Twofold Preferences under Uncertainty*

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Abstract

In the standard Anscombe-Aumann setting, we study twofold preferences, i.e., incomplete preferences that compare different acts based on two possibly different utility functions. Twofold preferences encapsulate the following form of status-quo bias: An alternative f can beat a status-quo g only when the pessimistic evaluation of f still exceeds the optimistic evaluation of g. We find that twofold preferences are useful to analyze economic problems for at least two reasons. From a behavioral perspective, this type of conservatism enables theoretical models to account for experimental phenomena such as failures in contingent reasoning. From a modeling perspective, our twofold approach can provide the unified framework for modeling various incomplete preferences that differ in additional postulates that the modeler wishes to impose. A series of axiomatization results in this paper provide the characterizations of the incomplete-counterparts of existing complete preferences in the literature.

Contents

*Acknowledgement will be added.

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1 Introduction

We study the preferences of a decision-maker (DM) that find it more difficult to evaluate uncertain prospects than deterministic outcomes. Specifically, while the DM can easily rank deterministic outcomes, she may sometimes be unable to compare ambiguous alternatives. We model these preferences by associating each act with a range of possible values and positing that the DM ranks different acts by comparing these ranges. Formally, in the standard Anscombe-Aumann setting, the DM associates to each act f an interval $U(f) = [U^\flat(f), U^\sharp(f)]$ of possible values, and she reveals to prefer f over g if and only if the lowest value attributed to f still exceeds the greatest value attributed to g. That is, $f \succ g$ if and only if $U^\flat(f) > U^\sharp(g)$. The derived ranking may be *incomplete* since, in general, the range of possible values $U(f)$ is not a singleton. On the other hand, we assume $U^\sharp(x) = U^\flat(x)$ holds when an act x involves no contingencies, capturing the idea that the DM feels it easy to evaluate deterministic outcomes.

We refer to this type of preference as *twofold preferences*. Twofold preferences are of interest as they are able to capture the following simple form of status-quo bias: An alternative f can beat a status-quo g only when the pessimistic evaluation of f still exceeds the optimistic evaluation of g. On top of that, we find twofold preferences are useful to model various incomplete preferences, thanks to its flexibility that makes the representation compatible with most existing axioms in decision-theory.[1] This paper takes an axiomatic approach to study various classes of twofold preferences. We explore the relationships between the functional representations of $U(f)$ and preference axioms that the modeler may impose on $\succ$ to postulate certain behavioral contents.

We begin our analysis by characterizing a general class of twofold preferences that we call *twofold MBA preferences*. This class admits a representation via interval orders of the form $U(f) = [I^\flat(u \circ f), I^\sharp(u \circ f)]$, where u is an affine vNM utility function over outcomes, and $I^\flat$ and $I^\sharp$ are monotonic functionals that satisfy $I^\flat \le I^\sharp$ and mild continuity assumptions.[2] This preference class is quite broad, and we show its axiomatization relies only on parsimonious structural assumptions. Yet, the axiomatization can serve as the benchmark throughout the latter analysis, as it is useful to clarify some basic properties common across most twofold preferences.

Although appealing for its generality, the above model can be too flexible for applications, e.g., when an econometrician wishes to match the observable choice data with modeling parameters. Therefore, we subsequently study several restricted classes by imposing additional axioms on $\succ$. Among several representation results we establish, one primary focus of the paper is what we call *twofold variational preferences*, which admit the following representation: For any pair of acts, we have $f \succ g$ if and only if

$$\min_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu + c^\flat(\mu) \right\} > \max_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ g) \mathrm{d}\mu - c^\sharp(\mu) \right\}, \tag{1}$$

where $c^\flat, c^\sharp : \Delta(\Sigma) \to [0, \infty]$ are convex functions that associate a cost to each belief μ over the underlying state space (S, Σ). Importantly, these cost functions must be *jointly grounded*, that is, there exists at least one belief μ^* to which both $c^\flat$ and $c^\sharp$ assign zero. One can naturally interpret such μ^* as the most plausible belief, serving as a reference point that implicitly affects how the DM evaluates every feasible alternative. But, the DM is also allowed to adopt different beliefs μ to become optimistic or pessimistic at the expense of some costs prescribed by $c^\flat$ or $c^\sharp$.

[1] We remark that the present paper is not the first attempt to model incomplete preferences via interval orders. See Section 1.1 for the discussion on related literature.

[2] Twofold MBA preferences can be seen as the incomplete preference counterpart of the MBA preferences first introduced by Cerreia-Vioglio et al. (2011). Indeed, their (complete) MBA preference can be obtained as a special case of our twofold MBA preference with $I^\flat = I^\sharp$.

Twofold variational preferences generalize the *twofold multi-prior preferences* of Miyashita and Nakamura (2020) and Echenique, Pomatto, and Vinson (2021) (henceforth, two papers are collectively referred to as EMNPV). As such, when $c^\flat$ and $c^\sharp$ are the indicator functions of some non-disjoint, convex and closed sets $C^\flat, C^\sharp \subseteq \Delta(\Sigma)$ (in the sense of convex analysis; see, e.g., Brezis, 2010), the representation (1) reduces to

$$\min_{\mu\in C^\flat}\int (u\circ f)\mathrm{d}\mu > \max_{\mu\in C^\sharp}\int (u\circ g)\mathrm{d}\mu. \tag{2}$$

Here, different acts are ranked based on their minimum vs. maximum expected utility obtained by optimizing over all beliefs considered as relevant by two selves, say "pessimism" and "optimism" of the DM. Twofold multi-prior preferences are of particular interest because of the connection with some solution concepts in game theory and mechanism design.[3]

Our Theorem 1 provides the characterization of twofold variational preferences. Behaviorally, the axiomatization is accomplished by weakening *C-independence* to *weak C-independence*, roughly paralleling with the shift from the *MEU representation* a lá Gilboa and Schmeidler (1989) to the *variational representation* a lá Maccheroni et al. (2006). In some parts, our proof then proceeds similarly to the one Maccheroni et al. (2006) employed to obtain their variational representation of complete preferences. However, some mathematical problems are specific to the present context because of distinctive feature of incomplete preferences, e.g., in (1), there is no need for $c^\flat$ and $c^\sharp$ to coincide, but they must be "consistent" in a sense to guarantee the basic property $I^\flat \leq I^\sharp$ of any interval orders. Our novel technical contribution lies in reducing this interval condition to a more primitive condition pertaining solely to the cost functions $c^\flat$ and $c^\sharp$. We accomplish this goal with the help of the Fenchel-Rockafeller theorem in convex duality and show that the aforementioned joint groundedness turns out to be key for the consistency between the DM's different selves.

We then use twofold variational preferences to conduct two exercises to clarify interesting relationships with existing decision models in the literature. In Section 4, we study the following "justifiability" counterpart of the decision rule (1); say, $f \succsim^\dagger g$ holds if and only if

$$\max_{\mu\in\Delta(\Sigma)}\left\{\int (u\circ f)\mathrm{d}\mu - c^\sharp(\mu)\right\} \geq \min_{\mu\in\Delta(\Sigma)}\left\{\int (u\circ g)\mathrm{d}\mu + c^\flat(\mu)\right\}. \tag{3}$$

Evidently, the only difference from (1) is that the exchanged roles of the DM's optimism and pessimism. The relation between the representations of $\succ$ and $\succsim^\dagger$ is roughly parallel with that between Bewley's (2002) *unanimity representation* and Lehrer and Teper's (2011) *justifiability representation*. Namely, the permissive relation $\succsim^\dagger$ is defined to rank f weakly higher than g if and only if the conservative relation $\succ$ does not rank f below g. Since one binary relation naturally defines the other in this way, we are

[3] For example, the *obvious dominance criterion* of Li (2017) will result when both $C^\flat$ and $C^\sharp$ coincide with the whole set of $\Delta(\Sigma)$. On the other hand, when $C^\flat$ is a singleton, the equation (2) can be related to the *intuitive criterion* of Cho and Kreps (1987), where the candidate equilibrium strategy, seen as the status quo, is challenged by the best possible outcome of a deviating strategy.

free to choose which to adopt, $\succ$ or $\succsim^{\dagger}$, as the model primitive. In our view, the two approaches are complementary rather than substitutable, as they differ in how we have to depart from the expected utility hypothesis. In this regard, the direct axiomatization of $\succsim^{\dagger}$, provided as Theorem 2, sheds an interesting light on the DM's conservatism by highlighting how $\succ$ (accepting the alternative) differs from $\succsim^{\dagger}$ (sticking to the status quo).

In Section 5, we clarify the relations between our twofold variational preferences and (complete) variational preferences of Maccheroni et al. (2006). As similar to the case of unanimity preferences and maximin preferences pointed out by Gilboa et al. (2010), we show that given a twofold variational preference $\succ^{\star}$ represented by $(u, c^{\flat}, c^{\sharp})$, if it is "pessimistically" extended to a complete preference $\succ$ following their *default to certainty* condition, then $\succ$ is the variational preference represented by $(u, c^{\flat})$. On the other hand, if $\succ^{\star}$ is "optimistically" extended to $\succ$ following the dual condition of default to certainty, then $\succ$ is the maximax-version of the variational preference represented by $(u, c^{\sharp})$. Hence, if the DM chooses being pessimistic (resp, optimistic) to make an actual choice, then only the pessimistic cost function $c^{\flat}$ (resp. $c^{\sharp}$) will survive as the ambiguity index in the representation of $\succ$, while the optimistic cost function $c^{\sharp}$ (resp. $c^{\flat}$) will be discarded. In this regard, the asymmetry between the DM's different selves can be seen as one defining feature of incompleteness.

Lastly, in Section 6, we provide some possible extensions. Specifically, we present three additional axiomatization results–Theorems 3, 4, and 5–each of which gives rise to the incomplete-preference analog of the *CEU representation* a lá Schmeidler (1989), and the *uncertainty averse* and *homothetic* representations both studied by Cerreia-Vioglio et al. (2011a), respectively. These results demonstrate the flexibility of our twofold approach to incorporate different behavioral postulates into incomplete preferences. Along with those we explicitly solve, in Section 7, we also outline some possible extensions left in this paper for the sake of potential future research. As usual, all mathematical proofs omitted from the main text are relegated to the appendices.

1.1 Related Literature

Ever since Anscombe and Aumann (1963), there have been many attempts to model preferences of economic agents in face of uncertainty. For an expositional purpose, we classify the existing models into three categories and discuss the connections with the present paper. As such, our categorization is roughly based on the "number" of utility functions used to represent the preferences.[4]

The complete model. The first category consists of models of complete preferences. Together with other basic axioms, i.e., transitivity and continuity, these preferences admit a representation via a *single* utility function. In the context of uncertainty, following the seminal work of Anscombe and Aumann (1963), the subsequent works have studied the weakening of their assumptions. In particular, the most controversial has been independence, and the vast majority of the literature proposes many ways to weaken it, e.g.,

[4] Needless to say, this is not a unique way of sensible classification. For example, if one focuses on axioms, Hara's (2021) multi-MEU representation would rather belong to the intersection of the first and third categories.

comonotonic-independence by Schmeidler (1989), C-independence by Gilboa and Schmeidler (1989), weak C-independence by Maccheroni et al. (2006), homotheticity by Cerreia-Vioglio et al. (2011a), and risk independence by Cerreia-Vioglio et al. (2011) to name a few. On the other hand, not only is it inaccurate as a description of real life, but we also find that completeness is hard to accept from a normative point of view, in particular when certain decisions that one is asked to make involve uncertainty. Indeed, Aumann (1962) himself argues that: *"Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable"* when it comes to making a decision in highly hypothetical situations, which will never be experienced.[5]

Fortunately, it turns out that much of the existing theory stays intact even when we depart from completeness, though we have to emphasize that the departure from completeness is accomplished at the expense of the simultaneous departure from monotonicity. As such, a series of axiomatization results presented in this paper has the exact counterparts in the literature of complete preferences under uncertainty.[6] In this regard, we think one contribution of our work is that it provides a systematic way to translate several postulates for complete preferences into the ones for incomplete preferences.

The twofold model. Our approach falls into this category, in which the incompleteness is modeled through *two* utility functions. While the fundamental idea of twofold preferences dates back to Luce (1956), Fishburn (1970), and Bridges and Mehta (1995), it is relatively new in the context of uncertainty. Miyashita and Nakamura (2020) and Echenique et al. (2021) are the first papers that introduce the notion of twofold representations in the Anscombe-Aumann setting.[7] As mentioned before, the present paper shows that the twofold approach is not limited to their multi-prior models, but rather it is quite flexible to capture other behavioral postulates as well. The formal connections with their works are intensively discussed in Section 3.3.

The unanimity model. The representations in this category also model incomplete preferences, but by means of *multiple* utility functions, whose unanimous agreement is required to derive a ranking between different acts. Following Bewley's (2002) seminal work, related papers include Faro (2015), Ok et al. (2012), and Hara (2021). In particular, Faro (2015) considers the extension of Bewley's (2002) multi-prior model to incorporate variational costs, in spirit, similarly to our extension of EMNPV's representation.

Though both unanimity and twofold approaches can be used to model incomplete preferences, there are many substantial differences. In terms of behavioral contents, the twofold representation tends to result in more conservative decisions than the unanimity representation (perhaps, contrary to the impression given by the number of utility functions used). Indeed, for an alternative to beat the status-quo, while the latter requires only the unanimous agreement in favor of the alternative, the former requires that the

[5] Another critic against completeness is the non-falsifiability from finite data sets, cf. Chambers et al. (2014).

[6] For example, our Theorems 1, 3, and 4 and 5 correspond to the incomplete-preference counterparts of the axiomatizations by Maccheroni et al. (2006), Schmeidler (1989), and Cerreia-Vioglio et al. (2011a), respectively.

[7] Valenzuela-Stookey (2020) also proposes a form of twofold representation in the Anscombe-Aumann model. His representation maintains monotonicity since it adds auxiliary state-wise comparisons, which is applied independently of the main twofold decision rule.

pessimistic evaluation of the alternative still exceeds the optimistic evaluation of the status-quo. This sort of conservatism enables us to model the empirical phenomena of *failures in contingent reasoning*, observed in laboratory experiments such as Esponda and Vespa (2014, 2019) and Li (2017).

Another difference lies in the analogy to the first-category representations. It may not be straightforward to come up with such analogs based on the unanimity model. Moreover, even if the analog is feasible, it may involve simultaneous departures from subordinate axioms, e.g., Faro's (2015) representation can incorporate the variational terms into Bewley's (2002) representation, but at the expense of the simultaneous violation of transitivity. On the other hand, the series of our axiomatizations are built on the same benchmark, but still, can differ in the behavioral content that the modeler may wish to postulate on top of the basic axioms common across all. In our view, this flexibility is one notable advantage of the twofold model.

2 Preliminaries

2.1 The Setting

Consider a set S of states of the world endowed with an algebra of events Σ. Let $B_0(\Sigma)$ denote the set of simple utility acts, i.e., Σ-measurable functions $a : S \to \mathbb{R}$ that take at most finitely many values. We endow $B_0(\Sigma)$ with the sup-norm topology, i.e., the topology generated by the norm $\|a\|_\infty = \sup_{s\in S} |a(s)|$. As is well-known, the norm dual of $B_0(\Sigma)$ is isometrically isomorphic to the space $ba(\Sigma)$ of all finitely additive and bounded set-functions $\lambda : \Sigma \to [-\infty, \infty]$ endowed with the total variation norm, where the duality is given by $\lambda(a) = \int a \mathrm{d}\lambda$. We denote by $\Delta(\Sigma) \subseteq ba(\Sigma)$ the set of finitely additive probability measures on (S, Σ), i.e., the set of non-negative elements of $ba(\Sigma)$ that assign measure 1 to S. A generic element $\mu \in \Delta(\Sigma)$ is called a ***belief***. The (relative) weak-* topology on $\Delta(\Sigma)$ induced by $B_0(\Sigma)$ coincides with the eventwise convergence topology. Other basic definitions and mathematical results, relevant to our analysis, are collected in Appendix A.

Let X be the set of outcomes, which is assumed to be a convex subset of a certain vector space. A ***(simple) act*** is a Σ-measurable function $f : S \to X$ that takes at most finitely many outcomes. Denote by $\mathcal{F}$ the set of all acts. Using the linear structure of X, we endow $\mathcal{F}$ with a mixture operation. Specifically, for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we denote with $\alpha f + (1 - \alpha)g \in \mathcal{F}$ the act whose outcome is $\alpha f(s) + (1 - \alpha)g(s) \in X$ in each state $s \in S$. With a slight abuse of notation, we denote with $x \in \mathcal{F}$ the ***constant act*** that yields the outcome x in every state. The set of all constant acts is then naturally identified with X.

The primitive of our analysis is a (strict) preference relation $\succ$ over $\mathcal{F}$. When $f \not\succ g$ and $g \not\succ f$, we say that f and g are ***incomparable*** and write as $f \bowtie g$. Throughout the paper, we only consider preference relations $\succ$ that, when restricted to constant acts, are the asymmetric part of a complete and transitive binary relation $\succsim$. We will then denote by $\sim$ the symmetric part of $\succsim$ over constant acts. Thus, $\succsim$ and $\sim$ are only used to describe the ranking among constant acts.

2.2 A General Class of Twofold Preferences

We shall begin our analysis with a general class of twofold preferences that we call twofold MBA preferences. (As a mnemonic, MBA refers to **M**onotonic, **B**ernoullian, **A**rchimedean.) These preferences admit a representation via interval orders of the form $[I^\flat(u \circ f), I^\sharp(u \circ f)] \subseteq \mathbb{R}$, where u is the standard vNM function, and only parsimonious topological and algebraic properties are imposed on preference functionals $I^\flat$ and $I^\sharp$.[8] This class will serve as the benchmark throughout the later analysis.

Definition 1. We say that a binary relation $\succ$ over $\mathcal{F}$ is a ***twofold MBA preference*** if there exist a surjective affine function $u : X \to \mathbb{R}$ and normalized, monotonic functionals $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ with $I^\flat \leq I^\sharp$, where $I^\flat$ is lower semicontinuous and $I^\sharp$ is upper semicontinuous, such that for all $f, g \in \mathcal{F}$

$$f \succ g \Longleftrightarrow I^\flat(u \circ f) > I^\sharp(u \circ g). \tag{4}$$

The first axiom is a collection of basic assumptions. Below in A1, the first three assumptions imply that the restriction of $\succsim$ to X satisfies all the vNM axioms. Hence, there exists an affine utility function $u : X \to \mathbb{R}$ such that $x \succ y$ if and only if $u(x) > u(y)$ for all $x, y \in X$. The last assumption is the unboundedness of $\succsim$, which also implies non-triviality. This assumption is used to guarantee that u is surjective, i.e., $u(X) = \mathbb{R}$.

Axiom A1 (Structural Assumptions)**.**

- (Strict Order)**.** $\succ$ is asymmetric and transitive. Moreover, the restriction of $\succ$ to X is negatively transitive.
- (Mixture Continuity)**.** For any $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succ h\}$ and $\{\alpha \in [0,1] : h \succ \alpha f + (1-\alpha)g\}$ are open relative to $[0,1]$.
- (Risk Independence)**.** For any $x, y, z \in X$ and $\alpha \in (0,1)$, $x \succ y$ if and only if $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$.
- (Unboundedness)**.** There exist $x, y \in X$ such that $x \succ y$. Moreover, for any $\alpha \in (0,1)$, there exist $\overline{z}, \underline{z} \in X$ such that $\alpha\overline{z} + (1-\alpha)y \succ x \succ y \succ \alpha\underline{z} + (1-\alpha)x$.

The next two axioms discipline those circumstances where the DM may be unable to rank alternative acts in the presence of uncertainty, and elucidate the essential role *constant acts* play in our twofold MBA axiomatization. We begin by providing the definitions of *Security* and *Potential* of acts. These notions, first introduced Kopylov (2009), characterize different ways of measuring the quality of uncertain prospects in comparison with unambiguous alternatives. Formally, we say that an act f is ***more secure*** than g if for every $x \in X$, $g \succ x$ implies that $f \succ x$, or equivalently, $f \not\succ x$ implies that $g \not\succ x$. Also, we say f has ***more potential*** than g if for every $x \in X$, $x \succ f$ implies that $x \succ g$, or equivalently, $x \not\succ g$ implies that $x \not\succ f$.

[8] See Appendix A.1 for the formal definitions of several functional properties.

The axiom **A2** demands that the notions of security and potential are consistent with the standard state-wise dominance relation. That is, if there exists a state-wise dominance relation between f and g, such relation should be preserved when the security and potential of f and g are compared. Instead, the axiom **A3** provides a sufficient condition for the incomparability of two ambiguous alternatives. Incidentally, it clarifies that the DM compares ambiguous acts only indirectly, using constant acts as reference points.

Axiom A2 (Secure-Potential Monotonicity)**.** For any $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then f is both more secure than g and has more potential than g.

Axiom A3 (Contagion of Incomparability)**.** For any $f, g \in \mathcal{F}$ and $x \in X$, if $f \bowtie x$ and $g \bowtie x$ then $f \bowtie g$.

Axioms **A1–A3** jointly characterize the whole class of twofold MBA preferences. The following proposition formally establishes this result. As a byproduct, it also clarifies what kind of uniqueness property our twofold MBA representation satisfies.

Lemma 1. *A preference relation $\succ$ satisfies* **A1–A3** *if and only if it a twofold MBA preference relation. Moreover, if $\succ$ is represented by the profile $(u, I^\flat, I^\sharp)$ according to* (4)*, then u is unique up to positive affine transformations, and $I^\flat, I^\sharp$ are unique given u fixed.*

It is interesting to remark that, despite mixture continuity must hold, neither $I^\flat$ nor $I^\sharp$ need be "fully" continuous. Instead, they are only required to satisfy symmetric semicontinuity properties. Roughly speaking, this is because $I^\sharp$ (resp. $I^\flat$) is responsible for determining only the lower-contour (resp. upper-contour) sets of $\succ$, hence, lower semicontinuity (resp. upper semicontinuity) is enough to make those sets open, as postulated by mixture continuity. Below, we present an example of a twofold MBA preference $\succ$ whose corresponding utility functionals violate continuity.

Example 1. Suppose that $S = 2$ and $X = \mathbb{R}$, so each act f is a member of $\mathbb{R}^2$. For simplicity, let $u : \mathbb{R} \to \mathbb{R}$ be the identity. Define utility functions $I^\flat, I^\sharp : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$I^\flat(f) = \begin{cases} \min\{f_1, f_2\} & \text{if } f_1 + f_2 \leq 2, \\ \frac{1}{3}\max\{f_1, f_2\} + \frac{2}{3}\min\{f_1, f_2\} & \text{if } f_1 + f_2 > 2; \end{cases}$$

$$I^\sharp(f) = \begin{cases} \frac{2}{3}\max\{f_1, f_2\} + \frac{1}{3}\min\{f_1, f_2\} & \text{if } f_1 + f_2 < 2, \\ \max\{f_1, f_2\} & \text{if } f_1 + f_2 \geq 2, \end{cases}$$

and let $f \succ g$ whenever $I^\flat(f) > I^\sharp(g)$. Observe that both $I^\flat$ and $I^\sharp$ are normalized and monotonic, and that the inequality $I^\flat \leq I^\sharp$ is satisfied. Moreover, $I^\flat$ is lower semicontinuous, and $I^\sharp$ is upper semicontinuous, but still, neither is continuous because of the jumps occurring on the line $f_1 + f_2 = 2$. See Figure 1 for illustration. ▲

In many cases, however, twofold MBA preferences of interest admit a representation with continuous functionals $I^\flat$ and $I^\sharp$. One instance of such case arises when $I^\flat$ and $I^\sharp$ are concave or convex. This

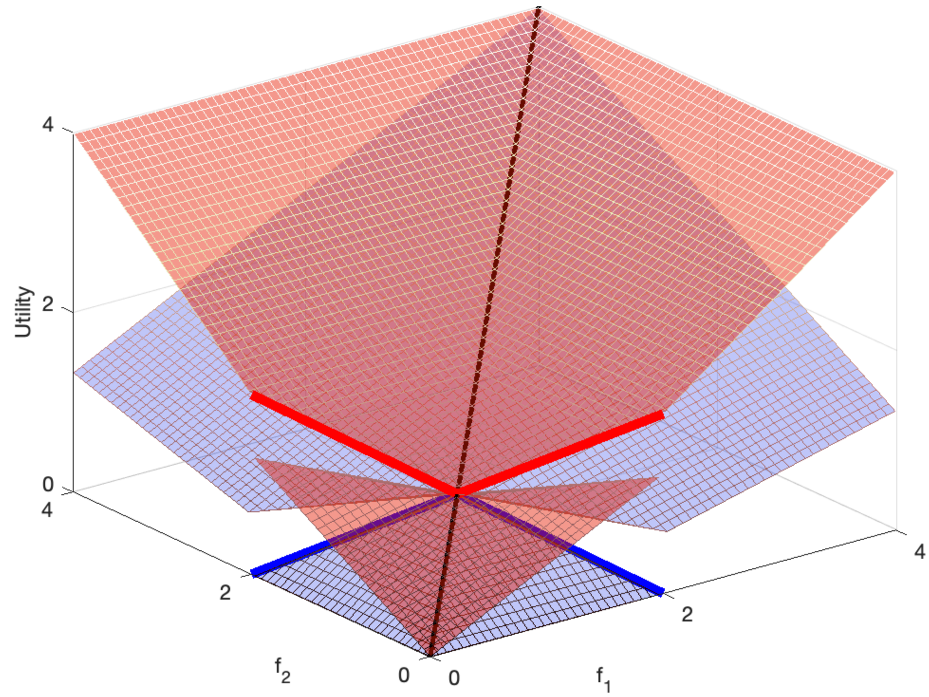

Figure 1: The graphs of $I^\flat$ (Blue) and $I^\sharp$ (Red) in Example 1 are displayed.

is exactly the objective of our focus in the next section. Another instance is when $I^\flat$ and $I^\sharp$ coincide with each other, in which case our representation boils down to the *MBA preferences* of Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) (henceforth, CGMMS). Specifically, CGMMS consider the following strengthening of our **A1** and **A2** to obtain their representation.[9]

Axiom B1 (Negative transitivity)**.** In addition to **A1**, assume further that $\succ$ is negatively transitive.

Axiom B2 (Monotonicity)**.** For any $f, g \in \mathcal{F}$, if $f(s) \succ g(s)$ for all $s \in S$, then $f \succ g$.

On top of **A1**, the axiom **B1** further assumes that $\succ$ is an asymmetric part of some complete preferences. The axiom **B2**, instead, is nothing more than the standard monotonicity axiom. Notice that under these axioms, our third axiom, **A3**, is automatically satisfied, which is therefore omitted from the statement of the next lemma.

Lemma 2 (CGMMS, Proposition 1)**.** *A preference relation $\succ$ satisfies* **B1** *and* **B2** *if and only if there exists a surjective affine function $u : X \to \mathbb{R}$ and a normalized, monotonic, and continuous functional $I : B_0(\Sigma) \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$, $f \succ g$ if and only if $I(u \circ f) > I(u \circ g)$. Moreover, u is unique up to positive affine transformations, and I is unique given u fixed.*

[9] More precisely, the axiom **B2** *per se* does not necessarily imply **A2** since the presumption of **B2** is now stated in terms of strict preferences. However, one can easily show that **B2** implies **A2** in the presence of **B1**. Also, we remark that the analysis of CGMMS is based on weak preference relations, but it is not difficult to obtain Lemma 2 below from their Proposition 1.

We conclude this section with one final observation. The next lemma shows that negative transitivity and monotonicity have the equivalent implications to twofold MBA preferences. In other words, adding *either* of these conditions forces the twofold MBA representation to satisfy $I^\flat = I^\sharp$. This implies that our attempt to weaken completeness is always accompanied with the simultaneous weakening of monotonicity.

Lemma 3. *Let $\succ$ be a twofold MBA preference represented. Then, $\succ$ satisfies negative transitivity if and only if it satisfies monotonicity. Consequently, $\succ$ satisfies either of these conditions if and only if it admits the representation as in Lemma* 2.

3 Twofold Variational Preferences

Thanks to its non-parametric feature, the twofold MBA representation is appealing for its generality. However, it can be too flexible for applications, e.g., an econometrician may wish to have more structures to identify meaningful preference parameters from the observable choice data. In this regard, we subsequently study several subclasses of twofold MBA by imposing additional axioms on $\succ$.

3.1 Representation Results

This section is devoted for the characterization of twofold variational preferences, which we see as the twofold-counterpart of variational preferences of Maccheroni, Marinacci, and Rustichini (2006) (henceforth, MMR). To this end, we consider the following two axioms.

Axiom A4 (Weak C-independence)**.** For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$ for some $x \in X$, then $\alpha f + (1-\alpha)y \succ \alpha g + (1-\alpha)y$ for all $y \in X$.

Axiom A5 (Quasi-convexity)**.** For any $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0,1)$, if $f \succ x$ and $g \succ x$, then $\alpha f + (1-\alpha)g \succ x$. Also, if $x \succ f$ and $x \succ g$, then $x \succ \alpha f + (1-\alpha)g$.

The axiom **A4** is adopted from MMR to our incomplete preference setting. It prescribes that when the DM is able to rank two acts f and g when mixed with the same constant act x, the same ranking results even when f and g are mixed with any other constant act $y \neq x$, as long as a uniform mixing proportion $\alpha \in (0,1)$ is used to make every comparison. As such, it greatly relaxes the C-independence axiom used by EMNPV to obtain their twofold multi-prior preference model. In particular, while C-independence requires that mixture operations with constant acts do not affect preference orders, our weak C-independence demands this only when mixing proportions are fixed. On the other hand, the axiom **A5** is the adoption of the convexity axiom in EMNPV. It clarifies that, in general, mixture operations always reduce the potential and increase the security of ambiguous acts.

Together with **A1–A3**, the axioms **A4** and **A5** characterize the class of preferences that admit a twofold variational representation as in (1). To introduce the formal statement of this result, we introduce some preliminary terminology. Fix any map of the form $c : \Delta(\Sigma) \to [0,\infty]$. For every $r \in [0,\infty]$, define the image set by $\{c = r\} \equiv \{\mu \in \Delta(\Sigma) : c(\mu) = r\}$. Likewise, the sets $\{c < r\}$ and $\{c \leq r\}$ can be defined.

In particular, we refer to the set $\{c < \infty\}$ as the ***domain*** of c. We say that c is ***grounded*** if $\{c = 0\} \neq \emptyset$, ***convex*** if $\{c \leq r\}$ is a convex set for all $r \in [0, \infty]$, and ***lower semicontinuous*** if $\{c \leq r\}$ is closed for all $r \in [0, \infty]$. Moreover, we say that two mappings $c_1, c_2 : \Delta(\Sigma) \to [0, \infty]$ are ***jointly grounded*** if $\{c_1 = 0\} \cap \{c_2 = 0\} \neq \emptyset$, i.e., there exists some $\mu^* \in \Delta(\Sigma)$ such that $c_1(\mu^*) = c_2(\mu^*) = 0$.

Theorem 1. *Let $\succ$ be a preference relation over $\mathcal{F}$. The following conditions are equivalent:*

i) *$\succ$ satisfies* **A1–A5***.*

ii) *There exist a surjective affine vNM utility function $u : X \to \mathbb{R}$ and two convex, lower semicontinuous, and jointly grounded cost functions $c^\flat, c^\sharp : \Delta(\Sigma) \to [0, \infty]$ such that, for all $f, g \in \mathcal{F}$, $f \succ g$ if and only if* (1) *holds. Moreover, u is unique up to positive affine transformations, and $c^\flat, c^\sharp$ are unique given u fixed.*

Definition 2. We say that $\succ$ is a ***twofold variational preference*** if it can be represented by a profile $(u, c^\flat, c^\sharp)$ satisfying all the requirements in Theorem 1.

To highlight the role played by the several assumptions imposed on the cost functions $c^\flat$ and $c^\sharp$, let us briefly discuss the necessity part of Theorem 1. First, it is instructive to notice that twofold variational preferences correspond to a special case of the twofold MBA preferences presented in the previous section, where the utility functionals are given by

$$I^\flat(u \circ f) = \min_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu + c^\flat(\mu) \right\}, \tag{5}$$

$$I^\sharp(u \circ g) = \max_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ g) \mathrm{d}\mu - c^\sharp(\mu) \right\}. \tag{6}$$

Notice that $I^\flat$ and $I^\sharp$ given as above are normalized, monotonic, and (fully) continuous.[10] Moreover, one can verify the inequality $I^\sharp \geq I^\flat$ from the fact that $c^\flat$ and $c^\sharp$ are jointly grounded.[11] By Lemma 1, these observations already verify the axioms **A1–A3**. In addition, the convexity of cost functions imply that

[10] Indeed, the mapping $a \mapsto \int a \mathrm{d}\mu + c^\flat(\mu)$ is continuous for every $\mu \in \Delta(\Sigma)$. Thus, being defined as the pointwise minimum, $I^\flat$ turns out to be continuous (cf. Lemma 2.41 in Aliprantis and Border, 2006). The same argument applies for $I^\sharp$ as well. In particular, the minimum and maximum in (5) and (6) are achievable since $c^\flat$ and $c^\sharp$ are lower semicontinuous, and since $\Delta(\Sigma)$ is a weak-* compact subset of $ba(\Sigma)$ due to the Banach-Alaoglu theorem.

[11] To see this, notice that $C^* \equiv \{c^\flat = 0\} \cap \{c^\sharp = 0\} \neq \emptyset$ when $c^\flat$ and $c^\sharp$ are jointly grounded. It then follows that

$$\begin{aligned} I^\sharp(a) = \max_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu - c^\sharp(\mu) \right\} &\geq \max_{\mu \in C^*} \int (u \circ f) \mathrm{d}\mu \\ &\geq \min_{\mu \in C^*} \int (u \circ f) \mathrm{d}\mu \geq \min_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu + c^\flat(\mu) \right\} = I^\flat(a) \end{aligned}$$

for all $a \in B_0(\Sigma)$. So, the joint groundedness of $c^\flat$ and $c^\sharp$ implies $I^\sharp \geq I^\flat$. In fact, the converse implication is also true, and the proof of it will be a crucial step in the entire proof of Theorem 1. In doing so, we will use the Fenchel-Rockafeller theorem in convex analysis, while the standard version of the theorem need be tailored to fit for our purpose. We provide an alternative in Appendix A.2.

$I^\flat$ and $I^\sharp$ are concave and convex, respectively. Together with the translation invariant properties, this readily implies the axioms **A4** and **A5** as well.

Similarly to the uniqueness of twofold MBA representations, when u is fixed the cost functions $c^\flat$ and $c^\sharp$ are uniquely identified.[12] Below we provide the explicit formulae that characterize these functions.

Proposition 1. *Let $\succ$ be a twofold variational preference represented by $(u, c^\flat, c^\sharp)$. The unique cost functions $c^\flat, c^\sharp$ are pinned down as follows: for all $\mu \in \Delta(\Sigma)$,*

$$c^\flat(\mu) = \sup_{(f,x)\in\mathcal{F}\times X} \left\{ u(x) - \int (u \circ f)\mathrm{d}\mu : f \succ x \right\}, \tag{7}$$

$$c^\sharp(\mu) = \sup_{(f,x)\in\mathcal{F}\times X} \left\{ \int (u \circ f)\mathrm{d}\mu - u(x) : x \succ f \right\}. \tag{8}$$

These expressions are similar to those obtained by Faro (2015) when axiomatizing his *variational Bewley preferences*, but here, suprema are taken only over pairs of general and constant acts. From an empirical point of view, this would be own advantages of twofold variational preferences. Indeed, when $X = \mathbb{R}$ so that constant acts are interpreted as a sure monetary outcomes, the expressions (7) and (8) imply that $c^\flat$ (resp. $c^\sharp$) can be identified by looking only at the *willingness to pay* (resp. *willingness to accept*) for uncertain prospects f. Put differently, in order to identify the DM's cost functions in twofold variational preferences, not necessarily needed is the rich choice data on all possible binary menus, but rather a strict subset of it will suffice.

3.2 The Symmetric Case

Twofold variational preferences allow the DM to employ two possibly different cost functions $(c^\flat, c^\sharp)$ to evaluate acts pessimistically and optimistically. It is therefore natural to wonder under which circumstances the representation exhibits symmetry, that is, $c^\flat = c^\sharp$.

Following the definition in Siniscalchi (2009), given any $f, g \in \mathcal{F}$, we say that f and g are ***complementary*** if $\frac{1}{2}f(s)+\frac{1}{2}g(s) \sim \frac{1}{2}(s')+\frac{1}{2}g(s')$ for all $s, s' \in S$. Namely, the DM can create an uncertainty-free portfolio by mixing two complementary acts at the fair ratio. The following axiom is due to EMNPV.

Axiom A6 (C-Betweenness)**.** For any complementary acts $f, g \in \mathcal{F}$, if $f \succ g$, then $f \succ \frac{1}{2}f + \frac{1}{2}g \succ g$.

This axiom prescribes that if the DM prefers f over g, and if the symmetric mixture between f and g is uncertainty-free, then such mixture should always be in relation with both f and g, and its ranking should lie in the middle of the preference order. EMNPV first introduced this axiom to characterize the symmetric case of their twofold multi-prior representation, i.e., the case where the two selves of the DM employ identical belief sets. Below we show that axiom **A6** has a similar role here, even when we consider a more general class of twofold representations.

[12] This uniqueness result is relying on the unboundedness of u, as similar to Proposition 6 of MMR.

Proposition 2. *Let $\succ$ be a twofold variational preference represented by $(u, c^\flat, c^\sharp)$. Then, $\succ$ satisfies* **A6** *if and only if $c^\flat(\mu) = c^\sharp(\mu)$ for all $\mu \in \Delta(\Sigma)$.*

3.3 Independence Axioms

Now we study the consequence of adding alternative independence axioms to the twofold variational representation. Consider the following variants, listed from the weakest to the strongest in order.

A4 (a) (Semi C-Independence)**.** For any $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0,1)$, if $f \succ g$ then $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$.

A4 (b) (C-Independence)**.** For any $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0,1)$, $f \succ g$ if and only if $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$.

A4 (c) (Independence)**.** For any $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \succ g$ if and only if $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

In general, twofold variational preferences may violate axioms **A4 (b)** and **A4 (c)**. This follows from the non-homogenous nature of utility functionals $I^\flat, I^\sharp$ given by (5) and (6). On the other hand, twofold variational preferences *do* satisfy axiom **A4 (a)**.

Corollary 1. *The axioms* **A1–A5** *imply* **A4 (a)***.*

Proof. By Theorem 1, any preference relation $\succ$ satisfying **A1–A5** admits a twofold variational representation $(u, c^\flat, c^\sharp)$, and the associated utility functionals $I^\flat$ and $I^\sharp$ are defined by (5) and (6), respectively. In particular, note that both are normalized, while $I^\flat$ is concave and $I^\sharp$ is convex. Now consider any pair of acts such that $f \succ g$, and let $x \in X$ and $\alpha \in (0,1)$. By the properties of $I^\flat$ and $I^\sharp$, it follows that

$$
\begin{aligned}
I^\flat(u \circ (\alpha f + (1-\alpha)x)) &\geq \alpha I^\flat(u \circ f) + (1-\alpha)u(x) \\
&> \alpha I^\sharp(u \circ g) + (1-\alpha)u(x) \geq I^\sharp(u \circ (\alpha f + (1-\alpha)x)),
\end{aligned}
$$

where the strict inequality follows from $f \succ g$ and $\alpha > 0$. Hence $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$, so $\succ$ satisfies **A4 (a)**. □

A close look at the proof reveals that Corollary 1 is relying on both that $I^\flat$ is concave, and that $I^\sharp$ is convex. As such, MMR's variational preference satisfies only the former, but the latter is generically violated because of the uncertainty aversion axiom. Therefore, the following implication of the axiom **A4 (a)**–pertaining to the DM's "monotonic" degree of decisiveness with respect to betting scale–is a distinctive feature of *twofold* variational preferences.

To illustrate, notice first that the "volatility" of an uncertain prospect will be reduced when it is mixed with a certain prospect. In view of this, **A4 (a)** posits that if the DM has managed to rank two ambiguous acts f and g, then she should maintain the same ranking when the volatility of these acts is reduced by the symmetric amounts captured by α and x. However, **A4 (a)** need not imply the converse, i.e., preference rankings may not be preserved when the volatility is increased. In other words, the DM may become more conservative as the scale of uncertainty gets larger. The following example illustrates this point.

Example 2. Consider an urn that contains 100 balls, either Red or Blues in unknown population, and the following bets:

	Red	Blue
f	\$20,000	\$5,000
g	\$5,000	\$10,000

	Red	Blue
f'	\$20	\$5
g'	\$5	\$10

Note that acts f' and g' are obtained by mixing with a ratio 1:999 the acts f and g with a constant act that pays \$0 with certainty. Thus, Semi C-independence prescribes that the DM must be able to rank f' and g' whenever he is able to rank f and g. On the other hand, the converse may not be true when the DM's preferences are captured by a twofold variational representation. Indeed, suppose that the DM is risk neutral, and the cost functions $c^\flat, c^\sharp$ are identical and given by $c(\mu) = \alpha|\mu - \mu^*|^2$, where $\alpha > 0$ and $\mu^* \in (0,1)$ is the least costly (i.e., the most plausible) probability that is assigned to the event of Red. If α is neither too large nor too small, the DM would be able to rank f' vs. g', whereas he would deem f and g incomparable. ▲

We conclude this section by clarifying the consequences of adding stronger independence axioms to our benchmark axioms **A1–A5**. When we require a twofold variational preference relation $\succ$ to satisfy C-independence, $\succ$ reduces to a ***twofold multi-prior preference*** that compares acts based on their worst-vs-best expected utility. This conservative criterion, which includes Li's (2017) *obvious dominanc* as a special case, was first studied by EMNPV.

Proposition 3. *Let $\succ$ be a twofold variational preference represented by $(u, c^\flat, c^\sharp)$. The following conditions are equivalent:*

i) *$\succ$ satisfies C-independence.*

ii) *$c^\flat$ and $c^\sharp$ take only the values 0 and ∞.*

iii) *For all $f, g \in \mathcal{F}$, $f \succ g$ if and only if (2) holds, with $C^\flat = \{c^\flat = 0\}$ and $C^\sharp = \{c^\sharp = 0\}$.*

Adding the standard independence axiom to a twofold variational preference has more dramatic effects. In such a case, the representation collapses to the standard ***subjective expected utility*** (SEU) model a lá Anscombe and Aumann (1963). As a result, the classic monotonicity and completeness axioms will be restored. Indeed, an even stronger statement can be made: Within the class of twofold variational preferences, monotonicity, completeness and independence are all equivalent properties.

Proposition 4. *Let* $\succ$ *be a twofold variational preference represented by* $(u, c^\flat, c^\sharp)$*. The following conditions are equivalent:*

i) $\succ$ *satisfies independence.*

ii) $\succ$ *satisfies monotonicity.*

iii) $\succ$ *satisfies negative transitivity.*

iv) *There exists* $\mu^* \in \Delta(\Sigma)$ *such that* $c^\flat$ *and* $c^\sharp$ *take the value 0 only at* μ^**, and* ∞ *otherwise.*

v) *There exists* $\mu^* \in \Delta(\Sigma)$ *such that for all* $f, g \in \mathcal{F}$*,* $f \succ g$ *if and only if*

$$\int (u \circ f)\mathrm{d}\mu^* > \int (u \circ g)\mathrm{d}\mu^*.$$

3.4 Comparative Statics: Prudence and Hope

In Section 2, we presented the notions of security and potential of ambiguous acts and used them to circumscribe the cases where a DM may be unable to compare different alternatives. Here, we use these notions to conduct comparative statics that involve different twofold variational preferences. Specifically, we compare the ambiguity attitudes of different decision-makers, say DM1 and DM2, whose preferences are modeled as twofold variational preferences $\succ_1$ and $\succ_2$ that share the same risk preferences over constant acts. As similar to Ghirardato and Marinacci (2002), we would interpret the preference of DM1 to be more ambiguity averse (resp. ambiguity loving) than that of DM2 when any ambiguous act f appears more secure for DM1 than for DM2. Formal definitions follow.

Definition 3. We say that $\succ_1$ is ***more prudent*** than $\succ_2$ if $\succ_1|_X = \succ_2|_X$ and $f \succ_1 x$ implies $f \succ_2 x$ for all $f \in \mathcal{F}$ and $x \in X$. Analogously, $\succ_1$ ***has more hope*** than $\succ_2$ if $\succ_1|_X = \succ_2|_X$ and $x \succ_1 f$ implies $x \succ_2 f$ for all $f \in \mathcal{F}$ and $x \in X$.

The next proposition characterizes the situations where DM1 turns out to be more prudent or have more hope than DM2. As will be apparent momentarily, the characterization relies on relating either of the pessimistic or optimistic cost functions of the two agents, provided that vNM utility functions are appropriately normalized.

Proposition 5. *Let* $\succ_1$ *and* $\succ_2$ *be twofold variational preferences that are represented by the profiles* $(u_1, c_1^\flat, c_1^\sharp)$ *and* $(u_2, c_2^\flat, c_2^\sharp)$*, respectively.*

i) $\succ_1$ *is more prudent than* $\succ_2$ *if and only if there exist constant numbers* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that* $u_2 = \alpha u_1 + \beta$ *and* $c_2^\flat \geq \alpha c_1^\flat$.

ii) $\succ_1$ *has more hope than* $\succ_2$ *if and only if there exist constant numbers* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that* $u_2 = \alpha u_1 + \beta$ *and* $c_2^\sharp \geq \alpha c_1^\sharp$.

The intuition behind Proposition 5 can be understood by looking at an appropriate limit case. Suppose that DM1's representation $(u_1, c_1^\flat, c_1^\sharp)$ employs the following extreme pessimistic cost function: $c_1^\flat \equiv 0$. Namely, the pessimistic self of DM1 conceives all beliefs are equally plausibly, therefore, she always assumes the worst case scenario whenever she contemplates a change from the status-quo. More precisely, DM1 will opt to change the status quo g in favor of an alternative act f only when the utility attributed to the worst contingent outcome that may result from f still exceeds his optimistic evaluation of g. As $I_2^\flat \geq I_1^\flat$ reveals that $I_2^\flat$ displays less ambiguity aversion than $I_1^\flat$, it is therefore obvious that DM2 with twofold preferences $\succ_2$ represented by $(u_2, c_2^\flat, c_2^\sharp)$ will display less prudence than DM1 whenever $u_1 = u_2$ and $c_1^\flat \leq c_2^\flat$.

We conclude this subsection with one last observation. From the above discussion, it may be natural to think that extreme prudence and hope are equivalent psychological features, as they both result in a stark affirmation of a status quo bias. It is worth remarking here that this is not the case. A very prudent DM sticks with the status quo because she under-evaluates alternative acts. As a result, she is more likely to indefinitely embrace a status quo that is well-balanced, i.e., that covers well all possible contingencies. Conversely, a very hopeful DM maintains his status quo configuration because she is highly optimistic about its future payoffs. In turn, this feature implies that she is more likely to keep a "risky" status quo, i.e., a status quo that is not well-balanced. Therefore, prudence and hope are pretty different psychological phenomena that can induce very different behavioral manifestations. It is then essential to separate them conceptually, and it is an advantage of our twofold variational representation to perform comparative statics involving such concepts by just considering the cost functions $c^\flat$ and $c^\sharp$.

4 Twofold Justifiable Preferences

Any twofold preference induces a complete preference in light of the "justifiability" notion. Specifically, given a twofold preference $\succ$, we define a binary relation $\succsim^\dagger$ over $\mathcal{F}$ by

$$f \succsim^\dagger g \Longleftrightarrow g \not\succ f,$$

for all $f, g \in \mathcal{F}$. In particular, when $\succ$ admits a twofold variational representation, the representation of $\succsim^\dagger$ is obtained by replacing the roles played by pessimism and optimism. The following corollary is a direct consequence of Theorem 1, and thus, the proof is omitted.

Corollary 2. *If (and only if)* $\succ$ *satisfies* **A1–A5***, there exist a surjective affine function* $u : X \to \mathbb{R}$ *and* $c^\flat, c^\sharp : \Delta(\Sigma) \to [0, \infty]$*, which are convex, lower semicontinuous, and jointly grounded, such that for all* $f, g \in \mathcal{F}$*,* $f \succsim^\dagger g$ *if and only if* (3) *holds.*

The relations between $\succ$ and $\succsim^\dagger$ can be understood analogously to the relations between Bewley's (2002) *unanimity preferences* and Lehrer and Teper's (2011) *justifiable preferences*. Specifically, the difference between $\succ$ and $\succsim^\dagger$ lies only in how an external observer interprets the situation where the DM does not exhibit any strict preference between two acts; namely, while $f \not\succ g$ and $g \not\succ f$ were interpreted as indicating that the DM is *indecisive* between f and g, we would interpret the same DM as being *indifferent* between f and g when $f \not\succ^\dagger g$ and $g \not\succ^\dagger f$.

While we have taken an incomplete preference $\succ$ as the primitive so far, it is also possible to derive an axiomatization result starting with a complete preference $\succsim^\dagger$. In our view, each approach has an advantage in its own right. For example, the axiom set for $\succ$ has clear behavioral contents in light of the so-called experimental phenomenon of *failures in contingent reasoning* (Esponda and Vespa, 2014; 2019). Specifically, EMNPV argue that the weakening of monotonicity **A2**, as well as the special treatment of constant acts in the axiom **A3**, formalize the DM's inability to invoke hypothetical consequences that would arise from her choice. On the other hand, the axiomatization of $\succ$ relies on simultaneous departures from completeness and monotonicity. Indeed, we demonstrate that using $\succsim^\dagger$ enables us to minimize departures from the SEU axioms by only deviating from transitivity. We believe that providing this alternative axiomatization sheds an interesting light on the DM's conservatism by highlighting how $\succ$ differs from $\succsim^\dagger$.

In what follows, we take a weak preference $\succsim^\dagger$ over $\mathcal{F}$ as the primitive. Let $\succ^\dagger$ and $\sim^\dagger$ denote the strict and indifferent parts of $\succsim^\dagger$, respectively. That is, $f \succ^\dagger g$ if and only if $f \succsim^\dagger g$ and $g \not\succsim^\dagger f$, and $f \sim^\dagger g$ if and only if $f \succsim^\dagger g$ and $g \succsim^\dagger f$. The first axiom of $\succsim^\dagger$ is a collection of basic assumptions pertaining to weak preferences. In particular, it includes the standard completeness and continuity in Anscombe and Aumann's (1963), whereas independence is assumed for constant acts.

Axiom J1 (Structural Assumptions for Weak Preferences)**.**

- (Completeness)**.** $\succsim^\dagger$ is a complete binary relation over $\mathcal{F}$.
- (Mixture Continuity)**.** For any $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim^\dagger h\}$ and $\{\alpha \in [0,1] : h \succsim^\dagger \alpha f + (1-\alpha)g\}$ are closed relative to $[0,1]$.
- (Risk Independence)**.** For any $x, y, z \in X$ and $\alpha \in (0,1)$, $x \succsim^\dagger y$ if and only if $\alpha x + (1-\alpha)z \succsim^\dagger \alpha y + (1-\alpha)z$.
- (Unboundedness)**.** There exist $x, y \in X$ such that for any $\alpha \in (0,1)$, there exist $\overline{z}, \underline{z} \in X$ for which $\alpha\overline{z} + (1-\alpha)y \succ^\dagger y \succ^\dagger x \succ^\dagger \alpha\underline{z} + (1-\alpha)x$.

As one key departure from expected utility theory, notice that our representation (3) need not satisfy transitivity due to the "thickness" of an indifference region: Observe that any act f such that $\max_{\mu\in\Delta(\Sigma)}\{\int(u \circ f)\mathrm{d}\mu - c^\sharp(\mu)\} > \min_{\mu\in\Delta(\Sigma)}\{\int(u \circ f)\mathrm{d}\mu + c^\flat(\mu)\}$ admits many certainty equivalents, which give rise to a continuum of distinct utility values $u(x)$. This implies the indifference part $\sim^\dagger$ of the DM's preference violates transitivity, i.e., $f \sim^\dagger x$ and $f \sim^\dagger y$, but $x \succ^\dagger y$ when $u(x) > u(y)$. In a sense, just like in Luce's (1956) "coffee and sugar" example where the DM could not perceive a gram difference of sugar, here intransitive indifference is attributed to the DM's inability of perceiving a small difference between uncertain prospects.

In contrast to this, $\succsim^{\dagger}$ would satisfy transitivity when the difference between uncertain prospects is sufficiently large. The next axiom collects several variants of weakened transitivity notions, which will turn out to characterize the representation (3). (As a mnemonic, Q is for **Q**uasi-transitivity, C is for **C**onstant-transitivity, and U is for **U**nambiguous-transitivity, respectively.) We write as $f \succsim^{\dagger}_S g$ when the outcomes of f state-wise dominate those of g, i.e., $f(s) \succsim^{\dagger} g(s)$ for all $s \in S$.

Axiom J2 (Transitivity Assumptions)**.**

- (Q-transitivity)**.** For any $f, g, h \in \mathcal{F}$, if $f \succ^{\dagger} g$ and $g \succ^{\dagger} h$, then $f \succ^{\dagger} h$.
- (C-transitivity)**.** For any $f, g \in \mathcal{F}$ and $x \in X$, if $f \succsim^{\dagger} x$ and $x \succsim^{\dagger} g$, then $f \succsim^{\dagger} g$.
- (U-transitivity)**.** For any $f, g, h \in \mathcal{F}$, if $f \succsim^{\dagger} g$ and $g \succsim^{\dagger}_S h$, then $f \succsim^{\dagger} h$. Also, if $f \succsim^{\dagger}_S g$ and $g \succsim^{\dagger} h$, then $f \succsim^{\dagger} h$.

Two implications from U-transitivity should be emphasized. First it ensures that $\succsim^{\dagger}$ is transitive over the set of constant acts X. Second U-transitivity implies that $\succsim^{\dagger}$ satisfied *monotonicity*, that is, for any $f, g \in \mathcal{F}$, $f \succsim^{\dagger}_S g$ implies that $f \succsim^{\dagger} g$. We omit the trivial proofs of those claims, but the reader is referred to Lemma 1 of Lehrer and Teper (2011).

The next theorem reveals that the axioms **J1** and **J2** are necessary and sufficient for $\succsim^{\dagger}$ to admit a justifiability counterpart of twofold MBA representation. Moreover, if (and only if) the strict part $\succ^{\dagger}$, in addition, satisfies the independence-type axioms presented in Section 3, the associated utility functionals admit variational representations.

Theorem 2. *A preference relation $\succsim^{\dagger}$ satisfies* **J1** *and* **J2** *if and only if there exist $(u, I^{\flat}, I^{\sharp})$, which satisfy all the assumptions in* Lemma 1*, such that for all $f, g \in \mathcal{F}$, $f \succsim^{\dagger} g$ if and only if*

$$I^{\sharp}(u \circ f) \geq I^{\flat}(u \circ f).$$

Moreover, for any such a preference relation $\succsim^{\dagger}$, the following conditions are equivalent:

i) $\succ^{\dagger}$ *satisfies* **A4** *and* **A5***.*

ii) *There exist a surjective affine function $u : X \to \mathbb{R}$ and $c^{\flat}, c^{\sharp} : \Delta(\Sigma) \to [0, \infty]$, which are convex, lower semicontinuous, and jointly grounded, such that for all $f, g \in \mathcal{F}$, $f \succsim^{\dagger} g$ if and only if* (3) *holds. Moreover, u is unique up to positive affine transformations, and $c^{\flat}, c^{\sharp}$ are unique given u fixed.*

5 Connections with Variational Preferences

This section investigates the formal relationships between our twofold variational preferences and other complete preferences studied in the literature. This exercise is motivated by the fact that external factors or events may sometimes force the DM to make a choice even though, according to his own preferences, he does not have an unambiguously preferred option to pick. To address this problem, we follow the analysis of Gilboa et al. (2010) and consider two preference relations $\succ^\star$ and $\succ$. The first relation $\succ^\star$ is a twofold variational preference, interpreted as the preference rankings that are solidly attached into the DM's mind. The second relation $\succ$, modeled as a MBA preference as in Lemma 2, is instead a possible complete extension of $\succ^\star$. It expresses a "breaking-ties" rule the DM may revert to whenever the relation $\succ^\star$ displays indecisiveness. Formally, we say that $\succ$ is a ***complete extension*** of $\succ^\star$ if $\succ$ is a MBA preference over $\mathcal{F}$ such that $\succ|_X \;= \succ^\star|_X$, and that $f \succ^\star g$ implies $f \succ g$ for all $f, g \in \mathcal{F}$.

There are many ways to extend a given incomplete preference to some complete order. In what follows we focus on two specific rules, one pessimistic and one optimistic. Both capture an opposite and extreme attitude towards ambiguity. Formally, we say that an extension $\succ$ of $\succ^\star$ is ***pessimistic*** if for every $f \in \mathcal{F}$ and $x \in X$, $f \not\succ^\star x$ implies $f \not\succ x$. Also, we say that an extension $\succ$ of $\succ^\star$ is ***optimistic*** if for every $f \in \mathcal{F}$ and $x \in X$, $x \not\succ^\star f$ implies $x \not\succ f$. In words, pessimistic extensions cannot *increase* the perceived level of security of any ambiguous acts. Conversely, an optimistic extension cannot *decrease* any perceived level of potential.[13]

In the next proposition, we show that the MMR's variational representation can be obtained when a twofold variational preference is extended pessimistically, provided that we require the extension to maintain at least continuity and monotonicity. Conversely, the dual representation involving a maximax-counterpart of variational preferences will result from optimistic extensions.

Proposition 6. *Let $\succ^\star$ be a twofold variational preference represented by a profile $(u, c^\flat, c^\sharp)$, and let $\succ$ be a MBA preference represented by a profile (v, I).*

i) *If $\succ$ is a pessimistic extension of $\succ^\star$, it is without loss to let $v = u$. Moreover, in that case, the functional $I : B_0(\Sigma) \to \mathbb{R}$ takes the following form:*

$$I(u \circ f) = \min_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu + c^\flat(\mu) \right\}, \; \forall f \in \mathcal{F}.$$

ii) *If $\succ$ is an optimistic extension of $\succ^\star$, it is without loss to let $v = u$. Moreover, in that case, the functional $I : B_0(\Sigma) \to \mathbb{R}$ takes the following form:*

$$I(u \circ f) = \max_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu - c^\sharp(\mu) \right\}, \; \forall f \in \mathcal{F}.$$

Two observations pertaining to Proposition 6 are in order. First, it shows how to derive a variational preference representation a lá MMR without imposing the classic uncertainty aversion axiom of Gilboa and Schmeidler (1989). Intuitively, the monotonicity and continuity properties embedded in $\succ$, the quasi-convexity axiom of $\succ^\star$, and the fact that $\succ^\star$ was extended pessimistically are enough to recover such

[13] Our definition of pessimistic extension is essentially equivalent to the notion of *default to certainty* used by Gilboa et al. (2010). It also resembles the classic comparative statics requirement often found in the literature that $\succ$ *is more ambiguity averse than* $\succ^\star$. Also, it is easy to see that optimistic extensions are the convex duals of the pessimistic extensions. Therefore, loosely speaking, we can interpret an optimistic extension $\succ$ of $\succ^\star$ as requiring that $\succ$ *is more ambiguity-loving than* $\succ^\star$.

an axiom. Also, it is essential to remark that when the DM chooses to extend pessimistically (resp. optimistically) her twofold variational preferences, only the pessimistic cost function $c^\flat$ (resp. optimistic cost function $c^\sharp$) will survive as the ambiguity index in the representation of $\succ$, while the other one will be entirely discarded. This indicates the asymmetry between the DM's different selves can be seen as one defining feature of incompleteness.

6 Extensions

As we have seen, our baseline models, twofold MBA and twofold variational preferences, correspond to the incomplete counterparts of complete preferences studied in CGMMS and MMR, respectively. Our representations utilize possibly different two utility functionals, suggesting an interpretation in terms of the tension between the DM's optimism and pessimism. In this section, we argue that the approaches via twofold representations can also be extended to "intermediate" classes as well, that is, narrower than twofold MBA, but still differ from twofold variational preferences.

In view of Lemma 1, we shall maintain the axioms **A1–A3** as the basic postulates that characterize dual-self perspective of the DM in face of uncertainty. In what follows, we instead try to weaken the additional axioms of **A4** and **A5**, used to characterize twofold variational preferences, to demonstrate that the class of twofold preferences is quite flexible to yield various counterparts of existing *complete* preferences in the literature.

6.1 Relaxing Quasi-convexity

We first relax the axiom **A5**, which we termed quasi-convexity. As such, we seek for an analogue of the *Choquet expected utility* (CEU) representation, introduced by Schmeidler (1989), as it does not depend on any axiom that posits the DM's attitude towards uncertainty. Following his definition, we say that two acts $f, g \in \mathcal{F}$ are ***comonotonic*** if for all $s, s' \in S$ we have $f(s) \succsim f(s')$ if and only if $g(s) \succsim g(s')$. The following is a well-known relaxation of independence.

Axiom A7 (Comonotonic Independence)**.** For any pairwise comonotonic acts $f, g, h \in \mathcal{F}$ and any $\alpha \in (0,1)$, $f \succ g$ if and only if $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

As is well-known, when it comes to complete preferences, Schmeidler's (1989) comonotonic independence implies Gilboa and Schmeidler's (1989) C-independence (together some other appropriate conditions). Indeed, the same is true in the present context as well, provided that $\succ$ satisfies **A1–A3**. This is formally proved as Claim 7 in Appendix B.10. In this sense, our attempt to weaken **A5** is relying on the strengthening of **A4**.

Let us review some standard terminology to introduce our representation. A function $\nu : \Sigma \to [0,1]$ is called a (probability) ***capacity*** if it satisfies $\nu(\emptyset) = 0$, $\nu(S) = 1$, and $\nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$ with $A \subseteq B$. In particular, a capacity ν is ***convex*** (resp. ***concave***) if $\nu(A \cup B) + \nu(A \cap B) \geq$ (resp. $\leq$) $\nu(A) + \nu(B)$ for all $A, B \in \Sigma$. We say that ν is ***conjugate*** to another capacity ν' if $\nu(A) = 1 - \nu'(S \setminus A)$ for all $A \in \Sigma$. Note that the conjugacy is a symmetric relation, and that a capacity is convex if and only if its conjugate is concave. We also remark that ν is a finitely additive probability measure (or, what we called a *belief* in Section 2) if and only if ν is both convex and concave if and only if the conjugate of ν is ν itself.

Definition 4. We say that $\succ$ is a ***twofold CEU preference*** if there exist a surjective affine vNM utility function $u : X \to \mathbb{R}$ and two probability capacities $\nu^\flat, \nu^\sharp : \Sigma \to [0,1]$ with $\nu^\sharp \geq \nu^\flat$ such that for all $f, g \in \mathcal{F}$, $f \succ g$ if and only if

$$\int (u \circ f) \mathrm{d}\nu^\flat > \int (u \circ g) \mathrm{d}\nu^\sharp. \tag{9}$$

Here, the integrals in (9) are performed in the sense of Choquet (1955); see also Schmeidler (1986). This representation is in spirit similar to the *Choquet expected utility* (CEU) representation of Schmeidler (1989), but differs in that it allows the DM to adopt possibly different capacities, each of which may be associated with the pessimistic and optimistic selves.

Theorem 3. *A preference relation $\succ$ over $\mathcal{F}$ satisfies* **A1–A3** *and* **A7** *if and only if it admits a twofold CEU representation by some profile $(u, \nu^\flat, \nu^\sharp)$. Moreover, u is unique up to positive affine transformations, and $\nu^\flat, \nu^\sharp$ are unique.*

We remark that twofold CEU preferences are compatible with **A5** and **A6**, while each of these axioms restricts the class of capacities the DM can adopt. It should also be emphasized that unlike twofold variational preferences, the full version of monotonicity or negative transitivity does not necessarily reduce twofold CEU representation to the SEU representation. Indeed, since we now depart from **A5**, the twofold CEU representation does not put any restriction on the DM's attitude towards uncertainty. As a result, it can encapsulate the Schmeidler's (1989) CEU representation as the special case of when $\nu^\flat = \nu^\sharp$. On the other hand, the SEU representation will emerge once after adding independence. The next proposition collects these observations.

Proposition 7. *Let $\succ$ be a twofold CEU preference represented by $(u, \nu^\flat, \nu^\sharp)$.*

i) *$\succ$ satisfies* **A5** *if and only if $\nu^\flat$ is convex and $\nu^\sharp$ is concave.*

ii) *$\succ$ satisfies* **A6** *if and only if $\nu^\flat$ is conjugate to $\nu^\sharp$.*

iii) *$\succ$ satisfies monotonicity or negative transitivity if and only if $\nu^\flat = \nu^\sharp$, i.e., $\succ$ is a CEU preference. In particular, for a twofold CEU preference, monotonicity and negative transitivity are equivalent.*

iv) *$\succ$ satisfies independence if and only if $\mu \equiv \nu^\flat = \nu^\sharp$ and μ satisfies finite additivity, i.e., $\succ$ is a SEU preference. In particular, for a twofold CEU preference, independence implies both monotonicity and negative transitivity.*

6.2 Relaxing Weak C-independence

Next we move on to the relaxation of **A4**. We first provide an axiomatization by assuming that $\succ$ satisfies only the minimal independence assumption, i.e., risk independence as in **A1**. In the context of complete preferences, the relevant analysis is conducted by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011a) (henceforth, CMMM). The following notion of twofold representation has the exact analog in their results.

Definition 5. We say that $\succ$ is a ***twofold convex preference*** if there exist a surjective affine vNM utility function $u : X \to \mathbb{R}$ and two functions $G^\flat, G^\sharp : \mathbb{R} \times \Delta(\Sigma) \to [-\infty, \infty]$ such that for all $f, g \in \mathcal{F}$, $f \succ g$ if and only if

$$\min_{\mu \in \Delta(\Sigma)} G^\flat \left(\int (u \circ f) \mathrm{d}\mu,\, \mu \right) > \max_{\mu \in \Delta(\Sigma)} G^\sharp \left(\int (u \circ g) \mathrm{d}\mu,\, \mu \right). \tag{10}$$

where $G^\flat$ and $G^\sharp$ satisfy the following assumptions:

- $G^\flat$ is quasiconvex and lower semicontinuous on $\mathbb{R} \times \Delta(\Sigma)$, increasing in the first component, and $\min_{\mu \in \Delta} G^\flat(r, \mu) = r$ for every $r \in \mathbb{R}$;
- $G^\sharp$ is quasiconcave and upper semicontinuous on $\mathbb{R} \times \Delta(\Sigma)$, increasing in the first component, and $\max_{\mu \in \Delta} G^\flat(r, \mu) = r$ for every $r \in \mathbb{R}$; and
- for all $f \in \mathcal{F}$, it holds that

$$\max_{\mu \in \Delta(\Sigma)} G^\sharp \left(\int (u \circ f) \mathrm{d}\mu,\, \mu \right) \geq \min_{\mu \in \Delta(\Sigma)} G^\flat \left(\int (u \circ f) \mathrm{d}\mu,\, \mu \right). \tag{11}$$

Note that twofold variational preferences are nested in twofold convex preferences, as the special case of when cost functions, $c^\flat$ and $c^\sharp$, are incorporated in $G^\flat$ and $G^\sharp$ in an additively separable manner. Indeed, when $\succ$ admits the representation (1), we can write as $G^\flat(r, \mu) = r + c^\flat(\mu)$. As such, it is evident that $G^\flat$ is increasing in r, and convex and lower semicontinuous because so is $c^\flat$. The condition $\min_{\mu \in \Delta} G^\flat(r, \mu)$ is then nothing more than that $c^\flat$ is grounded. Moreover, the condition (11) corresponds to the universal inequality $I^\sharp \geq I^\flat$, which is needed to assure the asymmetry of $\succ$. In particular, this condition is satisfied if there exists at least one $\mu^* \in \Delta(\Sigma)$ such that

$$G^\sharp \left(\int (u \circ f) \mathrm{d}\mu^*,\, \mu^* \right) = G^\flat \left(\int (u \circ f) \mathrm{d}\mu^*,\, \mu^* \right) = \int (u \circ f) \mathrm{d}\mu^* \tag{12}$$

for all $f \in \mathcal{F}$. We think that it is quite natural to expect the converse, i.e., (12) is necessary for (11) as well, but unfortunately, we do not have a proof. Indeed, in the twofold variational representation, notice that the equality (12) holds with the belief $\mu^* \in \{c^\flat = 0\} \cap \{c^\sharp = 0\}$.

The next theorem shows that the above representation is necessary and sufficient when we drop weak C-independence from the twofold variational representation.

Theorem 4. *A preference relation $\succ$ over $\mathcal{F}$ satisfies* **A1–A3** *and* **A5** *if and only if it admits a twofold convex representation by some profile $(u, G^\flat, G^\sharp)$.*

We next consider the subclass of twofold convex preferences where cost functions are incorporated in a multiplicatively separable manner. To this end, we consider the following weakening of C-independence, introduced by CMMM, with respect to a given reference outcome $x_e \in X$. One natural interpretation of x_e is the DM's initial endowment.

Axiom A8 (Homotheticity)**.** There exists $x_e \in X$ such that for any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \succ g$ if and only if $\alpha f + (1-\alpha)x_e \succ \alpha g + (1-\alpha)x_e$.

As the weakening of Gilboa and Schmeidler's (1989) C-independence axiom, homotheticity is somewhat *complementary* to weak C-independence. As such, notice that in **A8**, we can vary the mixture ratio of α in an arbitrary manner, but the mixed constant act x_e must be fixed. This is contrary to **A4**, where the choice of x is free, but α must be fixed. Therefore, in a formal sense, $\succ$ satisfies C-independence if and only if it satisfies both **A4** and **A8**. It is interesting to note that this complementarity between the axioms will be reflected in the representation in the form of whether cost functions are incorporated additively or multiplicatively.

Given any $r \in \mathbb{R}$, we define $[r]_+ = \max\{r, 0\}$ and $[r]_- = \max\{-r, 0\}$. Note that $[r]_+, [r]_- \geq 0$, and that $[-r]_+ = [r]_-$ and $[-r]_- = [r]_+$ hold for every $r \in \mathbb{R}$.

Definition 6. We say that $\succ$ is a ***twofold homothetic preference*** if there exist a surjective affine vNM utility function $u : X \to \mathbb{R}$, two non-empty closed convex sets $C^\flat, C^\sharp \subseteq \Delta(\Sigma)$, and four functions $(m^\flat, m^\sharp, M^\flat, M^\sharp)$ such that for all $f, g \in \mathcal{F}$, $f \succ g$ if and only if

$$\min_{\mu \in C^\flat} \left\{ \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_+}{m^\flat(\mu)} - \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_-}{M^\flat(\mu)} \right\} > \max_{\mu \in C^\sharp} \left\{ \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_+}{M^\sharp(\mu)} - \frac{\left[\int (u \circ g) \mathrm{d}\mu\right]_-}{m^\sharp(\mu)} \right\}, \tag{13}$$

where $(m^\flat, m^\sharp, M^\flat, M^\sharp)$ satisfy the following assumptions: for $i \in \{\flat, \sharp\}$,

- $m^i : C^i \to (0,1]$ is upper semicontinuous, concave and bounded away from 0;
- $M^i : C^i \to [1, \infty]$ is lower semicontinuous and convex; and
- for all $f \in \mathcal{F}$,

$$\max_{\mu \in C^\sharp} \left\{ \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_+}{M^\sharp(\mu)} - \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_-}{m^\sharp(\mu)} \right\} \geq \min_{\mu \in C^\flat} \left\{ \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_+}{m^\flat(\mu)} - \frac{\left[\int (u \circ f) \mathrm{d}\mu\right]_-}{M^\flat(\mu)} \right\}. \tag{14}$$

In the context of complete preferences, the related models are studied by CMMM and Chateauneuf and Faro (2009). In particular, the above representation is exactly the incomplete-counterpart of CMMM's homothetic preferences. Note that Definition 6 is nested in Definition 5, and thus, in Definition 1 as well.[14] Again, the condition (14) is needed to assure the asymmetry of $\succ$, and it is satisfied if there exist some $\mu^* \in C^\flat \cap C^\sharp$ such that $m^\flat(\mu^*) = m^\sharp(\mu^*) = M^\flat(\mu^*) = M^\sharp(\mu^*) = 1$.

Theorem 5. *A preference relation $\succ$ over $\mathcal{F}$ satisfies* **A1–A3***,* **A5***, and* **A8** *if and only if it admits a twofold homothetic representation by some profile $(u, m^\flat, m^\sharp, M^\flat, M^\sharp)$.*

Recall that in the twofold variational representation, i.e., when costs are additively separable, the associated utility functionals satisfy translation invariance, but may fail positive homogeneity. As a result, the DM's choice is potentially affected by betting scale. On the other hand, in the twofold homothetic representation, i.e., when costs are multiplicatively separable, positive homogeneity is satisfied, but translation invariance may fail. This in turn indicates the effect of *initial position*, i.e., the dependence of the DM's choice on where she was initially placed.

7 Summary and Concluding Remarks

This paper has introduced the unified approach for modeling incomplete preferences, which we termed twofold preferences. Compared with the models of unanimity preferences a lá Bewley (2002)–an alternative framework for modeling incomplete preferences–one notable advantage of our approach is the modeling flexibility, which has enabled us to study various incomplete-analogs of the existing complete preferences. We have provided a series of axiomatization results for various twofold preferences that differ in the behavioral content the modeler may wish to postulate on top of the basic axioms common across all. The inclusion relationships among different models can be summarized by Figure 2, where "2-" stands for the abbreviation of twofold.

Where could one locate existing complete preferences or Bewley preferences in this diagram? As for the latter, the intersection with the diagram solely consists of the red circle in the center. This is because the unanimity representation does not maintain our core axiom–contagion of incomparability–unless it exhibits completeness. On the other hand, the most complete preferences mentioned in this paper constitute the subclasses of twofold MBA. However, since our quasi-convexity is generically incompatible with Gilboa and Schmeidler's (1989) uncertainty aversion, our second most general class–twofold convex preferences–even intersect with most of them only at SEU.

However, we think that it is feasible to extend our twofold approach beyond the axiomatizations presented in this paper, so as to seek the "richer" intersections with the existing models. In this regard, our twofold CEU indicates some potential in this direction, as it is free from any assumption about the DM's attitudes towards uncertainty. Therefore, one particularly interesting subject for future research may be the subclass of twofold MBA that does not impose quasi-convexity.[15] In the context of complete preferences, the relevant models include, but are not limited to, invariant biseparable preferences by

[14] For the formal connection, see the footnote 17 in Appendix B.13.

[15] In contrast, it seems more difficult to obtain the intersection with Bewley preferences because we then have to depart from contagion of incomparability. In this regard, the Hara's (2021) representation may be informative for this sort of extensions, as it succeeds in capturing the sensible intersection of MEU and Bewley preferences.

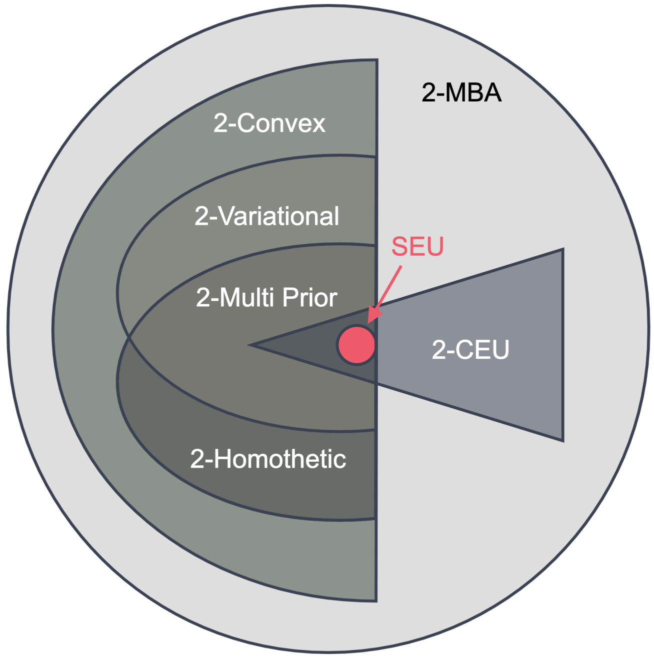

Figure 2: Inclusion relationships among different twofold preferences.

Ghirardato et al. (2004), Amarante (2009), and Chandrasekher et al. (2021), and smooth ambiguity preferences by Klibanoff et al. (2005) and Denti and Pomatto (2021).

The axiomatization of twofold preferences can be partly guided by the existing results in the literature of complete preferences. However, some problems are specific to the present context. Specifically, the representation of a complete preference has no counterpart of the inequality condition $I^\flat \leq I^\sharp$ that regulates the consistency between the DM's different selves. From a practical perspective, it is desirable if we can rewrite this non-parametric condition in terms of modeling parameters. In some cases, the answers to this exercise are feasible, e.g., in the twofold variational representation, the joint groundedness of cost functions serves this purpose. Yet, in other cases, including twofold convex or homothetic preferences, we, unfortunately, do not know what conditions on primitives are necessary and sufficient for having $I^\flat \leq I^\sharp$. Put differently, our questions can be phrased as: What are the minimal conditions on $G^\flat$ and $G^\sharp$ (resp. $m^\flat$, $m^\sharp$, $M^\flat$, and $M^\sharp$) to guarantee $I^\flat \leq I^\sharp$ in Definition 5 (resp. Definition 6)? Mathematically, this question seems related to the problem of generalizing the Fenchel-Rockafeller theorem from convex duality to quasiconvex duality.

Appendix A

A.1 Properties of Functionals

We introduce a few notations that will be used throughout the appendices. For $r \in \mathbb{R}$, let $r\mathbf{1}_S$ denote the constant function that takes r for every $s \in S$. For any sequence $\{r_n\} \subseteq \mathbb{R}$, we write as $r_n \uparrow r \in \mathbb{R}$ when the convergence occurs from below, i.e., $r_1 < r_2 < \cdots$ and $\lim_{n\to\infty} r_n = r$. Likewise, the notation $r_n \downarrow r$ is defined to mean the convergence from above. For functions $a, b \in B_0(\Sigma)$, we write as $a \geq b$ if $a(s) \geq b(s)$ for all $s \in S$, $a \gg b$ if $a(s) > b(s)$ for all $s \in S$, and $a \gneq b$ if $a \geq b$ and $a \neq b$. Given a functional of the form $I : B_0(\Sigma) \to \mathbb{R}$, some fundamental properties for our analysis follows.

- I is ***normalized*** if $I(r\mathbf{1}_S) = r$ for all $r \in \mathbb{R}$.
- I is ***monotonic*** if $I(a) \geq I(b)$ for all $a, b \in B_0(\Sigma)$ with $a \geq b$.
- I is ***translation invariant*** if $I(a + r\mathbf{1}_S) = I(a) + r$ for all $a \in B_0(\Sigma)$ and $r \in \mathbb{R}$.
- I is ***positively homogenous*** if $I(\lambda a) = \lambda I(a)$ for all $a \in B_0(\Sigma)$ and $\lambda > 0$.
- I is ***convex*** if $I(\lambda a + (1-\lambda)b) \leq \lambda I(a) + (1-\lambda)I(b)$ for all $a, b \in B_0(\Sigma)$ and $\lambda \in (0,1)$.
- I is ***concave*** if $I(\lambda a + (1-\lambda)b) \geq \lambda I(a) + (1-\lambda)I(b)$ for all $a, b \in B_0(\Sigma)$ and $\lambda \in (0,1)$.
- I is ***quasiconvex*** if $I(\lambda a + (1-\lambda)b) \leq \max\{I(a), I(b)\}$ for all $a, b \in B_0(\Sigma)$ and $\lambda \in (0,1)$.
- I is ***quasiconcave*** if $I(\lambda a + (1-\lambda)b) \geq \min\{I(a), I(b)\}$ for all $a, b \in B_0(\Sigma)$ and $\lambda \in (0,1)$.
- I is ***lower semicontinuous*** if for any $r \in \mathbb{R}$ and any sequence $\{a_n\} \subseteq B_0(\Sigma)$ such that $a_n \to a \in B_0(\Sigma)$, we have $I(a) \leq r$ whenever $I(a_n) \leq r$ for all $n \in \mathbb{N}$.
- I is ***upper semicontinuous*** if for any $r \in \mathbb{R}$ and any sequence $\{a_n\} \subseteq B_0(\Sigma)$ such that $a_n \to a \in B_0(\Sigma)$, we have $I(a) \geq r$ whenever $I(a_n) \geq r$ for all $n \in \mathbb{N}$.
- I is ***continuous*** if it is both upper and lower semicontinuous.

Note that if I is translation invariant, then it is normalized. In particular, I is called a ***niveloid*** if it is both monotonic and translation invariant. The next lemma, whose proof can be found in Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2014), will be useful to deal with niveloids.

Lemma 4. *A functional I is a niveloid if and only if it is monotonic and $I(\lambda a + (1-\lambda) r\mathbf{1}_S) = I(\lambda a) + (1-\lambda)r$ holds for all $a \in B_0(\Sigma)$, $r \in \mathbb{R}$ and $\lambda \in (0,1)$. Moreover, when I is a niveloid:*

i) *I is convex if and only if $I(a) = I(b)$ implies $I(\lambda a + (1-\lambda)b) \leq I(a)$.*

ii) *I is concave if and only if $I(a) = I(b)$ implies $I(\lambda a + (1-\lambda)b) \geq I(a)$.*

A.2 A Quick Review of Convex Conjugates

In this subsection, we provide some results in the theory of conjugate convex functions, which will be useful to prove the results in the main text.

We adopt the standard mathematical notation in this section. For a normed vector space X, denote by X^* the norm dual of X. Typical elements are written as $x \in X$ or $f \in X^*$, and by $\langle f, x \rangle$ we denote the duality pairing. Remark that these symbols are nothing to do with what we used for (constant) acts in our decision-theoretic framework, but our primary examples are $X = B_0(\Sigma)$ and $X^* = ba(\Sigma)$. Given a subset A of a normed space, we denote by $\bar{A}$ and A° to express the closure and interior of A, respectively.

A function of the form $\varphi : X \to (-\infty, \infty]$ will be central to our discussion. The notions of convexity and (semi-)continuity are defined in analogue with Appendix A.1. We say that φ is ***proper*** if $\{\varphi < \infty\} \neq \emptyset$. For a proper function φ, the ***Fenchel conjugate*** $\varphi^* : X^* \to (-\infty, \infty]$ is defined to be

$$\varphi^*(f) = \sup_{x \in X} \{\langle f, x \rangle - \varphi(x)\}, \forall f \in X^*. \tag{15}$$

Note that φ^* is convex and lower semicontinuous by construction. Indeed, the function $f \mapsto \langle f, x \rangle - \varphi(x)$ is convex and continuous on X^* for each fixed $x \in X$, hence, the upper envelops of these functions must be convex and lower semicontinuous. Likewise, we can define the Fenchel conjugate of φ^*, written as φ^{**}, by iterating the definition (15).

Lemma 5 (Fenchel-Moreau Theorem)**.** *Let $\varphi : X \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function. Then $\varphi^{**} = \varphi$.*

The proof of the Fenchel-Moreau theorem can be found, for example, in pp. 13–14 of Brezis (2010). Another useful tool is the so-called Fenchel-Rockafeller theorem. However, its textbook version is not directly applied to our analysis because its continuity assumption may be violated in some case. Instead, we present and prove an alternative version that fits for our purpose.

Lemma 6 (Fenchel-Rockafeller Theorem)**.** *Let $\varphi, \psi : X \to [0, \infty]$ be two proper, convex and lower semicontinuous functions. Assume that $\overline{\{\varphi < \infty\}} \cap \overline{\{\psi < \infty\}} \neq \emptyset$, and that $\overline{\{\varphi < \infty\}}$ is a compact subset of X. Then,*

$$\inf_{x \in X} \{\varphi(x) + \psi(x)\} = \sup_{f \in X^*} \{-\varphi^*(-f) - \psi^*(f)\}. \tag{16}$$

Proof. Denote by α and β to express the left and right sides of (16). By the definition of Fenchel conjugates, it is evident that $\alpha \geq \beta$. For the converse direction, suppose by contradiction that $\alpha > \beta$. Given any $\bar{r} \in \mathbb{R}$ fixed, we define two subsets $A, B \subseteq X \times \mathbb{R}_+$ as follows:

$$A = \{(x, r) \in X \times \mathbb{R}_+ : \varphi(x) \leq r \leq \bar{r}\},$$
$$B = \{(x, r) \in X \times \mathbb{R}_+ : \psi(x) + r \leq \beta\}.$$

Note that by the assumptions on φ and ψ, both A and B are non-empty, closed and convex. In addition, in holds that

$$A \subseteq \{\varphi \leq \bar{r}\} \times [0, \bar{r}] \subseteq \overline{\{\varphi < \infty\}} \times [0, \bar{r}],$$

where the right-side is compact since the domain of φ is precompact. Hence, as being a closed subset of a compact set, A is compact . Moreover, if there exists $(x, r) \in A \cap B$, then $\varphi(x) \leq r$ and $\psi(x) + r \leq \beta$, so adding these inequalities would yield $\varphi(x) + \psi(x) \leq \beta$, a contradiction to $\alpha > \beta$. Hence, we have $A \cap B = \emptyset$, so we can apply the strong separating hyperplane theorem (Theorem 5.79 in Aliprantis and Border, 2006) to get $f \in X^*$, $k \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$ such that

$$\inf_{(x,r)\in A} \big\{\langle f, x\rangle + kr\big\} \geq c_1 > c_2 \geq \sup_{(x,r)\in B} \big\{\langle f, x\rangle + kr\big\}. \tag{17}$$

Now we claim that $k > 0$. Since $\bar{r}$ is arbitrary, we can take $(x, r) \in A$ with an arbitrarily large r, so $k \geq 0$ must hold. Suppose not, $k = 0$. Given $x \in \overline{\{\varphi < \infty\}} \cap \overline{\{\psi < \infty\}}$, we can find two sequences $\{x_n\} \subseteq \overline{\{\varphi < \infty\}}$ and $\{x'_n\} \subseteq \overline{\{\psi < \infty\}}$, both of which converge to x. When $\bar{r}$ is large enough, $(x_n, r_n) \in A$ with some $r_n \in \mathbb{R}$, for each n. Also, since $\beta < \alpha \leq \infty$, $(x'_n, r'_n) \in B$ with some $r'_n \in \mathbb{R}$, for each n. Thus, (17) implies that $\langle f, x_n\rangle \geq c_1$ and $c_2 \geq \langle f, x'_n\rangle$ for every n, but then, letting $n \to \infty$ yields $c_2 \geq \langle f, x\rangle \geq c_1$, a contradiction. Therefore, we have shown that $k > 0$.

By (17), and by the construction of A, it holds for every $x \in X$ that $\langle f, x\rangle + k\varphi(x) \geq c_1$, or equivalently, $\langle -\frac{f}{k}, x\rangle - \varphi(x) \leq -\frac{c_1}{k}$. This implies that

$$\varphi^*(-f/k) \leq -c_1/k. \tag{18}$$

On the other hand, by (17), and by the construction of B, it holds for every $x \in X$ that $c_2 \geq \langle f, x\rangle + k(\beta - \psi(x))$, or equivalently, $\langle \frac{f}{k}, x\rangle - \psi(x) \leq \frac{c_2}{k} - \beta$. This implies that

$$\psi^*(f/k) \leq c_2/k - \beta. \tag{19}$$

Combining (18) and (19) yields $-\varphi^*(-\frac{f}{k}) - \psi^*(\frac{f}{k}) \geq \frac{c_1}{k} - (\frac{c_2}{k} - \beta) > \beta$, but this contradicts to the definition of β. Therefore, we have shown that $\alpha = \beta$. □

The last tool we present in this section is a minimax theorem. Specifically, the following version is due to Theorem 2 of Fan (1953).

Lemma 7 (Fan's Minimax Theorem)**.** *Let X and Y be convex subsets of two vector spaces, and let $f : X \times Y \to \mathbb{R}$. Suppose that X is compact Hausdorff, and $f(\cdot, y)$ is lower semicontinuous for all $y \in Y$. If f is convex on X and concave on Y, then*

$$\min_{x\in X} \sup_{y\in Y} f(x, y) = \sup_{y\in Y} \min_{x\in X} f(x, y).$$

In our proof of Theorem 1, we will apply the lemma by taking X to be $\Delta(\Sigma)$, endowed with the weak-* topology. Indeed, $\Delta(\Sigma)$ is known to be Hausdorff under this topology, and more, compact by the Banach-Alaoglu theorem (Theorem 6.21 in Aliprantis and Border, 2006).

Appendix B

B.1 Preliminaries for the Proofs of Main Results

B.1.1 Proof of Lemma 1

Necessity. Suppose that a profile $(u, I^\flat, I^\sharp)$ represents $\succ$ as in the statement. We only check mixture continuity since other axioms follow from routine arguments. Fix any $f, g, h \in \mathcal{F}$, and let $\alpha \in [0,1]$ be such that $\alpha f + (1-\alpha)g \succ h$. (If there exists no such α, we are done.) By (4), and by the affinity of u, it follows that

$$I^\flat(\alpha u \circ f + (1-\alpha)u \circ g) > I^\sharp(u \circ h). \tag{20}$$

Notice that the right side is constant in terms of α. Also, observe that

$$\|(\alpha u \circ f + (1-\alpha)u \circ g) - (\alpha_n u \circ f + (1-\alpha_n)u \circ g)\|_\infty = (\alpha - \alpha_n)\,\|u \circ f - u \circ g\|_\infty$$

which tends to 0 as $\alpha_n \to \alpha$. Hence, since $I^\flat$ is lower semicontinuous, there exists a neighborhood of α on which the equation (20) is maintained. The other implication of mixture continuity can be checked similarly by using the upper semicontinuity of $I^\sharp$.

Sufficiency. Suppose that $\succ$ satisfies **A1–A3**. As is discussed in the main text, the first three assumptions in **A1** guarantee that the restriction of $\succ$ to X satisfies all the vNM axioms.[16] Hence, there exists an affine function $u : X \to \mathbb{R}$ such that $x \succ y$ if and only if $u(x) > u(y)$ for all $x, y \in X$, cf. Kreps (1988, Theorem 5.11). Moreover, the last assumption of unboundedness implies $u(X) = \mathbb{R}$.

Claim 1. *For any $f \in \mathcal{F}$, the subsets $\{x : f \succ x\}$, $\{x : f \not\succ x\}$, $\{x : x \succ f\}$, and $\{x : x \not\succ f\}$ of X are non-empty.*

Proof. Since f is simple and since $\succsim$ is a weak order, we can pick the best outcome $\overline{x} \in f(S)$ in the image of f. That is, $\overline{x} \succsim f(s)$ for all $s \in S$, so **A2** implies that $\overline{x}$ is more secure than f. It follows that $f \not\succ \overline{x}$ since $\overline{x} \not\succ \overline{x}$. Moreover, since u is unbounded, we can pick some x for which $u(x) > u(\overline{x})$, or $x \succ \overline{x}$. Since $\overline{x}$ has more potential than f, it follows that $x \succ f$. The rest can be proved similarly. □

[16] More precisely, we would say $\succsim$ over X defined by $x \succsim y$ if and only if $y \not\succ x$ satisfies completeness, transitivity, independence, and mixture continuity (defined in terms of "closed" sets). Indeed, the completeness and transitivity of $\succsim$ follow from the asymmetry and negative transitivity of $\succ$, while the latter two postulates are trivially obtained from the independence and mixture continuity of $\succ$.

Define two utility functions $U^\flat, U^\sharp : \mathcal{F} \to \mathbb{R}$ by

$$U^\flat(f) = \sup\{u(x) : f \succ x\} \text{ and } U^\sharp(f) = \inf\{u(x) : x \succ f\}. \tag{21}$$

Notice that they are well-defined real-valued functions by Claim 1. Note also that $U^\sharp(x) = U^\flat(x) = u(x)$ holds for each $x \in X$. Then, let us show that $(U^\flat, U^\sharp)$ jointly represent $\succ$ in the sense of (4). Suppose that $U^\flat(f) > U^\sharp(g)$. By (21), we have $\sup\{u(x) : f \succ x\} > \inf\{u(x) : x \succ g\}$, so there exist some $x, y \in X$ such that $f \succ x$, $y \succ g$, and $u(x) > u(y)$. Thus, transitivity yields $f \succ g$. The converse direction relies on the next observation.

Claim 2. *For any $f, g \in \mathcal{F}$ with $f \succ g$, there exists $x \in X$ such that $f \succ x \succ g$.*

Proof. By Claim 1, we can take some $\overline{x}, \underline{x} \in X$ such that $\overline{x} \succ f \succ g \succ \underline{x}$. For notational simplicity, we write as $x_\alpha = \alpha\overline{x} + (1-\alpha)\underline{x}$ for each $\alpha \in [0,1]$. We then define $A = \{\alpha \in [0,1] : f \succ x_\alpha\}$ and $B = \{\alpha \in [0,1] : x_\alpha \succ g\}$. By mixture continuity, they are open relative to $[0,1]$. Also, they are non-empty since $0 \in A$ and $1 \in B$. Let $\alpha \notin A$, i.e., $f \not\succ x_\alpha$. If $x_\alpha \succ f$, transitivity yields $x_\alpha \succ g$, so $\alpha \in B$. Otherwise, we have $f \bowtie x_\alpha$, so $f \succ g$ and **A3** imply that g and x_α are comparable. If $g \succ x_\alpha$, transitivity yields $f \succ x_\alpha$, a contradiction to our assumption. Hence, it follows that $x_\alpha \succ g$, or $\alpha \in B$. Therefore, we obtain $A \cup B = [0,1]$. The connectedness of $[0,1]$ implies that there exists some $\alpha^* \in A \cap B$, whence $f \succ x_{\alpha^*} \succ g$ holds by construction. □

Suppose that $f \succ g$. By Claim 2, there exists some $x \in X$ for which $f \succ x \succ g$. By construction, it follows that $U^\flat(f) \geq u(x) \geq U^\sharp(g)$, so our remaining task is to prove strictness. Suppose not, $U^\flat(f) = u(x)$. By Claim 1, there exists $y \in X$ such that $y \succ f \succ x$. Then, observe that $A = \{\alpha \in [0,1] : f \succ (1-\alpha)x + \alpha y\}$ is open relative to $[0,1]$. In particular, since $0 \in A$, we have $f \succ (1-\alpha)x + \alpha y$ for a small $\alpha > 0$. On the other hand, the affinity of u implies that $u((1-\alpha)x + \alpha y) > u(x) = U^\flat(f)$, which contradicts to the construction of $U^\flat$. Hence, it follows that $U^\flat(f) > U^\sharp(g)$, and therefore, we have confirmed that $(U^\flat, U^\sharp)$ jointly represent $\succ$.

We convert $(U^\sharp, U^\flat)$ into the functionals on utility acts. Notice that $B_0(\Sigma) = \{u \circ f : f \in \mathcal{F}\}$ since $u(X) = \mathbb{R}$. Then, define $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ by

$$I^\flat(a) = U^\flat(f) \text{ and } I^\sharp(a) = U^\sharp(f) \text{ where } f \in \mathcal{F} \text{ and } u \circ f = a. \tag{22}$$

These functionals are well-defined. Indeed, if $u \circ f = u \circ g$, then $f(s) \sim g(s)$ for all $s \in S$. Thus, **A2** implies that f is as secure as g, from which $U^\flat(f) = U^\flat(g)$ follows. Also, **A2** implies that f has as much potential as g, hence $U^\sharp(f) = U^\sharp(g)$ holds as well. Therefore, the profile $(u, I^\flat, I^\sharp)$ provides the desired representation, as $(U^\flat, U^\sharp)$ have represented $\succ$.

Our remaining task is to check the properties of $(I^\flat, I^\sharp)$. As such, monotonicity follows immediately from **A2**, and normalization comes from the fact that $U^\flat(x) = U^\sharp(x) = u(x)$ for each $x \in X$. Let us show that $I^\sharp \geq I^\flat$. Suppose not, there exists some $a \in B_0(\Sigma)$ for which $I^\flat(a) > I^\sharp(a)$, and let $f \in \mathcal{F}$ be such that $u \circ f = a$. Since $(u, I^\flat, I^\sharp)$ represent $\succ$, it follows that $f \succ f$, which violates asymmetry. Therefore, we must have $I^\sharp \geq I^\flat$.

Finally, we shall establish the semicontinuity properties. Take any $a, b \in B_0(\Sigma)$ and $\lambda \in \mathbb{R}$ with $a \geq b$ and $I^\flat(a) > \lambda$. We consider the following two cases.

- $I^\flat(b) > \lambda$. In this case, $\alpha a + (1-\alpha)b \geq b$ holds for all $\alpha \in (0,1)$. Thus, monotonicity implies that $I^\flat(\alpha a + (1-\alpha)b) \geq I^\flat(b) > \lambda$.

- $I^\flat(b) \leq \lambda$. Since $b \in B_0(\Sigma)$, we can take some $\delta \in \mathbb{R}$ such that $b(s) \geq \delta$ for all $s \in S$. Moreover, since $I^\flat(a) > \lambda$, we can pick some $\lambda' \in \mathbb{R}$ such that $I^\flat(a) > \lambda' > \lambda \geq I^\flat(b) \geq \delta$. By $u(X) = \mathbb{R}$, there exist $x, y \in X$ for which $u(x) = \delta$ and $u(y) = \lambda'$. Also, let $f \in \mathcal{F}$ be such that $u \circ f = a$. By construction, $f \succ y \succ x$ holds. Thus, by mixture continuity, we have $(1-\alpha)f + \alpha x \succ y$ for a small $\alpha \in (0,1)$. Since $I^\flat$ is monotone, and since u is affine, it follows that

$$\begin{aligned} I^\flat(\alpha a + (1-\alpha)b) &\geq I^\flat(\alpha a + (1-\alpha)\delta \mathbf{1}_S) \\ &= I^\flat(u \circ ((1-\alpha)f + \alpha x)) > I^\sharp(u \circ y) = u(y) = \lambda' > \lambda. \end{aligned}$$

Hence, $I^\flat$ satisfies the condition (iv) in Lemma 42 of Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio (2011), thereby it is lower semicontinuous. A symmetric argument shows that $I^\sharp$ is upper semicontinuous.

Uniqueness. By the standard argument, u is unique up to positive affine transformations. Now fix any u, and consider profiles $(u, I^\flat, I^\sharp)$ and $(u, J^\flat, J^\sharp)$ that represent the same preference order $\succ$. Suppose not, $(I^\flat, I^\sharp) \neq (J^\flat, J^\sharp)$, so there exists $f \in \mathcal{F}$ for which either $I^\flat(u \circ f) \neq J^\flat(u \circ f)$ or $I^\sharp(u \circ f) \neq J^\sharp(u \circ f)$. Specifically, assume that $I^\flat(u \circ f) > J^\flat(u \circ f)$ without loss. Since u is surjective, we can pick $x \in X$ for which $I^\flat(u \circ f) > u(x) > J^\flat(u \circ f)$. However, since $(u, I^\flat, I^\sharp)$ and $(u, J^\flat, J^\sharp)$ represent $\succ$, and since $i^\sharp$ and $J^\sharp$ are normalized, it follows that $f \succ x$ and $f \not\succ x$, a contradiction. *Q.E.D.*

B.1.2 Proof of Lemma 3

Let $\succ$ be a twofold MBA preference represented by a profile $(u, I^\flat, I^\sharp)$. Clearly, if $I^\flat = I^\sharp$, then $\succ$ satisfies negative transitivity and monotonicity, cf. Lemma 2. Furthermore, we claim that when this inequality is violated, i.e., if there exists some $a \in B_0(\Sigma)$ such that $I^\sharp(a) > I^\flat(a)$, then $\succ$ violates both negative transitivity and monotonicity. Let $f \in \mathcal{F}$ be such that $u \circ f = a$. Since u is surjective, we can take $x, y \in X$ for which $I^\sharp(a) > u(x) > u(y) > I^\flat(a)$. It follows that $x \not\succ f$, $f \not\succ y$, but $x \succ y$. This implies the violation of negative transitivity. Moreover, since $I^\sharp$ is upper semicontinuous and monotonic, we can take $\epsilon > 0$ small enough that $I^\sharp(a) \geq I^\sharp(a - \epsilon \mathbf{1}_S) > I^\flat(a)$. If we let g be such that $u \circ g = a - \epsilon \mathbf{1}_S$, then $f(s) \succ g(s)$ for every $s \in S$, but the previous inequalities imply $f \not\succ g$. Therefore, $\succ$ violates monotonicity.

To sum up, the above arguments show that the following three conditions are equivalent for twofold

MBA preferences: (i) $\succ$ satisfies monotonicity, (ii) $\succ$ satisfies negative transitivity, and (iii) the representation of $\succ$ satisfies $I^\flat = I^\sharp$. The desired conclusion now follows. *Q.E.D.*

B.1.3 From Preference Axioms to Functional Properties

Given Lemma 1 at hand, we subsequently provide a few more preliminary lemmas to clarify how the axioms on $\succ$ are translated to additional properties of $I^\flat$ and $I^\sharp$. Throughout this subsection, assume that $\succ$ satisfies **A1–A3**, and hence, it admits the twofold MBA representation by $(u, I^\flat, I^\sharp)$. Specifically, the functionals $I^\flat, I^\sharp : \mathcal{F} \to \mathbb{R}$ are constructed as in (21), and they are normalized, monotonic, and lower/upper-semicontinuous, respectively. In what follows, we only prove the properties of $I^\flat$, since symmetric arguments can be applied to $I^\sharp$ as well.

Lemma 8. *If $\succ$ satisfies* **A4***, then $I^\flat$ and $I^\sharp$ are niveloids, i.e., monotonic and translation invariant.*

Proof. Fix any $a \in B_0(\Sigma)$, $r \in \mathbb{R}$, and $\lambda \in (0,1)$. By Lemma 4, it is enough to show that $I^\flat(\lambda a + (1-\lambda) r \mathbf{1}_S) = I^\flat(\lambda a) + (1-\lambda) r$. Since u is surjective, we can take $f \in \mathcal{F}$ and $x \in X$ so that $u \circ f = a$ and $u(x) = r$, as well as a sequence $\{x_n\} \subseteq X$ such that $u(x_n) \uparrow \frac{1}{\lambda} I^\flat(\lambda a)$. Also, let $x_0 \in X$ be such that $u(x_0) = 0$. Since $I^\flat(\lambda a) > u(\lambda x_n)$ by the affinity of u, the definitions of a and x_0 imply $\lambda f + (1-\lambda) x_0 \succ \lambda x_n + (1-\lambda) x_0$. Then, **A4** implies $\lambda f + (1-\lambda) x \succ \lambda x_n + (1-\lambda) x$. Again by the affinity of u, it follows that

$$I^\flat(\lambda a + (1-\lambda) r \mathbf{1}_S) > \lambda u(x_n) + (1-\lambda) u(x) \to I^\flat(\lambda a) + (1-\lambda) r,$$

where the convergence is due to the construction of the sequence $\{x_n\}$. Hence, we obtain $I^\flat(\lambda a + (1-\lambda) r \mathbf{1}_S) \geq I^\flat(\lambda a) + (1-\lambda) r$. As for the converse, just notice that this inequality holds for all $a \in B_0(\Sigma)$, $r \in \mathbb{R}$, and $\lambda \in (0,1)$. So, we have

$$I^\flat(\lambda a) = I^\flat(\lambda a + (1-\lambda) r \mathbf{1}_S - (1-\lambda) r \mathbf{1}_S) \leq I^\flat(\lambda a + (1-\lambda) r \mathbf{1}_S) - (1-\lambda) r,$$

from which the desired opposite inequality is obtained. This concludes the proof. □

Lemma 9. *If $\succ$ satisfies* **A4** *and* **A5***, then $I^\flat$ is concave, $I^\sharp$ is convex, and both are continuous.*

Proof. Fix any $a, b \in B_0(\Sigma)$ and $\lambda \in (0,1)$, and let $f, g \in \mathcal{F}$ be acts such that $a = u \circ f$ and $b = u \circ g$. Assume that $I^\flat(a) = I^\flat(b)$. Consider any sequences $\{x_n\} \subseteq X$ such that $u(x_n) \uparrow I^\flat(a)$. By construction, $f \succ x_n$ and $g \succ x_n$ for all n. Thus, axiom **A5** implies that $\lambda f + (1-\lambda) g \succ x_n$. So, it follows that $I^\flat(\lambda a + (1-\lambda) b) > u(x_n)$ for all $n \in \mathbb{N}$. Thus, letting $n \to \infty$ yields $I^\flat(\lambda a + (1-\lambda) b) \geq I^\flat(a)$. Since $I^\flat$ is a niveloid, this shows that $I^\flat$ is concave by Lemma 4. Finally, since $I^\flat$ is monotonic, any point $a \in B_0(\Sigma)$ has a neighborhood on which $I^\flat$ is bounded. The continuity then follows from Theorem 5.42 in Aliprantis and Border (2006). □

Lemma 10. *If $\succ$ satisfies* **A5***, then $I^\flat$ is quasi-concave and $I^\sharp$ is quasi-convex.*

Proof. Let all the variables be given as in the proof of Lemma 9, except that we assume $I^\flat(a) \geq I^\flat(b)$ without loss of generality, and that $u(x_n) \uparrow I^\flat(b)$. By construction, $I^\flat(u \circ f) > I^\flat(u \circ g) > u(x_n)$ holds, so we have $f \succ x_n$ and $g \succ x_n$ for all n. By **A5**, it follows that $\alpha f + (1-\alpha)g \succ x_n$, whence $I^\flat(\alpha a + (1-\alpha)b) > u(x_n)$ holds. Hence, letting $n \to \infty$ yields $I^\flat(\alpha a + (1-\alpha)b) \geq I^\flat(b)$, where $I^\flat(b) = \min\{I^\flat(a), I^\flat(b)\}$. So, $I^\flat$ is quasi-concave. □

Lemma 11. *If $\succ$ satisfies* **A8***, then $I^\flat$ and $I^\sharp$ are positively homogeneous.*

Proof. Fix any $a \in B_0(\Sigma)$ and $\lambda > 0$, and let $f \in \mathcal{F}$ be such that $u \circ f = a$. The claim is trivial when $\lambda = 1$. Also, notice that it is without loss to assume that $\lambda \in (0,1)$. Indeed, if we could prove the claim $\lambda \in (0,1)$, it follows for any $\lambda' > 1$ that $I^\flat(a) = I^\flat(\frac{1}{\lambda'} \cdot \lambda')a = \frac{1}{\lambda'} I^\flat(\lambda' a)$, whence positive homogeneity is confirmed for $\lambda' > 1$ as well.

Let $\{x_n\} \subseteq X$ be a sequence such that $u(x_n) \uparrow I^\flat(a)$. Since $a = u \circ f$, it follows that $f \succ x_n$ for all n, thereby **A8** implies that $\lambda f + (1-\lambda)x_e \succ \lambda x_n + (1-\lambda)x_e$ for all n. By this and $u(x_e) = 0$, we see that

$$I^\flat(\lambda a) = I^\flat(u \circ (\lambda f + (1-\lambda)x_e)) > u(\lambda x_n + (1-\lambda)x_e) = \lambda u(x_n),$$

from which letting $n \to \infty$ yields $I^\flat(\lambda a) \geq \lambda I^\flat(a)$. For the converse direction, let $\{y_n\} \subseteq X$ be a sequence such that $u(y_n) \uparrow \frac{1}{\lambda} I^\flat(\lambda a)$. Again, since $u(x_e) = 0$, it follows that $u(\lambda y_n + (1-\lambda)x_e) < I^\flat(u \circ (\lambda f + (1-\lambda)x_e))$, which means that $\lambda f + (1-\lambda)x_e \succ \lambda y_n + (1-\lambda)x_e$ for all n. Hence, **A8** implies that $f \succ y_n$, or $I^\flat(a) > u(y_n)$ holds for all n. By the construction of $\{y_n\}$, letting $n \to \infty$ leads to $I^\flat(a) \geq \frac{1}{\lambda} I^\flat(\lambda a)$, or $\lambda I^\flat(a) \geq I^\flat(\lambda a)$ as desired. □

Lemma 12. *Assume that $\succ$ satisfies* **A4***. Then, $\succ$ satisfies* **A6** *if and only if $I^\flat(a) = -I^\sharp(-a)$ holds for all $a \in B_0(\Sigma)$.*

Proof. Suppose that $I^\sharp(a) = -I^\flat(-a)$ holds for all $a \in B_0(\Sigma)$. Consider any complementary acts $f, g \in \mathcal{F}$ such that $f \succ g$, and let $a = u \circ f$ and $b = u \circ g$. By the definition of complementarity, we have $\frac{1}{2}a + \frac{1}{2}b = r\mathbf{1}_S$ for some $r \in \mathbb{R}$. Equivalently, $-b = a - 2r\mathbf{1}_S$ holds. Moreover, since $\succ$ satisfies **A4**, $I^\flat$ is translation invariance by Lemma 8. Hence, it follows that

$$r - I^\sharp(b) = r + I^\flat(-b) = r + I^\flat(a - 2r\mathbf{1}_S) = I^\flat(a) - r.$$

This implies $f \succ \frac{1}{2}f + \frac{1}{2}g$ if and only if $\frac{1}{2}f + \frac{1}{2}g \succ g$, whereas at least one of these relations always holds since $f \succ g$, and since $\succ$ satisfies **A3**. Therefore, $\succ$ satisfies **A6**.

For the converse direction, we want to show that $\succ$ violates **A6** if either $I^\flat(a) < -I^\sharp(-a)$ or $I^\sharp(a) > -I^\flat(-a)$ for some $a \in B_0(\Sigma)$. Since arguments are symmetric, here we just focus on the former case. Thus, assume that there exists $a \in B_0(\Sigma)$ such that

$$I^\flat(a) < \frac{1}{2}\left(I^\flat(a) - I^\sharp(-a)\right) \equiv k.$$

Let $\epsilon > 0$ be arbitrarily small, and define $b = -a + 2(k-\epsilon)\mathbf{1}_S$. It follows that

$$\begin{aligned} I^\flat(a) - I^\sharp(b) &= I^\flat(a) - I^\sharp(-a + 2(k-\epsilon)\mathbf{1}_S) \\ &= I^\flat(a) - I^\sharp(-a) - 2(k-\epsilon) \\ &= 2\epsilon, \end{aligned}$$

where the second line uses translation invariance, and the last line is due to the definition of k. Hence, we have $f \succ g$, where $f, g \in \mathcal{F}$ are such that $u \circ f = a$ and $u \circ g = b$. In particular, since $\frac{1}{2}a + \frac{1}{2}b = (k-\epsilon)\mathbf{1}_S$ by construction, f and g are complementary. Moreover, the definition of k implies $I^\sharp(\frac{1}{2}a + \frac{1}{2}b) = k - \epsilon \geq I^\flat(a)$ when ϵ is small enough. Therefore, $f \not\succ \frac{1}{2}f + \frac{1}{2}g$, and the axiom **A6** is falsified. □

B.2 Proof of Theorem 1

In the main text, we already discussed the necessity direction, so we only proves the sufficiency direction. Suppose that $\succ$ satisfies **A1**–**A5**. By Lemma 1, we know that $\succ$ can be represented by some profile $(u, I^\flat, I^\sharp)$, where $u : X \to \mathbb{R}$ is surjective and affine, and $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ are normalized and monotonic functionals such that $I^\sharp \geq I^\flat$. Moreover, in the presence of **A4** and **A5**, Lemmas 8 and 9 imply that $I^\flat$ and $I^\sharp$ are translation invariant, continuous, and concave/convex, respectively.

The key step is to construct the cost functions $c^\flat, c^\sharp$ as the convex conjugates associated to the functionals $I^\flat, I^\sharp$ (see Appendix A.2 for definitions and relevant mathematical results). Specifically, we define the function $c^\sharp : ba(\Sigma) \to \mathbb{R} \cup \{\infty\}$ to be the Fenchel conjugate of $I^\sharp$, that is,

$$c^\sharp(\mu) = \sup_{a \in B_0(\Sigma)} \left\{ \int a \mathrm{d}\mu - I^\sharp(a) \right\}, \ \forall \mu \in ba(\Sigma).$$

Note that $c^\sharp$ is convex and lower semicontinuous. Moreover, since $I^\sharp$ is convex and continuous, the Fenchel-Moreau theorem implies

$$I^\sharp(a) = \sup_{\mu \in ba(\Sigma)} \left\{ \int a \mathrm{d}\mu - c^\sharp(\mu) \right\}, \ \forall a \in B_0(\Sigma).$$

For what concerns $I^\flat$, we define $c^\flat$ to be the Fenchel conjugate of the auxiliary functional $\tilde{I}^\flat : a \mapsto -I^\flat(-a)$. Namely, we let

$$c^\flat(\mu) = \sup_{a \in B_0(\Sigma)} \left\{ \int a \mathrm{d}\mu + I^\flat(-a) \right\} = \sup_{a \in B_0(\Sigma)} \left\{ I^\flat(a) - \int a \mathrm{d}\mu \right\}, \ \forall \mu \in ba(\Sigma).$$

Again, note that $c^\flat$ is convex and lower semicontinuous. Moreover, since $\tilde{I}^\flat$ is convex and continuous, we can apply the Fenchel-Moreau theorem again to conclude that

$$I^\flat(a) = -\tilde{I}^\flat(-a) = -\sup_{\mu\in ba(\Sigma)} \left\{ \int (-a)\mathrm{d}\mu - c^\flat(\mu) \right\} = \inf_{\mu\in ba(\Sigma)} \left\{ \int a\mathrm{d}\mu + c^\flat(\mu) \right\}, \forall a \in B_0(\Sigma).$$

It follows that $c^\flat(\mu), c^\sharp(\mu) \geq 0$ hold for all $\mu \in \Delta(\Sigma)$ because $I^\flat$ and $I^\sharp$ are normalized. We next show that when expressing $I^\flat$ and $I^\sharp$ in terms of their convex conjugates $c^\flat, c^\sharp$, it is without loss of generality to focus on optimizing over $\Delta(\Sigma)$ rather than the whole $ba(\Sigma)$.

Claim 3. *If either μ is not positive or $\mu(S) \neq 1$, then $c^\flat(\mu), c^\sharp(\mu) = \infty$. Consequently, it holds for any $a \in B_0(\Sigma)$ that*

$$\begin{aligned} I^\flat(a) &= \inf_{\mu\in ba(\Sigma)} \left\{ \int a\mathrm{d}\mu + c^\flat(\mu) \right\} = \min_{\mu\in\Delta(\Sigma)} \left\{ \int a\mathrm{d}\mu + c^\flat(\mu) \right\}, \\ I^\sharp(a) &= \sup_{\mu\in ba(\Sigma)} \left\{ \int a\mathrm{d}\mu - c^\sharp(\mu) \right\} = \max_{\mu\in\Delta(\Sigma)} \left\{ \int a\mathrm{d}\mu - c^\sharp(\mu) \right\}. \end{aligned}$$

Proof. Since $I^\flat$ is normalized, we have $c^\flat(\mu) \geq r(1-\mu(S))$ for every $r \in \mathbb{R}$. It follows that $c^\flat(\mu) = \infty$ if $\mu(S) \neq 1$. Similarly, since $I^\sharp$ is normalized, we have $c^\sharp(\mu) \geq r(\mu(S)-1)$ for every $r \in \mathbb{R}$, from which we obtain $c^\sharp(\mu) = \infty$ if $\mu(S) \neq 1$. Next consider any $\mu \in ba(\Sigma)$ such that $\mu(E) < 0$ for some $E \in \Sigma$. Let $a = r\mathbf{1}_E$ and $b = -r\mathbf{1}_E$ for $r > 0$. Since $I^\flat, I^\sharp$ are monotonic, we know that $I^\flat(a) \geq 0 \geq I^\sharp(b)$. Hence, it follows that

$$\begin{aligned} c^\flat(\mu) &\geq I^\flat(a) - \int a\mathrm{d}\mu \geq -r\mu(E), \\ c^\sharp(\mu) &\geq \int b\mathrm{d}\mu - I^\sharp(b) \ \geq -r\mu(E), \end{aligned}$$

but since r is arbitrary and $\mu(E) < 0$, we must have $c^\flat(\mu), c^\sharp(\mu) = \infty$.

Notice that given a fixed $a \in B_0(\Sigma)$, the mappings $\mu \mapsto \int a\mathrm{d}\mu + c^\flat(\mu)$ and $\mu \mapsto \int a\mathrm{d}\mu - c^\sharp(\mu)$ are lower/upper semicontinuous, respectively. Therefore, the desired expressions for $I^\flat$ and $I^\sharp$ are obtained from the previous results, together with the fact that $\Delta(\Sigma)$ is weak-* compact. □

The following lemma summarizes the discussion obtained so far.

Lemma 13. *Let $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ be continuous niveloids. If $I^\flat$ is concave and $I^\sharp$ is convex, then*

$$I^\flat(a) = \min_{\mu\in\Delta(\Sigma)} \left\{ \int a\mathrm{d}\mu + c^\flat(\mu) \right\} \quad \textit{and} \quad I^\sharp(a) = \max_{\mu\in\Delta(\Sigma)} \left\{ \int a\mathrm{d}\mu - c^\sharp(\mu) \right\},$$

where $c^\flat, c^\sharp : \Delta(\Sigma) \to [0,\infty]$ are convex and lower semicontinuous functions given by

$$c^\flat(\mu) = \sup_{a\in B_0(\Sigma)} \left\{ I^\flat(a) - \int a\mathrm{d}\mu \right\} \quad \textit{and} \quad c^\sharp(\mu) = \sup_{a\in B_0(\Sigma)} \left\{ \int a\mathrm{d}\mu - I^\sharp(a) \right\}.$$

Our remaining task is to show that $c^\flat$ and $c^\sharp$ are jointly grounded, i.e., the existence of a belief $\mu^* \in \Delta(\Sigma)$ such that $c^\flat(\mu^*) = c^\sharp(\mu^*) = 0$. To this end, recall that the Fenchel conjugate of $c^\sharp$ is given by $I^\sharp$, and that of $c^\flat$ is given by $\tilde{I}^\flat : a \mapsto -I^\flat(-a)$. Also, note that $\overline{\{c^\flat < \infty\}}$ is a weak-* compact subset of $ba(\Sigma)$ because of Claim 3. Next, we prove that the essential domains of $c^\flat$ and $c^\sharp$ intersect.

Claim 4. *Both* $\{c^\flat = 0\}$ *and* $\{c^\sharp = 0\}$ *are non-empty. Moreover,* $\overline{\{c^\flat < \infty\}} \cap \overline{\{c^\sharp < \infty\}} \neq \emptyset$.

Proof. For the first part, we focus on $c^\flat$ because arguments are symmetric for $c^\sharp$. Since $c^\flat$ take only non-negative values and is lower semicontinuous, and since $\Delta(\Sigma)$ is weak-* compact, it is enough to show that $\min_{\mu\in\Delta(\Sigma)} c^\flat(\mu) \leq 0$. To this end, consider the mapping $(a,\mu) \mapsto I^\flat(a) - \int a \mathrm{d}\mu$ defined on the convex set $B_0(\Sigma) \times \Delta(\Sigma)$. Clearly, this mapping is continuous in both components, concave in a, and convex in μ. Therefore, together with the formula in Lemma 13, the application of Lemma 7 yields

$$\min_{\mu\in\Delta(\Sigma)} c^\flat(\mu) = \min_{\mu\in\Delta(\Sigma)} \sup_{a\in B_0(\Sigma)} \left\{ I^\flat(a) - \int a \mathrm{d}\mu \right\} = \sup_{a\in B_0(\Sigma)} \underbrace{\min_{\mu\in\Delta(\Sigma)} \left\{ I^\flat(a) - \int a \mathrm{d}\mu \right\}}_{\equiv \varphi(a)}. \tag{23}$$

We claim that $\varphi(a) \leq 0$ for every $a \in B_0(\Sigma)$. Let $M \equiv \max_{s\in S} a(s)$. Since $I^\flat$ is normalized and monotonic, we have $M \geq I^\flat(a)$. Now let μ be any belief supported on the set $\{s \in S : a(s) = M\}$. It follows that

$$\varphi(a) \leq I^\flat(a) - \int a \mathrm{d}\mu = I^\flat(a) - M \leq 0.$$

Since $a \in B_0(\Sigma)$ is arbitrary, we conclude that $\min_{\mu\in\Delta(\Sigma)} c^\flat(\mu) \leq 0$ from (23).

Let us now turn to the second statement of the claim. Suppose not, $\overline{\{c^\flat < \infty\}} \cap \overline{\{c^\sharp < \infty\}} = \emptyset$ holds. Note that $\overline{\{c^\flat < \infty\}}$ and $\overline{\{c^\sharp < \infty\}}$ are weak-* compact, convex, and non-empty because of the previous argument. Therefore, we can apply the strong separating hyperplane theorem to claim that there exists $a \in B_0(\Sigma)$ such that

$$\min\left\{ \int a \mathrm{d}\mu : \mu \in \overline{\{c^\flat < \infty\}} \right\} > \max\left\{ \int a \mathrm{d}\mu : \mu \in \overline{\{c^\sharp < \infty\}} \right\}.$$

By Lemma 13, it then follows that

$$\begin{aligned} I^\flat(a) = \min_{\mu\in\Delta(\Sigma)} \left\{ \int a \mathrm{d}\mu + c^\flat(\mu) \right\} &\geq \min\left\{ \int a \mathrm{d}\mu : \mu \in \overline{\{c^\flat < \infty\}} \right\} \\ &> \max\left\{ \int a \mathrm{d}\mu : \mu \in \overline{\{c^\sharp < \infty\}} \right\} \geq \max_{\mu\in\Delta(\Sigma)} \left\{ \int a \mathrm{d}\mu - c^\sharp(\mu) \right\} = I^\sharp(a), \end{aligned}$$

which is a contradiction to $I^\sharp \geq I^\flat$. □

We are now ready to apply Lemma 6 to two convex functions $c^\flat, c^\sharp$ to conclude that

$$\inf_{\mu\in ba(\Sigma)} \left\{ c^\flat(\mu) + c^\sharp(\mu) \right\} = \sup_{a\in B_0(\Sigma)} \left\{ I^\flat(a) - I^\sharp(a) \right\}.$$

Observe that the left-side is non-negative, while the right-side is non-positive because $I^\sharp \geq I^\flat$. Therefore, together with Claim 3, it follows that

$$\inf_{\mu\in ba(\Sigma)} \left\{ c^\flat(\mu) + c^\sharp(\mu) \right\} = \min_{\mu\in\Delta(\Sigma)} \left\{ c^\flat(\mu) + c^\sharp(\mu) \right\} = 0.$$

Since $c^\flat, c^\sharp$ are lower semicontinuous, and since $\Delta(\Sigma)$ is weak-* compact, we have $c^\flat(\mu^*) + c^\sharp(\mu^*) = 0$ for some $\mu^* \in \Delta(\Sigma)$. This, in turn, implies $c^\flat(\mu^*) = c^\sharp(\mu^*) = 0$ because $c^\flat, c^\sharp \geq 0$.

Finally, let us discuss the uniqueness of twofold variational representations. By Lemma 1, we already know that u is unique up to positive affine transformations, and the utility functionals $(I^\flat, I^\sharp)$ are unique when u is fixed. Then, the uniqueness of $c^\flat$ and $c^\sharp$ immediately follows from the uniqueness assertion of the Fenchel-Moreau theorem. *Q.E.D.*

B.3 Proof of Proposition 1

Let $\succ$ be a twofold variational preference represented by $(u, c^\flat, c^\sharp)$, and let $(I^\flat, I^\sharp)$ be the associated utility functionals. Below, we prove only (7). A symmetric argument can be used for (8). Given any $f \in \mathcal{F}$, recall that $I^\flat(u \circ f) = \sup_{x\in X}\{u(x) : f \succ x\}$. Then, Lemma 13 implies that

$$\begin{aligned} c^\flat(\mu) &= \sup_{a\in B_0(\Sigma)} \left\{ I^\flat(a) - \int a \mathrm{d}\mu \right\} \\ &= \sup_{f\in\mathcal{F}} \left\{ \sup_{x\in X} \{u(x) : f \succ x\} - \int (u\circ f) d\mu \right\} \\ &= \sup_{(f,x)\in\mathcal{F}\times X} \left\{ u(x) - \int (u\circ f)\mathrm{d}\mu : f \succ x \right\}, \end{aligned}$$

as desired. *Q.E.D.*

B.4 Proof of Proposition 2

Note that $c^\flat = c^\sharp$ if and only if $I^\flat(a) = -I^\sharp(-a)$ for all $a \in B_0(\Sigma)$ because $c^\flat$ and $c^\sharp$ are given as the Fenchel conjugates of $\tilde{I}^\flat : a \mapsto -I^\flat(-a)$ and $I^\sharp$, respectively. The proposition then follows directly from Lemma 12. *Q.E.D.*

B.5 Proof of Proposition 3

Clearly, (ii) and (iii) are equivalent, and (iii) implies (i). To show that (i) implies (ii), let $\succ$ be a twofold variational preference satisfying C-independence. Consider any $\mu \in \Delta(\Sigma)$ such that $c^\flat(\mu) > 0$. By Proposition 1, there exists $f \in \mathcal{F}$ and $x \in X$ with $f \succ x$ such that $u(x) - \int (u \circ f) \mathrm{d}\mu > 0$. For each $n \geq 2$, let $f_n \in \mathcal{F}$ be such that $u \circ f_n = n \cdot u \circ f$, and let $x_n \in X$ be such that $u(x_n) = n \cdot u(x)$. In addition, take an arbitrary $x_0 \in X$ for which $u(x_0) = 0$. Since $f \succ x$, we see that

$$I^\flat\left(u \circ \left(\frac{1}{n} f_n + \frac{n-1}{n} x_0\right)\right) = I^\flat(u \circ f) > u(x) = u\left(\frac{1}{n} x_n + \frac{n-1}{n} x_0\right),$$

from which $\frac{1}{n} f_n + \frac{n-1}{n} x_0 \succ \frac{1}{n} x_n + \frac{n-1}{n} x_0$. Since $\succ$ satisfies C-independence, this implies that $f_n \succ x_n$ for every $n \geq 2$. Hence, Proposition 1 implies that

$$c^\flat(\mu) \geq \sup_{n \geq 2} \left\{ u(x_n) - \int (u \circ f_n) \mathrm{d}\mu \right\} = \sup_{n \geq 2} \left\{ n \cdot \underbrace{\left(u(x) - \int (u \circ f) \mathrm{d}\mu \right)}_{>0} \right\} = \infty.$$

Therefore, $c^\flat$ takes only the value 0 and ∞, and similar arguments verify so does $c^\sharp$ as well. *Q.E.D.*

B.6 Proof of Proposition 4

Clearly, (iv) and (v) are equivalent, while (v) is equivalent to saying that $I^\flat = I^\sharp$. Hence, Lemma 3 implies that both (ii) and (iii) are equivalent to (v). Note also that (v) implies (i). So, it remains to show that (i) implies at least one of (ii)–(v).

To this end, we prove that a twofold variational preference $\succ$ satisfies monotonicity whenever it satisfies independence. Fix any $f, g \in \mathcal{F}$ with $f(s) \succ g(s)$ for all $s \in S$. Take some $h \in \mathcal{F}$ that makes $u \circ (\frac{1}{2} g + \frac{1}{2} h)$ be a constant function, e.g., the one such that $u \circ h = -u \circ g$. By the affinity of u, it follows that $(\frac{1}{2} f + \frac{1}{2} h)(s) \succ (\frac{1}{2} g + \frac{1}{2} h)(s)$ for all $s \in S$. In particular, since the right-side is constant across s, and since all the acts are simple, we have

$$\begin{aligned} I^\flat(u \circ (f/2 + h/2)) &\geq \min_{s \in S} \{u(f(s))/2 + u(h(s))/2\} \\ &> \max_{s \in S} \{u(g(s))/2 + u(h(s))/2\} \geq I^\sharp(u \circ (g/2 + h/2)), \end{aligned}$$

where the weak inequalities are due to the monotonicity of $I^\flat$ and $I^\sharp$. This implies $\frac{1}{2} f + \frac{1}{2} h \succ \frac{1}{2} g + \frac{1}{2} h$, from which independence implies $f \succ g$. Hence, $\succ$ satisfies monotonicity. *Q.E.D.*

B.7 Proof of Proposition 5

Here, we only prove part (i), since the proof of part (ii) is essentially the same. Let $\succ_1$ and $\succ_2$ be a pair of twofold variational preferences, represented by profiles $(u_1, c_1^\flat, c_1^\sharp)$ and $(u_2, c_2^\flat, c_2^\sharp)$, respectively. First, suppose that there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u_2 = \alpha u_1 + \beta$ and $c_2^\flat \geq \alpha c_1^\flat$. Note that for any $f \in \mathcal{F}$ and $x \in X$, it holds that $u_2 \circ f - u_2(x)\mathbf{1}_S = (\alpha u_1 \circ f + \beta \mathbf{1}_S) - (\alpha u_2(x) + \beta)\mathbf{1}_S = \alpha(u_1 \circ f - u_1(x)\mathbf{1}_S)$. Together $c_2^\flat \geq \alpha c_1^\flat$, hence, we have

$$\begin{aligned} I_2^\flat(u_2 \circ f) - u_2(x) &= \min_{\mu \in \Delta(\Sigma)} \left\{ \int (u_2 \circ f - u_2(x)\mathbf{1}_S) \mathrm{d}\mu + c_2^\flat(\mu) \right\} \\ &\geq \alpha \min_{\mu \in \Delta(\Sigma)} \left\{ \int (u_1 \circ f - u_1(x)\mathbf{1}_S) \mathrm{d}\mu + c_1^\flat(\mu) \right\} = \alpha(I_1^\flat(u_1 \circ f) - u_1(x)), \end{aligned}$$

from which $f \succ_2 x$ whenever $f \succ_1 x$. Therefore, $\succ_1$ is more prudent than $\succ_2$.

Conversely, suppose that $\succ_1$ is more prudent than $\succ_2$, in which case $\succ_1^c$ coincides with $c_2^\flat$ by the definition. By the uniqueness of vNM functions, it follows that $u_2 = \alpha u_1 + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then, the inequality $\alpha c_1^\flat \leq c_2^\flat$ follows immediately from Proposition 1; using the characterization of pessimistic cost functions (7), we have for any $\mu \in \Delta(\Sigma)$,

$$\begin{aligned} \alpha c_1^\flat(\mu) &= \alpha \sup_{(f,x) \in \mathcal{F} \times X} \left\{ u_1(x) - \int (u_1 \circ f) \mathrm{d}\mu : f \succ_1 x \right\} \\ &= \sup_{(f,x) \in \mathcal{F} \times X} \left\{ u_2(x) - \int (u_2 \circ f) \mathrm{d}\mu : f \succ_1 x \right\} \\ &\leq \sup_{(f,x) \in \mathcal{F} \times X} \left\{ u_2(x) - \int (u_2 \circ f) \mathrm{d}\mu : f \succ_2 x \right\} = c_2^\flat(\mu), \end{aligned}$$

where the inequality follows from the inclusion $\{(f,x) \in \mathcal{F} \times X : f \succ_1 x\} \subseteq \{(f,x) \in \mathcal{F} \times X : f \succ_2 x\}$, nothing more than the definition of prudence. *Q.E.D.*

B.8 Proof of Theorem 2

Since the necessity part is straightforward, we focus on the sufficiency part. Suppose that $\succsim^\dagger$ satisfies **J1** and **J2**. Note that **J1** implies that the restriction of $\succsim^\dagger$ on X satisfies all the vNM axioms, so it is represented by some affine function $u : X \to \mathbb{R}$. In particular, the unboundedness condition in **J1** assures that $u(X) = \mathbb{R}$.

The rest of our strategy follows the proof of Theorem 1. Specifically, we first define the utility functions $U^\flat, U^\sharp : \mathcal{F} \to \mathbb{R}$ by

$$U^\flat(f) = \sup\{u(x) : f \succ^\dagger x\} \text{ and } U^\sharp(f) = \inf\{u(x) : x \succ^\dagger f\},$$

for every $f \in \mathcal{F}$. Note that these functions are well-defined because $\succsim^\dagger$ is monotonic (and because each act f is simple). It is straightforward to see that $u(x) = U^\flat(x) = U^\sharp(x)$ for every $x \in X$. Moreover, using unambiguous transitivity (**J2**), one can show that

$$U^\flat(f) \leq \inf\{u(x) : f \sim^\dagger x\} \leq \sup\{u(x) : f \sim^\dagger x\} \leq U^\sharp(f). \tag{24}$$

Indeed, take any $x, y \in X$ such that $f \succ^\dagger x$ and $f \sim^\dagger y$. If $x \succsim^\dagger y$, together $f \sim^\dagger y$, unambiguous transitivity implies that $x \succsim^\dagger f$, a contradiction to $f \succ^\dagger x$. So, $y \succ^\dagger x$ since $\succsim^\dagger$ is complete. By the arbitrariness of x, y, the first inequality of (24) is verified, and the proof for the third inequality is similar. Hence, we have shown that $U^\sharp \geq U^\flat$.

Now we want to show that $(U^\flat, U^\sharp)$ jointly represent $\succsim^\dagger$ in such a way that $f \succsim^\dagger g$ if and only if $U^\sharp(f) \geq U^\flat(g)$. This is proven in Claim 6 with the help of an auxiliary result given as Claim 5.

Claim 5. *For any $f, g \in \mathcal{F}$ with $f \succ^\dagger g$, there exist $x, y \in X$ with $x \succ^\dagger y$ such that $f \succ^\dagger x$ and $y \succ^\dagger g$.*

Proof. Let $f \succ^\dagger g$. Using monotonicity and unboundedness, we can take outcomes $\overline{z}, \underline{z} \in X$ such that $\overline{z} \succ^\dagger f$ and $g \succ^\dagger \underline{z}$. By quasi transitivity (**J2**), $f \succ^\dagger \underline{z}$ and $\overline{z} \succ^\dagger g$. Set $A = \{\alpha \in [0,1] : f \succ^\dagger \alpha\overline{z} + (1-\alpha)\underline{z}\}$ and $B = \{\alpha \in [0,1] : \alpha\overline{z} + (1-\alpha)\underline{z} \succ^\dagger g\}$. Note that $0 \in A$ and $1 \in B$. Also, A and B are open by completeness and mixture continuity. Moreover, we claim that $A \cup B = [0,1]$. Indeed, if there exists $\alpha \in [0,1] \setminus (A \cup B)$, then completeness implies that $\alpha\overline{z} + (1-\alpha)\underline{z} \succsim^\dagger f$ and $g \succsim^\dagger \alpha\overline{z} + (1-\alpha)\underline{z}$. By C-transitivity (**J2**), it follows that $\alpha\overline{z} + (1-\alpha)\underline{z} \succ^\dagger \alpha\overline{z} + (1-\alpha)\underline{z}$, a contradiction. Therefore $A \cup B = [0,1]$, and thus, the connectedness of $[0,1]$ implies that $A \cap B \neq \emptyset$. In particular, since $A \cap B$ is an open set, we can pick distinct $\alpha, \beta \in A \cap B$. Set $x = \alpha\overline{z} + (1-\alpha)\underline{z}$ and $y = \beta\overline{z} + (1-\beta)\underline{z}$. Since $\overline{z} \succ^\dagger \underline{z}$, and since $(\succ^\dagger)^c$ over X is represented by an affine function u, we can let $x \succ^\dagger y$ without loss. Moreover, $f \succ^\dagger x$ and $y \succ^\dagger g$ holds by the constructions of A and B, so we have found the desired constant acts. □

Claim 6. *For any $f, g \in \mathcal{F}$, $f \succsim^\dagger g$ if and only if $U^\sharp(f) \geq U^\flat(g)$.*

Proof. Since $\succsim^\dagger$ is complete, it suffices to show that $f \succ^\dagger g$ if and only if $U^\flat(f) > U^\sharp(g)$. Suppose that $f \succ^\dagger g$. By Claim 5, there exist $x, y \in X$ with $x \succ^\dagger y$ such that $f \succ^\dagger x$ and $y \succ^\dagger g$. By the constructions of $U^\flat$ and $U^\sharp$, we have $U^\flat(f) \geq u(x)$ and $u(y) \geq U^\sharp(g)$. Together $u(x) > u(y)$, it follows that $U^\flat(f) > U^\sharp(g)$. Conversely, suppose that $U^\flat(f) > U^\sharp(g)$. By the constructions of $U^\flat$ and $U^\sharp$, we can take some $x, y \in X$ with $u(x) > u(y)$ such that $f \succ^\dagger x$ and $y \succ^\dagger g$. In particular, this implies $x \succ^\dagger y$, so applying quasi-transitivity (**J2**) twice, we get $f \succ^\dagger g$. □

Define $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ by

$$I^\flat(a) = U^\flat(f) \text{ and } I^\sharp(a) = U^\sharp(f) \text{ where } f \in \mathcal{F} \text{ and } u \circ f = a. \tag{25}$$

Note that $I^\flat$ and $I^\sharp$ are well-defined thanks to the monotonicity of $\succsim^\dagger$. Hence, by Claim 6, it follows that $f \succsim^\dagger g$ if and only if $I^\sharp(u \circ f) \geq I^\flat(u \circ g)$. Clearly, $I^\flat$ and $I^\sharp$ are normalized and monotonic. Moreover, by mimicking the proof of Lemma 1, we can show that $I^\flat$ is lower semicontinuous, and $I^\sharp$ is upper semicontinuous.

Lastly, observe that $f \succ^\dagger g$ if and only if $I^\flat(u \circ f) > I^\sharp(u \circ g)$, whereas $(I^\flat, I^\sharp)$ satisfy the same assumptions as Lemma 1. Therefore, Theorem 1 implies that $\succ^\dagger$ satisfies **A4** and **A5** if and only if $I^\flat$ and $I^\sharp$ admit variational representations. *Q.E.D.*

B.9 Proof of Proposition 6

Let $\succ^\star$ be a twofold variational preference represented by $(u, c^\flat, c^\sharp)$, and let $\succ$ be a MBA preference represented by (v, I). Assume that $\succ$ is a pessimistic extension of $\succ^\star$. Note that $x \succ^\star y$ implies $x \succ y$ for all $x, y \in X$. Moreover, since $\succ$ is pessimistic, $x \not\succ^\star y$ implies $x \not\succ y$. Hence, $\succ$ and $\succ^\star$ entail the same preference over X, from which we can let $v = u$ without loss of generality.

Now, for each $f \in \mathcal{F}$, let $x_f \in X$ be such that $I(u \circ f) = u(x_f)$. Note that such x_f does exist because $u(X) = \mathbb{R}$. Since I represents $\succ$, we have $f \not\succ x$, which in turn implies $f \not\succ^\star x$ because $\succ$ is an extension of $\succ$. Hence, it follows that

$$\min_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu + c^\flat(\mu) \right\} \leq u(x_f) = I(u \circ f).$$

In what follows, we show that the above weak inequality indeed holds with equality. By way of contradiction, suppose that the above inequality is instead strict. Then, there exists $y \in X$ for which

$$\min_{\mu \in \Delta(\Sigma)} \left\{ \int (u \circ f) \mathrm{d}\mu + c^\flat(\mu) \right\} < u(y) < u(x_f),$$

meaning that $f \not\succ^\star y$. Since $\succ^\star$ is pessimistic, it the follows that $f \not\succ y$. On the other hand, since $I(u \circ f) = u(x_f) > u(y)$, we must have $f \succ y$. This yields a contradiction. Therefore, we have shown that $\succ$ admits the representation as in part (i) of the proposition. The proof of part (ii) is symmetric and therefore omitted. *Q.E.D.*

B.10 Proof of Theorem 3

Note that twofold CEU preferences are nested in twofold MBA preferences, as the special case of when utility functionals take the form:

$$I^\flat(a) = \int a \mathrm{d}\nu^\flat \quad \text{and} \quad I^\sharp(a) = \int a \mathrm{d}\nu^\sharp. \tag{26}$$

Indeed, these functionals are normalized, continuous, and monotonic by the properties of Choquet integrals. Moreover, to show $I^\sharp \geq I^\flat$, we invoke the definition of Choquet integrals, stated in terms of the usual Riemann integrals; see Schmeidler (1986). For any $a \in B_0(\Sigma)$, it follows that

$$\begin{aligned}
&I^\sharp(a) - I^\flat(a) \\
&= \left(\int_0^\infty \nu^\sharp(a \geq t) \mathrm{d}t - \int_{-\infty}^0 (1 - \nu^\sharp(a \geq t)) \mathrm{d}t \right) - \left(\int_0^\infty \nu^\flat(a \geq t) \mathrm{d}t + \int_{-\infty}^0 (1 - \nu^\flat(a \geq t)) \mathrm{d}t \right) \\
&= \int_0^\infty \left(\nu^\sharp(a \geq t) - \nu^\flat(a \geq t) \right) \mathrm{d}t.
\end{aligned}$$

Clearly, the expression is weakly positive given that $\nu^\sharp \geq \nu^\flat$. Hence, Lemma 1 implies **A1–A3** are necessary for the representation. Moreover, **A7** must also be satisfied because Choquet integrals satisfy comonotonic additivity.

Conversely, suppose that $\succ$ satisfies **A1–A3** and **A7**. By Lemma 1, we know that $\succ$ admits a twofold MBA representation $(u, I^\flat, I^\sharp)$. We first remark that these axioms imply the axiom of C-independence, i.e., **A4 (b)**.

Claim 7. *Under* **A1–A3***, the axiom* **A7** *implies C-independence. Consequently,* $I^\flat$ *and* $I^\sharp$ *are positively homogenous niveloids.*

Proof. Fix any $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0,1)$. By Claim 2, we can take a constant act $z \in X$ such that $f \succ z \succ g$. Since f, x, z are pairwise comonotonic, **A7** implies $\alpha f + (1-\alpha)x \succ \alpha x + (1-\alpha)z$. Similarly, since g, x, z are pairwise comonotonic, $\alpha z + (1-\alpha)x \succ \alpha g + (1-\alpha)z$. Hence, it follows that $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)z$ by transitivity. Conversely, assuming that $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$, Claim 2 yields $y \in X$ such that $\alpha f + (1-\alpha)x \succ y \succ \alpha g + (1-\alpha)x$. Since u is surjective, we can take $y' \in X$ such that $u(y') = (u(y) - (1-\alpha)u(x))/\alpha$ so that $\alpha y' + (1-\alpha)x \sim y$ holds. This implies $\alpha f + (1-\alpha)x \succ \alpha y' + (1-\alpha)x \succ \alpha g + (1-\alpha)x$. Now **A7** yields $f \succ x$ and $x \succ g$, from which $f \succ g$. Hence, $\succ$ satisfies C-independence. It follows that $I^\flat$ and $I^\sharp$ are translation inviariant from Lemma 8. Moreover, since C-independence implies **A8**, Lemma 11 implies positive homogeneity. □

As is standard, we say that functions $a, b \in B_0(\Sigma)$ are comonotonic if $(a(s) - a(s'))(b(s) - b(s')) \geq 0$ for all $s, s' \in S$. We next show that our functionals are additive on comonotonic functions.

Claim 8. *For any* $a, b \in B_0(\Sigma)$ *and* $\lambda \in (0,1)$*, if* a *and* b *are comonotonic, then* $I^\flat(\lambda a + (1-\lambda)b) = \lambda I^\flat(a) + (1-\lambda)I^\flat(b)$ *and* $I^\sharp(\lambda a + (1-\lambda)b) = \lambda I^\sharp(a) + (1-\lambda)I^\sharp(b)$.

Proof. As a matter of course, arguments are symmetric, so we work only with $I^\flat$. Fix any comonotonic $a, b \in B_0(\sigma)$ and $\lambda \in (0,1)$, and let $f, g \in \mathcal{F}$ be such that $u \circ f = a$ and $u \circ g = b$. Take sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $x_n \uparrow I^\flat(a)$ and $y_n \uparrow I^\flat(b)$. By construction, we have $f \succ x_n$ and $g \succ y_n$ for all n. Since a are b comonotonic, the acts f, g, x are pairwise comonotonic. So, **A7** and $f \succ x_n$ imply $\lambda f + (1-\lambda)g \succ \lambda x_n + (1-\lambda)g$. Similarly, **A7** and $g \succ y_n$ imply $\lambda x_n + (1-\lambda)g \succ \lambda x_n + (1-\lambda)y_n$, from which transitivity yields $\lambda f + (1-\lambda)g \succ \lambda x_n + (1-\lambda)y_n$. It follows that

$$I^\flat(\lambda a + (1-\lambda)b) > \lambda u(x_n) + (1-\lambda)u(y_n) \to \lambda I^\flat(a) + (1-\lambda)I^\flat(b),$$

as n tends to ∞. For the converse direction, assume without loss that $I^\flat(a) \geq I^\flat(b)$, and take a sequence $\{z_n\} \subseteq X$ such that $u(z_n) \downarrow I^\flat(b)$. This implies $g \not\succ z_n$, so **A7** implies $\lambda f + (1-\lambda)g \not\succ \lambda f + (1-\lambda)z_n$ for all n. By Claim 7, it follows that

$$\begin{aligned} I^\flat(\lambda a + (1-\lambda)b) &\leq I^\flat(\lambda a + (1-\lambda)u(z_n)\mathbf{1}_S) \\ &= \lambda I^\flat(a) + (1-\lambda)u(z_n) \to \lambda I^\flat(a) + (1-\lambda)I^\flat(b), \end{aligned}$$

as n tends to ∞. Hence, we have completed the proof. $\square$

Since normalization and monotonicity are guaranteed by the definition of twofold MBA preferences, Claim 8 allows us to apply Schmeidler's representation theorem to $I^\flat$ and $I^\sharp$ (cf. the corollary in Section 3 of Schmeidler, 1986): There exist unique capacities $\nu^\flat, \nu^\sharp : \Sigma \to [0,1]$ that give rise the Choquet integral representations as in (26). As such, if $\nu^\flat(A) > \nu^\sharp(A)$ for some event $A \in \Sigma$, we would have $I^\flat(\mathbf{1}_A) = \nu^\flat(A) > \nu^\sharp(A) = I^\sharp(\mathbf{1}_A)$, a contradiction to $I^\sharp \geq I^\flat$. Therefore, we must have $\nu^\flat \geq \nu^\sharp$. Finally, the uniqueness of $\nu^\flat$ and $\nu^\sharp$ follows from the standard argument. *Q.E.D.*

B.11 Proof of Proposition 7

Let $\succ$ be a twofold CEU preference represented by a profile $(u, \nu^\flat, \nu^\sharp)$. For the sake of brevity, we may omit the trivial proof of the necessity directions of (i)–(iv). In what follows, we instead show that the each axiom in (i)–(iv) of $\succ$ can be translated to the corresponding properties of $\nu^\flat$ and $\nu^\sharp$.

(i). Suppose that $\succ$ satisfies **A5**. Recall that $\succ$ satisfies C-independence by Claim 7, so it satisfies **A4** and **A8** as well. Hence, Lemmas 9 and 11 imply that $I^\flat$ and $I^\sharp$ are positively homogeneous, as well as concave/convex, respectively. Now, Proposition 3 of Schmeidler (1986) implies $\nu^\flat$ is convex and $\nu^\sharp$ is concave.

(ii). Suppose that $\succ$ satisfies **A6**. Again since $\succ$ satisfies **A4**, Lemma 12 implies $I^\flat(a) = -I^\sharp(-a)$. Given any $A \in \Sigma$, taking $a = \mathbf{1}_A$ yields

$$\int \mathbf{1}_A \mathrm{d}\nu^\flat = -\int (-\mathbf{1}_A)\mathrm{d}\nu^\sharp \iff \int \mathbf{1}_A \mathrm{d}\nu^\flat = 1 - \int (\mathbf{1}_S - \mathbf{1}_A)\mathrm{d}\nu^\sharp$$
$$\iff \int \mathbf{1}_A \mathrm{d}\nu^\flat = 1 - \int \mathbf{1}_{S\setminus A} \mathrm{d}\nu^\sharp \iff \nu^\flat(A) = 1 - \nu^\sharp(S \setminus A),$$

where the first equivalence is due to the translation invariance of Choquet integrals, the second is just rewriting of $\mathbf{1}_{S\setminus A} = \mathbf{1}_S - \mathbf{1}_A$, and the third is the definition of Choquet integrals. Since A is arbitrary, we conclude that $\nu^\flat$ is conjugate to $\nu^\sharp$.

(iii) and (iv). By Lemma 3, $\succ$ satisfies either monotonicity or negative transitivity if and only if $I^\flat = I^\sharp$. By the uniqueness of capacities, it then follows that $\nu^\flat = \nu^\sharp$, so (iii) is obtained. Moreover, one can easily show that independence implies monotonicity by mimicking the proof of Proposition 4. Hence, if $\succ$ satisfies independence, it follows from (iii) that $\mu \equiv \nu^\flat = \nu^\sharp$, i.e., $\succ$ admits Schmeidler's (1989) CEU representation. As is well-known, independence then leads to the SEU representation, i.e., μ is finitely additive probability measure. *Q.E.D.*

B.12 Proof of Theorem 4

Let $\succ$ be a twofold MBA preference represented by some profile $(u, I^\flat, I^\sharp)$, where $u : X \to \mathbb{R}$ is surjective and affine, and $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ are normalized and monotonic functionals such that $I^\sharp \geq I^\flat$. Also, $I^\flat$

and $I^\sharp$ are lower/upper semicontinuous, respectively. When $\succ$ further satisfies **A5**, Lemma 10 implies that $I^\flat$ and $I^\sharp$ are quasi-concave/convex, respectively. Hence, the direct application of Theorem 4.1 in Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011b) yields the following duality representations.

Claim 9. *For every $a \in B_0(\Sigma)$,*

$$I^\flat(a) = \min_{\mu\in\Delta(\Sigma)} G^\flat\left(\int a\mathrm{d}\mu,\, \mu\right), \text{ where } G^\flat(r,\mu) = \sup_{b\in B_0(\Sigma)}\left\{I^\flat(b) : \int b\mathrm{d}\mu \le r\right\}, \tag{27}$$

$$I^\sharp(a) = \max_{\mu\in\Delta(\Sigma)} G^\sharp\left(\int a\mathrm{d}\mu,\, \mu\right), \text{ where } G^\sharp(r,\mu) = \inf_{b\in B_0(\Sigma)}\left\{I^\sharp(b) : \int b\mathrm{d}\mu \ge r\right\}. \tag{28}$$

It is worth mentioning that the above representations of $I^\flat$ and $I^\sharp$ admit natural interpretations in light of price theory, which are useful to understand how $G^\flat$ and $G^\sharp$ satisfy the necessary conditions. To illustrate, observe that $G^\flat(r,\mu)$ is interpreted as the value of the problem:

$$\max_{b\in B_0(\Sigma)} I^\flat(b) \text{ sub. to } \int b\mathrm{d}\mu \le r,$$

in which the economic agent tries to maximize a quasiconcave utility function $I^\flat$ over consumption bundles b subject to the budget constraint associated with a normalized price vector μ and the level of wealth r. In other words, $G^\flat$ can be seen as the indirect utility function. Hence, the standard arguments guarantee that $G^\flat$ is increasing in wealth r and satisfies quasiconcavity and lower semicontinuity, provided that $I^\flat$ is quasiconcave and continuous. Similarly, interpreting $G^\sharp$ as the value of the cost minimization problem, we see that it is increasing in r and satisfies quasiconvexity and upper semicontinuity. Lastly, it is clear that $\inf_{\mu\in\Delta(\Sigma)} G^\flat(r,\mu) = \sup_{\mu\in\Delta(\Sigma)} G^\sharp = r$ by the constructions (27) and (28). *Q.E.D.*

B.13 Proof of Theorem 5

Let $\succ$ be a twofold MBA preference represented by some profile $(u, I^\flat, I^\sharp)$, where $u : X \to \mathbb{R}$ is surjective and affine, and $I^\flat, I^\sharp : B_0(\Sigma) \to \mathbb{R}$ are normalized and monotonic functionals such that $I^\sharp \ge I^\flat$. Also, $I^\flat$ and $I^\sharp$ are lower/upper semicontinuous, respectively. When $\succ$ further satisfies **A5**, Lemma 10 and Lemma 11 imply that $I^\flat$ and $I^\sharp$ are positively homogenous and quasi-concave/convex, respectively. Hence, we can apply Proposition 8.1 of Cerreia-Vioglio et al. (2011b); for any $a \in B_0(\Sigma)$, the value $I^\flat(a)$ can be written in the following form:

$$I^\flat(a) = \min_{\mu\in C^\flat}\left\{\frac{\left[\int a\mathrm{d}\mu\right]_+}{m^\flat(\mu)} - \frac{\left[\int a\mathrm{d}\mu\right]_-}{M^\flat(\mu)}\right\}, \tag{29}$$

where $\emptyset \ne C^\flat \subseteq \Delta(\Sigma)$ is closed and convex, $m^\flat : C^\flat \to [0,\infty)$ is upper semicontinuous and concave, and $M^\flat : C^\flat \to (0,\infty]$ is lower semicontinuous and convex. Moreover, if we define $\tilde{I}^\sharp : a \mapsto -I^\sharp(-a)$, then $\tilde{I}^\sharp$

has the same properties as $I^\flat$, and hence, there exist $C^\sharp$, $m^\sharp$ and $M^\sharp$, as before, such that

$$\tilde{I}^\sharp(a) = \min_{\mu \in C^\sharp} \left\{ \frac{\left[\int a \mathrm{d}\mu\right]_+}{m^\sharp(\mu)} - \frac{\left[\int a \mathrm{d}\mu\right]_-}{M^\sharp(\mu)} \right\}.$$

By the construction of $\tilde{I}^\sharp$, it follows that

$$I^\sharp(a) = -\tilde{I}^\sharp(-a) = -\min_{\mu \in C^\sharp} \left\{ \frac{\left[-\int a \mathrm{d}\mu\right]_+}{m^\sharp(\mu)} - \frac{\left[-\int a \mathrm{d}\mu\right]_-}{M^\sharp(\mu)} \right\} = \max_{\mu \in C^\sharp} \left\{ \frac{\left[\int a \mathrm{d}\mu\right]_+}{M^\sharp(\mu)} - \frac{\left[\int a \mathrm{d}\mu\right]_-}{m^\sharp(\mu)} \right\}. \tag{30}$$

To conclude the proof, we shall show that for $i \in \{\flat, \sharp\}$, $0 < \inf_{\mu \in C^i} m^i(\mu) \leq \max_{\mu \in C^i} m^i(\mu) = 1$ and $\min_{\mu \in C^i} M^i(\mu) = 1$. Since $I^\flat$ and $I^\sharp$ are normalized, the statement (iv) of Proposition 8.1 in Cerreia implies that $\max_{\mu \in C^i} m^i(\mu) = \min_{\mu \in C^i} M^i(\mu) = 1$ for each $i \in \{\flat, \sharp\}$. Also, notice that $I^\flat$ and $I^\sharp$ are real-valued, as well as they are uniformly continuous due to Theorem 6.1 of Cerreia-Vioglio et al. (2011b).[17] Hence, the statement (iii) of Proposition 8.1 of Cerreia-Vioglio et al. (2011b) implies that $\inf_{\mu \in C^i} m^i(\mu) > 0$ for each $i \in \{\flat, \sharp\}$. *Q.E.D.*

[17] Specifically, our $\succ$ belongs to the class of preferences considered in Theorem 4, where $I^\flat$ can be written as

$$I^\flat(a) = \min_{\mu \in \Delta(\Sigma)} G^\flat\left(\int a \mathrm{d}\mu,\, \mu\right), \text{ where } G^\flat(r, \mu) = \begin{cases} \frac{[\int a \mathrm{d}\mu]_+}{m^\flat(\mu)} - \frac{[\int a \mathrm{d}\mu]_-}{M^\flat(\mu)} & \text{if } \mu \in C^\flat, \\ \infty & \text{if } \mu \notin C^\flat. \end{cases}$$

Since $G^\flat$ is lower semicontinuous, it follows from Theorem 6.1 of Cerreia-Vioglio et al. (2011b) that $I^\flat$ is uniformly continuous. A similar argument is applied for $I^\sharp$ as well.

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