a weaker version of MLRP in this environment. In fact, these sufficient conditions for information aggregation in large elections have a strong parallel in Siga (2013), which obtains similar results in the context of auctions with multidimensional signals and discrete states.

There is a recent literature showing that information aggregation can fail to obtain in elections, but these papers rely either on preference diversity (Bhattacharya (2013), Acharya (2013) or residual uncertainty, i.e., uncertainty about probability distributions over preferences (Feddersen and Pesendorfer (1997) or information (Mandler 2012)). Our paper is the first to show that aggregation can fail in an environment where voters have the same preferences, and there is no residual uncertainty. Mandler (2012) shows that, in a common preference environment, uncertainty over the probability distribution over signals may lead to “wrong” equilibrium assessments about the state due the peculiar logic of pivotality. On the

other hand, the logic for aggregation failure that we unveil is not based on any equilibrium reasoning at all.

Feddersen and Pesendorfer (1997) consider a setting with diverse preference along with the restriction that, for every voter, the utility difference between A and B is increasing in the state. In this setting, they show that every sequence of equilibrium aggregates information (provided there is no residual uncertainty over the distribution of preferences). On the other hand, Bhattacharya (2013) shows that if we relax the monotonicity assumption on utility differences, there generically exist equilibrium sequences that do not aggregate information. However, Bhattacharya (2013) is silent about whether information aggregating equilibria do exist in such settings.

In a separate section, we allow voters to have diverse preferences in addition to diverse information. The feasibility result (Theorem 1) generalizes to a case with diverse preferences, with the only modification that \mathcal{A} (resp. \mathcal{B}) is defined as the set of states in which the alternative A (resp. B) would win under full information. Our environment allows both Feddersen and Pesendorfer (1997) and Bhattacharya (2013) as special cases. We can show that in each of these cases, there exist strategies that do aggregate information. However, our proof of Theorem 2 does not directly extend to a setting with diverse preferences. Therefore, the existence of an information aggregating strategy does not automatically imply information aggregation in equilibrium. We are currently working on conditions that guarantee the existence of some equilibrium sequence that aggregates information in the diverse preference case.

The rest of the paper is organized as follows. Section 2 lays out the model with common voter preferences. Section 3 provides the main theorem that establishes conditions under which an environment allows full information aggregation, and discusses the implications for some specific environments. Section 4 shows that existence of a feasible strategy profile that fully aggregates information implies the existence of an equilibrium sequence of profiles that does the same. Section 5 discusses two extensions: one of these shows that our main results do not change if we consider continuous signal spaces, and the other one derives the conditions for existence of information-aggregating strategy profile in the case where voters may have preference heterogeneity. Section 6 concludes.

2. MODEL

In the model, there are n voters choosing between two alternatives A and B . Alternative A wins if it receives more than $q \in (0, 1)$ share of votes and loses if it receives less than q share. If A receives exactly nq votes, then we assume that tie is broken randomly.

In this section, we treat every voter as having the same preferences. In a later section, we show that our feasibility result can be extended to a setting where voters in an electorate may have different preferences. The utility of a voter from an alternative depends on an unobserved state variable $\theta \in \Theta$, where Θ is a compact, separable metric space. The utility

of each voter is given by a bounded and measurable function $u : \Theta \times \{A, B\} \rightarrow \mathbb{R}$. Let the three sets \mathcal{A} , \mathcal{B} and \mathcal{I} denote the respective regions in Θ where A is preferred to B , B is preferred to A and the voters are indifferent.

$$\begin{aligned}\mathcal{A} &= \{\theta \in \Theta : u(\theta, A) > u(\theta, B)\} \\ \mathcal{B} &= \{\theta \in \Theta : u(\theta, A) < u(\theta, B)\} \\ \mathcal{I} &= \{\theta \in \Theta : u(\theta, A) = u(\theta, B)\}\end{aligned}$$

Each voter i receives a private signal $x \in X$, where X is a compact, separable, metric space. Profiles of signals are denoted by $x^n \in X^n$. The information structure is captured by a probability measure η on $\Theta \times X$, as follows. When a voter gets a signal $x \in X$, he makes inferences about the true state θ using the conditional $\eta(\cdot | x) \in \Delta(\Theta)$, where $\Delta(\cdot)$ denotes the space of probability measures over “ \cdot ”, endowed with the weak star topology. Likewise, for a given θ , $\eta(\cdot | \theta) \in \Delta(X)$ is the conditional on the signal received by an individual voter. Hence we assume that voters’ signals are independent and identically distributed conditional on θ . We assume of $\theta \mapsto \eta(\cdot | \theta)$ is surjective and strongly continuous: for each Borel measurable $E \subset X$, $\eta(E | \theta_k) \rightarrow \eta(E | \theta)$ as $\theta_k \rightarrow \theta$. Let $m \in \Delta(\Theta)$ denote the marginal of η on Θ . That is, m is the prior on Θ . We assume that the support of m is Θ , and that the m -measure of the interior of Θ is one. We also assume that while both sets \mathcal{A} and \mathcal{B} have positive probability ex-ante, indifference occurs with zero probability, i.e. $m(\mathcal{A}) > 0$, $m(\mathcal{B}) > 0$ and $m(\mathcal{I}) = 0$. In particular, we allow for the case where all rankings are strict, i.e. \mathcal{I} is empty.

A tuple $\{u, \Theta, X, \eta, q\}$ is defined as an environment. An environment in addition to an electorate size n defines a game. In a game, a strategy for voter i is a measurable function $s_i : X \rightarrow \{0, 1\}$, with $s_i(x) = 1$ meaning that i votes for A at signal x . A behavioral strategy is a measurable function $\sigma_i : X \rightarrow [0, 1]$ with $\sigma_i(x)$ being the probability of a vote for A at signal x . Unless mentioned otherwise, we consider only symmetric strategies, i.e. voters with the same signal play the same strategy. Hence we can drop the index i and use $s(\cdot)$, $\sigma(\cdot)$ to denote individual strategies. We sometimes abuse terminology and refer to s or σ as a profile of strategies, with the understanding that every player uses the same s or σ , as the case may be.

In the main body of the paper, we shall consider the case where the space of signals X is countable (possibly finite). However, all our results holds in the case were X is an infinite. We deal with the infinite case in a separate section.

In what follows, we define the standard for information aggregation for a given strategy profile.

2.1. Full Information Equivalence. Given a state θ , strategy σ , electorate size n , and signal profile x^n , we say that the election leads to a wrong outcome if, for $\theta \in \mathcal{A}$, the alternative A fails to win, or for $\theta \in \mathcal{B}$, the alternative B fails to win. For a given strategy, a wrong outcome may occur due to two reasons: (i) randomness in the signal profile and (ii) randomness in the vote tally due to mixing between actions. We take an ex-ante perspective

and consider the probability of a wrong outcome in a given state due to either of these two sources of randomness. For a given environment, we then take the ex-ante likelihood of error by integrating the probability of error at each state with respect to the prior distribution over states. We say that in an environment $\{u, \Theta, X, \eta, q\}$, the strategy $\sigma(\cdot)$ aggregates information asymptotically if for any $\delta > 0$, there exists some n large enough such that the ex-ante likelihood of a wrong outcome is less than δ .

Suppose that the actual proportion of votes for A in state θ given a strategy profile σ in an electorate of size n is denoted by the random variable $z_n^\sigma(\theta)$. Denote the expected likelihood of a wrong outcome by

$$(1) \quad W_n^\sigma = \int_{\mathcal{A}} 1\{\theta : z_n^\sigma(\theta) \leq q\}m(d\theta) + \int_{\mathcal{B}} 1\{\theta : z_n^\sigma(\theta) \geq q\}m(d\theta) + \int_{\mathcal{I}} 1\{\theta : z_n^\sigma(\theta) = q\}m(d\theta)$$

We say that a strategy profile σ aggregates information if for every $\delta > 0$, there exists some n such that $W_n^\sigma < \delta$. More formally, we say that such a strategy profile achieves Full Information Equivalence.

Define the expected share of votes for A in state θ under symmetric strategy σ as

$$z^\sigma(\theta) \equiv \sum_{x \in X} \sigma(x)\eta(x|\theta)$$

For an asymmetric strategy profile $\sigma = (\sigma_i)_{i \geq 1}$ is made of not necessarily equal behavioral strategies, we define

$$z^\sigma(\theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mu_i^\theta,$$

where $\mu_i^\theta = \sum_{x \in X} \sigma_i(x)\eta(x|\theta)$. Observe that $z^\sigma(\theta)$ is a linear function of the conditional probability vection $\eta(\cdot|\theta)$ and is continuous in θ .

By the Strong Law of Large numbers,

$$z_n^\sigma(\theta) \rightarrow z^\sigma(\theta)$$

$\eta(\cdot|\theta)$ -almost surely as $n \rightarrow \infty$. In other words, as n becomes large, the realized share of votes for A is very close to the expected share $z^\sigma(\theta)$ with a high probability.

Given σ , let

$$\begin{aligned} \mathcal{A}^\sigma &= \{\theta \in \Theta : z^\sigma(\theta) > q\} \\ \mathcal{B}^\sigma &= \{\theta \in \Theta : z^\sigma(\theta) < q\} \\ \mathcal{I}^\sigma &= \{\theta \in \Theta : z^\sigma(\theta) = q\} \end{aligned}$$

denote the regions in Θ where the expected share of votes for A is higher than, lower than, or equal to q respectively. From continuity arguments, it is easy to see that if the electorate is sufficiently large and every voter uses the same strategy σ , alternative A (resp. B) wins with an arbitrarily high probability in states in \mathcal{A}^σ (resp. \mathcal{B}^σ). The outcome can go either way in the set \mathcal{I}^σ .

We say that σ achieves Full Information Equivalence (FIE) if the set of states where an alternative is preferred but fails to almost surely win is of measure zero.

Definition 1 (Full Information Equivalence). *In a given environment (u, Θ, X, η, q) , a strategy σ achieves Full Information Equivalence (FIE) if*

$$m(\mathcal{A} \setminus \mathcal{A}^\sigma) = m(\mathcal{B} \setminus \mathcal{B}^\sigma) = 0.$$

We say that an environment (u, Θ, X, η, q) allows FIE when there exists a profile σ that achieves FIE.

It is easy to check that this definition is equivalent to the expected probability of error $W_n^\sigma \rightarrow 0$. For somewhat technical purposes, we also need another definition. For some $\epsilon > 0$, we say that a strategy σ achieves ϵ -FIE if

$$m(\mathcal{A} \setminus \mathcal{A}^\sigma) = m(\mathcal{B} \setminus \mathcal{B}^\sigma) < \epsilon.$$

If all voters follow a strategy that achieves ϵ -FIE, for a large enough electorate, the probability of error is arbitrarily close to ϵ . Clearly, a strategy that achieves FIE also achieves ϵ -FIE for any $\epsilon > 0$.

The next section discusses properties of the environment that allows FIE. Notice that an environment allowing FIE is necessary but not sufficient for information aggregation in equilibrium.

3. FEASIBILITY OF INFORMATION AGGREGATION

To demonstrate the condition that determines whether an environment allows FIE or not, we start with a pair of examples.

Example 1. *Suppose there are two candidates A and B and the election follows majority rule, i.e., $q = \frac{1}{2}$. The quality of each candidate is given by a number between 0 and 1. Voters prefer the higher quality candidate. Candidate B is known to have a quality of $t \in (0, 1)$. The quality of candidate A is a random variable θ following a non-atomic distribution F over $[0, 1]$. In this case, A is preferred in the “high” states $\theta > t$, B is preferred in “low” states $\theta < t$, and the voters are indifferent in $\{t\}$. The signal space is $X = \{a, b\}$, and $\Pr(a|\theta) = \theta$. The environment allows FIE.*

Example 2. *Suppose there are two candidates A and B and the election follows majority rule, i.e., $q = \frac{1}{2}$. Each candidate has a location on the policy space $[0, 1]$. Voters prefer the candidate with location closest to $\frac{1}{2}$. Candidate B is known to have a location of $t \in (0, \frac{1}{2})$. The location of candidate A is a random variable θ following a non-atomic distribution F over $[0, 1]$. In this case, A is preferred in “moderate” states $\theta \in (\frac{1}{2} - t, \frac{1}{2} + t)$, while B is preferred in “extreme” states $\theta < \frac{1}{2} - t$ and $\theta > \frac{1}{2} + t$. Voters are indifferent at states $\frac{1}{2} - t$ and $\frac{1}{2} + t$. The signal space is $X = \{a, b\}$, and $\Pr(a|\theta) = \theta$. The environment does not allow FIE.*

In each of these cases, any symmetric strategy is given by a pair $\sigma = (\sigma_a, \sigma_b)$, where σ_x is the probability of voting A at signal $x \in \{a, b\}$. The expected vote share at state θ is

$z^\sigma(\theta) = \theta\sigma_a + (1 - \theta)\sigma_b$. The actual vote share is very close to the expected vote share with a high probability in a large electorate.

In example 1, a strategy σ allows FIE if (i) $z^\sigma(t) = \frac{1}{2}$ and (ii) $\sigma_a > \sigma_b$. Sufficiency is obvious to check. Condition (ii) is necessary to ensure that the vote share is higher in high states than in low states. To check necessity of (i), first suppose that (ii) holds but $z^\sigma(t) = \frac{1}{2} - \epsilon$. Now, by continuity of $z^\sigma(\cdot)$, for small δ , $z^\sigma(t + \delta) < \frac{1}{2}$, i.e., for a large enough electorate, in states between t and $t - \delta$, A loses with large probability. Notice that the feature of state t that drives this equality is not that voters are indifferent at t , but that t is the “border” of the set of states where A is preferred and those where B is preferred. We are going to call such states “pivotal states”.

In example 2, there are two such pivotal states: $\frac{1}{2} - t$ and $\frac{1}{2} + t$. By the same logic as example 1, we must have $z^\sigma(\frac{1}{2} - t) = z^\sigma(\frac{1}{2} + t) = \frac{1}{2}$ for σ to achieve FIE. Since $z^\sigma(\theta)$ is linear in θ , the only way this can be achieved is if $\sigma_a = \sigma_b$, but then $z^\sigma(\theta) = \frac{1}{2}$ for all states, leading to large probabilities of errors in almost all states. Thus, there is no symmetric strategy profile that achieves FIE in this case.

These two examples show us that it is the convexity of the sets \mathcal{A} and \mathcal{B} that ensures that we can find a strategy that allows FIE. The strategy that achieves FIE has the property that the vote share at the pivotal state is exactly equal to the threshold voting rule q . The failure of FIE occurs in example 2 because there are two such pivotal states because of the non-convexity of \mathcal{B} . In the next section, we generalize the idea contained in these two examples to a multidimensional state and signal space.

3.1. Main Result. To see the main result in the multidimensional setting, first notice that for each state, the conditional probability distribution is simply a vector on the simplex $\Delta(X)$ over the signal space, and the expected vote share for a given strategy is simply a linear function of that vector. Therefore, the level sets of the expected vote share function are hyperplanes in the simplex. An FIE strategy produces vote shares higher than q at states in \mathcal{A} (where A is preferred) and lower than q at states in \mathcal{B} (where B is preferred). Therefore an FIE strategy must produce a hyperplane that separates the conditional probability vectors arising from states in \mathcal{A} from those arising from states in \mathcal{B} . By continuity, all vectors arising from the pivotal states must lie on the hyperplane corresponding to the level set for q . Thus, the main result states that the conditional probability vectors arising from states in \mathcal{A} (resp. \mathcal{B}) must belong to a convex set for an FIE strategy to exist. On the other hand, if indeed these two sets of conditional probability vectors are both convex, we can find an FIE strategy that produces a hyperplane that separates the two sets.

Before stating the main result, we need some more definitions.

3.1.1. Definitions: Pivotal states and hyperplanes. Let M denote the interior of Θ . By our assumptions on m , we have $m(M) = 1$. We say that a state $\theta \in \Theta$ is *pivotal* if, for each $\epsilon > 0$, $m(\mathcal{A} \cap B_\epsilon(\theta))$ and $m(\mathcal{B} \cap B_\epsilon(\theta))$ are positive, where $B_\epsilon(\theta)$ is the ϵ open ball around θ . Let $M^{piv} \subset \Theta$ denote the set of pivotal states. Likewise, let M^A (resp. M^B) denote the

set of states for which there exists a $\varepsilon > 0$ with $m(B_\varepsilon(\theta) \cap \mathcal{B}) = 0$ (resp. $m(B_\varepsilon(\theta) \cap \mathcal{A}) = 0$.) Observe that (M^A, M^{piv}, M^B) forms a partition of M .

A hyperplane in $\Delta(X)$ is denoted by

$$H = \{\mu \in \Delta(X) : \sum_{x \in X} h(x)\mu(x) = \ell\},$$

for a given measurable function $h : X \rightarrow \mathbb{R}$, and a number $\ell \in \mathbb{R}$. Given a hyperplane H , we use

$$H^+ = \{\mu \in \Delta(X) : \sum_{x \in X} h(x)\mu(x) > \ell\} \text{ and}$$

$$H^- = \{\mu \in \Delta(X) : \sum_{x \in X} h(x)\mu(x) < \ell\}$$

to denote the two associated half-spaces.

The following theorem states that information is aggregated by some strategy if and only if there is a hyperplane on the simplex over signals that separates the conditional probability vectors arising from states in the interior of \mathcal{A} from those arising from states in the interior of \mathcal{B} ; and contains all such vectors arising from pivotal states. The proof relies heavily on the linearity of the vote share function in the conditional probabilities of signals.

Theorem 1. *An environment (u, Θ, X, η, q) allows FIE if and only if there exists a hyperplane H in $\Delta(X)$ such that $\eta(\cdot|\theta) \in H^+$ for $\theta \in M^A$, $\eta(\cdot|\theta) \in H^-$ for $\theta \in M^B$, and, if $M^{piv} \neq \emptyset$, $\eta(\cdot|\theta) \in H$ for $\theta \in M^{piv}$.*

Proof. Let σ be a profile that achieves FIE. We first show that, if $M^{piv} \neq \emptyset$, then $M^{piv} \subset \mathcal{I}^\sigma$. Assume to the contrary and pick $\theta \in M^{piv} \cap \mathcal{A}^\sigma$. As $z^\sigma(\cdot)$ is continuous, there is $\varepsilon > 0$ with $B_\varepsilon(\theta) \subset \mathcal{A}^\sigma$. Because θ is pivotal, we have $m(B_\varepsilon(\theta) \cap \mathcal{B}) > 0$. Hence $m(\mathcal{B} \cap \mathcal{A}^\sigma) \geq m((B_\varepsilon(\theta) \cap \mathcal{B}) \cap \mathcal{A}^\sigma) = m(B_\varepsilon(\theta) \cap \mathcal{B}) > 0$. But this means that $m(\mathcal{B} \setminus \mathcal{B}^\sigma) > 0$, contradicting FIE.

Next, we claim that $M^A \subset \mathcal{A}^\sigma$. Again, by contradiction, pick $\theta \in M^A$ and $\theta \notin \mathcal{A}^\sigma$. Observe that because σ achieves FIE, there must exist at least one pair x, x' with $\sigma(x) < q$ and $\sigma(x') > q$. For each integer k , let $\mu_k \in \Delta(X)$ be given by $\mu_k(E) = (1-1/k)\eta(E|\theta) + (1/k)\mu(E)$, for each measurable $E \subset X$, where μ satisfies $\sum_{x \in X} \sigma(x)\mu(x) < q$. Then $\mu_k \rightarrow \eta(\cdot|\theta)$ strongly as $k \rightarrow \infty$, and $\sum_{x \in X} \sigma(x)\mu_k(x) < q$ for every k . Let $\varepsilon > 0$ satisfy $m(B_\varepsilon(\theta) \cap \mathcal{B}) = 0$, which must exist because $\theta \in M^A$. Because the range of $\theta \mapsto \eta(\cdot|\theta)$ is $\Delta(X)$ and $\theta \mapsto \eta(\cdot|\theta)$ is strongly continuous, there exists $\theta' \in B_\varepsilon(\theta)$ and k large such that $\eta(\cdot|\theta') = \mu_k(\cdot)$. Observe that $z^\sigma(\theta') < q$. By continuity of z^σ , we can find $\varepsilon' > 0$ with $B_{\varepsilon'}(\theta') \subset B_\varepsilon(\theta)$ and $z^\sigma(\theta'') < q$ for all $\theta'' \in B_{\varepsilon'}(\theta')$. Hence $m(B_\varepsilon(\theta) \cap \mathcal{B}) = 0$ while $m(B_\varepsilon(\theta) \cap \mathcal{B}^\sigma) > 0$, meaning that $m(\mathcal{A} \setminus \mathcal{A}^\sigma) > 0$, contradicting FIE.

Similarly, we obtain $M^B \subset \mathcal{B}^\sigma$. Setting $H = \{\mu \in \Delta(X) : \sum_{x \in X} \sigma(x)\mu(x) = q\}$, the ‘‘only if’’ direction is verified.

For the ‘‘if’’ part, let $H = \{\mu \in \Delta(X) : \sum_{x \in X} h(x)\mu(x) = \ell\}$ denote the required hyperplane. Let σ be a behavioral strategy such that $\sigma(x) = q + \varepsilon(h(x) - \ell)$, where $\varepsilon > 0$ ensures

that $\sigma(x) \in [0, 1]$ (observe that, by re-scaling, it is without loss to have $|h(x)| \leq 1$ for every $x \in X$, and hence $|h(x) - \ell| \leq 2$ for every $x \in X$.) If $M^{piv} \neq \emptyset$, pick $\theta \in M^{piv}$. As $\eta(\cdot|\theta) \in H$, we have $z^\sigma(\theta) = \sum_{x \in X} (q + \varepsilon(h(x) - \ell))\eta(x|\theta) = q$, so $M^{piv} \subset \mathcal{I}^\sigma$. Likewise, for $\theta \in M^A$ (resp. $\theta \in M^B$) we have $\eta(\cdot|\theta) \in H^+$ (resp. $\eta(\cdot|\theta) \in H^-$), readily showing that $M^A \subset \mathcal{A}^\sigma$ and $M^B \subset \mathcal{B}^\sigma$. As $(\mathcal{A}^\sigma \cap M, \mathcal{B}^\sigma \cap M, \mathcal{I}^\sigma \cap M)$ is a partition of M as well, we must have $M^A = \mathcal{A}^\sigma \cap M$, $M^B = \mathcal{B}^\sigma \cap M$ and $M^{piv} = \mathcal{I}^\sigma \cap M$. As $\mathcal{A} \cap M \subset M^A$ and $\mathcal{B} \cap M \subset M^B$, we have $\mathcal{A} \subset \mathcal{A}^\sigma$ and $\mathcal{B} \subset \mathcal{B}^\sigma$ for m -almost all θ (whether M^{piv} is empty or not), and FIE is verified. \square

While the analysis above was restricted to symmetric strategies, The exact same results hold true when we consider FIE under asymmetric strategies. If there exist an asymmetric profile that achieves FIE in an environment (u, θ, X, η, q) , then exactly as in the symmetric case there will exist a hyperplane H in $\Delta(X)$ as in the statement of Theorem 1. Again setting $\sigma(x) = q + \varepsilon(h(x) - \ell)$ and following the steps of the “if” part above shows that the symmetric profile σ achieves FIE. Hence we have the following corollary, which says that it is without loss of generality to restrict our attention to symmetric strategies.

Corollary 1. *If an environment (u, θ, X, η, q) allows FIE with an asymmetric profile $\hat{\sigma} = (\hat{\sigma}_i)_{i \geq 1}$ then there exists a strategy σ such that the symmetric strategy profile where every voters uses σ also achieves FIE.*

Next, we show that the existence of an information aggregating strategy depends only on how preference interacts with distribution of information and not on the voting rule in use. If there is a strategy that achieves FIE for a given voting rule, then, for each non-unanimous threshold voting rule, there exists some strategy that achieves FIE.

Corollary 2. *If an environment (u, θ, X, η, q) allows FIE then for any $\hat{q} \in (0, 1)$, $(u, \theta, X, \eta, \hat{q})$ allows FIE.*

Proof. Let H be the hyperplane in $\Delta(X)$ associated with the environment (u, θ, X, η, q) , and pick $\hat{q} \in (0, 1)$. As in the “if” part above, set $\hat{\sigma}(x) = \hat{q} + \varepsilon(h(x) - \ell)$, where $\varepsilon > 0$ again ensures that $\hat{\sigma}(x) \in [0, 1]$, and follow the same steps to establish that $\hat{\sigma}$ achieves FIE. \square

It must be noted that while we cannot uniquely identify the strategies that aggregate information, all strategy profiles that aggregate information are characterized by a specific property: in pivotal states, they lead to a vote share exactly equal to the threshold necessary for the alternative A to win. If there are multiple pivotal states, the conditional distributions arising in such states must lie on a hyperplane on the simplex. As one moves in the simplex along the normal to that hyperplane, the vote share for A produced by any information-aggregating strategy changes monotonically. Later, we show that in environments that allow FIE, there also exists an equilibrium strategy profile with the same properties. McMurray (2014) finds a result with a similar flavor and interprets it as the endogenous emergence of a single dimension of political conflict in a multidimensional world..

3.2. Some Special Environments. In the previous section we have shown that in general environments, FIE can obtain only in rather special cases. Now, we turn to some specific environments that are of interest to us.

3.2.1. State space isomorphic to the simplex over signals. Suppose that the state space is convex and the conditional probability distribution is a linear function of the state. In other words, $\eta(\cdot|\alpha\theta + (1 - \alpha)\theta') = \alpha\eta(\cdot|\theta) + (1 - \alpha)\eta(\cdot|\theta')$ for all pairs θ, θ' and $\alpha \in [0, 1]$. Additionally, assume that there exist some pivotal state(s). In this setting, the condition for existence of FIE strategies simply boils down to all probability distributions corresponding to pivotal states lying on the same hyperplane in the simplex. By the linearity assumption, this condition is equivalent to there being a hyperplane in the state space such that all pivotal states lie on that hyperplane.

A special case of the above is when X is finite, θ is continuous and $\Theta = \Delta(X)$. Additionally, for any θ , $\eta(\cdot|\theta) = \theta$. The situation considered in examples 1 and 2 are special instances of this case. In this situation, the voter preferences are simply defined over the probability distribution over signals rather than a different state variable. In other words, in a large electorate, the preferences depends only on the profile of signals, i.e., the distribution of information in the electorate. In this case, the condition for information aggregation is simply that the set of states in which a particular alternative is preferred is convex.

Corollary 3. *If Θ is a convex subset of a vector space, $\eta(\cdot|\alpha\theta + (1 - \alpha)\theta') = \alpha\eta(\cdot|\theta) + (1 - \alpha)\eta(\cdot|\theta')$ for all pairs θ, θ' and $\alpha \in [0, 1]$, and $M^{piv} \neq \emptyset$, then an environment allows FIE if and only if there exists a hyperplane $H \in \Delta(X)$ such that $\eta(\cdot|\theta) \in H$ for $\theta \in M^{piv}$.*

Proof. There are two steps to this proof. First, we show that under the conditions, an environment allows FIE if and only if there exists a hyperplane H in $\Delta(X)$ such that $\eta(\cdot|\theta) \in H$ for $\theta \in M^{piv}$.

As in the proof above, we have $M^{piv} \subset \mathcal{I}^\sigma$ for the profile σ defined by $\sigma(x) = q + \varepsilon(h(x) - \ell)$ for all x . Say that there are $\theta^A \in M^A$ and $\theta^B \in M^B$ with θ^A, θ^B in \mathcal{A}^σ . There must exist $\alpha \in (0, 1)$ and $\bar{\theta} \in M^{piv}$ with $\bar{\theta} = \alpha\theta^A + (1 - \alpha)\theta^B$. By linearity of $\theta \mapsto \eta(\cdot|\theta)$, we would have $z^\sigma(\bar{\theta}) > q$, contradicting $M^{piv} \subset \mathcal{I}^\sigma$. Hence the sets M^A and M^B are each mapped to one of the halfspaces determined by the hyperplane associated with \mathcal{I}^σ . Changing signs if necessary, we have $M^A \subset \mathcal{A}^\sigma$ and $M^B \subset \mathcal{B}^\sigma$, and the rest follows as in the proof above. Observe that the existence of a hyperplane H in $\Delta(X)$ necessarily implies the existence of a hyperplane H' in Θ satisfying: $\theta \in H'$ for $\theta \in M^{piv}$. Indeed, by linearity of η , the mapping $\theta \mapsto \sum_{x \in X} h(x)\eta(x|\theta)$ defines the required hyperplane in Θ . \square

3.2.2. Finite state and signal space. Another case that is of interest to us is when both the state and signals are finite. Suppose that there are r states and k signals. By our assumption that $m(\mathcal{I}) = 0$, the ranking over alternatives is strict in every state. In this setting, any environment allows FIE if the signal space is sufficiently rich vis-a-vis the state space. As long as there are at least as many signals as there are states, and none of the

conditional probability vectors on the simplex can be expressed as a convex combination of the conditional probability distributions in the other states, there is a strategy that achieves FIE. As a particular case, when there are only two states, as long as the conditional probability distributions are not the same in the two states, there exists some strategy that achieves FIE. Notice that under this richness condition on signals, the utility function assigning states to rankings is immaterial for FIE.

Corollary 4. *Suppose there are r states and k signals, i.e., $\Theta = \{\theta_1, \dots, \theta_r\}$ and $X = \{x_1, \dots, x_k\}$ with $r \leq k$. Moreover, assume that there exists no state $\theta_t \in \Theta$ that satisfies the following: for some set of non-negative real numbers $\lambda_1, \dots, \lambda_r$ with $\lambda_t = 0$ and $\sum_{j=1}^r \lambda_j = 1$,*

$$\eta(\cdot|\theta_t) = \sum_{j=1}^r \lambda_j \eta(\cdot|\theta_j).$$

In such an environment, there exists a strategy that achieves FIE.

Proof. Consider any partition of Θ into two nonempty sets \mathcal{A} and \mathcal{B} . Denote by A_μ the set $\{\mu \in \Delta(X) : \theta \in \mathcal{A} \text{ and } \eta(\cdot|\theta) = \mu\}$, i.e., the set of conditional probability vectors arising in states in \mathcal{A} . Similarly, denote by B_μ the set $\{\mu \in \Delta(X) : \theta \in \mathcal{B} \text{ and } \eta(\cdot|\theta) = \mu\}$. Denote the respective convex hulls by $\text{co}(A_\mu)$ and $\text{co}(B_\mu)$. If $\text{co}(A_\mu)$ and $\text{co}(B_\mu)$ are disjoint, there exists some hyperplane H in $\Delta(X)$ that separates A_μ and B_μ , which is sufficient for the existence of a strategy that achieves FIE. Suppose now that $\text{co}(A_\mu)$ and $\text{co}(B_\mu)$ are not disjoint. By the linear independence assumption, each $\mu \in A_\mu$ is a vertex of $\text{co}(A_\mu)$ and each vertex of $\text{co}(A_\mu)$ belongs to A_μ . If $\text{co}(A_\mu)$ and $\text{co}(B_\mu)$ intersect, there must be some vertex of $\text{co}(A_\mu)$ that is contained in $\text{co}(B_\mu)$. Hence, there must be some $\mu \in A_\mu$ that is contained in $\text{co}(B_\mu)$. But then, such a vector μ is a convex combination of the vectors in B_μ , which is a contradiction. \square

We interpret corollary 4 to mean that if the signal space is sufficiently rich compared to the state space, then, generically information aggregation is obtained. More precisely, in such settings, full information aggregation is possible in some equilibrium sequence provided a full-rank condition is satisfied on the conditional probability vectors. This condition is essentially the Cremer-McClean condition on bidder types sufficient for full extraction of rents in auctions (Cremer and Mclean 1998).

3.2.3. Monotone Likelihood Ratio Property. Suppose that both signals and states have a natural order. A standard informativeness assumption on signals in this setting is the Monotone Likelihood Ratio Property (MLRP), which ensures that a signal is a “sufficient statistic” of the state (Milgrom 1981) in the sense that higher signals indicate higher states. Feddersen and Pesendorfer (1997) assumes strict MLRP condition on signals and shows (albeit in a model of diverse preferences) that information is aggregated in equilibrium. We obtain a sufficient condition for an environment to allow FIE which entertains MLRP as a specific case.

Definition 2 (Monotone Likelihood Ratio Property). *Suppose $\Theta = [0, 1]$ and $X = \{x_1, \dots, x_k\} \in [0, 1]^k$, with $x_1 < x_2 < \dots < x_k$. The signals are said to satisfy strict MLRP if, for any two signals $x < x'$, the likelihood ratio $\frac{\eta(x|\theta)}{\eta(x'|\theta)}$ is a decreasing function of θ .*

We obtain a sufficient condition for the existence of a strategy that achieves FIE in this environment that is weaker than MLRP. Assume that the prior m is non-atomic and has full support over $[0, 1]$. Moreover, suppose that for some $\theta^* \in (0, 1)$, A is preferred for $\theta > \theta^*$ and B is preferred for $\theta < \theta^*$. In other words, $M^A = (\theta^*, 1]$ and $M^B = [0, \theta^*)$.

Let $F(x|\theta) = \sum_{x_j \leq x} \mu_j(x|\theta)$ denote the cumulative distribution function of $\eta(\cdot|\theta)$. Strict MLRP implies that for every x , the cumulative distribution $F(x|\cdot)$ is a decreasing function. Now consider the following property: For each $\theta^a \in M^A$ and each $\theta^b \in M^B$, we have for all $x \in X$

$$(2) \quad F(x|\theta^a) < F(x|\theta^{piv}) < F(x|\theta^b)$$

As long as the property (2) is satisfied, there exists a strategy that achieves FIE. To see that, let x^* be the smallest $x \in X$ such that $1 - F(x|\theta^*) \geq q$. Now, set $\sigma(x) = 0$ for $x \leq x^*$ and $\sigma(x) = 1$ for $x > x^*$. It is easy to verify that the strategy profile σ achieves FIE.¹

Note that the property (2) is weaker than strict MLRP. While strict MLRP implies that $F(x|\cdot)$ is decreasing over the entire interval $[0, 1]$, property (2) does not require $F(x|\cdot)$ to be decreasing within M^A or within M^B .

4. EQUILIBRIUM ANALYSIS

From the previous section, it is clear that only special environments allow full information equivalence in the sense that there exist strategies that achieve FIE. However, it is not clear whether, even in such environments, voters have an incentive to use such strategies. In order to check whether voters find it in their interest to use such strategies, we consider voting as a game played in such environments. A game is defined as an environment (u, θ, X, η, q) along with a number of players n . We fix an environment and consider a sequence of games by letting the number of voters grow. Following the logic in McLennan (1998), we show that under common preferences, any environment that allows FIE also has a sequence of Nash equilibrium profiles that achieves FIE.

First, we define the game derived from the environment (u, θ, X, η, q) along with a number of players n more formally. We provide the analysis for the case where θ is a continuous random variable, the case where θ is discrete is exactly analogous.

It will be necessary to distinguish finite electorates, so let us use the notation $x^n = (x_1, \dots, x_n)$, $s^n = (s_1, \dots, s_n)$, and $\sigma^n = (\sigma_1, \dots, \sigma_n)$ for profiles of signals and strategies in a finite electorate $\{1, \dots, n\}$. Given η on $\Theta \times X$, construct ν on $\Theta \times X^n$ as $d\nu = dm \otimes (\otimes_{i=1}^n d\eta(\cdot|\theta))$

¹When the state space is not $[0, 1]$, a sufficient condition for obtaining an FIE strategy with signals $x_1 < \dots < x_k$ is that (1) each $\theta^{piv} \in M^{piv}$ leads to the same cumulative distribution $F(x|\theta^{piv})$ for all $x \in X$, and (2) for any $\theta^a \in M^A$ and $\theta^b \in M^B$, we obtain $F(x|\theta^a) < F(x|\theta^{piv}) < F(x|\theta^b)$ for all $x \in X$

and let η_x be the marginal of ν on X^n . Denote by $\eta(\cdot|x^n)$ the conditional of ν on Θ given a profile x^n .² For a given x^n , let

$$u^n(a, x^n) = \int_{\Theta} u(a, \theta) \eta(d\theta|x^n),$$

for $a \in \{A, B\}$. Given a profile of pure strategies s^n , let $u(s^n(x^n), x^n)$ denote the utility at profile x^n at the outcome induced by $s^n(x^n)$ (it will be A (resp. B) if $\frac{1}{n} \sum_{i=1}^n s_i(x_i) > q$ (resp. $\frac{1}{n} \sum_{i=1}^n s_i(x_i) < q$) with ties broken by a coin flip.) The multilinear extension at a profile of behavioral strategies is denoted $u(\sigma^n(x^n), x^n)$. Finally, the ex ante expected utility for a given profile σ^n is

$$u^n(\sigma^n) = \sum_{x^n \in X^n} u(\sigma^n(x^n), x^n) \eta_x(x^n).$$

The Bayesian game G^n played by the n voters is the game where each voter has the same space of behavioral strategies, $\Sigma_i = \{\sigma_i : X \rightarrow [0, 1]\}$, endowed with the narrow topology that makes it a compact, convex LCTVS, and the payoffs are the ones we just derived.

Suppose that σ_n^* is a maximizer of $u^n(\sigma^n)$. The existence of such a maximizer follows from compactness of the domain and continuity of u on σ^n .³ Following McLennan (1998), σ_n^* is a Bayesian Nash equilibrium of the game G^n . It is straightforward to restrict to profiles of symmetric strategies and ensure existence of a symmetric BNE. The next theorem tells us that the sequence σ_n^* achieves FIE as long as the environment (u, θ, X, η, q) allows FIE.

Theorem 2. *If the environment (u, θ, X, η, q) allows FIE, there exists a sequence σ^n of Nash equilibria of the game G^n that achieves FIE., i.e., $W_n^{\sigma^n} \rightarrow 0$.*

Proof. Observe that $u(\sigma^n)$ can be written as

$$\int_{\Theta} \sum_a u(a, \theta) \sum_{x^n \in X^n} \varphi_a^{\sigma^n}(x^n) \eta(x^n|\theta) m(d\theta)$$

where $a \in \{A, B, D\}$, “ D ” standing for a draw (and the associated coin flip to decide the winner), and $\varphi_a^{\sigma^n}(x^n)$ is the probability that a is the outcome of the election at the profile x^n . For each size n of electorate, consider a symmetric profile of strategies $\sigma^n = (\sigma, \dots, \sigma)$, so that $\sigma^\infty = (\sigma, \sigma, \dots)$. For each $\theta \in \Theta$, the proportion of votes for A converges to $z^\sigma(\theta)$ $\eta(\cdot|\theta)$ -almost surely as $n \rightarrow \infty$. Hence $\sum_{x^n \in X^n} \varphi_a^{\sigma^n}(x^n) \eta(x^n|\theta)$ converges, so Lebesgue Dominated Convergence implies that $u(\sigma^\infty) = \lim_{n \rightarrow \infty} u(\sigma^n)$ is well defined.

Observe that if the symmetric profile $\hat{\sigma}^\infty$ achieves FIE, then $u(\hat{\sigma}^\infty)$ is the maximum attainable value: for m -almost every $\theta \in \mathcal{A}$, A wins, and for m -almost every $\theta \in \mathcal{B}$, B wins. States in \mathcal{I} are irrelevant for the evaluation above because they are of m -measure zero. So, given that $u(\sigma^n)$ is linear in $u(a, \theta)$, the claim is verified.

²The construction is presented in general form so that it also covers the case of uncountable X that we will deal later.

³By construction, for each given θ , the probability measure on profiles is the product $\otimes_{i=1}^n \eta(\cdot|\theta)$; hence the integral of the utility function is a continuous function of the profile σ^n ; integrating out θ then recovers the ex ante utility, which must then be a continuous function of σ^n .

For each finite electorate $\{1, \dots, n\}$, choose σ^n as a maximizer of $u(\sigma^n)$. We know such profile is an equilibrium of the corresponding game G^n . We also know that $u(\hat{\sigma}^\infty)$ is the maximum feasible value of the ex ante utility. Hence

$$u(\hat{\sigma}^\infty) \geq u(\sigma^\infty) = \lim_n u(\sigma^n) \geq \lim_n u(\hat{\sigma}^n) = u(\hat{\sigma}^\infty),$$

establishing the result. In fact, if $W_n^{\sigma^n}$ were not to converge to zero, then we would have to have, say, $m(\mathcal{A} \setminus \mathcal{A}^{\sigma^\infty}) > 0$. That is, a set of positive measure in \mathcal{A} where B wins under σ^∞ , whereas we know that no such set exists for $\hat{\sigma}^\infty$. But then $u(\hat{\sigma}^\infty) > u(\sigma^\infty)$, contradicting what we just established. \square

5. EXTENSIONS

In this section, we study two extensions to the setup in which the main results are presented. First, we show that all our results go through if we consider a signal space X that is uncountable rather than finite or countable. Then, we study a relaxation of the common preference assumption. In this setting, we obtain a generalization of Theorem 1.

5.1. Uncountable signal space. The analysis above focused on the case that X was either finite or countable. In other words, voters have discrete pieces of information. On the other hand, we may also be interested in situations where the signal space is continuous, i.e., it is very unlikely that any two voters have the same private information. We show that while we have to use a different set of tools for this case, all our results in the previous section go through.

The setting where the signal space is uncountable has some technical disadvantages. In particular, we can no longer use the useful property that, for a given behavioral strategy $\sigma : X \rightarrow [0, 1]$ and distribution $\eta(\cdot|\theta) \in \Delta(X)$, the asymptotic relative frequency of votes for A is equal to the expectation of σ with respect to $\eta(\cdot|\theta)$, for almost all sample paths. In the uncountable case, the property certainly goes through with pure strategies. Therefore, we use purification ideas to tackle this case. To distinguish between pure and behavioral strategy profiles, we will use notations $\sigma(\cdot)$ and $s(\cdot)$ respectively. Unless otherwise stated, when we refer to a pure strategy profile $s(\cdot)$, we will imply that all voters use the pure strategy $s(\cdot)$. Also, a hyperplane in $\Delta(X)$ now is defined as $H = \{\mu \in \Delta(X) : \int h(x)\mu(dx) = \ell\}$.

We make the following regularity assumption in addition to the maintained assumptions above.

A1. The conditional distribution $\eta(\cdot|\theta)$ is absolutely continuous with respect to a fixed non-atomic measure $\mu \in \Delta(X)$, with strictly positive density, for every $\theta \in \Theta$.

Next, we define an approximate measure of FIE. Recall that the expected likelihood of a wrong outcome is denoted by W_n^σ , given by equation (1). We say that a strategy profile

σ achieves ε -FIE if there is some $\varepsilon > 0$ such that $W_n^\sigma < \varepsilon$ for all n sufficiently large. This definition is equivalent to

$$m(\mathcal{A} \setminus \mathcal{A}^\sigma) = m(\mathcal{B} \setminus \mathcal{B}^\sigma) < \varepsilon.$$

We show that if there is some behavioral strategy σ that satisfies the hyperplane condition, then, for any $\varepsilon > 0$, there is some profile of pure strategies that achieves ε -FIE. The following proposition extends Theorem 1 in the context of the infinite signal space.

Proposition 1. *Consider an environment (u, Θ, X, η, q) where the signal space X is uncountable, and suppose that assumption A1 holds. For each $\varepsilon > 0$, there exists a profile of pure strategies s^ε that achieves ε -FIE if there is a hyperplane H in $\Delta(X)$ such that $\eta(\cdot|\theta) \in H^+$ for $\theta \in M^A$, $\eta(\cdot|\theta) \in H^-$ for $\theta \in M^B$, and, if $M^{piv} \neq \emptyset$, $\eta(\cdot|\theta) \in H$ for $\theta \in M^{piv}$. If in addition Θ is a finite set, then there exists a profile of pure strategies s that achieves FIE if the above condition is satisfied. If the above condition fails, then there exists no strategy profile that achieves FIE.*

Proof. A straightforward adaptation of Theorem 1 shows that the existence of a hyperplane H satisfying the conditions listed in the statement of the Proposition are equivalent to the existence of a behavioral strategy σ such that $m(\mathcal{A} \setminus \mathcal{A}^\sigma) = m(\mathcal{B} \setminus \mathcal{B}^\sigma) = 0$. Consider first the case of Θ finite. Applying Lyapunov's theorem establishes the existence of a function $g : X \rightarrow \{0, 1\}$ such that

$$\int \sigma(x) \eta(dx|\theta) = \int g(x) \eta(dx|\theta)$$

for every $\theta \in \Theta$. Hence, setting $s(x) = g(x)$, we are able to replace σ with a pure strategy s with the property that $z^s(\theta) = z^\sigma(\theta)$ for every $\theta \in \Theta$. By the SLLN, $\frac{1}{n} \sum_{i=1}^n s \rightarrow z^s(\theta)$, $\eta(\cdot|\theta)$ -a.e. Hence $m(\mathcal{A} \setminus \mathcal{A}^s) = m(\mathcal{B} \setminus \mathcal{B}^s) = 0$ establishes that s achieves FIE.

The case that Θ is infinite is complicated by the failure of Lyapunov's theorem in infinite dimensions. We will resort to an approximation result to bypass this issue. First, let Σ denote the sigma-algebra of μ -measurable sets of X . Let $Y = \{\eta(E|\cdot)\}_{E \in \Sigma}$, a subset of the separable Banach space space $C(\Theta, [0, 1])$ of continuous functions endowed with the sup norm. Realizing Σ as a complete metric space on its own as a subspace of $L_1(\mu)$ (Aliprantis and Border (2006), Lemma 13.13), Y is also complete: a Cauchy sequence $(\eta(E_n|\cdot))_n$ in Y induces a Cauchy sequence $(E_n)_n$ in Σ , which must converge. In fact, for each $\varepsilon > 0$ and all θ , we have $|\eta(E_n|\theta) - \eta(E_m|\theta)| < \varepsilon$ for n, m large, so $|\mu(E_n) - \mu(E_m)| < \varepsilon$ as well, as the densities are positive. This means that there exist E such that $E_n \rightarrow E$. Now take a convergent sequence in Y , so that $\eta(E_n|\cdot)$ converges uniformly. As it also converges pointwise to $\eta(E|\cdot)$, it must be that $\eta(E_n|\cdot)$ converges to $\eta(E|\cdot)$ uniformly. That is, Y is closed subset of $C(\Theta, [0, 1])$, and hence a separable Banach space with the subspace metric.

Now, by the extension of Lyapunov's theorem to the infinite dimensional case (Aubin and Frankowska (1989), Theorem 8.7.4), the closure of the set $\{\eta(\cdot|\theta)\}_{\theta \in \Theta}$ is convex and compact. Hence, for a given FIE σ , there exists a sequence $g^n : X \rightarrow \{0, 1\}$ such that

$$\sup_{\theta \in \Theta} \left| \int g^n(x) \eta(dx|\theta) - \int \sigma(x) \eta(dx|\theta) \right| \rightarrow 0.$$

For a given $\delta > 0$, let $V_\delta = \{\theta : z^\sigma(\theta) \in [q - \delta, q + \delta]\}$, a closed neighborhood of \mathcal{I}^σ that converges to \mathcal{I}^σ as $\delta \rightarrow 0$. It follows that $m(V_\delta) \rightarrow 0$ as well. That is, there exists $\varepsilon(\delta) > 0$ such that $m(V_\delta) < \varepsilon(\delta)$ and $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. For a given $\varepsilon > 0$, let $\delta > 0$ be such that $\varepsilon(\delta) \leq \varepsilon$. Let g^ε be an element of the sequence g^n such that $\sup_{\theta \in \Theta} |\int g^\varepsilon(x)\eta(dx|\theta) - \int \sigma(x)\eta(dx|\theta)| < \delta$. Set $s^\varepsilon(\cdot) = g^\varepsilon(\cdot)$ as the common strategy for all voters, and conclude that outside of a set Θ^ε with $m(\Theta^\varepsilon) < \varepsilon$, $\int s^\varepsilon(x)\eta(dx|\theta) > q$ (resp. $\int s^\varepsilon(x)\eta(dx|\theta) < q$) whenever $\int \sigma(x)\eta(dx|\theta) > q$ (resp. $\int \sigma(x)\eta(dx|\theta) < q$).

Hence, for each $\varepsilon > 0$, outside a set Θ^ε with $m(\Theta^\varepsilon) < \varepsilon$, $\mathcal{A}^{s^\varepsilon} = \mathcal{A}^\sigma$ and $\mathcal{B}^{s^\varepsilon} = \mathcal{B}^\sigma$, so s^ε achieves ε -FIE. \square

Next, we turn to equilibrium analysis for this case. First note that the game with an uncountable X is defined in the same way as above, using integrals over X^n rather than summations. While proposition 1 is slightly weaker than theorem 1 in the sense that we only obtain ε -FIE, that is enough to guarantee a sequence of strategies that achieve FIE in the limit.

Proposition 2. *If an environment allows ε -FIE for all $\varepsilon > 0$, then there exists a sequence σ^n of Nash equilibria of the game G^n that achieves FIE, i.e., $W_n^{\sigma^n} \rightarrow 0$.*

Proof. For each $\varepsilon > 0$ there exists an ε -FIE, s^ε , which achieves the maximum feasible ex ante utility outside a set Θ^ε with $m(\Theta^\varepsilon) < \varepsilon$. Let u^* be the maximum feasible ex ante utility. As in the arguments establishing Theorem 2 above, let σ^n maximize $u(\cdot)$ in the finite electorate game. Then $u^* \geq \lim_{n \rightarrow \infty} u(\sigma^n) \geq u(s^\varepsilon) \geq u^* - \varepsilon$. As $\varepsilon > 0$ is arbitrary, $\lim_n u(\sigma^n) = u^*$, so the same argument as in Theorem 2 establishes that σ^∞ achieves FIE. \square

5.2. Diverse Preferences. So far we have assumed that all voters have the same preferences. In this section we extend our results to a case where the voters in the electorate may have different preferences. We maintain the assumption that all voters are ex ante identical, and draw their information and preferences from some distribution conditional on the state. To do so, we retain the elements of the set-up in the main section and assume in addition that the private signal x is also payoff relevant. Thus, the private draw of an individual serves two functions: it is a view about the outcomes and it provides information about how others view the outcomes. We may think of $x_i = (s_i, t_i)$, where s_i is the common value component and t_i is the private value component of the preference. Notice that this is a general setting that can encompass many different environments. In particular, it admits the environments studied in Feddersen and Pesendorfer (1997) with continuous state space and Bhattacharya (2013) with just two states.

Consider, therefore, that voters preferences are given by $u : \Theta \times X \times \{A, B\} \rightarrow \mathbb{R}$. We deal only with the case that X is at most countable.

We now normalize $u(\theta, x) = 1$ if $u(\theta, x, A) > u(\theta, x, B)$, $u(\theta, x) = 0$ if $u(\theta, x, A) < u(\theta, x, B)$, $u(\theta, x) = \frac{1}{2}$ if $u(\theta, x, A) = u(\theta, x, B)$. This normalization is innocuous for the feasibility result.

Since there is no commonly preferred candidate under diverse preferences, we have to be careful while defining the standard for information aggregation. We say that a strategy profile aggregates information if, for a large electorate, the outcome is the same as the outcome when the state is commonly known. Notice that in a large electorate with finite signals, knowing the state can be interpreted as knowing the entire profile of private signals.

By the SLLN, asymptotically the proportion of voters that prefer A to B , given a state θ , is

$$u(\theta) = \sum_{x \in X} u(\theta, x) \eta(x|\theta).$$

In a large electorate, A would get a vote share very close to $u(\theta)$ if the state were known to be θ . Therefore, under full information A wins depending on whether $u(\theta)$ is greater or less than the threshold q . Now fix $q \in (0, 1)$ and redefine the sets \mathcal{A} , \mathcal{B} , and \mathcal{I} as

$$\begin{aligned} \mathcal{A}_q &= \{\theta \in \Theta : u(\theta) > q\} \\ \mathcal{B}_q &= \{\theta \in \Theta : u(\theta) < q\} \\ \mathcal{I}_q &= \{\theta \in \Theta : u(\theta) = q.\} \end{aligned}$$

In states \mathcal{A}_q (resp. \mathcal{B}_q), alternative A (resp. B) wins under full information and large electorates. We then say that an environment (u, Θ, X, η, q) allows FIE if there exists a strategy σ such that

$$m(\mathcal{A}_q \setminus \mathcal{A}^\sigma) = m(\mathcal{B}_q \setminus \mathcal{B}^\sigma) = 0.$$

Define M_q^A and M_q^B as the respective interiors of \mathcal{A}_q and \mathcal{B}_q .

It is simple to verify now that the argument in the proof of Theorem 1 follows through line-by-line, so FIE is again characterized by the hyperplane condition. More formally, we have the following result,

Proposition 3. *An environment (u, Θ, X, η, q) and diverse preference allows FIE if and only if there exists a hyperplane H in $\Delta(X)$ such that $\eta(\cdot|\theta) \in H^+$ for $\theta \in M_q^A$, $\eta(\cdot|\theta) \in H^-$ for $\theta \in M_q^B$, and, if $M^{piv} \neq \emptyset$, $\eta(\cdot|\theta) \in H$ for $\theta \in M_q^{piv}$.*

It is also simple to verify that the arguments for Corollaries 1 and 3 remain valid. Corollaries 2 and 4 on the other hand, do not go through any longer: changing q changes the sets \mathcal{A}_q , \mathcal{B}_q , and \mathcal{I}_q , so the arguments given above do not show that FIE is obtained for the new level \hat{q} .

Notice that since Feddersen and Pesendorfer (1997) result already tells us that information is aggregated in equilibrium, the existence of FIE strategies is trivial in their setting. More interestingly, while Bhattacharya (2013) concentrates on showing that, for any consequential rule, there exists an equilibrium that fails to aggregate information, it can be checked that in Bhattacharya's two-state setting, there always exists some feasible strategy that achieves FIE. It would therefore be very interesting to examine the conditions under which, in a general setting with diverse preferences, there exists some equilibrium sequence that aggregates information.

However, the proof of Theorem 2 explicitly utilizes the common value setting, and therefore does not automatically generalize to an environment with diverse preferences. In particular, we do not know yet whether, given preference diversity in the electorate, the existence of a feasible strategy profile guaranteeing FIE also implies that FIE is achieved in equilibrium. Our efforts are currently focussed on analyzing conditions under which the hyperplane result also implies that information is aggregated in equilibrium in presence of preference heterogeneity.

6. CONCLUSION

The existing literature on information aggregation in large elections has largely focussed on specific preference and information environments. In this paper, we consider general environments in order to analyze conditions under which information is aggregated. Preferences depend on the state of the world, and each state of the world is synonymous with a probability distribution over private signals. Therefore, preferences are simply mappings from allowable probability distributions over private information to rankings over the two alternatives A and B . In a large electorate, the frequency distribution over signals is approximately the same as the probability distribution. Thus, our question is whether the election achieves the outcome that would obtain if the entire profile of private signals were publicly known. If an environment permits a feasible strategy profile that can induce the full information outcome with a high probability in almost all states, we say that the environment allows Full Information Equivalence (FIE). Moreover, we are interested in whether such a strategy profile is incentive compatible, i.e., it constitutes a Nash equilibrium in the underlying game.

We study both environments with and without preference heterogeneity. In both cases, we find that an environment allows FIE if and only if a hyperplane on the simplex over signals separates the probability distributions arising in states where A is preferred from those arising in states where B is preferred. Therefore, if the information environment is sufficiently complex, there is no strategy profile that aggregates information. We like to stress here that the failure of FIE has nothing to do with equilibrium assessments over the states based on the criterion of one's vote being pivotal in deciding the election.

We obtain sharper positive results in the common preference case where all voters would have agreed on their rankings if they had known the profile of signals. In this case, voting aggregates information alone (and not preference). We find that as long as an environment allows FIE, there is a sequence of equilibria that achieves FIE. We must mention here that there may be other equilibrium sequences that do not aggregate information - but ours is only a possibility result. As implications of this result, we provide several examples of common preference environments where information will be aggregated. We show that if there are only a finite number of states and signals, FIE holds in equilibrium for any preference mapping from states to alternatives as long as the signal space is sufficiently rich compared to the space of states. As a special case, we show that whenever there are two states, FIE is generically achieved in equilibrium. We also show that in the common preference environment, the

voting rule does not matter for information aggregation: as long as FIE is achieved under some voting rule, FIE is achieved under every other non-unanimous voting rule.

Feasibility of FIE does not automatically imply FIE in equilibrium when voting has the burden of aggregating information and preferences simultaneously, i.e., in the diverse preference case. However, we conjecture that under some conditions, feasibility of FIE indeed implies the existence of an equilibrium sequence that also achieves FIE in the case with diverse preferences. Our current research efforts are focussed on unveiling these conditions.

7. REFERENCES

- (1) Acharya, Avidit. "Equilibrium False Consciousness." mimeo, Stanford University (2014).
- (2) Aliprantis, C. D., and K. C. Border. "Infinite Dimensional Analysis", Springer-Verlag. (2006).
- (3) Aubin, Jean-Pierre, and Hélène Frankowska. "Set-Valued Analysis", Birkhäuser (2009).
- (4) Austen-Smith, David, and Jeffrey S. Banks. "Information aggregation, rationality, and the Condorcet jury theorem." *American Political Science Review* (1996): 34-45.
- (5) Berg, Sven. "Condorcet's jury theorem, dependency among jurors." *Social Choice and Welfare* 10, no. 1 (1993): 87-95.
- (6) Bhattacharya, Sourav. "Preference Monotonicity and Information Aggregation in Elections." *Econometrica* 81, no. 3 (2013): 1229-1247
- (7) Cremer, Jaques, and Richard P. McLean, "Full Extraction of Surplus in Bayesian and Dominant Strategy Auctions", *Econometrica* 56, no. 6, (1988), 1247-1257.
- (8) Duggan, John, and César Martinelli. "A Bayesian model of voting in juries." *Games and Economic Behavior* 37, no. 2 (2001): 259-294.
- (9) Feddersen, Timothy, and Wolfgang Pesendorfer. "Voting behavior and information aggregation in elections with private information." *Econometrica: Journal of the Econometric Society* (1997): 1029-1058.
- (10) Ladha, Krishna K. "The Condorcet jury theorem, free speech, and correlated votes." *American Journal of Political Science* (1992): 617-634.
- (11) Mandler, Michael. "The fragility of information aggregation in large elections." *Games and Economic Behavior* 74, no. 1 (2012): 257-268.
- (12) McLennan, Andrew. "Consequences of the Condorcet jury theorem for beneficial information aggregation by rational agents." *American Political Science Review* (1998): 413-418.
- (13) McMurray, Joseph: "Why the Political World is Flat: An Endogenous Left-Right Spectrum in Multidimensional Political Conflict", mimeo, Brigham Young University
- (14) Milgrom, Paul R. "Good news and bad news: Representation theorems and applications." *The Bell Journal of Economics* (1981): 380-391.

- (15) Myerson, Roger B. "Extended Poisson games and the Condorcet jury theorem." Games and Economic Behavior 25, no. 1 (1998): 111-131.
- (16) Siga, Lucas. "Complex Signals and Information Aggregation in Large Markets" mimeo, New York University (2013)
- (17) Wit, Jürgen. "Rational choice and the Condorcet jury theorem." Games and Economic Behavior 22, no. 2 (1998): 364-376.

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