Vertical Integration with Incomplete Information*

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Preliminary and incomplete

Abstract

Using an independent private values model, we analyze the effects of and incentives for vertical integration. Each firm is characterized by an amount of internally held resources and a maximum demand for these resources as inputs, which determine the firm's extent of vertical integration, and a distribution for the firm's private marginal value for resources. After private information is realized, firms trade resources and payoffs are realized. Depending on type realizations, a vertically integrated firm may sell resources to others, buy resources from others, and/or consume resources held internally. A certain extent of vertical integration is necessary and sufficient for the first-best to be possible. With two firms, equilibrium vertical integration is socially optimal but with more firms, vertical integration is typically excessive because of externalities from bilateral transactions. The model provides both rationale and guidance for divestiture after a vertical merger.

Keywords: vertical integration, boundaries of the firm, bargaining power, investment
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1 Introduction

Of longstanding interest in economics and antitrust, vertical integration has received renewed attention in recent merger cases and in public policy debates related to big tech.[1] Traditionally, it has been viewed favorably by competition authorities largely because, in settings with complete information and linear prices, it has the procompetitive effect of eliminating double marginalization. However, efficiencies that hinge on restrictions of the contracting space cannot be shown to be merger specific because they could be achieved without the merger if those restrictions were lifted and, for example, two-part tariffs were permitted.[2]

As a matter of principle, within the independent private values framework, incomplete information models embody an inherent tradeoff between rent extraction and social surplus without any restrictions on the contracting space and thus offer a possible remedy. Paraphrasing one astute observer, economists should take incomplete information models to heart because, in models with complete information, two-part tariffs are simply too powerful. The challenge encountered by the incomplete information approach to vertical integration to date is that, away from the rather special cases of markets with one buyer (seller) and possibly multiple sellers (buyers) with single-unit demand and supply prior to integration, the approach lacks tractability and poses modeling questions that appear difficult to resolve.[3]

[1]See, e.g., Baker et al. (2019); Luco and Marshall (2020); Kang and Muir (2022). The new *Vertical Merger Guidelines* released by the U.S. DOJ and FTC in 2020 (replacing the 1984 *Non-Horizontal Merger Guidelines*) paint a largely favorable view of vertical mergers, emphasizing the possibility that a vertical merger eliminates a double markup and thereby has procompetitive effects. For example, they state that "vertical mergers often benefit consumers through the elimination of double marginalization, which tends to lessen the risks of competitive harm" (p. 2) Consistent with that generally positive view of vertical mergers, the 2018 vertical merger of AT&T and Time Warner was approved by the Courts amidst various objections. The traditional overall favorable view of vertical mergers is currently being challenged, partly in light of concerns over acquisitions by big tech companies. In September of 2021, the FTC rescinded its support for the 2020 *Vertical Merger Guidelines*, with commissioners commenting that the guidelines did not appreciate various mechanisms by which vertical mergers could be harmful to consumers and competition (FTC 2021). The wisdom of the AT&T–Time Warner merger is itself in question as AT&T is in the process of divesting the Time Warner assets that they acquired in 2018.

[2]Antitrust practice generally looks for any claimed merger efficiencies to be merger specific (see, e.g., U.S. *Horizontal Merger Guidelines* (https://www.justice.gov/atr/public/guidelines/hmg-2010.pdf), p. 28; *Vertical Merger Guidelines*, p. 11). The *Vertical Merger Guidelines* (p. 12) elaborate that the agencies will consider "contracts between similarly situated firms in the same industry and contracting efforts considered by the merging firms," but that "The Agencies do not, however, reject the merger specificity of the elimination of double marginalization solely because it could theoretically be achieved but for the merger, if such practices are not reflected in documentary evidence." See also Choné et al. (2021) for an in-depth discussion of the problems associated with arguments based on the elimination of double markups when double markups arise because of restrictions on the contracting space. Further, Loertscher and Muir (2022) show that a monopolist may optimally use linear prices prior to market consolidation but not after.

[3]See Loertscher and Marx (2022) for specifications of generalized Myerson and Satterthwaite (1983) settings in which vertical integration increases or decreases expected social surplus. The difficult modeling choices faced by this approach for generalizations stem from the need to transform the type distributions from two independent entities pre integration into a single distribution post integration in order to maintain

In this paper, we analyze the effects of and incentives for vertical integration in an independent private values model that sidesteps the aforementioned problems. Firms strategically interact in an input market. Each firm requires a resource—for example, mobile telephony companies need spectrum licenses, furniture producers require wood, electric vehicle manufacturers require batteries—as an input to serving a downstream market, to which it has exclusive access among the firms interacting in the input market. Firms have constant marginal values for the input up to some commonly known maximum demand. These values are the private information of the firms and are drawn from continuous distributions with identical support, which is also commonly known. We allow for general ownership structures over the scarce input. Firms that own as much as their maximum demand will only ever trade as *sellers* in the input market, where their value is the opportunity cost for selling. Likewise, firms with zero ownership will only ever trade as *buyers* in the input market. In contrast, firms that have positive ownership that is less than their maximum demand are (partially) *vertically integrated* and under the first-best will trade as buyers if their values are high, as sellers if their values are low, or not trade at all if their values are equal to the Walrasian price. As in Loertscher and Marx (2022), we model the market as an incentive compatible and individually rational mechanism that must not run a deficit. For most of the paper, we assume that the market maximizes equally weighted expected social surplus, subject to these constraints. Vertical integration is modeled as a transfer of resources that occurs before private information is realized.

We show that vertical integration is inherently neither good nor bad because a certain degree of vertical integration is necessary and sufficient for the first-best to possible. This resonates with and reinforces the message from Loertscher and Marx (2022) that incomplete information settings provide no basis for a presumption that vertical integration is socially desirable. The model also provides a rationale and guidance for divestitures following vertical integration.[4] The model also allows us to shed new light on the question of the boundaries of firms. We show that the boundaries of the firms relate to incentives for raising rivals costs. Assuming that vertical integration, modeled as a bilateral transfer of resources prior to the realization of private information, continues as long as such transfers are profitable, we first show that with two firms, the equilibrium level of vertical integration corresponds to the levels of vertical integration, or ownership structures, that permit the first-best.[5] For

a single dimension of private information.

[4]For example, in the U.S. Department of Justice's review of the Halliburton-Baker Hughes merger, consideration was given to requiring divestitures, but emphasis was placed on the need for assets to be divested to a firm, such as GE, that already had a presence in the market ("Halliburton, Baker Hughes in Talks to Sell $7 Billion of Assets to Carlyle Group," *Wall Street Journal*, April 14, 2016, https://www.wsj.com/articles/halliburton-baker-hughes-in-talks-to-sell-assets-to-carlyle-group-1460658525).

[5]This resonates with an insight from oligopoly models that vertical integration is beneficial in highly

the case with two firms, we also completely characterize the first-best permitting levels of vertical integration and show that an increase in either firm's maximum demand increases the range of first-best permitting ownership structures. With more than two firms, bilateral transfers of resources involve externalities that can harm rivals. Because of the possibility of raising rivals' costs, the alignment of first-best permitting ownership structures and those that arise in equilibrium does not extend to three or more firms, suggesting a fundamental tension between equilibrium and the planner's optimum and a role for ongoing antitrust vigilance.[6]

In an extension, we generalize the model by allowing the agents' bargaining power to differ, which permits us to analyze the effects of vertical integration when, as is often hypothesized, vertical integration increases agents' bargaining power.

This paper contributes to an emerging literature on incomplete information industrial organization with and without vertical integration. Loertscher and Marx (2019) studies a procurement setup with a single buyer to define buyer power and analyze how the unilateral effects of horizontal mergers of suppliers are affected by buyer power.[7] Loertscher and Marx (2022) sets up an incomplete information bargaining model with multiple buyers and multiple suppliers and takes an as-if, mechanism-design based approach to modeling how the agents interacting in the market, allowing each agent to have a potentially different underlying type distribution and different bargaining power.

The recent upsurge of interest in vertical integration is reflected, for example, in the incomplete information models of Choné et al. (2021), Kang and Muir (2022), and Loertscher and Marx (2022). Choné et al. (2021) show that in an incomplete information setup, vertical integration eliminates double markups even in the absence of contract restrictions in the sense of reducing information rents in transactions between upstream and downstream firms—elimination of double marginalization is not an artefact of contractual restrictions and so can be merger specific. Kang and Muir (2022), who use an incomplete information model to analyze the effects of a dominant platform, is another case in point. Using a mechanism design approach, they show that a monopoly platform that can sell its own product downstream optimally price discriminates against independent suppliers, who are harmed by the platform's in-house production, but that in equilibrium in-house production benefits consumers. Without vertical integration being its focus, Loertscher and Marx (2022) make

concentrated markets (see e.g. Loertscher and Reisinger, 2014).

[6] This is reminiscent of the example of Aivazian and Callen (1981), who show that in contrast to the case of two players, with three players, one can have an empty core and, therefore, a breakdown of the Coase theorem even with complete information.

[7] Coordinated effects in a procurement setting are analyzed in Loertscher and Marx (2021a) while Loertscher and Marx (2021b) provides an overview and background for incomplete information models in Industrial Organization.

the observation that incomplete information models provide no basis for a presumption that vertical integration is inherently good or bad from a social surplus perspective.Our paper differs from and complements these in various ways, with possibly the most prominent point of departure being that the present frameworks admits an arbitrary number of vertically integrated firms.[8] Larsen and Zhang (2021) show how one can consistently estimate the underlying distributions and bargaining weights given data on market outcomes, assuming only that the allocations and payments derive from an incentive compatible mechanism, without restrictions on the underlying extensive form. There are a number of empirical papers looking at bargaining problems in incomplete information setups, including Backus et al. (2020), Backus et al. (2019), Larsen (2021), and Byrne et al. (2021).

Our approach builds on results and methodology developed in the literature on the efficiency of bilateral and multilateral trade from Myerson and Satterthwaite (1983), Williams (1987), and Gresik and Satterthwaite (1989). That some vertical integration is necessary and sufficient for the first-best in our setting relates to the partnership literature initiated by Cramton et al. (1987), with subsequent contributions by Che (2006), Figueroa and Skreta (2012), and Liu et al. (2022), among others. The second-best mechanism that is used when the first-best is not possible builds on work by Lu and Robert (2001) and Loertscher and Wasser (2019). With differential bargaining weights, our paper combines the optimal dissolution mechanism for a partnership problem of Loertscher and Wasser (2019) with the incomplete information bargaining approach for two-sided problems of Williams (1987) and Loertscher and Marx (2022).

The remainder of the paper is organized as follows. In Section 2, we provide the setup, and in Section 3, we define the market mechanism. In Section 4, we analyze vertical integration as it relates to the first-best. In Section 5, we consider implications for the boundaries of the firm. Extensions in Section 6 allow for investment and for incomplete information bargaining with differential bargaining weights. Section 7 provides additional discussion. Section 8 concludes the paper.

2 Setup

We consider a setup with $n \geq 2$ firms and denote the set of firms by $\mathcal{N} \equiv \{1, \ldots, n\}$. Each firm i is endowed with resources $r_i \geq 0$, where the total supply of resources is denoted by $R \equiv \sum_{i \in \mathcal{N}} r_i$, and has positive maximum demand of $k_i \in [r_i, R]$ for those resources for use

[8] An oligopoly model precursor is Farrell and Shapiro (1990), which examines a Cournot setup with complete information in which firms' asset holdings reduce their marginal costs. In that setup, increases in the asset holdings of the largest firm shift production towards the largest, and therefore lowest-cost, firm and so increase welfare; whereas, in our setup, increases in the resources of the largest firm can reduce welfare.

in a downstream market to which it has exclusive access.[9] We denote the total demand by $K \equiv \sum_{i \in \mathcal{N}} k_i$. Our assumption that $0 \le r_i \le k_i$ for all i implies that $0 \le R \le K$. To focus on the interesting case, we assume in addition that $0 < R < K$ so that there is positive supply and excess demand.

Firm i's constant marginal value for the good, denoted θ_i, is an independent draw from distribution F_i with support $[\underline{\theta}, \overline{\theta}]$, where $0 \le \underline{\theta} < \overline{\theta}$, and density f_i that is positive on the interior of the support.[10] To begin, we take firms' resource endowments as fixed and focus on trades among firms after the realization of types. Then we turn to the issue of ex ante transactions of endowments.

We categorize firms as buyers, sellers, or vertically integrated firms. Firm i is a *buyer* if it has no resources of its own to sell, $r_i = 0$, or if there is no external demand for any resources it has, for all $j \in \mathcal{N} \backslash \{i\}$, $k_j = r_j$; firm i is a *seller* if it has no demand for additional resources, $k_i = r_i$ or the are no available resources to purchase, $\sum_{j \in \mathcal{N} \backslash \{i\}} r_j = 0$; and otherwise firm i is said to be *vertically integrated*. Thus, for any vertically integrated firm i, we have $0 < r_i < k_i$. Further, if firm i is vertically integrated, then there is some firm j with $r_j < k_j$ to whom firm i might sell, and some firm j' (possibly with $j = j'$) with $0 < r_{j'}$ from whom firm i might buy.

Related setups with the restriction that $k_i = R$ for all $i \in \mathcal{N}$ are used to analyze when the first-best is possible in a partnership context, including Cramton et al. (1987), who have $F_i = F$ for all i, and Che (2006), who allows distributions to differ. Also related are setups used to analyze profit-maximizing and second-best outcomes, including Lu and Robert (2001), who have $F_i = F$ and $k_i < R$, and Loertscher and Wasser (2019), who allow distributions to differ, but with $k_i = R$. In contrast, we analyze both first-best and second-best outcomes in a setting that allows, but does not require, differing distributions and $k_i < R$.

3 Market mechanism

We model the market mechanism as a direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ that consists of an allocation rule $\mathbf{Q} : [\underline{\theta}, \overline{\theta}]^n \to [0, R]^n$ and payment rule $\mathbf{M} : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}^n$, where for reports $\boldsymbol{\theta}$, $Q_i(\boldsymbol{\theta})$ specifies the quantity allocated to firm i and $M_i(\boldsymbol{\theta})$ specifies the payment from firm

[9] If, contrary to our assumption, $k_i < r_i$, then firm i is has no value for $r_i - k_i$ units, which it is willing to sell for free. And if, contrary to our assumption, $k_i > R$, then firm i has demand that cannot be met and so can be ignored. It is without loss to assume, as we do, that $k_i > 0$, for otherwise, the assumption that $k_i \ge r_i$ would imply that $r_i = k_i = 0$, and so we could just eliminate i from the market.

[10] We can accommodate differing/non-overlapping supports. In that case, for example, if a firm whose value for the resource is drawn from a distribution with a higher, non-overlapping support has resources equal to its maximum demand, then that firm essentially leaves the market.

i to the mechanism.[11] Feasibility requires that $\sum_{i\in\mathcal{N}} Q_i(\boldsymbol{\theta}) \leq R$. We focus on incentive compatible, individually rational mechanisms (see Appendix A.1 for formal definitions) that have no deficit in expectation, i.e., $\mathbb{E}_{\boldsymbol{\theta}}[\sum_{i\in\mathcal{N}} M_i(\boldsymbol{\theta})] \geq 0$.[12] Firm i's (interim) outside option associated with not participating in the mechanism is $\theta_i r_i$.

Given mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$, denote firm i's interim expected allocation by

$$q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\theta_i, \boldsymbol{\theta}_{-i})]$$

and firm i's interim expected payment by

$$m_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\theta_i, \boldsymbol{\theta}_{-i})].$$

Firm i's interim expected payoff from participation in the mechanism is

$$u_i(\theta_i) \equiv q_i(\theta_i)\theta_i - m_i(\theta_i).$$

Given $\langle \mathbf{Q}, \mathbf{M}\rangle$, firm i's *worst-off type* is $\hat{\theta}_i \in [\underline{\theta}, \overline{\theta}]$ that minimizes the interim expected payoff of firm i net of the firm's outside option:

$$\hat{\theta}_i \in \arg\min_{\theta_i \in [\underline{\theta}, \overline{\theta}]} q_i(\theta_i)\theta_i - m_i(\theta_i) - r_i\theta_i. \tag{1}$$

While $\hat{\theta}_i$ depends on the mechanism and endowment, to conserve notation we drop these arguments where the meaning is clear. Standard arguments (see, e.g., Cramton et al., 1987) imply that:

Lemma 1. *Given an incentive compatible, individually rational mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$, if $q_i(\theta_i)$ is continuous and satisfies $q_i(\underline{\theta}) \leq r_i \leq q_i(\overline{\theta})$, then firm i's worst-off type $\hat{\theta}_i$ satisfies*

$$q_i(\hat{\theta}_i) = r_i.$$

As explained in (Cramton et al., 1987, p. 618), intuitively, the worst-off type expects on average to be neither a buyer nor a seller, and therefore a firm with the worst-off type has no incentive to overstate or understate its valuation and so does not need to be compensated

[11] By the Revelation Principle, a focus on direct mechanisms is without loss of generality.

[12] In our independent private values setting, any Bayesian incentive compatible and interim individually rational mechanism can be implemented with dominant strategies and ex post individual rationality. By construction, it yields the same interim and hence ex ante expected payoffs and revenue. Thus, while we formally state our assumptions in Appendix A.1 in terms of Bayesian incentive compatibility and interim individual rationality, one could also use the ex post versions of those constraints. However, under what conditions the no-deficit constraint can be allowed to hold ex post remains an open question.

to induce truthful reporting, which is why it is the worst-off type.

It follows from Lemma 1 that under the first-best allocation rule, which allocates R to the firms with the highest types,[13] if firm i is a seller, then $\hat{\theta}_i = \overline{\theta}$; if firm i is a buyer, then $\hat{\theta}_i = \underline{\theta}$; and if firm i is vertically integrated then $\hat{\theta}_i$ is interior.

By appropriately defining virtual type functions for our setting, we can use standard mechanism design arguments (see, e.g., Krishna, 2002, Chapter 5.1) to write each firm's expected payment to the mechanism in terms of the virtual types and the allocation rule. Specifically, define the *virtual type* functions associated with net buyers and suppliers, respectively, by

$$\Psi_i^B(\theta) \equiv \theta - \frac{1 - F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi_i^S(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)}.$$

In addition, define the overall virtual type function with cutoff type x by:[14]

$$\Psi_i(\theta, x) \equiv \begin{cases} \Psi_i^S(\theta) & \text{if } \theta < x, \\ \Psi_i^B(\theta) & \text{otherwise.} \end{cases}$$

Using these virtual type functions, we then have:

Proposition 1. *Given an incentive compatible, individually rational mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$, *firm i's expected payment to the mechanism is*

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}\left[\Psi_i(\theta_i, \hat{\theta}_i) q_i(\theta_i)\right] - u_i(\hat{\theta}_i), \tag{2}$$

where $u_i(\hat{\theta}_i) \geq r_i \hat{\theta}_i$ *(and* $u_i(\hat{\theta}_i) = r_i \hat{\theta}_i$ *if individual rationality binds for firm i).*

Proof. See Appendix A.2.

For buyers and sellers, the expression for firm i's expected payment to the mechanism in Proposition 1 reduces to the usual expressions. If firm i is a buyer, then $\hat{\theta}_i = \underline{\theta}$, and so by Proposition 1, firm i's expected payment to the mechanism when individual rationality binds is the expectation of the buyer's virtual value times its allocation:

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_i^B(\theta_i) q_i(\theta_i)].$$

[13] Under the first-best allocation rule, the interim expected allocation of firm i with type $\underline{\theta}$ is $R - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j$, which is less than or equal to r_i by our assumption that $r_j \leq k_j$ for all $j \in \mathcal{N}$, and the interim expected allocation of firm i with type $\overline{\theta}$ is k_i, which is greater than or equal to r_i by assumption.

[14] Virtual type function $\Psi_i(\theta, x)$ is not increasing in θ if x is interior—in particular, it jumps down at $\theta = x$. As a result, when considering the second-best allocation rule, we will need to define the ironed version of Ψ_i to address this nonmonotonicity, but we can postpone that for now.

If firm i is a seller, then $\hat{\theta}_i = \overline{\theta}$, and so by Proposition 1, firm i's expected payment from the mechanism when individual rationality binds is the expectation of the seller's virtual cost times the quantity that it sells:[15]

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_i^S(\theta_i)(r_i - q_i(\theta_i))].$$

4 Socially optimal degrees of vertical integration

We begin our analysis of vertical integration by examining when the first-best is possible.

Let $\mathbf{Q}^e$ denote the first-best allocation rule, i.e., $\mathbf{Q}^e(\boldsymbol{\theta})$ allocates the total resources R to the firms so as to maximize the sum of the firms' types multiplied by their allocations (as mentioned above, we ignore ties between firms' types, which occur with probability zero). We let $q_i^e(\theta_i)$ denote firm i's interim expected allocation under the efficient allocation rule:[16]

$$q_i^e(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^e(\boldsymbol{\theta})].$$

Then, using Lemma 1, firm i's worst-off type under the efficient allocation rule, $\hat{\theta}_i^e$, satisfies

$$q_i^e(\hat{\theta}_i^e) = r_i.$$

To see that such a $\hat{\theta}_i^e$ is well defined, note that q_i^e is continuous and increasing with $q_i^e(\underline{\theta}) = 0$ and $q_i^e(\overline{\theta}) = \min\{k_i, R\} = k_i \geq r_i$.

We say that the first-best is *possible* if there exists an incentive compatible, individually

[15]To see this, note that $\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_i^S(\theta_i)q_i(\theta_i)] - \overline{\theta}r_i$, where q_i is the quantity retained by the seller, so $r_i - q_i$ is the quantity sold. Rewriting this using $\mathbb{E}_{\theta_i}[\Psi_i^S(\theta_i)] = \overline{\theta}$, we have the result.

[16]For the case of symmetric maximum demands, i.e., for all $i \in \mathcal{N}$, $k_i = k$, the first-best allocates the total supply of R to the firms with the highest types, "filling up" each firm up to its maximum demand before allocating units to a firm with a lower type. Assuming symmetric distributions, i.e., for all $i \in \mathcal{N}$, $F_i = F$, the probability of the ℓ-th highest out of $\boldsymbol{\theta}_{-i}$ being greater than or equal to θ_i is

$$\sum_{j=\ell}^{n-1} \binom{n-1}{j} (1 - F(\theta_i))^j F(\theta_i)^{n-1-j}.$$

Using $\lceil x \rceil$ to denote the smallest integer greater than or equal to x and temporarily letting $y \equiv \lceil R/k \rceil$ to make the expression below easier to read, it follows that with symmetric maximum demands and distributions,

$$\begin{aligned} q_i^e(\theta_i) \quad &= \quad k \left(1 - \sum_{j=y-1}^{n-1} \binom{n-1}{j} (1 - F(\theta_i))^j F(\theta_i)^{n-1-j} \right) \\ &\quad + (R - (y-1)k) \binom{n-1}{y-1} (1 - F(\theta_i))^{y-1} F(\theta_i)^{n-y}. \end{aligned}$$

rational, no-deficit mechanism that achieves the first-best allocation, i.e., that has allocation rule $\mathbf{Q}^e$. Let $\Pi^e(\mathbf{r})$ denote the maximal expected revenue of an incentive compatible, individually rational mechanism with a first-best allocation rule, given resource endowment $\mathbf{r}$. Because $\Pi^e(\mathbf{r})$ is the maximal revenue, it follows that the individual rationality constraint binds for firms' worst-off types in the associated mechanism, so, using Proposition 1, we have

$$\Pi^e(\mathbf{r}) = \sum_{i \in \mathcal{N}} \left(\mathbb{E}_{\theta_i} \left[\Psi_i(\theta_i, \hat{\theta}_i^e) q_i^e(\theta_i) \right] - r_i \hat{\theta}_i^e \right). \tag{3}$$

It then follows that the first-best is possible if and only if

$$\Pi^e(\mathbf{r}) \geq 0.$$

For example, if $\Pi^e(\mathbf{r}_1) > \Pi^e(\mathbf{r}_0) \geq 0$, then the first-best is possible under both $\mathbf{r}_1$ and $\mathbf{r}_0$, and there is more slack to achieve the first-best under $\mathbf{r}_1$ than under $\mathbf{r}_0$.

One can show that there exists an endowment such that the first-best is possible and that the unique endowment vector that maximizes the expected budget surplus, denoted $\mathbf{r}^*$, is the unique one that equalizes the firms' worst-off types:

Lemma 2. (Liu et al., 2022, Lemma 2 and Theorem 1) *There exists a unique resource vector* $\mathbf{r}^*$ *such that* $\hat{\theta}_1^e(r_1^*) = \cdots = \hat{\theta}_n^e(r_n^*) \equiv \hat{\theta}^e$*, where* $\hat{\theta}^e \in (\underline{\theta}, \overline{\theta})$*; moreover,* $\mathbf{r}^*$ *is the unique maximizer of* Π^e*, with*

$$\Pi^e(\mathbf{r}^*) = \max_{\mathbf{r}} \Pi^e(\mathbf{r}) > 0.$$

4.1 Role of vertical integration in achieving the first-best

To derive implications for vertical integration, observe first that the result of Lemma 2 that $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ implies that $r_i^* \in (0, k_i)$ for all i. Thus, maximizing $\Pi^e(\mathbf{r})$ requires that all firms be vertically integrated. And, further, an appropriate degree of vertical integration, namely $\mathbf{r}^*$, is *sufficient* for the first-best to be possible.

In addition, vertical integration is *necessary* for the first-best to be possible. To see this, suppose that for all i, $r_i \in \{0, k_i\}$, so that all agents are either buyers or sellers, i.e., there is no vertical integration. Then the impossibility results imply that first-best trade is not possible without running a deficit.[17] Thus, we have the following result:

[17]For the case with $n = 2$, see Myerson and Satterthwaite (1983), and for generalizations, see, for example, Gresik and Satterthwaite (1989), Williams (1999), Segal and Whinston (2011) or Delacrétaz et al. (2019). The proof of Delacrétaz et al. applies directly to the present setting because since agents with payoff functions with constant marginal values are "decomposable" as defined there.

Proposition 2. *The first-best is not possible in the absence of vertical integration and is possible when all firms are vertically integrated with endowments* $\mathbf{r}^*$.

By Proposition 2, some degree of vertical integration is necessary for the first-best to be possible, and an appropriate level of vertical integration is sufficient for the first-best to be possible. Further, using Lemma 2, shifts in endowments towards $\mathbf{r}^*$, support achieving the first-best. For example, in the case of symmetric distributions and capacities, i.e., $r_i^* = R/n$ for all $i \in \mathcal{N}$, so movement towards symmetric endowments increases Π^e and supports achieving the first-best.

The connection between whether the first-best is possible and the endowment vector $\mathbf{r}^*$ brings to the forefront the question of how $\mathbf{r}^*$ varies with the size and strength of firms in the market. The possibility of differences in the maximum demands, k_1 and k_2, allow for differences in firm sizes, and any differences in F_1 and F_2 can be thought of as differences in productivity across the two firms. Because, all else equal, $k_i > k_j$ implies $q_i(\theta) > q_j(\theta)$ for $\theta \in (\underline{\theta}, \overline{\theta})$, and F_i first-order stochastically dominates F_j implies $q_i(\theta) > q_j(\theta)$ for $\theta \in (\underline{\theta}, \overline{\theta})$, we have the following result:[18]

Proposition 3. *If* $F_i = F_j$, *then* $k_i > k_j$ *implies* $r_i^* > r_j^*$; *and if* $k_i = k_j$, *then* $F_i(\theta) < F_j(\theta)$ *for all* $\theta \in (\underline{\theta}, \overline{\theta})$ *implies* $r_i^* > r_j^*$.

Thus, "larger" firms with larger maximum demands and "stronger" firms with better distributions in the sense of first-order stochastic dominance have more resources under $\mathbf{r}^*$.

Focusing on the case of $n = 2$, we can provide a full characterization of $\mathbf{r}^*$. Firm i's interim expected allocation is $q_i(\theta) = R - k_j + (K - R)F_j(\theta)$, where $j \neq i$. For both firms to have the same worst-off type $\hat{\theta}$, $\hat{\theta}$ has to satisfy $q_1(\hat{\theta}) + q_2(\hat{\theta}) = R$. Straightforward algebra then reveals that $\hat{\theta}$ satisfies

$$F_1(\hat{\theta}) + F_2(\hat{\theta}) = 1.$$

In other words, the common worst-off type $\hat{\theta}$ does not vary with k. Using $q_i(\hat{\theta}) = r_i^*$, we have, for $i \neq j$,

$$r_i^* = (K - R)F_j(\hat{\theta}) + R - k_j,$$

which implies that

$$\frac{\partial r_i^*}{\partial k_i} = F_j(\hat{\theta}) > 0 > -(1 - F_j(\hat{\theta})) = \frac{\partial r_i^*}{\partial k_j}. \tag{4}$$

For example, if the firms' distributions are symmetric, then $F_i(\hat{\theta}) = \frac{1}{2}$ and so $r^* > R - r^*$ if and only if $k_1 > k_2$, which says that the endowment that maximizes $\Pi^e(r)$ gives relatively

[18] See Liu et al. (2022). In addition, Proposition 3 is related to the result obtained by Che (2006) for the case with $k_i = 1$ for all $i \in \mathcal{N}$ and distributions ranked by first-order stochastic dominance.

more to a firm with the greater maximum demand. As another example, if $k_1 = k_2 = R$, then $r^* > R - r^*$ if and only if $F_1(\hat{\theta}) < F_2(\hat{\theta})$, which says that the endowment that maximizes $\Pi^e(r)$ gives a relatively more to a firm whose type distribution first-order stochastically dominates that of the other firm.

Summarizing, we have the following result:

Proposition 4. *For $n = 2$, a firm's endowment under $\mathbf{r}^*$:*

(i) *increases in its own maximum demand,*

(ii) *decreases in its rival's maximum demand,*

(iii) *increases with FOSD shifts in its distribution, and*

(iv) *decreases with FOSD shifts in its rival's distribution.*

Proof. The results for changes in maximum demands follow from 4. For the proof of results for changes in distributions, see Appendix B.

Turning to the *relative degree of vertical integration* of each firm i, defined as $r_i^R \equiv r_i^*/k_i$, we have

$$r_i^R = \frac{K - R}{k_i} F_j(\hat{\theta}) + \frac{R - k_j}{k_i},$$

where $\hat{\theta}$ is the firms' common worst-off type under $\mathbf{r}^*$. It then follows that $\frac{\partial r_i^R}{\partial k_i} = -\frac{R - k_j}{k_i^2}\big(1 - F_j(\hat{\theta})\big)$ and $\frac{\partial r_i^R}{\partial k_j} = -\frac{1}{k_i}\big(1 - F_j(\hat{\theta})\big)$, which implies that $\frac{\partial r_i^R}{\partial k_j} < \frac{\partial r_i^R}{\partial k_i} \leq 0$, where the final inequality is strict unless $k_j = R$, giving us the following result:

Proposition 5. *For $n = 2$, a firm's relative degree of vertical integration decreases in both firms' maximum demands.*

Thus, even though an increase in a firm's maximum demand increases its endowment under $\mathbf{r}^*$, it decreases its relative degree of vertical integration under $\mathbf{r}^*$.

4.2 Rationale and guidance for divestitures

With these results in hand, we can consider the effects of shifts in endowments.

Letting $\Pi^{\mathbf{Q}}$ be the expected budget surplus associated with allocation rule $\mathbf{Q}$ and binding individual rationality, and letting $\hat{\theta}_i^{\mathbf{Q}}(r_i)$ be the worst-off type of firm i with endowment r_i under allocation rule $\mathbf{Q}$, we have:

Proposition 6. *Given incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, $\mathbf{r}$ with $\hat{\theta}_1^{\mathbf{Q}}(r_1) \geq \hat{\theta}_2^{\mathbf{Q}}(r_2)$, and $\Delta \in (0, \min\{k_1 - r_1, r_2\}]$,*

$$\Pi^{\mathbf{Q}}(r_1 + \Delta, r_2 - \Delta, r_3, \ldots, r_n) - \Pi^{\mathbf{Q}}(\mathbf{r}) < 0. \tag{5}$$

Proof. See Appendix B.

Thus, holding fixed the allocation rule, the budget surplus under binding individual rationality decreases as endowment is shifted from a firm with weakly lower endowment to one with weakly greater endowment. As intuition for why this holds, note that the left side of (5) depends only on payments from firm 1 and firm 2. The allocation is fixed, so changes in their payments relate only to the values of their outside options, $r_1\hat{\theta}_1^{\mathbf{Q}}(r_1)$ and $r_2\hat{\theta}_2^{\mathbf{Q}}(r_2)$, where the mapping $\hat{\theta}_i^{\mathbf{Q}}(\cdot)$ is increasing and does not vary with the shift in endowments. Shifting endowment to the firm with the weakly larger worst-off type improves the total outside option for the firms' worst-off types and thus reduces payments to the mechanism by firms 1 and 2.

We now use Proposition 6 as a stepping stone to prove an additional result. We say that endowment vector $\mathbf{r}'$ *majorizes* $\mathbf{r}$ if for all $k \in \mathcal{N}$, $\sum_{i=1}^{k} r'_{[i]} \geq \sum_{i=1}^{k} r_{[i]}$, where $r_{[i]}$ is the i-th largest element of $\mathbf{r}$, with a strict inequality for some k and equality for $k = n$. As shown in Marshall et al. (2011, Lemma B.1), if $\mathbf{r}'$ *majorizes* $\mathbf{r}$, then $\mathbf{r}$ can be obtained from $\mathbf{r}'$ by a finite number of transforms of the form described in Proposition 6.[19] Thus, we have the following result:[20]

Proposition 7. *Assuming symmetric distributions and maximum demands, given incentive compatible mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$*, if* $\mathbf{r}'$ *majorizes* $\mathbf{r}$*, then* $\Pi^{\mathbf{Q}}(\mathbf{r}') < \Pi^{\mathbf{Q}}(\mathbf{r})$*.*

Proof. See Appendix B.

The assumption in Proposition 7 of symmetric distributions and maximum demands implies that $\hat{\theta}_i^{\mathbf{Q}}(\cdot) = \hat{\theta}_j^{\mathbf{Q}}(\cdot)$ for all i and j. Thus, the condition of Proposition 6 that $\hat{\theta}_1^{\mathbf{Q}}(r_1) \geq \hat{\theta}_2^{\mathbf{Q}}(r_2)$ reduces to $r_1 \geq r_2$, and so a majorization of the endowment vector is sufficient to reduce $\Pi^{\mathbf{Q}}$.

Proposition 7 implies that with symmetric distributions and maximum demands, the acquisition by one firm of the production capacity of another firm that has a weakly smaller capacity has the potential to cause a market that is efficient to become inefficient.[21] We illustrate this in Figure 1. In the example of panel (a), there are four symmetric firms, and the first-best is possible when each firm has an endowment of 1/4, but not following the acquisition by firm 1 of all of firm 2's endowment, which corresponds to $r_1 = 2/n$ in the figure.

[19] Such transforms are known as T-transforms: given vector $(x_1, \ldots, x_n)$, a T-transform of $\mathbf{x}$ is a vector with two coordinates x_j and x_k replaced by $\lambda x_j + (1-\lambda)x_k$ and $\lambda x_k + (1-\lambda)x_j$ for some $\lambda \in (0,1)$.

[20] Proposition 7 can also be stated as saying that $\Pi^{\mathbf{Q}}$ is Schur-concave (Marshall et al., 2011, p. 80).

[21] This result is reminiscent of the finding of Farrell and Shapiro (1990) that in a Cournot setup, under certain conditions, "any capital sale raises price if and only if the acquiring firm is larger than the divesting firm" (Farrell and Shapiro, 1990, p. 282).

This is apparent from the figure because $\Pi^e(\mathbf{r})$ is negative when $r_1 = 2/n$. In panel (b), there are five symmetric firms. The first-best is possible when each firm has an endowment of 1/5, and it remains possible after firm 1 has acquired all of firm 2's endowment ($r_1 = 2/n$) because there $\Pi^e(\mathbf{r})$ remains positive, illustrating that even when some firms' endowments are zero, that does not necessarily imply that the first-best becomes impossible. But the first-best is no longer possible after firm 1 has acquired all of the endowments of both firm 2 and firm 3 ($r_1 = 3/n$). The "kink" in the figure corresponds to the point at which the endowment for firm 2 reaches 0 and firm 1 starts to draw from the endowment of firm 3.

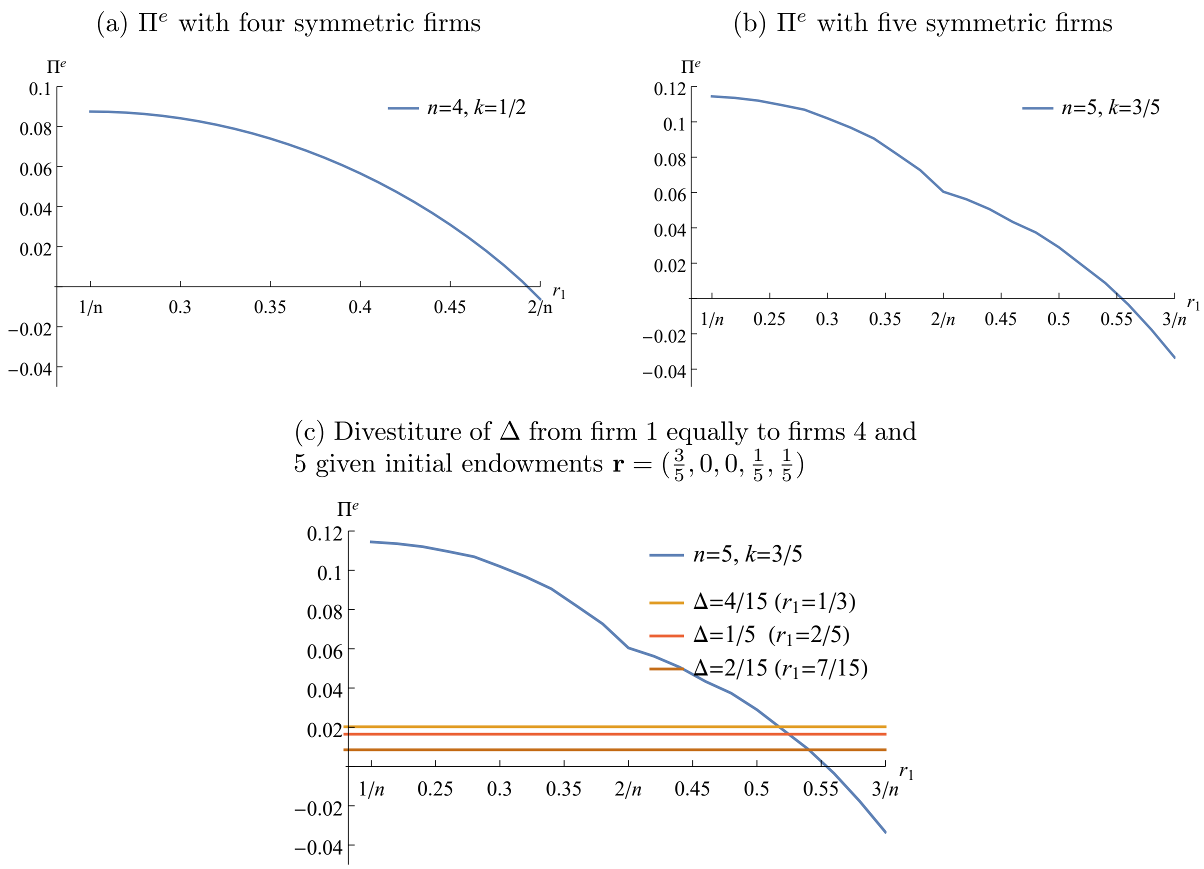

Figure 1: Expected budget surplus with binding individual rationality under the efficient mechanism, $\Pi^e(\mathbf{r})$. Panel (a) assumes that $n = 4$, $k = 1/2$, and $R = 1$ and shows $\Pi^e(r_1, 2/n - r_1, 1/n, 1/n)$ for $r_1 \in [1/n, 2/n]$. Panel (b) assumes that $n = 5$, $k = 3/5$, and $R = 1$ and shows $\Pi^e(r_1, 2/n - r_1, 1/n, 1/n, 1/n)$ for $r_1 \in [1/n, 2/n]$ and $\Pi^e(r_1, 0, 3/n - r_1, 1/n, 1/n)$ for $r_1 \in [2/n, 3/n]$. Panel (c) adds to panel (b) the values for Π^e associated with divestitures of Δ from firm 1 equally to firms 4 and 5 starting from endowments $\mathbf{r} = (\frac{3}{5}, 0, 0, \frac{1}{5}, \frac{1}{5})$. All panels assume that types are uniformly distributed on $[0, 1]$ and $R = 1$.

Of course, a majorization of the firms' resources corresponds to an increase in the resource-based Herfindahl-Hirschman Index (HHI), defined as the sum of the squared resource shares, $\sum_{i \in \mathcal{N}} (r_i/R)^2$. As Proposition 7 shows, majorization-induced increases in the resource-based HHI can cause a market that is efficient to become inefficient. This provides some support for spectrum aggregation limits that restrict the share of spectrum licenses for a geographic area that can be held by a single mobile wireless carrier.[22]

These results offer guidance for efficiency-enhancing divestitures. Specifically, divestitures that move a market towards having endowments $\mathbf{r}^*$ offer the potential for efficiency gains. If firms are symmetric in terms of their maximum demands and distributions, then divestitures that increase the symmetry of endowments do no harm in the sense that they weakly relax the budget constraint (and the first-best remains possible if it was possible prior to the change), and such divestitures potentially cause the first-best to be possible when it was not prior to the change.

Applying Proposition 7 to the question of divestitures, we have the following corollary:

Corollary 1. *With symmetric distributions and maximum demands, divestitures that increase the symmetry of endowments do no harm and potentially cause the first-best to be possible when it was not prior to the change.*

Further, if a competition authority can only require a divestiture of resources from one particular firm, perhaps because that firm has taken actions that require approval from the authority, to no more than one other firm, where that other firm must already have some upstream activity, i.e., it has a positive resource endowment, then we can identify a divestiture that maximally relaxes the no-deficit constraint. Specifically, if the designated firm has the j-th highest endowment, then only possible divestitures that one need consider are those that equalize the endowments of the designated firm and a firm with a lower, positive endowment (Thon and Wallace, 2004), as long as the maximum demands of the other firms allow that. That is, in considering a divestiture from firm 1 to firm 2 with $r_2 \in (0, r_1)$, one would consider a shift of resources from firm 1 to firm 2 of

$$\min\left\{\frac{r_1 - r_2}{2}, \quad k_2 - r_2\right\}.$$

Any divestiture would need to go to a vertically integrated firm because those are the firms with positive endowments and “room” to increase their endowments.

Figure 1(c) illustrates divestiture effects. It takes as the starting scenario the acquisition by firm 1 of all of the resources of firms 2 and 3, with resulting resource vector $\mathbf{r} = (\frac{3}{5}, 0, 0, \frac{1}{5}, \frac{1}{5})$. We then consider divestitures from firm 1, which, under the requirement that divestitures must go to other vertically integrated firms, can only go to firms 4 and 5.

[22]U.S. limits are embodied in 47 CFR §20.6 “CMRS spectrum aggregation limit.” Related to U.S. limits, see Cramton (2013), related to Europe, see Gretschko et al. (2013), and related to Australia, see ACCC (2018).

If divestitures are made equally to firms 4 and 5, then as shown in Figure 1(c), a divestiture of a fraction of the amount that firm 1 previously acquired from firm 3 (e.g., $\Delta = \frac{2}{15}$) is sufficient to restore the possibility of the first-best. Indeed, that divestiture produces a value for Π^e that is not far from what is achieved by an equal reallocation of resources among the three remaining vertically integrated firms, which would maximally relax the no-deficit constraint given that firms 2 and 3 are no longer vertically integrated.

Turning to comparative statics with respect to the total supply R, maximum demands k_i, and number of firms n, we find that unlike the redistribution of endowment considered above, those changes affect the interim expected allocation rule and so also affect the firms' worst-off type functions. Of course, if symmetry is maintained both before and after the change, the first-best remains possible both before and after. For example, given a symmetric setup, if R increases, with the additional supply being distributed equally to the firms, then the first-best remains possible. Similarly, if the common maximum demand k increases, then again the first-best remains possible. And, if one of the firms is replicated, then symmetry is again maintained, and so the first-best remains possible.

5 Boundaries of the firms

We now analyze what sets of ownership structures are stable in the sense that they would be the ones one expect to see when all gains from bilateral transfers of resources at the ex ante stage, that is, prior to the realization of private information, are exhausted.

The section is structured as follows. We first derive the second-best mechanism required to determine payoffs for $\mathbf{r}$ such the first-best is not possible. Second, we analyze what degrees of vertical integration are both consistent with exhausting gains from trade and the first-best. Then we analyze in some detail the case with two firms, before turning to the case with three or more firms. Readers primarily interested in the case of the two firms can skip the section on second-best mechanisms because, as we show, only a characterization of the first-best is required for that case. Before embarking, we provide background on the theory of firm and on theories of raising rivals' costs.

5.1 Background

Determinants of firm boundaries is a longstanding topic of study in economics, with a key question being why we observe so much economic activity inside firms if markets are such powerful and effective mechanisms for allocating scarce resources (Holmström and Roberts, 1998). Coase (1937) argues that firm boundaries can be explained by efficiency considerations, including coordination problems and transaction costs. Others, including Klein et al. (1978), Williamson (1975), Williamson (1985), Grossman and Hart (1986), and Hart and Moore (1990), explain firm boundaries by incentives, including hold-up problems. In all of these cases, the focus has been on explaining the extent of vertical integration. See Atalay et al. (2019) for empirical evidence that firm boundaries are an economically significant barrier to trade.

A "raising rivals' costs" theory of harm argues that following vertical integration, the integrated firm will charge more to external buyers for the inputs that it controls. The profitability of such a strategy usually relies on diversion of downstream customers to the integrated firm (see, e.g., Salop and Scheffman, 1983, 1987; Ordover et al., 1990). Raising rivals' costs theories have played a prominent, and sometimes controversial, role in antitrust practice (see, e.g., Coate and Kleit, 1990; Salop, 2017)

5.2 Second-best program

To derive the stable sets of ownership structures, or equivalently, the boundaries of the firms, one needs in general to consider $\mathbf{r}$ such that the first-best is not possible. In other words, one needs to know what is the second-best. In what follows, therefore, we derive the second-best mechanism, which builds on Lu and Robert (2001) and Loertscher and Wasser (2019).

If $\Pi^e(\mathbf{r}) < 0$, then the market only achieves the second-best. The relevant Lagrangian for the problem of maximizing expected social surplus subject to incentive compatibility and no deficit, assuming that individual rationality binds, is

$$\mathcal{L} \equiv \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \overbrace{\Big(\theta_i Q_i(\boldsymbol{\theta}) - (\Psi_i(\theta_i,\hat{\theta}_i)Q_i(\boldsymbol{\theta}) - \hat{\theta}_i r_i)\Big)}^{\text{firm } i\text{'s surplus}} + \rho \sum_{i\in\mathcal{N}} \Big(\overbrace{\underbrace{\Psi_i(\theta_i,\hat{\theta}_i)}_{\text{virtual type}} Q_i(\boldsymbol{\theta}) - \hat{\theta}_i r_i}^{\text{payment by firm } i}\Big)\right],$$

where ρ is the Lagrange multiplier on the no-deficit constraint.

Given worst-off types $\hat{\boldsymbol{\theta}}$ and ρ, one can solve for the allocation rule pointwise

$$Q_i^*(\boldsymbol{\theta};\hat{\boldsymbol{\theta}},\rho).$$

This involves working with the ironed virtual type functions for any vertically integrated firms (see Appendix A.3). Using $\mathbf{Q}^*$, we can define the expected budget surplus $\Pi^*(\mathbf{r};\hat{\boldsymbol{\theta}},\rho)$, and then we can solve for $\hat{\boldsymbol{\theta}}$ and ρ that deliver $q_i^*(\hat{\theta}_i;\hat{\boldsymbol{\theta}},\rho) = r_i$ and $\Pi^*(\mathbf{r};\hat{\boldsymbol{\theta}},\rho) = 0$.

5.3 Vertical integration consistent with profitability and the first-best

In this section, we consider how market structure affects the profitability of ex ante resource acquisitions and which acquisitions are most profitable for the firms involved, focusing on acquisitions that do not disrupt the feasibility of an efficient allocation. As we show, an acquisition that does not disrupt the efficiency of the allocation is profitable for the acquiring firm improves the firm's outside option and so improves the firm's expected payoff from participation in the mechanism.

For $i \in \mathcal{N}$, let $\eta_i \in [0,1]$ denote firm i's share of any budget surplus, where $\sum_{i\in\mathcal{N}} \eta_i = 1$. We focus on the mechanism $\langle \mathbf{Q}^e, \mathbf{M}^e \rangle$, where $\mathbf{Q}^e$ is the first-best allocation rule and $\mathbf{M}^e$ is a payment rule such the budget surplus is divided among the firms according to their shares:

$$\mathbb{E}_{\theta_i}[m_i^e(\theta_i;\mathbf{r})] = \mathbb{E}_{\theta_i}[\Psi_i(\theta_i, \hat{\theta}_i^e) q_i^e(\theta_i)] - r_i \hat{\theta}_i^e - \eta_i \Pi^e(\mathbf{r}),$$

which implies that the mechanism satisfies incentive compatibility, individual rationality, and no deficit if and only if $\Pi^e(\mathbf{r}) \geq 0$.

Denote firm i's expected payoff from participation in the mechanism, as a function of $\mathbf{r}$, as

$$u_i^e(\mathbf{r}) \equiv \mathbb{E}_{\theta_i}[\theta_i q_i^e(\theta_i) - m_i^e(\theta_i;\mathbf{r})].$$

If $\mathbf{r}$ is such that $\Pi^e(\mathbf{r}) \geq 0$ and firm 1 acquires the endowment of firm 2 (assuming that $r_1 + r_2 \leq k$) and for $\hat{\mathbf{r}} \equiv (r_1 + r_2, 0, r_3, \ldots, r_n)$ we have $\Pi^e(\hat{\mathbf{r}}) \geq 0$, then the change in firm 1's expected payoff is

$$\begin{aligned}
u_1^e(\hat{\mathbf{r}}) - u_1^e(\mathbf{r}) &= \mathbb{E}_{\theta_1}[m_1^e(\theta_1;\mathbf{r}) - m_1^e(\theta_1;\hat{\mathbf{r}})] \\
&= (r_1 + r_2)\hat{\theta}_1^e(\hat{r}_1) - r_1 \hat{\theta}_1^e(r_1) + \eta_1(\Pi^e(\hat{\mathbf{r}}) - \Pi^e(\mathbf{r})) \\
&= r_1(\hat{\theta}_1^e(\hat{r}_1) - \hat{\theta}_1^e(r_1)) + r_2 \hat{\theta}_1^e(\hat{r}_1) + \eta_1(\Pi^e(\hat{\mathbf{r}}) - \Pi^e(\mathbf{r})),
\end{aligned}$$

which is positive if $\eta_1 = 0$ because $\hat{\theta}_i^e(\cdot)$ is increasing in the relevant range.

Thus, an acquisition that does not disrupt the efficiency of the allocation is profitable for an acquiring firm that does not capture any share of the budget surplus. In that case, the acquisition improves the firm's outside option and so improves the firm's expected payoff from participation in the mechanism. Further, acquisitions of larger endowments are more profitable.

For firm 2, the firm whose endowment is being acquired, we have

$$\begin{aligned} u_2^e(\hat{\mathbf{r}}) - u_2^e(\mathbf{r}) &= \mathbb{E}_{\theta_2}[m_2^e(\theta_2; \mathbf{r}) - m_2^e(\theta_2; \hat{\mathbf{r}})] \\ &= -r_2 \hat{\theta}_2^e(r_2) + \eta_2(\Pi^e(\hat{\mathbf{r}}) - \Pi^e(\mathbf{r})), \end{aligned}$$

which is negative if $\eta_2 = 0$ and if also if $\eta_2 > 0$ and $r_1 \geq r_2$ because then $\Pi^e(\hat{\mathbf{r}}) - \Pi^e(\mathbf{r}) < 0$.

For the transaction to occur, we need the gain to the acquirer to be greater than or equal to the loss to the acquiree. That is, assuming that $\eta_1 = \eta_2 = 0$, we need

$$0 \leq r_1(\hat{\theta}_1^e(\hat{r}_1) - \hat{\theta}_1^e(r_1)) + r_2(\hat{\theta}_1^e(\hat{r}_1) - \hat{\theta}_2^e(r_2)),$$

which holds if, for example, the firms have symmetric distributions and maximum demands because then the functions $\hat{\theta}_1^e$ and $\hat{\theta}_2^e$ are the same (and nondecreasing), so $\hat{\theta}_1(\hat{r}_1) \geq \hat{\theta}_2(r_2)$.

Defining endowments $\mathbf{r}$ to be *stable* if there are no mutually beneficial pairwise transactions and *first-best permitting* if $\Pi^e(\mathbf{r}) \geq 0$, then we have the following contrasting results, the first for the case of two firms and the second for the case of more than two firms:

Proposition 8. *For $n = 2$, mutually beneficial transactions of endowments exist if and only if $\Pi^e(\mathbf{r}) < 0$; $\mathbf{r}$ is stable if and only if it is first-best permitting.*

Proof. See Appendix B.

And, now for more that two firms, we have:

Proposition 9. *For $n \in \{3, 4, \dots\}$, a mutually beneficial pairwise transaction of endowments exists if $\Pi^e(\mathbf{r}) > 0$ and there are two vertically integrated firms i and j with $\eta_i + \eta_j < 1$; that is, $\mathbf{r}$ is stable and strictly first-best permitting only if either (a) there is exactly one vertically integrated firm or (b) there are exactly two vertically integrated firms i and j with $\eta_i + \eta_j = 1$.*

Proof. See Appendix B.

Propositions 8 and 9 imply that that with only two firms, one expects mutually beneficial resource transactions to allow the first-best to be achieved, but that with more than two firms, the first-best need not be achieved.

5.4 Firm boundaries with two firms

Given Proposition 8, for the case of two firms, the set of stable firm boundaries is the same as the set of first-best permitting endowments. Thus, the case with $n = 2$ agents offers reasonably clean and transparent results.

With only two firms, the ex ante resource allocation is also one-dimensional, and we let r denote firm 1's initial resource allocation, implying that firm 2's is $R-r$. Using Proposition 1 and the definitions of the virtual type functions, expected revenue under the first-best with binding individual rationality, $\hat{\Pi}^e(r) \equiv \Pi^e(r, R-r)$, can be written as

$$\begin{aligned}\hat{\Pi}^e(r) = & \int_{\underline{\theta}}^{\hat{\theta}_1(r)} \Psi_1^S(x) q_1(x) dF_1(x) + \int_{\hat{\theta}_1(r)}^{\overline{\theta}} \Psi_1^B(x) q_1(x) dF_1(x) - r\hat{\theta}_1(r) \\ & + \int_{\underline{\theta}}^{\hat{\theta}_2(r)} \Psi_2^S(x) q_2(x) dF_2(x) + \int_{\hat{\theta}_2(r)}^{\overline{\theta}} \Psi_2^B(x) q_2(x) dF_2(x) - (R-r)\hat{\theta}_2(r),\end{aligned}$$

where here we write firm 2's worst-off type, $\hat{\theta}_2(r)$, as a function of firm 1's endowment, which means that $\hat{\theta}_2(r)$ is decreasing in r. Given r, firm 1 worst-off type $\hat{\theta}_1(r)$ satisfies $q_1(\hat{\theta}_1(r)) = r$ and firm 2's worst-off type $\hat{\theta}_2(r)$ satisfies $q_2(\hat{\theta}_2(r)) = R - r$.

Differentiating $\hat{\Pi}^e(r)$ with respect to r, it is straightforward to show that $\hat{\Pi}^e(r)$ is strictly concave for r in the relevant range.

Lemma 3. *For $n = 2$, $\hat{\Pi}^e(r)$ is strictly concave, $\frac{d^2\hat{\Pi}^e(r)}{dr^2} < 0$, for $r \in (R - k_2, k_1)$.*

Proof. See Appendix B.

To define a set of benchmark values, take the case of $k_1 = k_2 = R$. Within that case, let $\underline{r} \in (0, R)$ and $\overline{r} \in (0, R)$ be the smallest and largest r such that $\Pi^e(r) = 0$, respectively. That these are bounded away from $\underline{\theta}$ and $\overline{\theta}$ follows from the impossibility theorem of Myerson-Satterthwaite, and that they are bounded away from r^* follows because, by Lemma 2, $\Pi^e(\mathbf{r}^*) > 0$. Further, let $\underline{\theta}_i = \hat{\theta}_i(\underline{r})$ and $\overline{\theta}_i = \hat{\theta}_i(\overline{r})$ be the worst-off types associated with $\underline{r}$ and $\overline{r}$, respectively (writing these as functions of firm 1's endowment). Because of the monotonicity properties of worst-off types and $\underline{r} < r^* < \overline{r}$, we have

$$\underline{\theta}_1 < \hat{\theta} < \underline{\theta}_2 \quad \text{and} \quad \overline{\theta}_2 < \hat{\theta} < \overline{\theta}_1. \tag{6}$$

For the more general setting with maximum demands that are possibly less than R and different from each other, we write $\underline{r}(\mathbf{k})$ and $\overline{r}(\mathbf{k})$ to make the dependence on $\mathbf{k}$ explicit. The values $\underline{r}(\mathbf{k})$ and $\overline{r}(\mathbf{k})$ are such that the first-best is possible if and only if $r \in [\underline{r}(\mathbf{k}), \overline{r}(\mathbf{k})]$. These are the boundaries of the firms insofar as these are the only ownership structures one should observe if firms transact at the ex ante stage until all gains from transactions are exhausted. We then have the following result:

Proposition 10. *For $n = 2$, the boundaries of the firms are*

$$\underline{r}(\mathbf{k}) = k_1 - (K - R)F_1(\underline{\theta}_2) \quad \text{and} \quad \overline{r}(\mathbf{k}) = k_1 - (K - R)F_1(\overline{\theta}_2).$$

Proof. See Appendix B.

The difference

$$\overline{r}(\mathbf{k}) - \underline{r}(\mathbf{k})$$

is naturally called *the range of first-best permitting endowments* since the first-best is possible if and only if r_1 is in the interval $[\underline{r}(\mathbf{k}), \overline{r}(\mathbf{k})]$. Together with (6), Proposition 10 implies that this range is an increasing function of the excess demand $K - R$ because $\overline{r}(\mathbf{k}) - \underline{r}(\mathbf{k}) = (K - R)(F_1(\underline{\theta}_2) - F_1(\overline{\theta}_2))$, where $\underline{\theta}_2 > \overline{\theta}_2$ and neither $\underline{\theta}_2$ nor $\overline{\theta}_2$ depends on K or R. This gives us the following corollary to Prop. 10:

Corollary 2. *For $n = 2$, the greater is the excess demand, $K - R$, the greater is the range of first-best permitting endowments.*

Proof. See Appendix B.

As shown in Proposition 2, firm boundaries depend on market structure, where greater excess demand results in a larger range of first-best permitting resource shares. For example, with $n = 2$, $k_1 = k_2 = R = 1$, and $F_1 = F_2$ uniform on $[0, 1]$, the first-best is possible if $r_1 \in [0.21, 0.79]$ (as shown by Cramton et al. (1987), this is the set of possible ownership shares for a "dissolvable partnership").[23] This interval of first-best permitting shares defines the stable set of degrees of vertical integration. Retaining the assumptions that $n = 2$, $k_1 = R = 1$, and common uniform distributions, we can allow firm 2's maximum demand k_2 to vary, with $k_2 \in (0, 1]$. We illustrate this in Figure 2(a).

[23] For this case, $\hat{\Pi}^e(r) = -r^2 + r - 1/6$ and so $\underline{r} = \frac{3-\sqrt{3}}{6} = 0.2113$ and $\overline{r} = \frac{3+\sqrt{3}}{6} = 0.7887$.

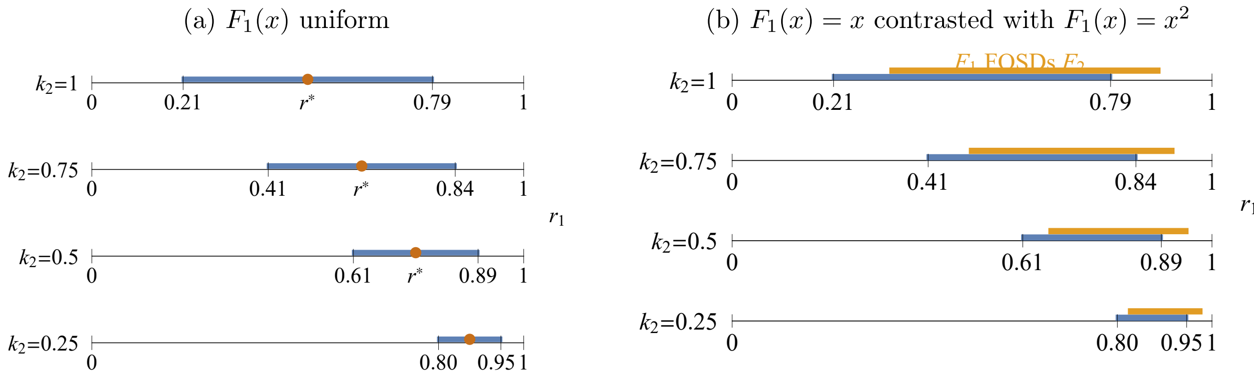

Figure 2: Range of first-best permitting endowments. Bars indicate the range of r_1 such that the first-best is possible, with $r_2 = 1 - r_1$. Assumes $n = 2$, $k_1 = R = 1$, and F_2 uniform on $[0, 1]$, with F_1 and k_2 varying as shown.

As illustrated in Figure 2(a), because with $n = 2$ profitable transactions move the market to first-best permitting endowments, we obtain the prediction that the market will be characterized by vertically integrated firms, with bounds on the extent of vertical integration as shown in the figure. The smaller is k_2 (lower is the overall excess demand), the tighter is the stable set of endowments.

In addition, Figure 2(b) illustrates the effect of a FOSD improvement in firm 1's distribution. For $F_1(x) = x^s$ and $F_2(x) = x$, both $\underline{r}(\mathbf{k})$ and $\overline{r}(\mathbf{k})$ increase with s, which represents a FOSD improvement in firm 1's distribution, as long as s is sufficiently close to 1. Thus, the effect of such an improvement in firm 1's distribution is to shift the stable set towards higher endowments for firm 1 and corresponding lower endowments for firm 2.

These comparative statics also offer a novel perspective on the classic case study of General Motor's acquisition of 60 percent of its supplier Fisher Body in 1919, which was followed by full vertical integration between the two firms in 1926. Of course, our model is stripped of many of the intricate details that have shaped the debate surrounding that episode (see e.g. Klein et al., 1978; Klein, 2000, 2007; Coase, 2000; Casadesus-Masanell and Spulber, 2000; Roider, 2006) and is therefore agnostic as to the relative merits of the explanations that have been put forth. What our model does suggest, however, is that growth in demand for General Motor's products, which was not matched by the growth of independent demand for Fisher Body's input, made a much larger degree of vertical integration optimal in 1926 than before to the 1920s.[24]

[24]Production by Chevrolet, GM's flagship division, increased by more than 25 percent per year between 1918 and 1929, while the production of GM's biggest rival, Ford, grew at an annual rate of less than 10 percent during that period (see U.S. Automobile Production Figures).

5.5 Firm boundaries with three or more firms

With $n > 2$, the implications for firm boundaries are more complex. For example, starting from strictly first-best permitting shares, the sharing of the positive budget surplus that arises in that case may create opportunities for profitable integration by two firms (harming the third). This takes us to the boundary of the first-best permitting endowments. From that point, "raising rivals costs" effects associated with the majorization of the endowment vector move the market to the second-best region. This can be expected to continue until no further profitable majorization is possible. This raises questions whether markets might be characterized by continued churn involving episodes of integration and disintegration, or whether the only stable sets involve boundary endowments—i.e., buyer and sellers—implying that the first-best is not possible.

To illustrate, assume $n = 3$, $k = 1$, $R = 1$, and types that are uniformly distributed on $[0, 1]$. Define $\hat{r}$ such that $\Pi^e(\hat{r}, \hat{r}, 1 - 2\hat{r}) = 0$, which implies that endowment vector $(\hat{r}, \hat{r}, 1 - 2\hat{r})$ is on the boundary of the first-best permitting endowments ($\hat{r} = 0.12$). Allow firm 1 to acquire portion Δ of firm 2's endowment, producing post-integration endowment vector

$$(\hat{r} + \Delta, \hat{r} - \Delta, 1 - 2\hat{r}).$$

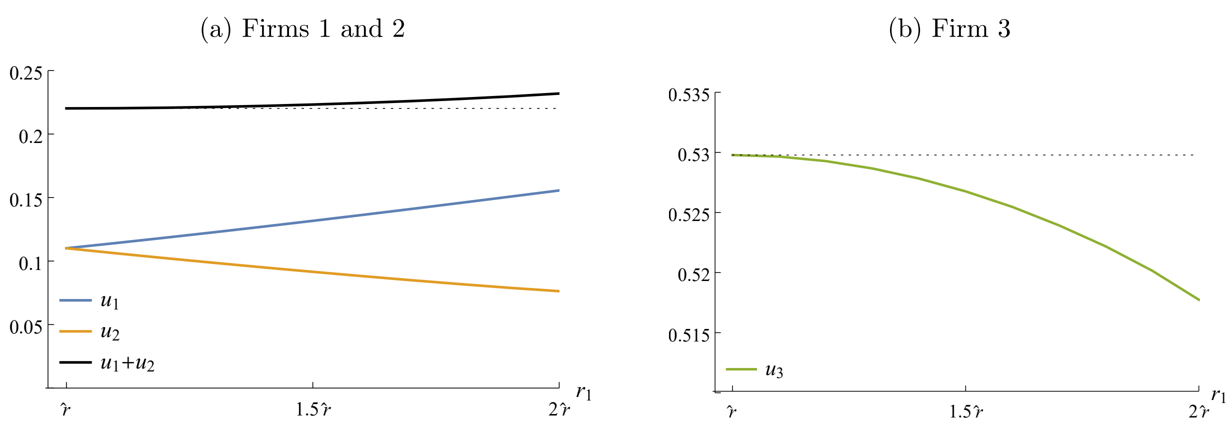

Figure 3: Effects of shifts in endowment from firm 2 to firm 1.

As shown in Figure 3, as the resource endowment is shifted from firm 2 to firm 1, firm 1 is better off, firm 2 is worse off, and firms 1 and 2 are jointly better off. Meanwhile, the external firm, firm 3, is worse off and, not shown in the figure, social surplus decreases.

Exploring further, consider the starting point $(\hat{r}, \hat{r}, 1 - 2\hat{r})$ discussed above. Firms 1 and 2 increase their joint surplus if firm 1 acquires all of firm 2's endowment, resulting in: $(2\hat{r}, 0, 1 - 2\hat{r})$. Then, firms 1 and 3 increase their joint surplus if firm 3 acquires $1.33\hat{r}$ units of firm 1's endowment, resulting in: $(0.67\hat{r}, 0, 1 - 0.67\hat{r})$. At this point, further endowment shifts are not jointly profitable.

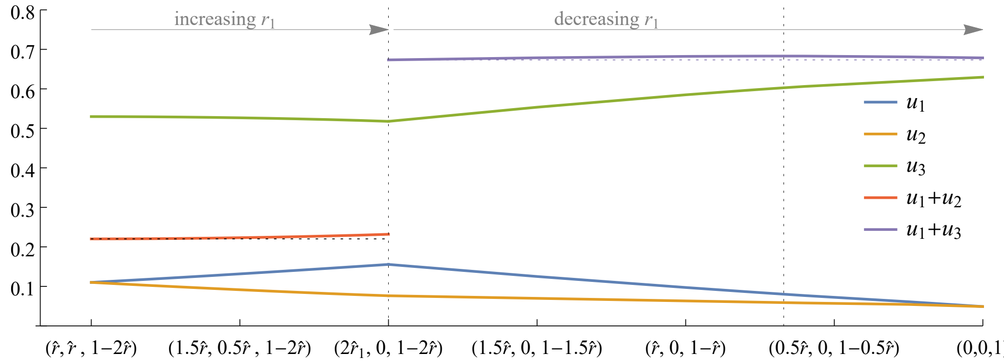

Figure 4: Illustration of shifts in endowments and implied firm boundaries.

In this example, market forces produce: 1 buyer and 2 asymmetric vertically integrated firms and outcomes away from the first-best.

Our results on the boundaries of the firms have implications for topics of interest in industrial organization and antitrust, particularly, as we discuss here, the notion of vertical integration raising rivals' costs.

Building on our prior analysis, we have the following result:

Proposition 11. *For* $n \in \{3, 4, \dots\}$, *if* $\Pi^e(\mathbf{r}) = 0$ *and there exist two vertically integrated firms, then a mutually beneficial pairwise transactions of endowments exists that results in resource vector* $\mathbf{r}'$ *with* $\Pi^e(\mathbf{r}') < 0$.

Proof. See Appendix B.

The requirement in Proposition 11 that there be two vertically integrated firms arises because the result relies on two firms being able to shift resources from the one with the weakly lower worst-off type to the one with the weakly higher worst-off type. The only way for this to occur is with two vertically integrated firms. To see this, note that a buyer's worst-off type is $\underline{\theta}$, so it always has the lowest worst-off type, but it has no resources to shift to another firm. A seller's worst-off type is $\overline{\theta}$, so it always has the highest worst-off type, but it has no demand for to additional resources. With two vertically integrated firms, each has resources to sell and capacity to buy more, so a shift of resources from the firm with weakly lower worst-off type to the one with the weakly higher worst-off type is feasible. A shift in resources of this type, reduces Π^e below zero by Proposition 6. To show that such a transactions between vertically integrated firms is mutually beneficial for those firms, we must account for the overall loss of efficiency in the market as a result of the first-best no longer being possible. But because the other firms absorb some of the impact of the loss of efficiency, the first-order effect of the shift in endowments for the two trading firms is to increase the sum of their outside options, which reduces their payments to the mechanism and leaves them better off.

Of course, the reduction in efficiency is the only effect on the rival firms, so they are worse off, which gives us the following corollary:

Corollary 3. *With $n > 2$ and $\Pi^e(\mathbf{r}) \geq 0$, vertically integrated firms have an incentive for transactions that harm rivals and reduce social surplus below the first-best.*

Because vertically integrated firms have an incentive for transactions that reduce the efficiency of the market below the first-best and that these transactions harm rivals. This suggests that concerns related to raising rivals' costs can be tied to the overall efficiency effects of shifts of resources among vertical integrated firms. This result contrasts with the finding of Farrell and Shapiro (1990) that in a Cournot setup, a profitable reallocation of capital that reduces welfare benefits the rivals, who will increase their output in response to the contraction in total output by the transacting firms. Thus, they do not get a "raising rivals' costs" effect because the welfare reduction is born entirely by the transacting firms. In contrast, in our setup, the reduction in the efficiency of the market affects all firms.

The contrast in the boundaries of the firms between the case with two firms and the one with more than two firms (see Propositions 8 and 11) is stark, and perhaps a little disturbing. After all, one may wonder, why should such a seemingly small difference make such a big difference? However, Propositions 8 and 11 have precursors in the complete-information literature and debate on the Coase Theorem, with Aivazian and Callen (1981) arguing that with more than two agents, the emptiness of the core may render any efficient bargaining impossible, and Coase (1981) countering that his argument (Coase, 1960) was based on the case with two agents. With that in mind, Propositions 8 and 11 provide an incomplete-information formalization of these opposing views and forces.

6 Extensions

In this section, we analyze various extensions of the model. We first study individual firms' incentives to increase their own resources at some cost from an outside supplier and compare these to what the social planner would want the firms to do. Second, allowing the firms to invest to improve their type distributions, we show that the endowments that permit efficient reallocation also induce efficient investments (which also allows us to reconsider the GM–Fisher Body case). Third, building on Loertscher and Marx (2022), we analyze incomplete information bargaining when the market mechanism may favor some agents relative to others.

6.1 Increasing total supply

In this section, we consider the effects of additions to the total supply of resources. We find that additions to the resources of the largest firm can render the first-best no longer possible. Specifically, for $n = 2$, an increase in r_1 means a decrease in excess demand and so, by Proposition 2, the range of first-best permitting r_1 shrinks. Thus, starting from the maximum r_1 in the first-best permitting interval, the increase in r_1 results in endowments outside the set of first-best permitting endowments, with negative implications for all firms.[25] Thus, the marginal cost at which a firm is willing to buy incremental endowment exceeds what the planner would be willing to pay to endow that firm with incremental endowment.

For endowments that are strictly first-best permitting, a marginal increase in one firm's endowment need not prevent the first-best, but it does change the first-best allocation rule itself because the additional resources must be allocated (recall that we assume that $K > R$ so that there is demand for additional resources). The change in the allocation rule weakly increases the worst-off types of all firms, with a further strict increase in the worst-off type of the firm receiving the additional resources. The additional resources are allocated efficiently and so increase social surplus. On net, the changes in the firms' worst-off types benefit the firm receiving the resources. It is straightforward to show that that the marginal cost at which a firm is willing to buy incremental endowment can exceed what the planner would be willing to pay to endow that firm with incremental endowment.[26] Thus, firms can have excess incentives to increase their own resources.

6.2 Investment

We now allow firms to make investments that affect their type distributions without changing the supports. Specifically, given investment $e_i \geq 0$, $f_i(\theta_i, e_i)$ with support $[\underline{\theta}, \overline{\theta}]$ denotes the density of i's type, and given $\mathbf{e} = (e_1, \ldots, e_n)$, we denote by $f(\boldsymbol{\theta}; \boldsymbol{e}) \equiv \times_{i \in \mathcal{N}} f_i(\theta_i; e_i)$ the joint density of $\boldsymbol{\theta}$. For each firm i, the cost of investment is given by $c_i(e_i)$. The social planner's

[25] This contrasts with the result of Farrell and Shapiro (1990) for an oligopoly setup that, under certain conditions, an increase in the assets held by the largest firm increases welfare, and an increase in the assets held by a sufficiently small firm reduces social surplus. The intuition for their result is that in their setup, an increase in a firm's assets shifts output towards that firm. If the firm is already the largest, then it has the lowest marginal cost, so this is a shift in production towards the lowest-cost firm, which increases social surplus. An increase in the assets of a sufficiently small firm shifts output towards a higher-cost firm, which reduces social surplus.

[26] For example, with $n = 2$, $k_1 = k_2 = 1$, and types that are uniformly distributed on $[0, 1]$, for $\mathbf{r}$ such that $\Pi^e(\mathbf{r}) > 0$, the derivative of firm 1's expected payoff with respect to r_1 is $5/12$ and the derivative of social surplus with respect to r_1 is only $1/3$.

problem is

$$\max_{\mathbf{e}} \int_{[\underline{\theta},\overline{\theta}]^n} \boldsymbol{\theta} \cdot \mathbf{Q}^e(\boldsymbol{\theta}) f(\boldsymbol{\theta}; \mathbf{e}) d\boldsymbol{\theta} - \sum_{i \in \mathcal{N}} c_i(e_i),$$

where $\mathbf{Q}^e(\boldsymbol{\theta}) = (Q_1^e(\boldsymbol{\theta}), \dots, Q_1^e(\boldsymbol{\theta}))$, $Q_i^e(\boldsymbol{\theta}) \in [0, k_i]$ is the efficient allocation of i given $\boldsymbol{\theta}$, and $\boldsymbol{\theta} \cdot \mathbf{Q}^e(\boldsymbol{\theta})$ denotes the inner product. We assume that the solution to the planner's problem $\overline{\mathbf{e}}$ is unique and for all i characterized by the first-order condition

$$\int_{[\underline{\theta},\overline{\theta}]^n} \boldsymbol{\theta} \cdot \mathbf{Q}^e(\boldsymbol{\theta}) \frac{f(\boldsymbol{\theta}; \overline{\mathbf{e}})}{f_i(\theta_i, \overline{e}_i)} \frac{\partial f_i(\theta_i, \overline{e}_i)}{\partial e_i} d\boldsymbol{\theta} - c_i'(\overline{e}_i).$$

As shown by Liu et al. (2022), given $\overline{\mathbf{e}}$, there exists $\mathbf{r}$ such that first-best is possible without running a deficit. Moreover, assuming that investments are not observable and not contractible, given $\mathbf{r}$, the game in which firms first simultaneously choose investments and then participate in the market mechanism has an efficient Nash equilibrium $\mathbf{e}^*$, that is, an equilibrium satisfying $e_i^* = \overline{e}_i$ for all i (see Liu et al., 2022, Proposition 8). Thus, in this, like in many other incomplete information models, there is no tension between efficient investments and efficient bargaining or transactions—efficient bargaining implies that there is an equilibrium with efficient investments.

Applied to the GM–Fisher Body case, this would mean that the degree of vertical integration that became necessary to permit efficient transactions as the demand for GM's cars grew also aligned with incentives for efficient investments in type distributions.

6.3 Allowing for differences in bargaining power

Thus far, we have assumed that all agents have the same bargaining power insofar as the market mechanism maximized the equally weighted sum of agents' payoffs, subject to the various constraints. But it is of course conceivable, and in practice often argued, that some agents have more bargaining power than others. To accommodate for this possibility, we now adopt the market mechanism based on incomplete information bargaining of Loertscher and Marx (2022), adapting it to the setting in which a firm's position as a buyer or seller is determined endogenously, as in Lu and Robert (2001) and Loertscher and Wasser (2019).

We assign to each firm i a bargaining weight $w_i \in [0, 1]$, and we assume that at least one firm has a positive weight. We assume that the market mechanism maximizes the weighted sum of the firms' expected surpluses. That is, we assume a market mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, with $\mathbf{Q} : [\underline{\theta}, \overline{\theta}]^n \to [0, R]^n$ and $\mathbf{M} : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}^n$, where for reports $\boldsymbol{\theta}$, $Q_i(\boldsymbol{\theta})$ specifies the quantity allocated to firm i and $M_i(\boldsymbol{\theta})$ specifies the payment from firm i to the mechanism.[27] Feasibility requires that $\sum_{i\in\mathcal{N}} Q_i(\boldsymbol{\theta}) \leq R$. We focus on incentive compatible, individually rational mechanisms (see the appendix for formal definitions) that have *no deficit* in expectation, i.e., $\mathbb{E}_{\boldsymbol{\theta}}[\sum_{i\in\mathcal{N}} M_i(\boldsymbol{\theta})] \geq 0$, and assume that, subject to those constraints, $\langle \mathbf{Q}, \mathbf{M}\rangle$ maximizes

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} w_i\left(\theta_i \min\{k_i, Q_i(\boldsymbol{\theta})\} - M_i(\boldsymbol{\theta})\right)\right],$$

where firm i's outside option associated with not participating in the mechanism is $\theta_i r_i$.

To incorporate bargaining weights, it is useful to define *weighted* virtual type functions:

$$\Psi^B_{i,\alpha}(\theta) \equiv \theta - (1-\alpha)\frac{1-F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi^S_{i,\alpha}(\theta) \equiv \theta + (1-\alpha)\frac{F_i(\theta)}{f_i(\theta)}$$

and the associated combined function $\Psi_{i,\alpha}(\theta, x)$.

The relevant Lagrangian for the problem of maximizing expected weighted surplus subject to no deficit, assuming that individual rationality binds for all firms, is

$$\mathcal{L} \equiv \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} w_i \overbrace{\left((\theta_i - \Psi_{i,0}(\theta_i, \hat{\theta}_i))Q_i(\boldsymbol{\theta}) + \hat{\theta}_i r_i\right)}^{\text{firm } i\text{'s surplus}} + \rho\left(\overbrace{\sum_{i\in\mathcal{N}} \Psi_{i,0}(\theta_i, \hat{\theta}_i)Q_i(\boldsymbol{\theta}) - \hat{\theta}_i r_i}^{\text{mechanism budget surplus}}\right)\right],$$

where ρ is the Lagrange multiplier on the no-deficit constraint. The solution value for the Lagrange multiplier must have $\rho \geq \max \mathbf{w}$ because otherwise the Lagrangian is maximized by running an infinite deficit, in violation of the no-deficit constraint.

We can rewrite this as

$$\mathcal{L} \equiv \rho\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \Psi_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)Q_i(\boldsymbol{\theta})\right] + \sum_{i\in\mathcal{N}}(w_i - \rho)\hat{\theta}_i r_i.$$

Given $\rho \geq \max \mathbf{w}$ and $(\hat{\theta}_i)_{i\in\mathcal{N}^V}$ (with $\hat{\theta}_i = \overline{\theta}$ for all $i \in \mathcal{N}^S$ and $\hat{\theta}_i = \underline{\theta}$ for all $i \in \mathcal{N}^B$), one can maximize $\mathcal{L}$ with respect to $\mathbf{Q}$ pointwise, subject to each Q_i being nondecreasing, which is necessary and sufficient for incentive compatibility, by choosing $\mathbf{Q}$ to maximize (see Loertscher and Wasser, 2019)

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)Q_i(\boldsymbol{\theta})\right],$$

where, because $\Psi_{i,\alpha}$ is not monotone, we employ standard ironing techniques to this function

[27] By the Revelation Principle, a focus on direct mechanisms is without loss of generality.

(see, e.g., Loertscher and Marx, 2020) to derive $\overline{\Psi}_{i,\alpha}$, which is the ironed version of $\Psi_{i,\alpha}$ and z_i is the ironing parameter (see Appendix A.3).

Specifically, for any $(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, \boldsymbol{\rho}, \mathbf{w})$, rank the ironed weighted virtual types $\overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)$ from largest to smallest, breaking ties at random so that individual rationality is satisfied. This gives us an ordering of firms, and we can define $\mathcal{I}$ to be the set of the first I firms in that order, where I is the maximum number of firms whose demand can be fulfilled, at least partially, starting with the firm with the highest ironed weighted virtual type and proceeding in sequence. Thus, I is defined so that the sum of the maximum demands for the first $I-1$ firms in the sequence is less than R and the sum of the maximum demands for the first I firms in the sequence is greater than or equal to R. Let I^* denote the index of the firm whose ironed weighted virtual type is I-th in the sequence. Thus, $\sum_{i\in\mathcal{I}\setminus\{I^*\}} k_i < R \leq \sum_{i\in\mathcal{I}} k_i$. (To reduce notation, we do not explicitly indicate the dependence of $\mathcal{I}$, I, and I^* on $(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, \rho, \mathbf{w})$.)

The allocation that maximizes $\mathcal{L}$ for a given $(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, \rho, \mathbf{w})$ is then given by

$$Q_i^*(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, \rho, \mathbf{w}) = \begin{cases} 0 & \text{if } i \in \mathcal{N}\setminus\mathcal{I}, \\ k_i & \text{if } i \in \mathcal{I}\setminus\{I^*\}, \\ R - \sum_{i\in\mathcal{I}\setminus\{\mathcal{I}^*\}} k_i & \text{if } i = I^*. \end{cases}$$

Given $Q_i^*(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, \rho, \mathbf{w})$, the worst-off types as a function of ρ and $\mathbf{w}$, $\hat{\boldsymbol{\theta}}^*(\rho, \mathbf{w})$, are given by for all $i \in \mathcal{N}$,

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^*(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^*(\rho, \mathbf{w}), \rho, \mathbf{w})] = r_i.$$

Then the solution value of the Lagrange multiplier, ρ^*, is the minimum value for ρ such that the no-deficit constraint is satisfied, i.e.,

$$\rho^* \equiv \min\left\{\rho \geq \max \mathbf{w} \mid \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \Psi_{i,0}(\theta_i, \hat{\theta}_i^*(\rho, \mathbf{w}))Q_i^*(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}^*(\rho), \rho, \mathbf{w}) - \hat{\theta}_i^*(\rho, \mathbf{w})r_i\right] \geq 0\right\}.$$

If there is a budget surplus at the first-best allocation, then that surplus is distributed to the firms with the largest bargaining weight. If more that one firm has the largest bargaining weight, we break ties by assuming that any surplus is allocated equally among those firms.

We illustrate in Figure 5 for the case of two firms how the payoffs under incomplete information bargaining vary with bargaining weights and endowments. For different values of r_1 and $r_2 = R - r_1$, the figure shows the expected gains from participation in the mechanism of firm 1 and firm 2 for the different possible bargaining weights. Letting $\Delta \equiv w_1 - w_2$, when $\Delta = 1$, firm 1 has all the bargaining power, and when $\Delta = -1$, firm 2 has all the bargaining power. The case of $\Delta = 0$ corresponds to equal bargaining weights.

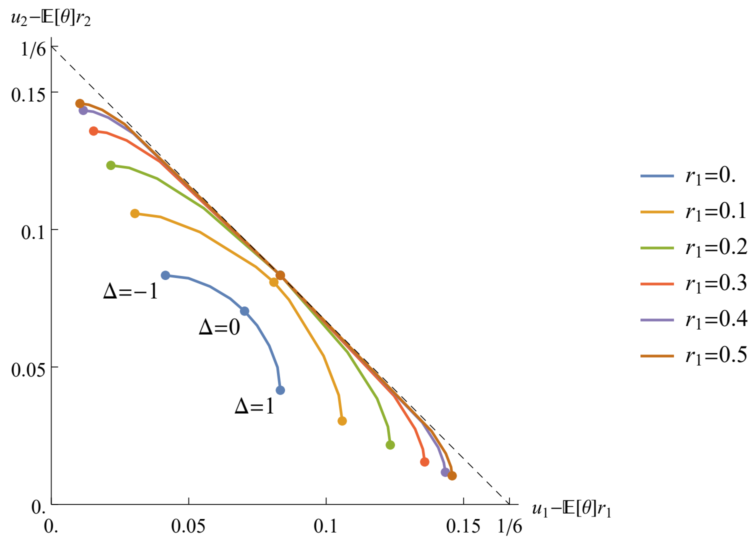

Figure 5: Frontier of gains from participation associated with the range of possible relative bargaining weights, with $\Delta \equiv w_1 - w_2$, for different values of r_1 as indicated and $r_2 = 1 - r_1$. Assumes $n = 2$, $k_1 = k_2 = 1$, and types that are uniformly distributed on $[0, 1]$.

As shown in Figure 5, an increase in a firm's relative bargaining weight increases its expected payoff, but can also decrease the social surplus. The first-best is possible if r_1 is sufficiently close to $1/2$ and Δ is sufficiently close to 0, i.e., the firms' endowments and bargaining weights are sufficiently symmetric. However, even with symmetric endowments, the first-best is not possible with extreme relative bargaining weights; and even with symmetric bargaining weights, the first-best is not possible with extreme endowments.

7 Discussion

Contract restrictions

The incomplete information setup assumes that transactions—horizontal or vertical mergers—occur before private information is realized. The only contracts available at that stage are simple transactions of property rights. In particular, no contracts are admissible that are contingent of future realizations of private information. So this is a restriction on the contracting space at the *ex ante* stage. Without it, the first-best would always be possible, but it would be also be hard to see how firms—that can bind themselves in these ways—are still independent entities.

Internal agency problems within vertically integrated firms

With the interpretation that every firm has demand for the input, and that costs for productive capacity only arise from the opportunity cost of *not* being able to serve that demand, vertical integration does *not* mean that the vertically integrated firm has to solve an internal agency problem: if firm i obtains some of firm j's productive capacity, i's opportunity cost for this capacity will be θ_i. This means that we are not simply assuming the agency problem internal to the vertically integrated firm away as does the literature, which is always subject to criticism—in particular, if one does not spell out how the firm solves this problem, it is actually a bit tricky to make the case that the (alleged) efficiencies are merger-specific because there is no agency problem internal to the firm. At the same time, vertical integration still eliminates a double-markup (or information rent) problem with the (newly) vertically integrated firm. Suppose that $k_i = k_j = 1 = r_j$ and $r_i = 0$ prior to integration. Prior to integration, firm i has to pay an information rent to firm j if it wants to consume a unit. Post integration, firm i does not have to pay an information rent because it owns the supply unit. But of course, now firm j is a buyer and has to pay an information rent if it wants to consume. In this setting, assuming that $n = 2$, transferring the supply unit from one firm to the other one does nothing to improve outcomes in the sense that the first-best remains elusive and the transfer merely changes to index of the firm who is the buyer and the firm who is the seller in the Myerson-Satterthwaite problem. The reason why partial vertical integration can be welfare improving, and indeed induce the first-best, is that it reduces overall information rents and thereby the aggregate value of outside options; in some sense, it reduces overall market power.

"Backward" vs "forward" vertical integration

It is tempting to interpret the differences arising in our setting between the cases where (i) a buyer acquires a supply unit (with no associated change in distributions) and (ii) a supplier acquires another buyer (in which case the distributions will have to adjust) as differences between "backward" and "forward" vertical integration. But this is not quite right because the direction of integration does not matter. In case (ii), it would be the same if a buyer acquired a supplier, which includes its downstream demand; and in case (i), it would be the same if a supply unit—defined as a productive capacity without access to downstream demand—acquired a buyer.

Mergers with both horizontal and vertical components

We have modeled vertical mergers as firm i acquiring the resources of firm j, producing a merged entity that has resources $r_i + r_j$. One could similarly define a horizontal merger that creates a merged entity has demand $k_i + k_j$ with an adjusted type distribution. Combined horizontal and vertical mergers can be viewed as reducing number of firms, in which case, in order to maintain comparability, one must adjust the designer's objective, including rescaling bargaining weights and market surplus shares.

Consumer surplus effects

Competition authority objectives may differ. In some settings, a focus on firm surplus is without loss of generality because firm surplus and consumer surplus are aligned. See Loertscher and Marx (2022) for a setup in which the resource input improves quality by a known multiplicative factor and thereby maintains alignment between firm surplus and consumer surplus.

8 Conclusions

There is a growing recognition of the need to derive theoretical predictions as well as policy prescriptions without arbitrary restrictions of the contracting space. With that in mind, the way results are derived can be as important as the results themselves. The independent private values model provides a framework that dispenses with arbitrary restrictions on the contracting space. By analyzing vertical integration in an incomplete information model, our paper contributes to the growing literature in Industrial Organization that uses the independent private values model to reconsider questions and problems of longstanding and concurrent interest.

Our model provides no basis for a presumption that vertical integration is either inherently good or bad as a certain degree of vertical integration is necessary and sufficient for the first-best. We also show that with two firms, the equilibrium level of vertical integration aligns with that which a benevolent social planner would choose. However, this goods news does not, generally, extend beyond two firms, suggesting that a fundamental tension between private and social incentives for vertical integration and an ongoing role for antitrust scrutiny. Moreover, our model provides a rationale as well as guidance for divestitures and a new perspective on the boundaries of the firms.

References

Aivazian, V. A. and J. L. Callen (1981): "The Coase Theorem and the Empty Core," *The Journal of Law & Economics*, 24, 175–181.

Atalay, E., A. Hortascu, M. J. Li, and C. Syverson (2019): "How Wide Is the Firm Border?" *Quarterly Journal of Economics*, 134, 1845–1882.

Australian Competition & Consumer Commission (2018): "Allocation Limits Advice for the 3.6 GHz Spectrum Allocation," Public Version.

Backus, M., T. Blake, B. Larsen, and S. Tadelis (2020): "Sequential Bargaining in the Field: Evidence from Millions of Online Bargaining Interactions," *Quarterly Journal of Economics*, 135, 1319–1361.

Backus, M., T. Blake, and S. Tadelis (2019): "On the Empirical Content of Cheap-Talk Signaling: An Application to Bargaining," *Journal of Political Economy*, 127, 1599–1628.

Baker, J. B., N. L. Rose, S. C. Salop, and F. Scott Morton (2019): "Five Principles for Vertical Merger Enforcement Policy," *Antitrust*, 33, 12–19.

Byrne, D. P., L. A. Martin, and J. S. Nah (2021): "Price Discrimination, Search, and Negotiation in an Oligopoly: A Field Experiment in Retail Electricity," University of Melbourne.

Casadesus-Masanell, R. and D. F. Spulber (2000): "The Fable of Fisher Body," *The Journal of Law & Economics*, 43, 67–104.

Che, Y.-K. (2006): "Beyond the Coasian Irrelevance: Asymmetric Information," Unpublished Lecture Notes, Columbia University.

Choné, P., L. Linnemer, and T. Vergé (2021): "Double Marginalization and Vertical Integration," CESifo Working Paper No. 8971.

Coase, R. (1960): "The Problem of Social Cost," *Journal of Law and Economics*, 3, 1–44.

Coase, R. H. (1937): "The Nature of the Firm," *Economica*, 4, 386–405.

——— (1981): "The Coase Theorem and the Empty Core: A Comment." *Journal of Law and Economics*, 24, 183–187.

——— (2000): "The Acquisition of Fisher Body By General Motors," *The Journal of Law & Economics*, 43, 15–32.

Coate, M. B. and A. N. Kleit (1990): "Exclusion, Collusion, and Confusion: The Limits of Raising Rivals' Costs," Federal Trade Commission, Working Paper No. 179.

Cramton, P. (2013): "The Rationale for Spectrum Limits and Their Impact on Auction Outcomes," Working Paper, University of Maryland.

Cramton, P., R. Gibbons, and P. Klemperer (1987): "Dissolving a Partnership Efficiently." *Econometrica*, 55, 615–632.

Delacrétaz, D., S. Loertscher, L. Marx, and T. Wilkening (2019): "Two-Sided Allocation Problems, Decomposability, and the Impossibility of Efficient Trade," *Journal of Economic Theory*, 179, 416–454.

Farrell, J. and C. Shapiro (1990): "Asset Ownership and Market Structure in Oligopoly," *RAND Journal of Economics*, 21, 275–292.

Federal Trade Commission (2021): "Federal Trade Commission Withdraws Vertical Merger

Guidelines and Commentary," Press Release, September 15, 2021.

Figueroa, N. and V. Skreta (2012): "Asymmetric partnerships," *Economics Letters*, 115, 268–271.

Gresik, T. and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48, 304–332.

Gretschko, V., S. Knapek, TWS Partners, and A. Wambach (2013): "Spectrum Aggregation Limits in Auctions with Spectrum below 1 GHz: The European Experience," Working Paper, University of Cologne.

Grossman, S. J. and O. D. Hart (1986): "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration," *Journal of Political Economy*, 94, 691–719.

Hart, O. and J. Moore (1990): "Property Rights and the Nature of the Firm." *Journal of Political Economy*, 98, 1119–1158.

Holmström, B. and J. Roberts (1998): "The Boundaries of the Firm Revisited," *Journal of Economic Perspectives*, 12, 73–94.

Kang, Z. Y. and E. V. Muir (2022): "Contracting and vertical control by a dominant platform," Working paper, Stanford University.

Klein, B. (2000): "Fisher-General Motors and the Nature of the Firm." *Journal of Law & Economics*, 43, 105–141.

——— (2007): "The Economic Lessons of Fisher Body-General Motors," *International Journal of the Economics of Business*, 14, 1–36.

Klein, B., R. G. Crawford, and A. A. Alchian (1978): "Vertical Integration, Appropriable Rents, and the Competitive Contracting Process," *Journal of Law and Economics*, 21, 297–326.

Krishna, V. (2002): *Auction Theory*, Elsevier Science, Academic Press.

Larsen, B. J. (2021): "The Efficiency of Real-World Bargaining: Evidence from Wholesale Used-Auto Auctions," *Review of Economic Studies*, 88, 851–882.

Larsen, B. J. and A. Zhang (2021): "Quantifying Bargaining Power Under Incomplete Information: A Supply-Side Analysis of the Used-Car Industry," Working Paper, Stanford University.

Liu, B., S. Loertscher, and L. M. Marx (2022): "Efficient Consignment Auctions," Working Paper, University of Melbourne.

Loertscher, S. and L. M. Marx (2019): "Merger Review for Markets with Buyer Power," *Journal of Political Economy*, 127, 2967–3017.

——— (2020): "Procedures and Properties for Ironed Virtual Types," University of Melbourne.

——— (2021a): "Coordinated Effects in Merger Review," *Journal of Law & Economics*, 64, 705–744.

——— (2021b): "Incomplete-Information Models for Industrial Organization," Working Paper, University of Melbourne.

——— (2022): "Incomplete Information Bargaining with Applications to Mergers, Investment, and Vertical Integration," *American Economic Review*, 112, 616–649.

Loertscher, S. and E. V. Muir (2022): "Monopoly Pricing, Optimal Randomization, and Resale," *Journal of Political Economy*, 130, 566–635.

Loertscher, S. and M. Reisinger (2014): "Market Structure and the Competitive Effects of Vertical Integration," *RAND Journal of Economics*, 45, 471–494.

Loertscher, S. and C. Wasser (2019): "Optimal Structure and Dissolution of Partnerships," *Theoretical Economics*, 14, 1063–1114.

Lu, H. and J. Robert (2001): "Optimal trading mechanisms with ex ante unidentified traders," *Journal of Economic Theory*, 97, 50–80.

Luco, F. and G. Marshall (2020): "The Competitive Impact of Vertical Integration by Multiproduct Firms," *American Economic Review*, 110, 2041–2064.

Marshall, A. M., I. Olkin, and B. Arnold (2011): *Inequalities: Theory of Majorization and Its Applications*, Springer, second ed.

Myerson, R. B. and M. A. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

Ordover, J. A., G. Saloner, and S. C. Salop (1990): "Equilibrium Vertical Foreclosure," *American Economic Review*, 80, 127–142.

Roider, A. (2006): "Fisher Body Revisited: Supply Contracts and Vertical Integration." *European Journal of Law and Economics*, 22, 181–196.

Salop, S. C. (2017): "The Raising Rivals' Cost Foreclosure Paradigm, Conditional Pricing Practices, and the Flawed Incremental Price-Cost Test," *Antitrust Law Journal*, 81, 371–421.

Salop, S. C. and D. T. Scheffman (1983): "Raising Rivals' Costs," *American Economic Review*, 2, 267–271.

——— (1987): "Cost-Raising Strategies," *Journal of Industrial Economics*, 36, 19–34.

Segal, I. and M. D. Whinston (2011): "A Simple Status Quo that Ensures Participation with Application to Efficient Bargaining," *Theoretical Economics*, 6, 109–125.

Thon, D. and S. W. Wallace (2004): "Dalton Transfers, Inequality and Altruism," *Social Choice and Welfare*, 22, 447–465.

Williams, S. R. (1987): "Efficient Performance in Two Agent Bargaining," *Journal of Economic Theory*, 41, 154–172.

——— (1999): "A Characterization of Efficient, Bayesian Incentive Compatible Mechanisms," *Economic Theory*, 14, 155–180.

Williamson, O. E. (1975): *Markets and Hierarchies: Analysis and Antitrust Implications*, New York: Free Press.

——— (1985): *The Economic Institutions of Capitalism*, New York: Free Press.

A Appendix: Concepts and derivations

A.1 Mechanisms and constraints

Take as given a direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, where $Q_i : [\underline{\theta}, \overline{\theta}]^n \to [0, k_i]$ and $M_i : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}$. Given reports $\boldsymbol{\theta}$, $Q_i(\boldsymbol{\theta})$ is the quantity allocated to firm i and $M_i(\boldsymbol{\theta})$ is the payment from firm i to the mechanism. By the Revelation Principle, a focus on direct mechanisms is without loss of generality.

Let $q_i(x)$ be firm i's expected quantity if it reports x and the other firms report truthfully, and let $m_i(x)$ be firm i's expected payment if it reports x and the other firms report truthfully:

$$q_i(x) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(x, \boldsymbol{\theta}_{-i})] \quad \text{and} \quad m_i(x) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(x, \boldsymbol{\theta}_{-i})].$$

Because we assume independent draws, $q_i(x)$ and $m_i(x)$ depend only on the report x and not on the reporting firm's true type. The expected payoff of a buyer with type θ that reports x is then $q_i(x)\theta - m_i(x)$.

The mechanism is *incentive compatible* for firm i if for all $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$,

$$u_i(\theta) \equiv q_i(\theta)\theta - m_i(\theta) \geq q_i(\theta')\theta - m_i(\theta'). \tag{7}$$

Individual rationality is satisfied for firm i if for all $\theta \in [\underline{\theta}, \overline{\theta}]$, $u_i(\theta) \geq r_i\theta_i$. The mechanism satisfies the *no-deficit* condition if

$$\mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i \in \mathcal{N}} M_i(\boldsymbol{\theta})\Big] \geq 0.$$

A.2 Interim expected payoffs and expected payments

Standard arguments (see, e.g., Krishna, 2002, Chapter 5.1) proceed as follows:

Incentive compatibility for firm i implies that

$$u_i(\theta) = \max_{x \in [\underline{\theta}, \overline{\theta}]} \{q_i(x)\theta - m_i(x)\},$$

i.e., u_i is a maximum of a family of affine functions, which implies that u_i is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain.[28] In

[28] A function $h : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is absolutely continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (θ_k, θ'_k) of $[\underline{\theta}, \overline{\theta}]$ satisfies $\sum_k (\theta'_k - \theta_k) < \delta$, then $\sum_k \left|h(\theta'_k) - h(\theta_k)\right| < \varepsilon$. One can show that absolute continuity on compact interval $[a, b]$ implies that h has a derivative h' almost everywhere, the derivative is Lebesgue integrable, and that $h(x) = h(a) + \int_a^x h'(t)dt$ for all $x \in [a, b]$.

addition, incentive compatibility implies that $u_i(x) \geq q_i(\theta)x - m_i(\theta) = u_i(\theta) + q_i(\theta)(x - \theta)$, which for $\varepsilon > 0$ implies

$$\frac{u_i(\theta + \varepsilon) - u_i(\theta)}{\varepsilon} \geq q_i(\theta)$$

and for $\varepsilon < 0$ implies

$$\frac{u_i(\theta + \varepsilon) - u_i(\theta)}{\varepsilon} \leq q_i(\theta),$$

so taking the limit as ε goes to zero, at every point θ where u_i is differentiable, $u_i'(\theta) = q_i(\theta)$. Because u_i is convex, this implies that $q_i(\theta)$ is nondecreasing. Because every absolutely continuous function is the definite integral of its derivative,

$$u_i(\theta) = u_i(\hat{\theta}_i) + \begin{cases} -\int_\theta^{\hat{\theta}_i} q_i(t)dt & \text{if } \theta < \hat{\theta}_i \\ \int_{\hat{\theta}_i}^{\theta} q_i(t)dt & \text{otherwise,} \end{cases}$$

which implies that, up to an additive constant, firm i's expected payoff in an incentive-compatible direct mechanism depends only on the allocation rule.

Using $u_i(\theta) = \theta q_i(\theta) - m_i(\theta)$, this gives us

$$m_i(\theta) = \theta q_i(\theta) - u_i(\hat{\theta}_i) + \begin{cases} \int_\theta^{\hat{\theta}_i} q_i(t)dt & \text{if } \theta < \hat{\theta}_i \\ -\int_{\hat{\theta}_i}^{\theta} q_i(t)dt & \text{otherwise.} \end{cases} \tag{8}$$

Individual rationality requires that $u_i(\hat{\theta}_i) \geq r_i\hat{\theta}_i$, which requires that $m_i(\hat{\theta}_i) \leq 0$. If individual rationality binds, i.e., $m_i(\hat{\theta}_i) = 0$, then evaluating (8) at $\theta = \hat{\theta}_i$ and using $q_i(\hat{\theta}_i) = r_i$, we obtain

$$u_i(\hat{\theta}_i) = \hat{\theta} r_i.$$

Using the definition of m_i in (8), the expected payment by firm i is then

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}}\left[m_i(\theta)\right] &= \int_{\underline{\theta}}^{\overline{\theta}} m_i(x) f_i(x) dx \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \left(x q_i(x) + \int_x^{\hat{\theta}_i} q_i(t) dt \cdot \mathbf{1}_{x<\hat{\theta}_i} - \int_{\hat{\theta}_i}^{x} q_i(t) dt \cdot \mathbf{1}_{x \geq \hat{\theta}_i} \right) f_i(x) dx - u_i(\hat{\theta}_i) \\
&= \int_{\underline{\theta}}^{\overline{\theta}} x q_i(x) f_i(x) dx + \int_{\underline{\theta}}^{\hat{\theta}_i} F_i(x) q_i(x) dx - \int_{\hat{\theta}_i}^{\overline{\theta}} (1 - F_i(x)) q_i(x) dx - u_i(\hat{\theta}_i) \\
&= \int_{\underline{\theta}}^{\hat{\theta}_i} \left(x + \frac{F_i(x)}{f_i(x)} \right) q_i(x) f_i(x) dx + \int_{\hat{\theta}_i}^{\overline{\theta}} \left(x - \frac{1 - F_i(x)}{f_i(x)} \right) q_i(x) f_i(x) dx - u_i(\hat{\theta}_i) \\
&= \int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_i^S(x) q_i(x) f_i(x) dx + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_i^B(x) q_i(x) f_i(x) dx - u_i(\hat{\theta}_i) \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \Psi_i(x, \hat{\theta}_i) q_i(x) f_i(x) dx - u_i(\hat{\theta}_i) \\
&= \mathbb{E}_{\theta_i}\left[\Psi_i(\theta_i, \hat{\theta}_i) q_i(\theta_i)\right] - u_i(\hat{\theta}_i),
\end{aligned}
$$

where the first equality uses the definition of the expectation, the second uses (8), the third switches the order of integration and integrates, the fourth collects terms, the fifth uses the definitions of Ψ_i^S and Ψ_i^B, the sixth uses the definition of Ψ_i, and the last equality uses the definition of the expectation.

A.3 Second-best mechanism and ironing

To define the second-best mechanism, we use the *weighted* virtual type functions, defined as:

$$\Psi_{i,\alpha}^B(\theta) \equiv \theta - (1-\alpha)\frac{1 - F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi_{i,\alpha}^S(\theta) \equiv \theta + (1-\alpha)\frac{F_i(\theta)}{f_i(\theta)}$$

and the associated combined function $\Psi_{i,\alpha}(\theta, x)$.

Then we can rewrite the Lagrangian as

$$\mathcal{L} \equiv \rho \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in \mathcal{N}} \Psi_{i,\frac{1}{\rho}}(\theta_i, \hat{\theta}_i) Q_i(\boldsymbol{\theta})\right] + \sum_{i \in \mathcal{N}} (1-\rho)\hat{\theta}_i r_i.$$

Let $\overline{\Psi}_{i,\alpha}$ denote the ironed weighted virtual type function. That is,

$$\overline{\Psi}_{i,\alpha}(\theta, x) \equiv \begin{cases} \Psi^S_{i,\alpha}(\theta) & \text{if } \Psi^S_{i,\alpha}(\theta) < z_i, \\ z_i & \text{if } \Psi^B_{i,\alpha}(\theta) \leq z_i \leq \Psi^S_{i,\alpha}(\theta) \\ \Psi^B_{i,\alpha}(\theta) & \text{otherwise,} \end{cases}$$

where the ironing parameter z_i is such that

$$\int_{\Psi^{S^{-1}}_{i,\alpha}(z_i)}^{x} (\Psi^S_{i,\alpha}(t) - z_i) dF_i(t) = \int_{x}^{\Psi^{B^{-1}}_{i,\alpha}(z_i)} (z_i - \Psi^B_{i,\alpha}(t)) dF_i(t).$$

We illustrate the ironed virtual type function in Figure A.1.

(a) Virtual type functions (b) Ironed virtual type function

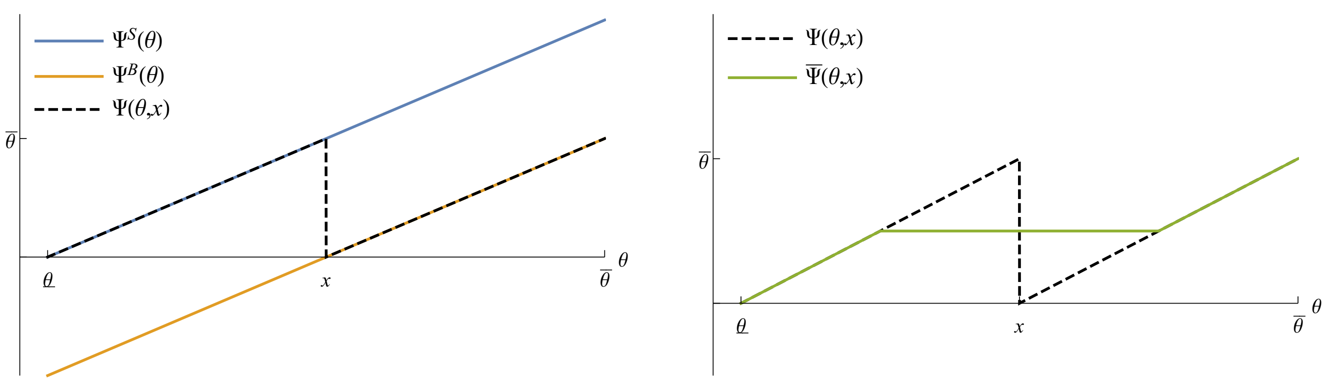

Figure A.1: Illustration of ironed virtual type functions. Assumes types are uniformly distributed on $[0, 1]$.

Using results of Loertscher and Wasser (2019), given ρ, $\hat{\boldsymbol{\theta}}$, and tie-breaking probabilities, one can maximize $\mathcal{L}$ with respect to $\mathbf{Q}$ pointwise, subject to each Q_i being nondecreasing, which is necessary and sufficient for incentive compatibility, by choosing $\mathbf{Q}$ to maximize

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in \mathcal{N}} \overline{\Psi}_{i,\frac{1}{\rho}}(\theta_i, \hat{\theta}_i) Q_i(\boldsymbol{\theta})\right],$$

and employing the tie-breaking probabilities to break ties among the ironed virtual types. Specifically, for each $\boldsymbol{\theta}$, rank the ironed weighted virtual types $\overline{\Psi}_{i,\frac{1}{\rho}}(\theta_i, \hat{\theta}_i)$ from largest to smallest, breaking ties according to the tie-breaking probabilities, and assign the supply to the highest-ranked agents.

Given this allocation rule and the associated interim expected allocation rule $\mathbf{q}$, which are functions of ρ, $\hat{\boldsymbol{\theta}}$, and the tie-breaking probabilities, one can then jointly determine ρ, $\hat{\boldsymbol{\theta}}$, and the tie-breaking probabilities so that: (i) ρ is the smallest value greater than or equal to 1 such that the no-deficit constraint is satisfied, i.e., $\Pi^{\mathbf{Q}}(\mathbf{r}) \geq 0$; and (ii) $\hat{\boldsymbol{\theta}}$ are the worst-off types, i.e., $q_i(\hat{\theta}_i) = r_i$ for all i. Using Loertscher and Wasser (2019, Theorem 1) such values exist, including tie-breaking probabilities that take the form of a hierarchical tie-breaking rule defined by $\mathbf{a} \equiv (a_1, \ldots, a_{n!}) \in \Delta^{n!-1}$ that specifies a probability distribution over the $n!$ possible orderings of the n agents. Given a tie among agents in $\mathcal{I}$, a hierarchy is chosen according to the probability distribution $\mathbf{a}$ and then tied agents are ordered according to the chosen hierarchy.

B Appendix: proofs

Proof of Proposition 4. Suppose that agent 1's distribution changes from F_1 to $\breve{F}_1$ in the sense of first-order stochastic dominance, that is, for all $\theta \in (\underline{\theta}, \overline{\theta})$, we have $\breve{F}_1(\theta) < F_1(\theta)$. Denoting the common worst-off type given $\breve{F}_1$ and F_2 by $\breve{\theta}$, we have (by the same arguments as in the proof of Proposition 10)

$$\breve{F}_1(\breve{\theta}) + F_2(\breve{\theta}) = 1.$$

Because of first-order stochastic dominance, we have

$$\hat{\theta} < \breve{\theta}.$$

Denoting by $\hat{\mathbf{r}}$ and $\breve{\mathbf{r}}$ the endowments that maximize Π^e before and after the change in firm 1's distribution, as shown in the discussion following Lemma 3, $\hat{r}_1 = R - k_2 + (K - R)\, F_2(\hat{\theta})$ and $\breve{r}_1 = R - k_2 + (K - R)\, F_2(\breve{\theta})$, which implies that $\hat{r}_1 < \breve{r}_1$. Of course, $\hat{r}_2 > \breve{r}_2$ because endowments add up to R. ■

Proof of Proposition 6. Take as given incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, endowment vector $\mathbf{r}$ with $\hat{\theta}_1^{\mathbf{Q}}(r_1) \geq \hat{\theta}_2^{\mathbf{Q}}(r_2)$, and $\Delta \in (0, \min\{k - r_1, r_2\}]$. To reduce notation, in what follows we write $\hat{\theta}_i(r_i)$ instead of $\hat{\theta}_i^{\mathbf{Q}}(r_i)$. Then we have

$$\begin{aligned}
& \Pi^{\mathbf{Q}}(r_1 + \Delta, r_2 - \Delta, r_3, \ldots, r_n) - \Pi^{\mathbf{Q}}(\mathbf{r}) \\
= \; & \mathbb{E}_\theta \left[\Psi_{1,0}(\theta, \hat{\theta}_1(r_1 + \Delta)) q_1(\theta) \right] - \hat{\theta}_1(r_1 + \Delta)(r_1 + \Delta) \\
& + \mathbb{E}_\theta \left[\Psi_{2,0}(\theta, \hat{\theta}_2(r_2 - \Delta)) q_2(\theta) \right] - \hat{\theta}_2(r_2 - \Delta)(r_2 - \Delta) \\
& - \mathbb{E}_\theta \left[\Psi_{1,0}(\theta, \hat{\theta}_1(r_1)) q_1(\theta) \right] + \hat{\theta}_1(r_1) r_1 - \mathbb{E}_\theta \left[\Psi_{2,0}(\theta, \hat{\theta}_2(r_2)) q_2(\theta) \right] + \hat{\theta}_2(r_2) r_2.
\end{aligned}$$

Using the definition of the expectation and $\Psi_{i,0}$, we can rewrite this as

$$\begin{aligned}
&\int_{\underline{\theta}}^{\hat{\theta}_1(r_1+\Delta)} \Psi_{1,0}^S(t) q_1(t) dF_1(t) + \int_{\hat{\theta}_1(r_1+\Delta)}^{\overline{\theta}} \Psi_{1,0}^B(t) q_1(t) dF_1(t) \\
&+ \int_{\underline{\theta}}^{\hat{\theta}_2(r_2-\Delta)} \Psi_{2,0}^S(t) q_2(t) dF_2(t) + \int_{\hat{\theta}_2(r_2-\Delta)}^{\overline{\theta}} \Psi_{2,0}^B(t) q_2(t) dF_2(t) \\
&-\hat{\theta}_1(r_1+\Delta)(r_1+\Delta) - \hat{\theta}_2(r_2-\Delta)(r_2-\Delta) + \hat{\theta}_1(r_1)r_1 + \hat{\theta}_2(r_2)r_2 \\
&- \int_{\underline{\theta}}^{\hat{\theta}_1(r_1+\Delta)} \Psi_{1,0}^S(t) q_1(t) dF_1(t) + \int_{\hat{\theta}_1(r_1)}^{\hat{\theta}_1(r_1+\Delta)} \Psi_{1,0}^S(t) q_1(t) dF_1(t) \\
&- \int_{\hat{\theta}_1(r_1)}^{\hat{\theta}_1(r_1+\Delta)} \Psi_{1,0}^B(t) q_1(t) dF_1(t) - \int_{\hat{\theta}_1(r_1+\Delta)}^{\overline{\theta}} \Psi_{1,0}^B(t) q_1(t) dF_1(t) \\
&- \int_{\underline{\theta}}^{\hat{\theta}_2(r_2-\Delta)} \Psi_{2,0}^S(t) q_2(t) dF_2(t) - \int_{\hat{\theta}_2(r_2-\Delta)}^{\hat{\theta}_2(r_2)} \Psi_{2,0}^S(t) q_2(t) dF_2(t) \\
&- \int_{\hat{\theta}_2(r_2-\Delta)}^{\overline{\theta}} \Psi_{2,0}^B(t) q_2(t) dF_2(t) + \int_{\hat{\theta}_2(r_2-\Delta)}^{\hat{\theta}_2(r_2)} \Psi_{2,0}^B(t) q_2(t) dF_2(t).
\end{aligned}$$

Rearranging, we have

$$\begin{aligned}
&\int_{\hat{\theta}_1(r_1)}^{\hat{\theta}_1(r_1+\Delta)} \left(\Psi_{1,0}^S(t) - \Psi_{1,0}^B(t)\right) q_1(t) dF_1(t) + \int_{\hat{\theta}_2(r_2-\Delta)}^{\hat{\theta}_2(r_2)} \left(\Psi_{2,0}^B(t) - \Psi_{2,0}^S(t)\right) q_2(t) dF_2(t) \\
&-\hat{\theta}_1(r_1+\Delta)(r_1+\Delta) - \hat{\theta}_2(r_2-\Delta)(r_2-\Delta) + \hat{\theta}_1(r_1)r_1 + \hat{\theta}_2(r_2)r_2 \\
= \; & \int_{\hat{\theta}_1(r_1)}^{\hat{\theta}_1(r_1+\Delta)} q_1(t) dt - \int_{\hat{\theta}_2(r_2-\Delta)}^{\hat{\theta}_2(r_2)} q_2(t) dt \\
&-\hat{\theta}_1(r_1+\Delta)(r_1+\Delta) - \hat{\theta}_2(r_2-\Delta)(r_2-\Delta) + \hat{\theta}_1(r_1)r_1 + \hat{\theta}_2(r_2)r_2 \\
< \; & \left(\hat{\theta}_1(r_1+\Delta) - \hat{\theta}_1(r_1)\right)(r_1+\Delta) - \left(\hat{\theta}_2(r_2) - \hat{\theta}_2(r_2-\Delta)\right)(r_2-\Delta) \\
&-\hat{\theta}_1(r_1+\Delta)(r_1+\Delta) - \hat{\theta}_2(r_2-\Delta)(r_2-\Delta) + \hat{\theta}_1(r_1)r_1 + \hat{\theta}_2(r_2)r_2 \\
= \; & \left(\hat{\theta}_2(r_2) - \hat{\theta}_1(r_1)\right)\Delta \\
\leq \; & 0,
\end{aligned}$$

where the first equality uses $\Psi_{i,0}^S(t) - \Psi_{i,0}^B(t) = \frac{1}{f_i(t)}$, the inequality uses the results that q_i and $\hat{\theta}_i$ are increasing and that $q_i(\hat{\theta}_i(x)) = x$, the second equality simplifies, and the final inequality uses our assumption that $\hat{\theta}_2(r_2) \leq \hat{\theta}_1(r_1)$. Thus, $\Pi^{\mathbf{Q}}(r_1+\Delta, r_2-\Delta, r_3, \ldots, r_n) - \Pi^{\mathbf{Q}}(\mathbf{r}) < 0$, which completes the proof. ■

Proof of Proposition 7. Given $\mathbf{y}$ that majorizes $\mathbf{x}$, where $\mathbf{y}$ and $\mathbf{x}$ are assumed to be sorted in descending order (the procedure applies equally to vectors that are not initially sorted by simply sorting them), the T-transform proceeds by letting j be the largest index such that $x_j < y_j$ and k be the smallest index greater than j such that $y_k < x_k$. Because $\mathbf{y}$ majorizes $\mathbf{x}$, such j and k exist. (It is necessarily the case that $x_1 \leq y_1$, and for the smallest index such that $\mathbf{x}$ and $\mathbf{y}$ differ, it must be that the element of $\mathbf{y}$ is larger. Given that the elements of $\mathbf{x}$ and $\mathbf{y}$ sum to the same amount, there then must be an index k such that $y_k < x_k$.) It then follows that

$$y_k < x_k \leq x_j < y_j.$$

Define $\Delta \equiv \min\{y_j - x_j, x_k - y_k\}$ and $1 - \lambda \equiv \frac{\Delta}{y_j - y_k}$ and

$$\mathbf{y}^* \equiv (y_1, \ldots, y_{j-1}, y_j - \Delta, y_{j+1}, \ldots, y_{k-1}, y_k + \Delta, y_{k+1}, \ldots, y_n).$$

Then $\mathbf{y}$ majorizes $\mathbf{y}^*$ and all elements of $\mathbf{y}^*$ are nonnegative. Further, by the definition of Δ, we have $\Delta > 0$, $y_k + \Delta \leq x_k \leq k$, which implies that $\Delta \leq k - y_k$, and $y_k + \Delta \leq y_k + y_j - x_j < y_j$, so $\Delta \in (0, \min\{k - y_k, y_j\}]$. Thus,

$$y = (y_1^*, \ldots, y_{j-1}^*, y_j^* + \Delta, y_{j+1}^*, \ldots, y_{k-1}^*, y_k^* - \Delta, y_{k+1}^*, \ldots, y_n^*),$$

with $y_j^* - y_k^* = y_j - \Delta - y_k - \Delta \geq y_j - y_k - (y_j - x_j) - (x_k - y_k) = x_j - x_k \geq 0$. Because $y_j^* \geq y_k^*$ and using the assumed symmetry (which implies that $\hat{\theta}_i^{\mathbf{Q}}(\cdot)$ is the same for all i and so we can drop the firm subscript), $\hat{\theta}^{\mathbf{Q}}(y_j^*) \geq \hat{\theta}^{\mathbf{Q}}(y_k^*)$, and so by Proposition 6, $\Pi^{\mathbf{Q}}(\mathbf{y}) < \Pi^{\mathbf{Q}}(\mathbf{y}^*)$.

To see that iteration of this process produces $\mathbf{x}$ in a finite number of iterations, for two vectors $\mathbf{u}$ and $\mathbf{v}$, $d(\mathbf{u}, \mathbf{v})$ be the number of nonzero differences $u_i - v_i$. Because $y_j^* = x_j$ if $\Delta = y_j - x_j$ and $y_k^* = x_k$ if $\Delta = x_k - y_k$, it follows that $d(\mathbf{x}, \mathbf{y}^*) \leq d(\mathbf{x}, \mathbf{y}) - 1$, implying that $\mathbf{x}$ can be derived from $\mathbf{y}$ in a finite number of iterations. ■

Proof of Proposition 8. If $\Pi^e(\mathbf{r}) < 0$, then the first-best is not achieved under $\mathbf{r}$. The two firms can increase their joint payoff through, for example, a transaction that shifts endowments to $\mathbf{r}^*$, where $\Pi^e(\mathbf{r}^*) \geq 0$ so the first-best is achieved (and the first-best surplus is divided between the two firms). If $\Pi^e(\mathbf{r}) \geq 0$, then the first-best is achieved prior to any transaction (and first-best surplus is divided between the two firms), so no further increases in joint surplus are possible. This completes the proof of the first part of the proposition.

Turning to the second part of the proposition, suppose that $\Pi^e(\mathbf{r}) > 0$ and there exist two vertically integrated firms indexed by 1 and 2 with $\eta_1 + \eta_2 < 1$. By virtue of the firms being vertically integrated, $0 < r_1 < k_1$ and $0 < r_2 < k_2$. Without loss of generality, we can assume that $\hat{\theta}_2^e(r_2) \leq \hat{\theta}_1^e(r_1)$. Because $r_1 < k_1$ and $0 < r_2$, there exists $\Delta > 0$ sufficiently small that the endowment vector $\mathbf{r}'$ defined by $r_1' = r_1 + \Delta$, $r_2' = r_2 - \Delta$, and $\mathbf{r}'_{-\{1,2\}} = \mathbf{r}_{-\{1,2\}}$ is a feasible endowment vector (i.e., $r_1 + \Delta \leq k_1$ and $0 \leq r_2 - \Delta$). Further, using the continuity of Π^e and the assumption that $\Pi^e(\mathbf{r}) > 0$, there exists $\Delta > 0$ sufficiently small that $\Pi^e(\mathbf{r}') > 0$. Taking Δ to satisfy these conditions, the first-best is achieved under both $\mathbf{r}$ and $\mathbf{r}'$, and by Proposition 6,

$$\Pi^e(\mathbf{r}') < \Pi^e(\mathbf{r}). \tag{9}$$

Defining $\hat{m}_i(r_i) \equiv \mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}_i^e(r_i))q_i^e(\theta_i)\right] - r_i\hat{\theta}_i^e(r_i)$ and noting that $\Pi^e(\mathbf{r}) = \sum_{i\in\mathcal{N}} \hat{m}_i(r_i)$, it follows that

$$\Pi^e(\mathbf{r}) - \Pi^e(\mathbf{r}') = \hat{m}_1(r_1) + \hat{m}_2(r_2) - \hat{m}_1(r_1 + \Delta) - \hat{m}_2(r_2 - \Delta). \tag{10}$$

Because firm i's expected surplus under the first-best is

$$u_i^e(\mathbf{r}) \equiv \mathbb{E}_{\theta_i}\left[\theta_i q_i^e(\theta_i)\right] - \hat{m}_i(r_i) + \eta_i \Pi^e(\mathbf{r}),$$

the change in the joint expected surplus of firms 1 and 2 from a change in endowment vector from $\mathbf{r}$ to $\mathbf{r}'$ is

$$\begin{aligned}
& u_1^e(\mathbf{r}') + u_2^e(\mathbf{r}') - u_1^e(\mathbf{r}) - u_2^e(\mathbf{r}) \\
= \;& -\hat{m}_1(r_1 + \Delta) + \eta_1\Pi^e(\mathbf{r}') - \hat{m}_2(r_2 - \Delta) + \eta_2\Pi^e(\mathbf{r}') + \hat{m}_1(r_1) - \eta_1\Pi^e(\mathbf{r}) + \hat{m}_2(r_2) - \eta_2\Pi^e(\mathbf{r}) \\
= \;& (1 - \eta_1 - \eta_2)\left(\Pi^e(\mathbf{r}) - \Pi^e(\mathbf{r}')\right) \\
> \;& 0,
\end{aligned}$$

where the first equality uses the definition of $u_i^e(\cdot)$, the second equality uses (10), and the inequality uses the assumption that $\eta_1 + \eta_2 < 1$ and (9). Thus, the joint expected payoff of firms 1 and 2 increases as a result of shifting amount Δ of firm 2's endowment to firm 1, which completes the proof. ■

Proof of Lemma 3. Taking the first derivative of $\Pi^e(r)$ we obtain

$$\begin{aligned}\frac{d\Pi^e(r)}{dr} &= \frac{d\hat{\theta}_1(r)}{dr}\left[\left(\Psi_1^S(\hat{\theta}_1(r)) - \Psi_1^B(\hat{\theta}_1(r))\right) q_1(\hat{\theta}_1(r)) f_1(\hat{\theta}_1(r)) - r\right] - \hat{\theta}_1(r) \\ &+ \frac{d\hat{\theta}_2(r)}{dr}\left[\left(\Psi_2^S(\hat{\theta}_2(r)) - \Psi_2^B(\hat{\theta}_2(r))\right) q_2(\hat{\theta}_2(r)) f_2(\hat{\theta}_1(r)) - (R-r)\right] + \hat{\theta}_2(r) \\ &= -\hat{\theta}_1(r) + \hat{\theta}_2(r),\end{aligned}$$

where the second equality uses $\Psi_i^S - \Psi_i^B = -1/f_i$, which implies that $(\Psi_i^S - \Psi_i^B) f_i q_i = q_i$, and that $q_1(\hat{\theta}_1(r)) = r$ and $q_2(\hat{\theta}_2(r)) = R - r$. Because $\hat{\theta}_1(r)$ increases in r and $\hat{\theta}_2(r)$ decreases in r, $\frac{d^2\Pi^e(r)}{dr^2} < 0$ follows. ■

Proof of Proposition 9. The result on instability follows from Proposition 8. Because $\Pi^e(\mathbf{r}) > 0$, which implies that the first-best is possible, Proposition 2 implies that there is at least one vertically integrated firm. If there is only one vertically integrated firm, then the only possible transactions are from a seller to a buyer, a seller to the vertically integrated firm, or the vertically integrated firm to a buyer. Because the sellers' worst-off types are $\overline{\theta}$, buyers' worst-off types are $\underline{\theta}$, and the vertically integrated firm's worst-off type is in $(\underline{\theta}, \overline{\theta})$, each of these transactions is from a firm with a higher worst-off type to a firm with a lower worst-off type, which by Proposition 6, increases Π^e, to the detriment of the trading firms' joint expected surplus. So such transactions are not mutually beneficial, implying that the endowment is stable. If there exist exactly two vertically integrated firms i and j with $\eta_i + \eta_j = 1$, then the only possible transactions from a firm with a weakly lower worst-off type to a firm with a weakly higher worst-off type are transactions between the two vertically integrated firms, but because $\eta_i + \eta_j = 1$, such transactions do not change the firms' expected joint surplus, implying that the endowment is stable. If there are more than two vertically integrated firms, then necessarily there are two of those vertically integrated firms, i and j, for which $\eta_i + \eta_j < 1$, so the condition of the proposition is satisfied for the endowment not to be stable. ■

Proof of Proposition 10. Recall the we assume that $\max\{k_1, k_2\} \leq R < k_1 + k_2 \equiv K$. For the purposes of the proof, we add $\mathbf{k}$ as an argument to q_i and $\hat{\Pi}^e$, so we have $q_i(\theta; \mathbf{k})$ and $\hat{\Pi}^e(r; \mathbf{k})$. Recall that firm i's interim expected allocation is

$$q_i(\theta; \mathbf{k}) = F_j(\theta)\,(K - R) + R - k_j, \tag{11}$$

where $j \neq i$. Recall that $\underline{\theta}_1$ and $\underline{\theta}_2$ are the worst-off types when $\mathbf{k} = (R, R)$ and firm 1's endowment is $\underline{r}$, which is the smallest endowment such that $\Pi^e(r;(R,R)) = 0$. Thus, we have

$$\hat{\Pi}^e(\underline{r};(R,R)) = 0. \tag{12}$$

From Lemma 1, we have $q_1(\underline{\theta}_1;(R,R)) = \underline{r}$ and $q_2(\underline{\theta}_2;(R,R)) = R - \underline{r}$, so

$$RF_2(\underline{\theta}_1) = \underline{r} \quad \text{and} \quad RF_1(\underline{\theta}_2) = R - \underline{r},$$

which implies that

$$\underline{r} = R(1 - F_1(\underline{\theta}_2)) \tag{13}$$

and

$$F_2(\underline{\theta}_1) + F_1(\underline{\theta}_2) = 1. \tag{14}$$

We show that when

$$\underline{r}(\mathbf{k}) \equiv k_1 - (K - R)\, F_1(\underline{\theta}_2), \tag{15}$$

as claimed in the statement of the lemma, the worst-off types continue to be $(\underline{\theta}_1, \underline{\theta}_2)$ and that $\underline{r}(\mathbf{k})$ is the smallest endowment r such that $\hat{\Pi}^e(r;\mathbf{k}) = 0$. On the latter point, by Lemma 3, $\hat{\Pi}^e(r;\mathbf{k})$ is strictly concave in r, so we need only show that $\hat{\Pi}^e(\underline{r}(\mathbf{k});\mathbf{k}) = 0$ and $\underline{r}(\mathbf{k}) < r^*(\mathbf{k})$.

Using (14) and (15), we have

$$q_1(\underline{\theta}_1;\mathbf{k}) - \underline{r}(\mathbf{k}) = 0$$

and

$$q_2(\underline{\theta}_2;\mathbf{k}) - (R - \underline{r}(\mathbf{k})) = 0,$$

which by Lemma 1 implies that $(\underline{\theta}_1, \underline{\theta}_2)$ are the worst-off types when firm 1's endowment is $\underline{r}(\mathbf{k})$.

Next we show that $\hat{\Pi}^e(\underline{r}(\mathbf{k});\mathbf{k})=0$. Using (3), (11), and (15), we have

$$
\begin{aligned}
&\hat{\Pi}^e(\underline{r}(\mathbf{k});\mathbf{k})\\
&=\int_{\underline{\theta}}^{\underline{\theta}_1}\Psi_1^S(x)q_1(x;\mathbf{k})dF_1(x)+\int_{\underline{\theta}_1}^{\overline{\theta}}\Psi_1^B(x)q_1(x;\mathbf{k})dF_1(x)-\underline{r}(\mathbf{k})\underline{\theta}_1\\
&\quad+\int_{\underline{\theta}}^{\underline{\theta}_2}\Psi_2^S(x)q_2(x;\mathbf{k})dF_2(x)+\int_{\underline{\theta}_2}^{\overline{\theta}}\Psi_2^B(x)q_2(x;\mathbf{k})dF_2(x)-(R-\underline{r}(\mathbf{k}))\underline{\theta}_2\\
&=\int_{\underline{\theta}}^{\underline{\theta}_1}\Psi_1^S(x)(F_2(x)\,(K-R)+R-k_2)dF_1(x)+\int_{\underline{\theta}_1}^{\overline{\theta}}\Psi_1^B(x)(F_2(x)\,(K-R)+R-k_2)dF_1(x)\\
&\quad+\int_{\underline{\theta}}^{\underline{\theta}_2}\Psi_2^S(x)\,(F_1(x)\,(K-R)+R-k_1)\,dF_2(x)+\int_{\underline{\theta}_2}^{\overline{\theta}}\Psi_2^B(x)\,(F_1(x)\,(K-R)+R-k_1)\,dF_2(x)\\
&\quad-(k_1-(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_1-(R-k_1+(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_2.
\end{aligned}
$$

Using $q_1(\theta;(R,R))=RF_2(\theta)$ and $q_2(\theta;(R,R))=RF_1(\theta)$, we can rewrite this as

$$
\begin{aligned}
\hat{\Pi}^e(\underline{r}(\mathbf{k});\mathbf{k}) \;=\;& (K-R)\int_{\underline{\theta}}^{\underline{\theta}_1}\Psi_1^S(x)\frac{1}{R}q_1(x;\mathbf{R})dF_1(x)+(K-R)\int_{\underline{\theta}_1}^{\overline{\theta}}\Psi_1^B(x)\frac{1}{R}q_1(x;\mathbf{R})dF_1(x)\\
&+(K-R)\int_{\underline{\theta}}^{\underline{\theta}_2}\Psi_2^S(x)\frac{1}{R}q_2(x;\mathbf{R})dF_2(x)+(K-R)\int_{\underline{\theta}_2}^{\overline{\theta}}\Psi_2^B(x)\frac{1}{R}q_2(x;\mathbf{R})dF_2(x)\\
&+(R-k_2)\int_{\underline{\theta}}^{\underline{\theta}_1}\Psi_1^S(x)dF_1(x)+(R-k_2)\int_{\underline{\theta}_1}^{\overline{\theta}}\Psi_1^B(x)dF_1(x)\\
&+(R-k_1)\int_{\underline{\theta}}^{\underline{\theta}_2}\Psi_2^S(x)dF_2(x)+(R-k_1)\int_{\underline{\theta}_2}^{\overline{\theta}}\Psi_2^B(x)dF_2(x)\\
&-(k_1-(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_1-(R-k_1+(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_2.
\end{aligned}
$$

Integrating and using (13), $\mathbb{E}[\Psi_i^B(x) \mid x \geq p] = p$, and $\mathbb{E}[\Psi_i^S(x) \mid x \leq p] = p$ yields

$$\begin{aligned}
& \hat{\Pi}^e(\underline{r}(\mathbf{k});\mathbf{k}) \\
= \;& \frac{K-R}{R}\hat{\Pi}^e(\underline{r};(R,R)) + \frac{K-R}{R}\underline{r}\underline{\theta}_1 + \frac{K-R}{R}(R-\underline{r})\,\underline{\theta}_2 \\
& +(R-k_2)\underline{\theta}_1 + (R-k_1)\,\underline{\theta}_2 - (k_1-(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_1 - (R-k_1+(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_2 \\
= \;& \frac{K-R}{R}\hat{\Pi}^e(\underline{r};(R,R)) + (K-R)\,(1-F_1(\underline{\theta}_2))\,\underline{\theta}_1 + (K-R)F_1(\underline{\theta}_2)\underline{\theta}_2 \\
& +(R-k_2)\underline{\theta}_1 + (R-k_1)\,\underline{\theta}_2 - (k_1-(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_1 - (R-k_1+(K-R)\,F_1(\underline{\theta}_2))\,\underline{\theta}_2 \\
= \;& \frac{K-R}{R}\hat{\Pi}^e(\underline{r};(R,R)) \\
= \;& 0,
\end{aligned}$$

where the final equality uses (12).

It remains to show that $\underline{r}(\mathbf{k}) < r^*(\mathbf{k})$. Note that $r^*(\mathbf{k})$ is such that

$$q_1(\hat{\theta}(\mathbf{k});\mathbf{k}) = r^*(\mathbf{k}) \;\text{ and }\; q_2(\hat{\theta}(\mathbf{k});\mathbf{k}) = 1 - r^*(\mathbf{k}),$$

which means that $F_2(\hat{\theta}(\mathbf{k}))\,(K-R) + R - k_2 = r^*(\mathbf{k})$ and $F_1(\hat{\theta}(\mathbf{k}))\,(K-R) + R - k_1 = R - r^*(\mathbf{k})$, and so

$$F_1(\hat{\theta}(\mathbf{k})) + F_2(\hat{\theta}(\mathbf{k})) = 1,$$

which implies that $\hat{\theta}(\mathbf{k})$ does not vary with $\mathbf{k}$. Dropping $\mathbf{k}$ as an argument of $\hat{\theta}$, we have

$$r^*(\mathbf{k}) = k_1 - (K-R)\,F_1(\hat{\theta}).$$

By the definition of $\underline{r}$ and Lemma 2, which implies that $\hat{\Pi}^e(r^*(\mathbf{k});\mathbf{k}) > 0$, we have $\underline{r} < r^*(R,R)$, which implies that $1 - F_1(\underline{\theta}_2) < 1 - F_1(\hat{\theta})$, and so $\hat{\theta} < \underline{\theta}_2$. It then follows that

$$\underline{r}(\mathbf{k}) = k_1 - (K-R)F_1(\underline{\theta}_2) < k_1 - (K-R)F_1(\hat{\theta}) = r^*(\mathbf{k}),$$

which completes the proof for $\underline{r}(\mathbf{k})$. The result that $\overline{r}(\mathbf{k}) = k_1 - (K-R)F_1(\overline{\theta}_2)$ follows from analogous arguments. ■

Proof of Proposition 2. Using the notation of the proof of Proposition 10, endowment $\underline{r}$ and associated worst-off types $\underline{\theta}_1$ and $\underline{\theta}_2$ are defined by $q_1(\underline{\theta}_1;(R,R)) = \underline{r}$, $q_2(\underline{\theta}_2;(R,R)) = R - \underline{r}$, and $\hat{\Pi}^e(\underline{r};(R,R))=0$. The first two equations amount to

$$F_2(\underline{\theta}_1)+F_1(\underline{\theta}_2)=1, \tag{16}$$

which does not depend on R or K. Using the result of Proposition 10 that $\underline{r}=R(1-F_1(\underline{\theta}_2))$ and noting that $q_i(\theta;(R,R))=RF_j(\theta)$, we have

$$\begin{aligned} &\frac{1}{R}\hat{\Pi}^e(\underline{r};(R,R)) \\ = &\int_{\underline{\theta}}^{\underline{\theta}_1}\Psi_1^S(x)F_2(x)dF_1(x)+\int_{\underline{\theta}_1}^{\overline{\theta}}\Psi_1^B(x)F_2(x)dF_1(x)-(1-F_1(\underline{\theta}_2))\underline{\theta}_1 \\ &+\int_{\underline{\theta}}^{\underline{\theta}_2}\Psi_2^S(x)F_1(x)dF_2(x)+\int_{\underline{\theta}_2}^{\overline{\theta}}\Psi_2^B(x)F_1(x)dF_2(x)-F_1(\underline{\theta}_2)\underline{\theta}_2, \end{aligned} \tag{17}$$

where the right side does not depend on R or K. Thus, $\underline{\theta}_1$ and $\underline{\theta}_2$ are defined by (16) and the condition that the right side of (17) is equal to zero, and so do not depend on R or K. Using this along with the arguments in the body of the paper prior to Proposition 2 completes the proof of the proposition. ■

Proof of Proposition 11. Assume, as in the statement of the proposition, that $n\in\{3,4,\dots\}$, $\Pi^e(\mathbf{r})=0$, and firms 1 and 2 are vertically integrated. Without loss of generality, assume that $\hat{\theta}_1^e(r_1)\geq\hat{\theta}_2^e(r_2)$.

Define resource vector $\tilde{\mathbf{r}}(\Delta)$ by $\tilde{r}_1(\Delta)\equiv r_1+\Delta$, $\tilde{r}_2(\Delta)\equiv r_2-\Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta)=\mathbf{r}_{-\{1,2\}}$, which is feasible for $\Delta\in[0,\min\{k_1-r_1,r_2\}]$, which is a nonempty interval because firms 1 and 2 are vertically integrated. Because we are considering a shift from the firm with the weakly lower worst-off type to the firm with the weakly higher worst-off type, by Proposition 6, for all $\Delta>0$ in the feasible range, we have

$$\Pi^e(\tilde{\mathbf{r}}(\Delta))<\Pi^e(\mathbf{r})=0. \tag{18}$$

Using the results of Appendix A3, for resources $\tilde{\mathbf{r}}(\Delta)$, the Lagrangian associated with designer's problem can be written as

$$\mathcal{L}=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}\Psi_{i,\frac{1}{\rho}}(\theta_i,\hat{\theta}_i)Q_i(\boldsymbol{\theta})\right]+\sum_{i\in\mathcal{N}}(1-\rho)\hat{\theta}_i\tilde{r}_i(\Delta),$$

where ρ is the Lagrange multiplier on the no-deficit constraint. Further, as discussed in Appendix A.3, given Δ and ρ, the worst-off types $\hat{\boldsymbol{\theta}}(\Delta,\rho)$ and second-best allocation rule

$\mathbf{Q}^*(\boldsymbol{\theta};\hat{\boldsymbol{\theta}},\rho)$ are jointly determined by the equality constraints $\mathbb{E}_{\theta_i}[Q_i^*(\hat{\theta}_i;\hat{\boldsymbol{\theta}},\rho)] = \tilde{r}_i(\Delta)$ and the requirement that $\mathbf{Q}^*(\boldsymbol{\theta};\hat{\boldsymbol{\theta}},\rho)$ assigns the supply R to the firms up to their maximum demands in order according to the ranking of their ironed weighted virtual types $\left(\overline{\Psi}_{j,\frac{1}{\rho}}(\theta_j,\hat{\theta}_j)\right)_{j\in\mathcal{N}}$, with ties occurring with probability zero for $\rho = 1$, which will be the relevant case for us. Thus, given ρ, the effect of Δ on $\mathbf{Q}^*(\boldsymbol{\theta};\hat{\boldsymbol{\theta}}(\Delta,\rho),\rho)$ comes only through its effect on $\hat{\boldsymbol{\theta}}(\Delta,\rho)$. For a given Δ and ρ, we can write the interim expected allocation rule of firm i as

$$q_i^*(\theta_i;\hat{\boldsymbol{\theta}}(\Delta,\rho),\rho),$$

where

$$q_i^*(\hat{\theta}_i(\Delta,\rho);\hat{\boldsymbol{\theta}}(\Delta,\rho),\rho) = \tilde{r}_i(\Delta).$$

To reduce notation, for the next part of the proof, we drop the arguments of $\hat{\boldsymbol{\theta}}(\Delta,\rho)$ and simply write $\hat{\boldsymbol{\theta}}$. It is to be understood that $\hat{\boldsymbol{\theta}}$ is evaluated at the relevant Δ and ρ.

Using Proposition 1, given Δ and ρ, the budget surplus under binding individual rationality is

$$\Pi^*(\tilde{\mathbf{r}}(\Delta);\hat{\boldsymbol{\theta}},\rho) \equiv \sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i,\hat{\theta}_i)q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho)\right] - \sum_{i\in\mathcal{N}} \hat{\theta}_i\tilde{r}_i(\Delta).$$

Letting $\rho^*(\Delta)$ be the solution value of the Lagrange multiplier given Δ, by (18), we have

$$\Pi^*(\tilde{\mathbf{r}}(\Delta);\hat{\boldsymbol{\theta}},\rho^*(\Delta)) = 0. \tag{19}$$

Further, because $\Pi^e(\tilde{\mathbf{r}}(0)) = \Pi^e(\mathbf{r}) = 0$, it follows that $\rho^*(0) = 1$ and that $\rho^*(\Delta)$ increases as Δ increases above zero,

$$\left.\frac{\partial\rho^*(\Delta)}{\partial\Delta}\right|_{\Delta\downarrow 0} \geq 0 \text{ and } \rho^*(\Delta) > \rho^*(0) \text{ for } \Delta > 0. \tag{20}$$

Using the results of Appendix A2, given Δ, the expected payoff of firm i is

$$\tilde{u}_i(\Delta) \equiv \mathbb{E}_{\theta_i}\left[\left(\theta_i - \Psi_{i,0}(\theta_i,\hat{\theta}_i)\right)q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho^*(\Delta))\right] + \hat{\theta}_i\tilde{r}_i(\Delta).$$

To establish that the envisioned transaction between firms 1 and 2 is mutually beneficial, we need to show that for $\Delta > 0$ sufficiently small,

$$\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) > \sum_{i\in\{1,2\}} \tilde{u}_i(0). \tag{21}$$

Using the definitions of $\tilde{u}_i(\Delta)$ and Π^*, we can write

$$\begin{aligned}
\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) &= \sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho^*(\Delta))\right] - \Pi^*(\tilde{\mathbf{r}}(\Delta);\hat{\boldsymbol{\theta}},\rho^*(\Delta)) \\
&\quad + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \left(\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j,\hat{\theta}_j) q_j^*(\theta_j;\hat{\boldsymbol{\theta}},\rho^*(\Delta))\right] - \hat{\theta}_j \tilde{r}_j(\Delta)\right) \qquad (22) \\
&= \sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho^*(\Delta))\right] \\
&\quad + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \left(\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j,\hat{\theta}_j) q_j^*(\theta_i;\hat{\boldsymbol{\theta}},\rho^*(\Delta))\right] - \hat{\theta}_j r_j\right),
\end{aligned}$$

where the second equality uses (19) and $\tilde{r}_j(\Delta) = r_j$ for $j \in \mathcal{N}\setminus\{1,2\}$. Thus, the joint expected payoff of firms 1 and 2 is equal to their expected utility from consumption plus the expected payments by their rivals.

As can be seen in (22), Δ only enters $\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta)$ through $\hat{\boldsymbol{\theta}}$ and $\rho^*(\Delta)$. Focus first on effects coming through $\hat{\boldsymbol{\theta}}$. Using the definitions of the virtual type functions and $q_i(\hat{\theta}_i;\hat{\boldsymbol{\theta}},\rho) = r_i$,

$$\begin{aligned}
&\frac{\partial}{\partial\hat{\theta}_i}\left(\mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i,\hat{\theta}_i) q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho)\right] - \hat{\theta}_i r_i\right), \\
=\ &\frac{\partial}{\partial\hat{\theta}_i}\left(\int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_{i,0}^S(\theta_i) q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho) dF_i(\theta_i) + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_{i,0}^B(\theta_i) q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho) dF_i(\theta_i)\right) - r_i \\
=\ &(\Psi_{i,0}^S(\hat{\theta}_i) - \Psi_{i,0}^B(\hat{\theta}_i)) q_i(\hat{\theta}_i;\hat{\boldsymbol{\theta}},\rho) f_i(\hat{\theta}_i) \\
&+ \int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_{i,0}^S(\theta_i) \frac{\partial q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho)}{\partial\hat{\theta}_i} dF_i(\theta_i) + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_{i,0}^B(\theta_i) \frac{\partial q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho)}{\partial\hat{\theta}_i} dF_i(\theta_i) \qquad (23) \\
=\ &q_i(\hat{\theta}_i;\hat{\boldsymbol{\theta}},\rho) + \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i,\hat{\theta}_i) \frac{\partial q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho)}{\partial\hat{\theta}_i} dF_i(\theta_i) - r_i \\
=\ &\int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i,\hat{\theta}_i) \frac{\partial q_i(\theta_i;\hat{\boldsymbol{\theta}},\rho)}{\partial\hat{\theta}_i} dF_i(\theta_i).
\end{aligned}$$

Using (23) and noting that $q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho)$ is independent of $\hat{\boldsymbol{\theta}}$ when $\rho = 1$, the derivative of the right side of (22) with respect to $\hat{\boldsymbol{\theta}}$ is zero when evaluated at $\Delta = 0$. Thus, when considering the effect of a marginal change in Δ on $\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta)$ at $\Delta = 0$, we need only consider effects that come through $\rho^*(\Delta)$.

It follows then that

$$\begin{aligned}
\sum_{i\in\{1,2\}} \tilde{u}_i'(0) &= \rho^{*\prime}(0)\left(\sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i \frac{\partial q_i^*(\theta_i;\hat{\boldsymbol{\theta}},1)}{\partial\rho}\right] + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j,\hat{\theta}_j)\frac{\partial q_j^*(\theta_i;\hat{\boldsymbol{\theta}},1)}{\partial\rho}\right]\right) \\
&= \rho^{*\prime}(0)\sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}\left[\theta_i \frac{\partial q_i^*(\theta_i;\hat{\boldsymbol{\theta}},1)}{\partial\rho}\right] \\
&\quad +\rho^{*\prime}(0)\sum_{j\in\mathcal{N}\setminus\{1,2\}}\left(\int_{\underline{\theta}}^{\hat{\theta}} F_j(\theta_j)\frac{\partial q_j^*(\theta_j;\hat{\boldsymbol{\theta}},1)}{\partial\rho}d\theta_j - \int_{\hat{\theta}_j}^{\overline{\theta}}(1-F_j(\theta_j))\frac{\partial q_j^*(\theta_j;\hat{\boldsymbol{\theta}},1)}{\partial\rho}d\theta_j\right) \\
&= \rho^{*\prime}(0)\sum_{j\in\mathcal{N}\setminus\{1,2\}}\left(\int_{\underline{\theta}}^{\hat{\theta}} F_j(\theta_j)\frac{\partial q_j^*(\theta_j;\hat{\boldsymbol{\theta}},1)}{\partial\rho}d\theta_j - \int_{\hat{\theta}_j}^{\overline{\theta}}(1-F_j(\theta_j))\frac{\partial q_j^*(\theta_j;\hat{\boldsymbol{\theta}},1)}{\partial\rho}d\theta_j\right),
\end{aligned}$$

where the second equality uses the definition of $\Psi_{i,0}$ and the third equality uses

$$\sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}\left[\theta_i \frac{\partial q_i^*(\theta_i;\hat{\boldsymbol{\theta}},1)}{\partial\rho}\right] = 0,$$

which follows from the envelope theorem. To see this, define $\tilde{\mathcal{L}}(\Delta,\rho)$ to be the Lagrangian evaluated at the optimal allocation rule and optimal worst-off types as a function of Δ and ρ:

$$\tilde{\mathcal{L}}(\Delta,\rho) = \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho)\right] - (1-\rho)\Pi^*(\tilde{\mathbf{r}}(\Delta);\hat{\boldsymbol{\theta}},\rho). \tag{24}$$

By the envelope theorem, $\frac{\partial\tilde{\mathcal{L}}(\Delta,\rho)}{\partial\rho} = \Pi^*(\tilde{\mathbf{r}}(\Delta);\hat{\boldsymbol{\theta}},\rho)$. Thus, using (19), $\frac{\partial\tilde{\mathcal{L}}(\Delta,\rho^*(\Delta))}{\partial\rho} = 0$, and so differentiating (24) with respect to ρ and evaluating at $\Delta = 0$ and $\rho^*(0) = 1$, we have

$$0 = \frac{\partial\tilde{\mathcal{L}}(0,1)}{\partial\rho} = \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \theta_i \frac{\partial q_i^*(\theta_i;\hat{\boldsymbol{\theta}},1)}{\partial\rho}\right], \tag{25}$$

which is the desired result.

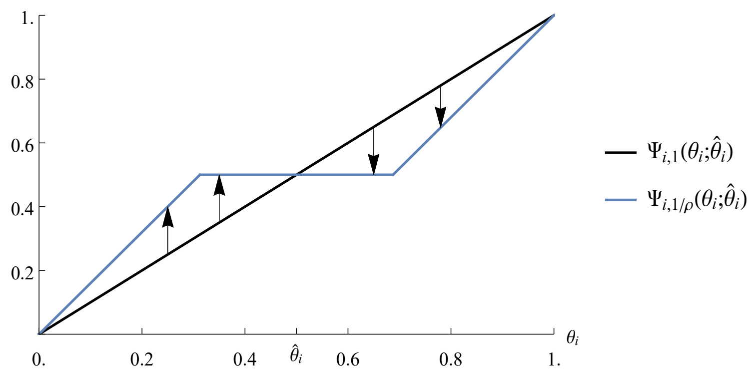

Figure B.2: Illustration of effects of increase in ρ on ironed weighted virtual types

By the definition of the ironed weighted virtual types that underpin q_i^*, $q_i^*(\theta_i; \hat{\boldsymbol{\theta}}, \rho)$ increases in ρ for $\theta_i < \hat{\theta}_i$ and decreases in ρ for $\theta_i > \hat{\theta}_i$. This is illustrated in Figure B.2. Thus, $\sum_{i \in \{1,2\}} \tilde{u}_i'(0) \geq 0$ and for $\Delta > 0$ sufficiently small $\sum_{i \in \{1,2\}} \tilde{u}_i(\Delta) > \sum_{i \in \{1,2\}} \tilde{u}_i(0)$, which implies that transactions between firms 1 and 2 of $\Delta > 0$ sufficiently small are mutually beneficial. By Proposition 6, such transactions result in $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < 0$, which completes the proof. ■