Multiproduct Pricing Made Simple

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Overview

- Multiproduct pricing important for:
  - unregulated monopoly
  - oligopoly
  - most efficient prices which cover fixed costs or generate tax revenue (Ramsey prices)
  - optimal regulation when costs are private information

- Key feature is that firm(s)/regulator must decide about price structure as well as overall price level

- This paper:
  - derives simple formulas using notion of consumer surplus as function of quantities
  - demonstrates equivalence between symmetric Cournot equilibria and Ramsey prices
  - describes generalized form of homothetic preferences so that pricing decisions can be decomposed into “relative” and “average” decisions
  - firms then have good incentives to choose relative quantities
Some (old) literature

- Baumol & Bradford (1970): principles of Ramsey pricing
  - “plausible that damage to welfare minimized if quantities are proportional to the efficient quantities”
- Gorman (1961): conditions on preferences to get linear Engel curves
  - we show it maximizes a Ramsey objective (and vice versa)
  - firms first decide how much surplus to offer customers, then solve Ramsey problem of maximizing profit subject to this constraint
- Marketing literature: patterns of cost passthrough in retailing
  - own-cost passthrough is positive, cross-cost passthrough ambiguous
- Baron & Myerson (1982): optimal regulation of single-product firm with unobserved costs
  - we can sometimes extend this to the multiproduct case
General framework

- There are $n$ products
  - quantity of product $i$ is $x_i$
  - vector of quantities is $x = (x_1, \ldots, x_n)$

- Consumers have quasi-linear preferences
  - there is representative consumer with concave gross utility $u(x)$, who maximizes $u(x) - p \cdot x$ when price vector is $p$
  - inverse demand function is $p_i(x) \equiv \partial u(x) / \partial x_i$ or in vector notation $p(x) \equiv \nabla u(x)$
  - total revenue with quantities $x$ is $r(x) = x \cdot \nabla u(x)$
  - so consumer surplus with quantities $x$ is
    \[ s(x) \equiv u(x) - x \cdot \nabla u(x) \]
Ramsey monopoly problem

- Products supplied by monopolist with convex cost function $c(x)$
- Ramsey objective with weight $0 \leq \alpha \leq 1$ is

$$[r(x) - c(x)] + \alpha s(x) = [u(x) - c(x)] - (1 - \alpha)s(x)$$

- $\alpha = 0$ corresponds to profit maximization
- $\alpha = 1$ corresponds to total surplus maximization

- First-order condition for maximizing Ramsey objective is

$$p(x) = \nabla c(x) + (1 - \alpha)\nabla s(x)$$

- price above [below] cost for product $i$ if $s$ is increasing [decreasing] in $x_i$
- when $c(x)$ is homogeneous degree 1 and $\alpha \approx 1$ Ramsey problem is solved by $x \approx ax^w$, where $x^w$ is efficient quantity vector (with $p = \nabla c$)
- so equiproportionate quantity reductions a good rule of thumb for small deviations
Ramsey quantities as weight on consumers varies

\[ x_1 \]

\[ x_2 \]

Armstrong & Vickers ( )

Multiproduct Pricing

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Consider symmetric Cournot market where each multiproduct firm has cost function \( c(x) \). Then symmetric equilibrium (if it exists) has first-order condition for total quantities \( x \)

\[
p(x) = \nabla c\left(\frac{1}{m}x\right) + \frac{1}{m} \nabla s(x)
\]

This coincides with optimal quantities in the Ramsey problem of maximizing

\[
u(x) - mc\left(\frac{1}{m}x\right) - (1 - \alpha)s(x)
\]

when \( \alpha = \frac{m-1}{m} \)

**Theorem**

*If m firms have the same convex cost function, there exists a symmetric Cournot equilibrium in which quantities maximize the Ramsey objective with \( \alpha = \frac{m-1}{m} \). There are no asymmetric equilibria.*

Comparative statics for \( m \) straightforward when \( c(x) \) is CRS.
Sketch proof of existence of Cournot equilibrium

- When $\alpha = \frac{m-1}{m}$, the Ramsey objective when firm $j$ chooses quantity vector $x^j$ is
  \[
  \frac{1}{m} r(\Sigma j x^j) + \frac{m-1}{m} u(\Sigma j x^j) - \Sigma j c(x^j)
  \]
  which has symmetric solution $x^j \equiv x$, say
  - In particular, choosing $y = x$ maximizes the function
    \[
    \rho(y) \equiv \frac{1}{m} r([m-1]x + y) + \frac{m-1}{m} u([m-1]x + y) - c(y)
    \]
  - A Cournot firm’s best response when its rivals each supply $x$ is to choose quantity vector $y$ to maximize
    \[
    \pi(y) \equiv y \cdot p([m-1]x + y) - c(y) \leq \rho(y) - \frac{m-1}{m} u(mx)
    \]
    (inequality follows from concavity of $u$)
    - Hence
      \[
      \pi(x) - \pi(y) \geq \rho(x) - \rho(y) \geq 0
      \]
      and it is an equilibrium for each firm to supply $x$
Homothetic consumer surplus

**Theorem**

*Consumer surplus* $s(x)$ *is homothetic in* $x$ *if and only if*

$$u(x) = h(x) + g(q(x))$$

*where* $h(x)$ *and* $q(x)$ *are both homogeneous degree 1*

- **"If":** We have

  $$p(x) = \nabla h(x) + g'(q(x)) \nabla q(x)$$

  so

  $$r(x) = h(x) + g'(q(x))q(x)$$

  and hence consumer surplus is

  $$s(x) = g(q(x)) - g'(q(x))q(x)$$

  which depends only on $q(x)$
Homothetic consumer surplus

- We can write quantities in “polar coordinates” form

\[ x = q(x) \cdot \frac{x}{q(x)} \]

- \( x/q(x) \) is homogeneous degree 0, depends only on the ray from origin
- \( q(x) \) measures how far along that ray \( x \) lies
- refer to \( q(x) \) as “composite quantity” and \( x/q(x) \) as “relative quantities”
- we know consumer surplus \( s(x) \) depends only on the \( q(x) \) coordinate

- Three degrees of freedom in the family: \( q(x), h(x) \) and \( g(q) \)
  - this is a much wider class than those where consumer surplus is homothetic in \textit{prices}
  - such preferences must have homothetic \( u(x) \), so \( h \equiv 0 \)
Examples

- **Linear demand**: An example with linear $h(x)$ is

  \[ u(x) = a \cdot x - \frac{1}{2} x^T M x \; ; \; p(x) = a - M x \]

  where $a > 0$ and $M$ is a positive definite matrix, so that

  \[ h(x) = a \cdot x \; , \; q(x) = \sqrt{x^T M x} \; , \; g(q) = -\frac{1}{2} q^2 \]

- **Logit demand**: An example with linear $q(x)$ has demand function

  \[ x_i(p) = \frac{e^{a_i - p_i}}{1 + \sum_j e^{a_j - p_j}} \]

  which corresponds to the utility function

  \[ u(x) = a \cdot x + \sum x_i \log \frac{q(x)}{x_i} + g(q(x)) \]

  \[ h(x) \]

  where $q(x) \equiv \sum x_j$ and $g(q) = -q \log q - (1 - q) \log(1 - q)$
Examples

**Strictly complementary products:**

- In some natural settings consumption requires prior purchase of a base product ("access")
- Suppose all who buy access (e.g. to theme park) get gross utility $U(y)$ from complementary services (rides) $y$.
- If $x_1$ consumers acquire access, and each then buys complementary services $y = x_2 / x_1$, where $x_2$ is the total supply of complementary services, gross utility has the form

\[
  u(x_1, x_2) = x_1 U(y) + g(x_1) = x_1 U(x_2 / x_1) + g(x_1)
\]

- This is an example with $q(x) = x_1$ so consumer surplus is simply a function of the number of consumers that buy access
- So services are priced at marginal cost
Consumer optimization

- Consumer surplus with price vector $p$ and quantities $x$ is

$$u(x) - p \cdot x = g(q(x)) - q(x) \frac{p \cdot x - h(x)}{q(x)}$$

- expressed in terms of the two coordinates $q(x)$ and $x/q(x)$
- for all $q(x)$ surplus is maximized by minimizing $[p \cdot x - h(x)]/q(x)$
- this determines the optimal relative quantities given price vector $p$, say $x^*(p)$
- let

$$\phi(p) \equiv \min_{x \geq 0} : \frac{p \cdot x - h(x)}{q(x)}$$

which is concave in $p$ and $x^*(p) \equiv \nabla \phi(p)$
The consumer then optimizes her composite quantity, say $Q$

$$Q(\phi) \text{ maximizes } g(Q) - Q\phi(p)$$

and so consumer demand as function of $p$ is

$$x(p) = Q(\phi(p))x^*(p)$$

Thus $\phi(p)$ is the “composite price”

- all prices with the same $\phi(p)$ induce the same composite quantity $q(x)$
- inverse demand for composite quantity is $\phi = g'(Q)$
Market analysis

- Suppose a monopoly or oligopoly supplies the $n$ products
  - with constant-returns-to-scale cost function $c(x)$
- We know Cournot equilibrium coincides with Ramsey optimum, so study the latter objective which is
  \[
  \alpha g(q(x)) + (1 - \alpha)q(x)g'(q(x)) - q(x)\frac{c(x) - h(x)}{q(x)}
  \]
  - again, a function of the two coordinates $q(x)$ and $x/q(x)$
  - regardless of composite quantity, choose relative quantities $x^*$ to minimize $[c(x) - h(x)]/q(x)$
  - so relative quantities the same for all $\alpha$ in Ramsey problem
  - this implies relative price-cost margins also the same for all $\alpha$

- Monopolist has good incentives to choose its relative quantities
  - sole inefficiency stems from it supplying too little composite quantity
Market analysis

- Let

\[ \kappa = \min_{x \geq 0} \frac{c(x) - h(x)}{q(x)} \]

- then optimal composite quantity \( Q \) maximizes

\[ \alpha g(Q) + (1 - \alpha) Qg'(Q) - \kappa Q \]

which satisfies the Lerner formula

\[ \frac{g'(Q) - \kappa}{g'(Q)} = (1 - \alpha) \eta(Q), \text{ where } \eta(Q) \equiv -\frac{Qg''(Q)}{g'(Q)} \]

**Theorem**

As \( \alpha \) increases (or number of Cournot competitors increases), composite quantity increases, composite price decreases, each individual quantity increases equiproportionately, and each price-cost margin contracts equiproportionately.
Cost passthrough

- Suppose \( c(x) \equiv c \cdot x \), so that \( \kappa = \phi(c) \) and \( x^* = x^*(c) \)
  - all vectors \( c \) with the same “composite cost” \( \phi(c) \) induce seller to supply same composite quantity and same composite price
- If \( c_i \) increases, \( \phi(c) \) increases, and so composite quantity decreases along with consumer surplus
  - so our class not rich enough to permit the “Edgeworth paradox”, where a higher cost for a product induces firm to reduce all prices
- When \( h(x) = a \cdot x \) the Ramsey prices are

\[
 p_i = \frac{c_i - (1 - \alpha)\eta(Q)a_i}{1 - (1 - \alpha)\eta(Q)}
\]

- So with constant \( \eta \) there is zero “cross-cost” passthrough in prices (though quantities are affected unless demands are independent)
- For instance, profit-maximizing \( (\alpha = 0) \) prices and quantities with linear demand \( (\eta = -1) \) are

\[
p = \frac{1}{2}(a + c)
\]
Optimal monopoly regulation

- Suppose monopolist has private information about its vector of constant marginal costs, \( c \)
  - regulator puts weight \( 0 \leq \beta \leq 1 \) on profit relative to consumer surplus
  - can make transfer to firm to encourage higher quantity

- Look for situations where firm is given discretion over choice of relative quantities

- Consider hypothetical case:
  - regulator knows the efficient relative quantities \( x^* \) corresponding to the firm’s costs, but not the (scalar) average level of costs, \( \kappa = \phi(c) \)
  - set of \( c \) with the same \( x^* = x^*(c) \) is a straight line
Optimal monopoly regulation

- This scalar screening problem can be solved as Baron & Myerson
  - suppose regulator’s prior for $\kappa$ on this iso-$x^*$ line has CDF $F(\kappa \mid x^*)$ and density $f(\kappa \mid x^*)$
  - then optimal quantities for type-$\kappa$ firm are
    $Q \left( \kappa + (1 - \beta) \frac{F(\kappa \mid x^*)}{f(\kappa \mid x^*)} \right) \times x^*$
    - composite price
  - each firm supplies the efficient relative quantities $x^*$
  - If this regulatory scheme does not depend on $x^*$ it is valid even when regulator cannot observe $x^*$
    - i.e., if distribution for cost vector $c$ is such that $\phi(c)$ and $x^*(c)$ are stochastically independent the regulation problem can be solved
    - incentive scheme depends only on the firm’s composite quantity
    - firm has freedom to choose its relative quantities
To illustrate, suppose \( u(x) = \sqrt{x_1} + \sqrt{x_2} \), so \( q(x) = (\sqrt{x_1} + \sqrt{x_2})^2 \), \( h(x) \equiv 0 \), \( g(Q) = \sqrt{Q} \) and

\[
\kappa = \phi(c) = \frac{1}{c_1 + \frac{1}{c_2}}
\]

- The method works if \( \frac{1}{c_1} + \frac{1}{c_2} \) and \( \frac{c_2}{c_1} \) are stochastically independent.
- E.g., if each \( \frac{1}{c_i} \) independently comes from exponential distribution (so \( c_i \) comes from inverse-\( \chi^2 \) distribution) optimal prices are

\[
p_i = c_i \times [1 + (1 - \beta)\kappa(1 + \kappa)]
\]

- Even very high-cost firms produce (so no “exclusion”), though regulated prices can be above unregulated monopoly prices.
- Price for one product increases with other product’s cost, even though costs are i.i.d.
Insightful to consider multiproduct pricing problems by way of consumer surplus as a function of quantities

Ramsey-Cournot equivalence result

(Surprisingly?) broad class of demand systems with consumer surplus as a homothetic function of quantities is quite tractable

Relative quantities are efficient

Multiproduct cost passthrough results

Conditions identified for optimal monopoly regulation to focus on the level but not the pattern of prices