Growing through Mergers and Acquisitions*

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Abstract

The paper studies how merger and acquisition (M&A) affects the aggregate growth rate with an endogenous growth model. We model M&A as a capital reallocation process which can increase both the productivity and growth rates of firms. The model is tractable and largely consistent with patterns observed in M&A at the micro level. Matching our model to the data, we find that prohibiting M&A would reduce the aggregate growth rate of the US by 0.8% and would reduce aggregate TFP by 10%. We use our model to address the M&A boom that began in the 1990s. The model implies that this boom could increase the aggregate growth rate by 0.2%. We find 18% of the increased M&A can be explained by a technological change that reduced the costs of M&A.

Keywords: Merger and Acquisition, Two-sided Matching, Complementarity, Growth, Capital Reallocation

JEL Codes: C78, E10, G34, O49

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1 Introduction

Firm growth is a key determinant of the macroeconomic growth (Luttmer (2007)). How do firms grow? They can either grow "in house" through internal investment or grow "externally" through merger and acquisition (M&A).[1] Macroeconomists often focus on the first channel, while only a few study the second channel. In contrast, growing through M&A is very common in the real world. In US, about 30% of firms are involved in M&A in the last few decades.[2] Expenditures on M&A have averaged about 5% of annual GDP.[3]

Macroeconomists typically neglect M&A, possibly because M&A is considered as a capital reallocation process in which talented managers acquire more assets or employees. Usually people assume new acquired firms directly get acquiring firms' productivity but do not specify how the mechanism works (Manne (1965), Lucas (1978)).[4] Hence M&A is not typically distinguished from other investments. However, as a report from Toyota says "(the target firm) is an integrated system and difficult to digest", acquiring firms get not only the target firms' machines but also their management systems, selling channels and so on. Acquirers need to absorb the "organization capital" of target firms in M&A, which distinguishes M&A from other investments.

In this paper, we would like to understand how acquiring firms digest targets and how M&A changes both the firm growth rate and the aggregate growth rate. Our strategy is to use a general M&A technology function, which predicts micro M&A patterns consistent with empirical observations. The key property of the M&A technology is that it is easier for acquiring firms to digest similar and small targets.[5]

We then incorporate this M&A technology into an endogenous growth model in which firms are allowed to choose to invest through M&A or internal investment. We model the M&A market as a frictionless market, as in Roy (1951). Acquirers take prices of targets as given and optimally acquire those firms. The existence of the M&A increases the growth rate of the firm hence improves the aggregate growth rate.

At the micro level, our model predicts that (1) There is a positive assortative match-

[1] In this paper, "internal investment" means creating new capital, while M&A is a process of ownership change of existing capital.

[2] Source: Compustat dataset from 1978-2012.

[3] Source: SDC M&A database from 1978-2012.

[4] This framework becomes standard now. Recent research explores how financial friction (Eisfeldt and Rampini (2006), Midrigan and Xu (2014)) and asymmetric information (Eisfeldt and Rampini (2008)) affect capital reallocation using this assumption.

[5] This assumption is consistent with both the theory and empirical observations. We will discuss it later.

ing pattern on firm productivity;[6] (2) Productive firms choose to become acquirers while unproductive firms become targets;[7] (3) Target firms are younger than acquiring firms; (4) Productive firms prefer growing "in house". All of them are consistent with patterns observed in M&A data.

To evaluate the impact of M&A on the aggregate economy, we decompose the aggregate growth into: internal capital accumulation and M&A. The model predicts that the aggregate growth rate would decrease by 0.8% if firms can only grow through internal capital accumulation, which accounts for 21% of the US growth rate. This finding is a complementary of Greenwood et al. (1997) which claims that the internal capital accumulation explains 60% of the US growth, but neglects M&A.

We apply our model to explain the M&A boom of US economy since 1990s. Previous research suggests deregulation accounts for the major part in the M&A boom (Boone & Mulherin (2000); Andrade et al., (2001)). However, there is anecdotal evidence suggesting that the boom may be driven by a decrease in the M&A cost as a result of information technology (IT) improvement. In the words of a Deloitte consulting report, "IT makes integration easy". We evaluate the impacts of decline of M&A cost through the lens of our model, and find that it accounts 18% change in the M&A boom. Moreover, our model implies that the boom can increase the aggregate growth rate by 0.2%.

The paper contributes to the existing literature in two aspects. First, we contribute to the growth literature. Should firms expand through investing internally or M&A? Most existing growth models neglect the second channel. In our model, we fill this gap: the model distinguishes M&A and internal investments by introducing the M&A technology. A possible explanation of this technology is the cost of transferring the organization capital in the M&A. Quoting from Prescott and Visscher (1980), "Organization capital is not costlesly moved, however, and this makes the capital organization specific. .... . Variety is the spice of life at some level of activity, but we resist major changes in life-style."[8]

[6] In the capital reallocation literature (Lucas 1978, Midrigan and Xu 2014), acquiring firms only make quantity decisions: how much capital should be purchased from target firms (all the capital is taken as homogeneous regardless where the capital comes from). Yet firms in our environment (maybe also in the real world) face a more complex problem: They should trade off between the quality and quantity of target firms. Should firms buy large but unproductive targets or small but productive targets? We provide conditions to guarantee the equilibrium has a positive sorting pattern.

[7] The first two implications are also noticed by other papers (David 2013).

[8] Atkeson and Kehoe (2005) claim the accumulation of organization capital within the firm can account 8% of US output. Our paper suggests that transferring organization capital across firms may be also important.

Moreover, Rob and Zemsky (2002) show the cost of transferring organization capital is low when two firms are similar. The model, taking these theories as the microfoundations, discusses the growth effect of M&A.

Second, the paper contributes to understand the driving forces of recent M&A boom. In the finance literature, lots of empirical papers have studied reasons of merger waves using event-study analysis (Harford (2005)). In the industrial organization literature, several structural empirical works have evaluated the primary driving forces of M&A (Jeziorski (2009) and Stahl (2009)). However, the M&A in these models is motivated by increasing the acquiring firms' monopoly power, which may subvert the aggregate efficiency. Our model is a complementary to these papers, as we emphasize the positive effect of M&A.

The following parts are organized as follows: section 2 discusses related literature; section 3 discusses our M&A digestion technology function; section 4 shows the model; section 5 provides some empirical evidence of the model; section 6 explores our model's quantitative predictions; section 7 applies the model to explain the recent M&A boom and section 8 concludes.

2 Related Literature

There are several other related papers in the literature that we have not mentioned yet. First, the paper relates to the "one to many" assignment research, such as Eeckhout and Kircher (2012) and Geerolf (2013). Both of them study static matching models and Eeckhout and Kircher (2012) is closer to our model. This paper distinguishes from their model from two aspects: (1) We solve a dynamic model; (2) We endogenize the status choice of the acquiring firm and the target firm.

Second, a small number of theoretical papers have modeled M&A and studied the associated benefits and costs.[9] Jovanovic and Rousseau (2002) explain M&A as a simple capital reallocation process. Rhodes-Kropf and Robinson (2005) build a theory of M&A based on an asset's complementarity assumption. The most related paper is David (2013), which develops a structural model that M&A gains come from both the complementarity between acquiring and target firms assets' and capital reallocation. We also combine com-

[9] Some empirical papers in the finance literature report that stock prices of acquirers fall on the M&A anoucement day and take this as evidence that M&A reduces efficiency. However, Braguinsky and Jovanovic (2004) show that even M&A increases efficiency, the acquirer's stock price may still fall. Furthermore, Masulis et al. (2007) show that stock prices increase if the M&A is a cash transaction or the target is a private firm.

plementarity and capital reallocation assumptions but go beyond the existing literature by exploring how M&A gains and costs vary with firms' productivity and size. Another difference is that David (2013) studies an M&A market with search frictions, and prices in his model are determined by bargaining. While we model the M&A market as a competitive market and prices are determined by market clearing conditions. In the real world, acquiring firms often buy targets from the stock market, which we believe is closer to the assumption in our model.

Third, the paper is related to a series of empirical papers studying productivity change after M&A. Schoar (2002) and Braguinsky et.al (2013) document that productivity of acquiring firms will drop temporarily during the M&A, while target firms productivity will increase. However, target firms productivity can not catch up with acquiring firms. The M&A technology assumption in our model fits all of these findings.

Fourth, considering M&A as a way of increasing targets' productivity, the paper relates to the recent literature on the spread of knowledge and economic growth. Perla and Tonetti (2014) and Lucas and Moll (2014) study how technology is spread by assuming that unproductive firms can raise productivity via imitating productive firms. We explore another channel of technology spread: M&A.

In addition, the paper relates to the literature on stock market and economic growth. Levine and Zervos (1998) finds a well functioning stock market can increase the economic growth rate. Our paper points to a possible channel: stock market can make M&A easier, leading to an increase in the economic growth rate.

Lastly, starting from the seminal paper by Hsieh and Klenow (2009) there is a huge literature arguing that resource reallocation can explain aggregate TFP differences across countries. This paper, by modeling a particular way of capital reallocation, points out capital reallocation can not only result in huge TFP differences but can also generate a large differences in growth rates.

3 M&A Technology

Each firm is endowed with a firm specific productivity z and some capital when it is born. Productivity z is fixed over time unless the firm is acquired. At time t if the firm has capital k on hand, the firm's output is $y = zk$. In the M&A, acquirers can change the productivity of targets.

Table 1: Output Change before and after M&A

	t	t+1	t+2
Target	zk_T		
Acquirer	$(1-s_t)\,zk$	$zk+\hat{z}_T k_T$	$zk+\hat{z}_T k_T$
Acquirer (Hayashi insight)	$(1-s_t)zk$	$z\,(k+k_M)$	$z\,(k+k_M)$

3.1 A Simple Example

Consider two firms (z,k) and (z_T,k_T). Suppose there are no depreciation, no further investment and $z>z_T$. In period t, z starts to acquire z_T. To do so, z needs to spend time $s_t\in[0,1]$ to digest the target firm. The output of the target and acquiring firm is presented in table 1. In period t, a forgone cost $s_t z$ needs to be paid in the M&A process and the output of the acquirer is $(1-s_t)\,zk$. At the end of period t, the acquirer owns the target.

Then in $t+1$, the productivity of the acquirer jumps back to its original level z, while the productivity of the target will be changed from z_T to $\hat{z}_T$. If the M&A process can create value, $\hat{z}_T$ should be greater than z_T. The target belongs to the acquirer and the output of the acquirer after M&A is $zk+\hat{z}_T k_T$. From period $t+2$, we assume the output is same as period $t+1$ and does not change in the future.

In the third row of table 1, we show another way of writing the output of the acquirer. To avoid tracking distribution of $\hat{z}_T$ within the acquiring firm, we use Hayashi insight (1982, 1991): we transform the contribution of target output into efficiency units of capital. The output of the acquirer after M&A can be rewritten as $zk+\hat{z}_T k_T=z\,(k+k_M)$, where k_M is the efficiency units of capital acquired from the target. $k_M=\frac{\hat{z}_T}{z}k_T$. Hence through M&A, the acquirer expands its capital from k units to $k+k_{M,t}$ units. This is what we call "growing through M&A".

3.2 The General Case

More generally, we assume an acquirer (z,k_t) can buy several target firms at the same time. We call target firm's name as j. Denote $k_{T,t}\,(j)$ as the capital acquired from target firm j and $z_T\,(j)$ as the productivity of target j. The total capital acquired is $k_{T,t}=\int k_{T,t}\,(j)\,dj$.

After M&A, the productivity of target j will increase to $\hat{z}_T\,(j)=\hat{z}_T\left(s,\frac{k_T}{k},z,z_T\,(j)\right)$. We assume $\hat{z}'_{T,s}>0,\hat{z}'_{T,\frac{k_T}{k}}<0$. In other words, the acquiring firm can spend more time s and increase z_T more. Or if the acquiring firm buys lots of capital, it is hard to change

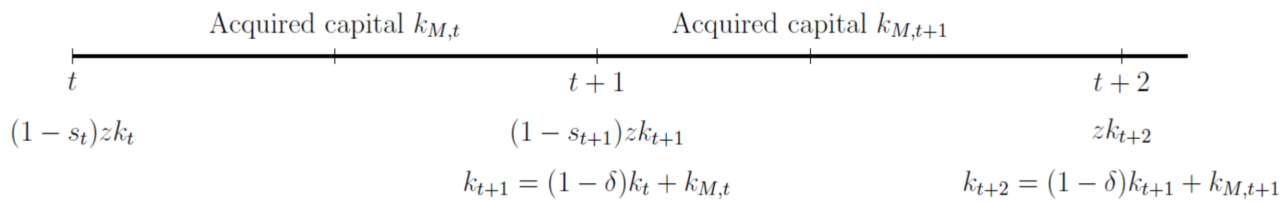


Figure 1: Capital Accumulation through M&A

the productivity of targets.

Similar as the previous simple example, we define the increase of acquirer's capital as

$$k_{M,t} = \int \frac{\hat{z}_T(j)}{z} k_{T,t}(j)\, dj \tag{1}$$

Figure 1 shows an example how the capital and output change through M&A. Consider firm z will acquire target firms in both period t and $t+1$. The output in t is $(1-s_t)k_t$ and firm z can get new acquired capital $k_{M,t}$ from target firms. In $t+1$, the capital of the acquirer will increase to $k_{t+1} = (1-\delta)k_t + k_{M,t}$ where δ is the depreciation rate. Firm z will acquire capital again in $t+1$, thus the output will be $(1-s_{t+1})zk_{t+1}$. Finally, in period $t+2$, firm z will have capital $k_{t+2} = (1-\delta)k_{t+1} + k_{M,t+1}$

When firms can expand through M&A and internal investment at the same time, the capital evolution rule is as follows

$$k_{t+1} = (1-\delta)\, k_t + i_t + k_{M,t} \tag{2}$$

$k_{M,t}$ is defined in equation (1) and i_t is the internal investment which is created from an increasing and convex technology $\Phi(i,k)$.

In this paper, we assume the functional form of $\hat{z}_T$ as

$$\hat{z}_T(j) = hs^{\theta}\left(\frac{k_T}{k}\right)^{-(1-\alpha)} f(z, z_T(j)) \tag{3}$$

where $h \in (0,1)$, $\theta \in (0,1)$ and $\alpha \in (0,1)$. We assume $f(z, z_T)$ is a CES function

$$f(z, z_T) = \left[(1-\varepsilon)\, z^{\psi} + \varepsilon z_T^{\psi}\right]^{\frac{1}{\psi}}, \quad \psi < 1 \tag{4}$$

Armed with the above functional forms, we have the M&A technology that transforms targets' capital into the acquirer's capital as

$$k_{M,t} = hs_t^{\theta} \left(\frac{k_{T,t}}{k_t}\right)^{-(1-\alpha)} \int \hat{f}\left(\frac{z_T(j)}{z}\right) k_{T,t}(j)\, dj \tag{5}$$

In equation (5), fix $k_{T,t}$, if the productivity of target firms is closer to the productivity of the acquirer, $\hat{f}\left(\frac{z_T(j)}{z}\right)$ will increase. It suggests that if the relative productivity distance $\frac{z_T}{z}$ is closer, the M&A technology is more efficient.

3.3 Relation to Existing Literature

It is helpful to think several special cases to understand the M&A technology.

Case 1 $\hat{z}_T = z$: In this case, the acquirer uses his productivity to replace the targets' productivity. It represents the M&A technology in many capital reallocation literature.[10]

Case 2 $\hat{z}_T = hs^{\theta} z_T$:[11] This function says that the acquiring firm can spend time s to increase the targets' productivity. This assumption is used broadly in human capital literature, such as Ben-Porath (1967).

Case 3 $\hat{z}_T = f(z, z_T)$ and f is a CES function: This assumption is consistent with papers by Rhodes-Kropf and Robinson (2005) and David (2013) which explore the complementarity property between theacquiring firm and the target firm. Rob and Zemsky (2001) study the optimal design of firms' organization and conclude that the cost of two firms merging together depends on the productivity distance between acquiring and target firms (equation (7)).[12]

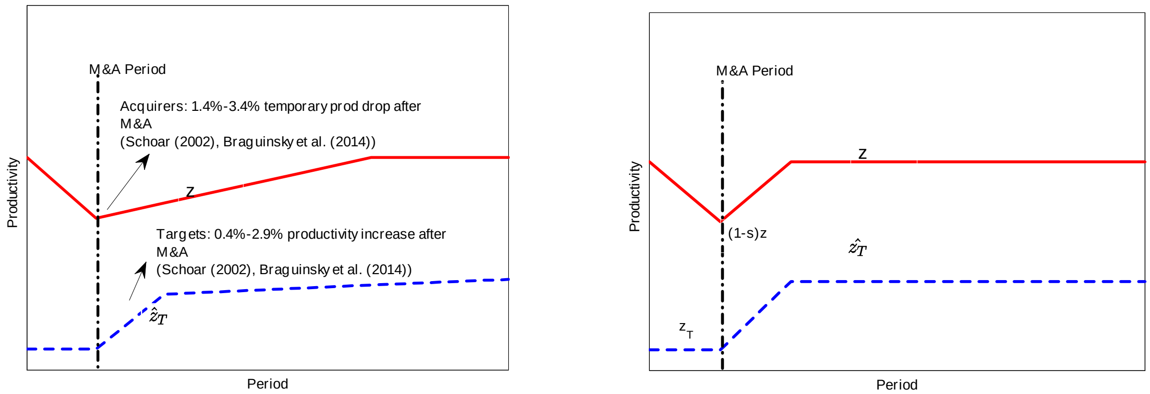
Thus the functional form assumption in (3) is quite general. Many existing models are nested as special cases of our model.

Empirical evidence supports the M&A technology assumption as well. Schoar (2002) and Braguinsky et al. (2013) study the productivity change after M&A. Their findings are summarized in the left graph of figure 2: (1) During the M&A process, the productivity of acquiring firms will drop and recover in a few years; (2) Targets' productivity $\hat{z}_T$ will increase but can not catch up with acquirer productivity. Both are consistent with our M&A technology. The right graph of figure 2 shows the prediction of our M&A technology.

[10] Look at footnote 4.

[11] In this case, we need to assume $h > 1$.

[12] More generally, this function is also used in human capital literature, such as Cunha et al. (2010, equation (2.3) and (2.4)). They study the complementarity between parents' and children's abilities.



Data Model

Figure 2: Productivity before and after M&A

Notes: This figure compares productivity of acquiring and acquired firms before and after M&A in the data and the model. Productivity change in the data comes from Schoar (2002) and Braguinsky et al. (2014). They can distinguish the target and the acquirer output after M&A because both of them use plant level data. Their main findings: (1) Acquiring firms productivity will temporarily drop by 1.4%-3.4%; (2) Targets productivity will increase 0.4%-2.9% but can not catch up with acquiring firms.

During the M&A period, the productivity of acquiring firm will drop temporarily due to the forgone cost sz and then will recover back. The productivity of target firms will increase but will not exceed z since f is a CES function and s is smaller than 1.[13]

Moreover, our M&A technology is also consistent with Carlin, et al. (2010) which finds that M&A is most valuable if one large firm acquires a similar but small target firm.

4 Model

In this section, we introduce the setup of our model. We organize this section in the following manner: We first describe the consumer and the firm problems and then define the equilibrium. Then we explore implications of the model for equilibrium existence, M&A pattern, aggregate efficiency and the growth rate.

[13]The recover of z and the increase of z_T in the model are in 1 period. It is not consistent with the data. However, assuming the changes take several periods, same as the data, does not change our results too much.

4.1 Household Problem

A representative consumer who consumes aggregate consumption C_t each period maximizes the lifetime utility

$$\max \sum_{t=0}^{\infty} \beta^t U\left(C_t\right), \quad \beta \in (0,1)$$

The optimal intertemporal optimization condition yields

$$\frac{1}{1+r_t} = \beta \frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)} \tag{6}$$

where r_t is the equilibrium interest rate at time t. We assume there is no aggregate uncertainty hence the consumer has a deterministic consumption path.

4.2 Firm Problem

There is a continuum of risk neutral firms who produce one homogeneous good. The firm's production function is same as in section 3. Each firm is initially endowed with a permanent productivity z and some capital. The productivity is fixed over time unless the firm is acquired. Each firm can expand by accumulating capital either through M&A as explained in section 3 or through internal capital accumulation.

In figure 3, we summarize the timing of the firm problem. At the beginning of each period, the firm needs to choose whether to become a target firm (sell his capital) or an acquiring firm (get new capital). If the firm chooses to sell its capital, it will produce first and then optimally choose the amount of capital Δ to sell. At the end of the period, there is a death shock: with probability $1-\omega$, it will die and all its capital will be burnt. If the firm chooses to become an acquirer, it receives an iid random shock: with probability λ the firm has a chance to acquire target firms. If it has access to M&A, the firm can choose the target firm's level of z_T, the amount of capital it wants to buy from the target, k_T, and the time s_t. If the acquiring firm does not have the opportunity to engage in M&A, it can only accumulate capital internally.

The M&A markets are organized in this way: there are a continuum of capital markets. Each capital market is indexed by the target firm's productivity on this market, z_T. At time t, the target firm can get a price $P_t\left(z_T\right)$ for each unit of capital. Hence if the target firm chooses to sell an amount Δ of its capital on market z_T, it can get $P_t\left(z_T\right)\Delta$.[14]

[14]Notice that we do not assume the capital markets are indexed by both target productivity and amount

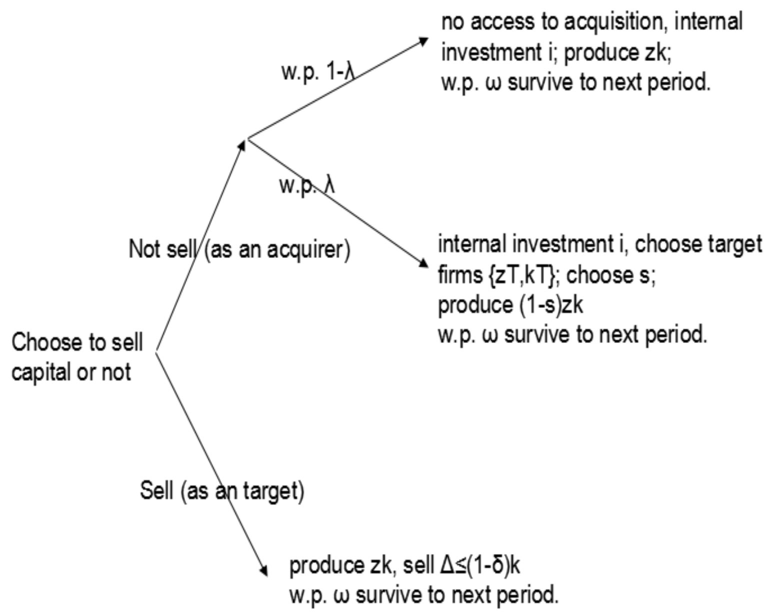


Figure 3: Timing

Define V_t^A as the acquiring firm's value, V_t^I as the value of a firm investing internally only and V_t^T as the value of a target firm at time t. Then if the acquiring firm has a chance to acquire targets, we have

$$V_t^A(z,k) = \max_{s, z_T(j), k_T(j), i} \left\{ \begin{array}{c} (1-s)\, zk - \int P_t(z_T(j))\, k_T(z_T(j))\, dj - \Phi(i,k) \\ + \frac{\omega}{1+r_t} \max\left[\lambda V_{t+1}^A(z,k') + (1-\lambda) V_{t+1}^I(z,k'), V_{t+1}^T(z,k')\right] \end{array} \right\} \tag{7}$$

$$\text{s.t. (2) and (5), } i \geq 0, k_T \geq 0, s \in [0,1]$$

Equation (7) says the acquiring firm optimally chooses the productivity of his target, z_T, the capital it buys from the target firm, k_T, the time it would like to spend on M&A, s and internal investment i. The current output is $(1-s)\, zk$ and the cost of investment is $\int P_t(z_T(j))\, k_T(z_T(j))\, dj + \Phi(i,k)$. Hence the first row in equation (7) is the current profit. The firm discounts future by $\frac{\omega}{1+r_t}$. In the next period, the firm needs to choose whether to become an acquirer or a target. If it becomes an acquirer, the firm will have

of capital. Hence targets with the same productivity will pool their capital in one market and the acquirer may choose the amount of capital desired.

a chance to acquire target firms with probability λ. With probability $1-\lambda$, the firm can expand only through internal capital accumulation. Hence the expected value of an acquirer is $\lambda V_{t+1}^{A} + (1-\lambda) V_{t+1}^{I}$. The firm optimally chooses between the maximum of $\lambda V_{t+1}^{A} + (1-\lambda) V_{t+1}^{I}$ and V_{t+1}^{T}.

If the acquiring firm does not have a chance to acquire targets, it optimally chooses internal investment and receives value:

$$V_t^I(z,k) = \max_i \left\{ \begin{array}{c} zk - \Phi(i,k) \\ +\frac{\omega}{1+r_t} \max\left[\lambda V_{t+1}^A(z,k') + (1-\lambda) V_{t+1}^I(z,k'), V_{t+1}^T(z,k')\right] \end{array} \right\} \tag{8}$$

$$\text{s.t. (2), } i \geq 0$$

Equation (8) is very similar as equation (7) except $k_T = 0$. It says that the acquiring firm can only invest through internal capital accumulation i.

If a firm chooses to become a target

$$V_t^T(z,k) = \max_{k' \geq 0} \left\{ \begin{array}{c} zk + P_t(z)\Delta \\ +\frac{\omega}{1+r_t} \max\left[\lambda V_{t+1}^A(z,k') + (1-\lambda) V_{t+1}^I(z,k'), V_{t+1}^T(z,k')\right] \end{array} \right\}) \tag{9}$$

$$\text{s.t } k' = (1-\delta)k - \Delta$$

Equation (9) defines the value of the target firm at time t. The firm's current profit at time t includes output zk and income from selling capital $P_t(z)(k' - (1-\delta)k)$. Capital next period will become to k'.

In period t, there is a mass of entrants e_{t+1} pay the entry cost and draw productivity from a distribution with PDF $m(z)$ whose support is $[z_{\min}, z_{\max}]$. There is one period of time-to-build: new entrants start to produce next period. Each new entrant is endowed with an initial capital $\tilde{k}_{t+1}$ which is a fixed fraction μ of average firm capital $\bar{K}_t$ in the economy. That is $\tilde{k}_{t+1} = \mu \bar{K}_t$. The cost of entry *per unit* of capital is q and the entry process satisfies the free entry condition

$$q\tilde{k}_{t+1} = \frac{1}{1+r_t} \int V_{t+1}\left(z, \tilde{k}_{t+1}\right) m(z)\, dz \tag{10}$$

We simplify the model by making the following assumption.

Assumption 1: $\Phi(i,k) = \phi\left(\frac{i}{k}\right)k$

Proposition 1 *Given assumption 1, then firm value functions are constant returns to scale on capital k:* $J_t^A(z) = \frac{V_t^A}{k}$, $J_t^T(z) = \frac{V_t^T}{k}$, $J_t^I(z) = \frac{V_t^I}{k}$

Proof. See appendix. ■

Define $\hat{x} = \frac{x}{k}$. Then the investment rate of the firm is $\hat{k} = \frac{k_M + i}{k}$. Equations (7) to (9) can be rewritten as

$$J_t^A(z) = \max_{\hat{k}\geq 0}\left\{z - c_t^A\left(z,\hat{k}\right) + \frac{\omega}{1+r_t}\left(1-\delta+\hat{k}\right)J_{t+1}(z)\right\} \tag{11}$$

$$\begin{aligned} s.t.\ \ c_t^A\left(z,\hat{k}\right) &= \min_{z_T(j),\hat{k}_T(j),s\in[0,1]}\left\{sz + \int P_t\left(z_T(j)\right)\hat{k}_T\left(z_T(j)\right)dj + \phi(\hat{\imath})\right\} \quad (12)\\ \hat{k}_M &= hs^{\theta}\hat{k}_T^{-(1-\alpha)}\int \hat{f}\left(\frac{z_T(j)}{z}\right)\hat{k}_T(j)\,dj,\ \ \hat{k}_M \in \left[0,\hat{k}\right], \hat{k} = \hat{\imath} + \hat{k}_M \end{aligned}$$

$$J_t^I(z) = \max_{\hat{k}\geq 0}\left\{z - \phi\left(\hat{k}\right) + \frac{\omega}{1+r_t}\left(1-\delta+\hat{k}\right)J_{t+1}(z)\right\} \tag{13}$$

$$J_t^T(z) = z + (1-\delta)P_t(z) \tag{14}$$

$$J_{t+1} = \max\left(\lambda J_{t+1}^A + (1-\lambda)J_{t+1}^I, J_{t+1}^T\right) \tag{15}$$

Equation (11) defines J_t^A. We decompose the firm problem into two steps. First, we solve the cost of firm z if investment, $c_t\left(z,\hat{k}\right)$. It is defined in (12). The first term in (12) sz is the forgone cost of M&A. The second term $\int P_t\left(z_T(j)\right)\hat{k}_T\left(z_T(j)\right)dj$ is the price paid to the target firms and the third term $\phi(\hat{\imath})$ is the cost of internal investment. In (12), we optimally choose target z_T, $\hat{k}_T$ and $\hat{\imath}$ to minimize the cost of investment. Second, we solve the optimal investment rate of firm z in equation (11). $z - c_t^A\left(z,\hat{k}\right)$ is the profit in t. In next period, the firm will expand by $1-\delta+\hat{k}$. It will survive with probability ω and the firm value will be $\left(1-\delta+\hat{k}\right)J_{t+1}$, otherwise the firm will die and gets 0. As we will show later, there is only one z_T that will be acquired for each firm z.

From (12), we can see how M&A can improves the firm growth rate. The M&A technology in section 2 will give us a endogenous and M&A cost $sz + \int P_t\left(z_T(j)\right)\hat{k}_T\left(z_T(j)\right)dj$.

It is increasing and convex in $\hat{k}_M$. In other words, firms have two technologies to expand: through M&A or through internal investment. Both of them have convex cost functions. The existence of M&A will help firms to smooth the cost of growth hence reduce the cost of growth, as shown in equation (12).

Equation (13) is similar except that the firm can not acquire capital from the target hence $\hat{k}_M = 0$. $\phi(\hat{\imath})$ is the cost of internal capital investment.

Equation (14) describes the value of a target firm. Notice that when the firm chooses to become a target, it will sell all its capital since the firm's value function is linear in k.

The free entry condition can be simplified to

$$q = \frac{1}{1+r_t} \int J_{t+1}(z)\, m(z)\, dz \tag{16}$$

The economic mechanism of the model can be seen from equation (12) and (16). Because the existence of M&A reduces cost of firm growth, the expected firm value $\int J_{t+1}(z)\, m(z)\, dz$ will increase. From household's Euler equation, we can see that interest rate is positively correlated with aggregate growth, hence the M&A will increase the aggregate growth rate.

4.3 Equilibrium

A competitive equilibrium can be defined as follows.

Definition 2 *A competitive equilibrium includes: (i) two occupation sets A_t, T_t, if $z \in A_t$ (or T_t) then firm will choose to be acquirer (target); (ii) a matching function $z_{T,t}(z)$; (iii) prices $P_t(z)$ and r_t; (iv) Number of entrants e_t; (v) distribution of firm size and productivity $\Gamma_t(k,z)$; (vi) aggregate consumption C_t, such that (a) firm and household problems are solved given prices; (b) distributions are consistent with firm decisions; (c) capital markets clear: $\forall$ measurable subset $A' \subseteq A_t$,its image set defined by the matching function $z_{T,t}$ is $z_{T,t}(A') \subseteq T_t$, then*

$$\lambda \int_{z \in A', k} \hat{k}_{T,t}(z)\, k d\Gamma_t(k,z) = \int_{z \in z_{T,t}(A'), k} (1-\delta)\, k d\Gamma_t(k,z) \quad \forall A' \subseteq A \tag{17}$$

(d) goods market clears

$$Y_t = C_t + \int \Phi_i d_i + q e_{t+1} \tilde{k}_{t+1} \tag{18}$$

To complete the definition of the equilibrium, we also need to define the off-equilibrium price. If the firm $z \notin T$ chooses to become a target, the deviation price is defined as

$$P_t(z) = \sup \left\{ \begin{array}{l} p : \text{there exists an acquirer } (z_A, k_A) \text{ if matched} \\ \text{with } z \text{ at price } p, \text{ payoff is same as } V_t^A(z_A, k_A) \end{array} \right\}$$

In other words, the deviation price is defined as the best price that firm z can get to make some acquiring firms indifferent.

In equation (17), the left hand side is the total demand for capital from acquirer $z \in A'$ at time t. $\hat{k}_{T,t}(z)$ is the demand of acquiring firm z *per unit* of capital. Among z, there are only a share λ that can acquire firms. Hence after multiplying $\hat{k}_{T,t}(z_{T,t}(z))$ by firm size k and λ, we have the demand for targets' capital of acquiring firms (z, k). Then we sum across all possible k and get the demand for targets' capital of acquiring firms conditional on productivity z. Integrating across all firms in set A', we get total demand for capital of acquiring firms whose productivity is in set A'. The right hand side of equation (17) is the total supply of the capital from target firms. The set of target productivity is given by the image set $z_{T,t}(A')$ and the total capital of those firms is given by the right hand side.

4.4 Model Solution

We define the static profits (per unit of capital) of firms as $\pi_t^A\left(z, \hat{k}\right) = z - c_t^A\left(z, \hat{k}\right), \pi_t^I\left(z, \hat{k}\right) = z - \phi\left(\hat{k}\right)$. The first equation is the profit function of acquiring firms given productivity z and firm growth rate $\hat{k}$. The second equation is the profit of internal accumulation firms. Notice that both profit functions are decreasing in capital $\hat{k}$. Let us define $\hat{k}_t^{A*}(z)$ and $\hat{k}_t^{I*}(z)$ as capital levels that drive the firm profits to be 0: $\pi_t^A\left(z, \hat{k}_t^{A*}(z)\right) = 0, \pi_t^I\left(z, \hat{k}_t^{I*}(z)\right) = 0$. We assume the following:

Assumption 2: $\hat{k}_t^{A*}(z_{\max}) < \frac{1+r_t}{\omega} + \delta - 1, \hat{k}_t^{I*}(z_{\max}) < \frac{1+r_t}{\omega} + \delta - 1$

The above assumption says that growth rate of the firm can not be too large. When profit is positive, the growth rate should be smaller than $\frac{1+r_t}{\omega}$. Intuitively, if the growth rate is greater than discount rate. firm value will be infinite.

Proposition 3 *Under assumption 2 and* $\pi_z > 0$, *we have (1) Equations (11)-(15) have a solution; (2)* $J_t(z)$ *is increasing and convex in* z*; (3)* $\hat{k}_t(z)$ *is increasing in* z*.*

Proof. See appendix. ■

From the definition of equilibrium, we can see that the capital market clearing condition is much more complicated than standard models: we have infinite capital markets and all of them should satisfy condition (17). The following two propositions show that we can simplify the capital market clearing conditions under some assumptions.

Proposition 4 *(Status Choice) There exists a cutoff value z_t^* such that $\lambda J_t^A(z_t^*)+(1-\lambda)J_t^I(z_t^*)=J_t^T(z_t^*)$ and if $z>z_t^*$ then firm will choose to be acquirer; if $z<z_t^*$ then it will choose to become target.*

Proof. See appendix ■

The above proposition says that acquiring firms productivity are higher than target firms productivity. Intuitively, in our M&A technology, there are two parts: $f(z, z_T)$ measures the productivity change after M&A while v is the efficiency of absorbing target firms. If an unproductive firm acquires a productive firm, then potential output of M&A, $f(z, z_T)k_T$, will be smaller than the target's initial output $z_T k_T$. Given the efficiency of absorbing v is smaller than 1, there is no gain when an unproductive firm acquires a productive target.

The next proposition gives us the condition for when we will see a sorting pattern in M&A.

Proposition 5 *(Sorting) If $\psi \leq 0$, then z_T increases on z.*

Proof. See appendix ■

Figure 4 shows the equilibrium matching pattern. When $\psi \leq 0$, our model equilibrium can be summarized as: in each period new entrants enter, then less productive firms will be acquired while productive firms will survive. More productive acquiring firms will buy more productive target firms.

In the following parts, we assume $\psi \leq 0$. From the market clearing condition (17) and positive sorting condition, we have

$$\lambda \int_{z}^{z_{\max}} \hat{k}_{T,t}(z)\, k d\Gamma_t(k,z) = \int_{z_{T,t}(z)}^{z_t^*} (1-\delta)\, k d\Gamma_t(k,z) \quad \forall z \geq z_t^* \tag{19}$$

Comparing the above equation and condition (17), we can see it is much simpler: first, z will only choose a unique target firm z_T; second, we do not need to solve market clearing conditions for any possible set A' but only need to check the subsets that above z.

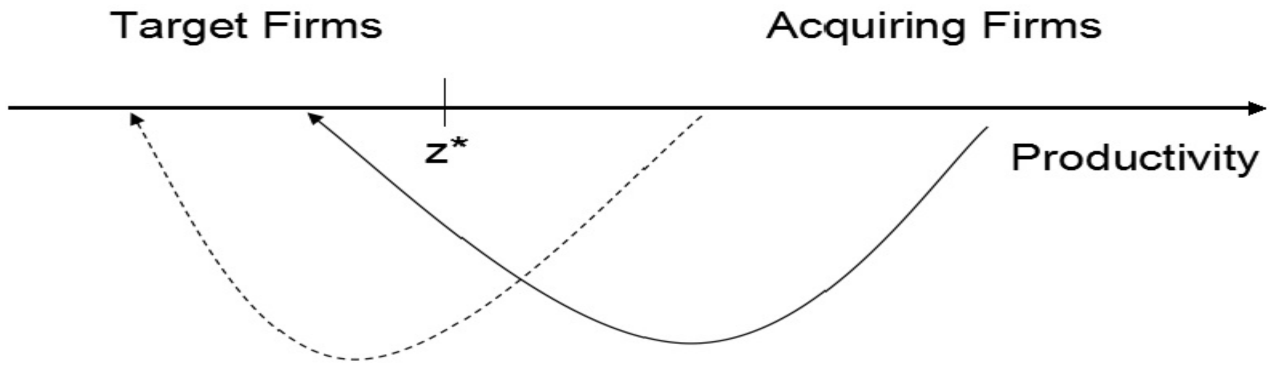


Figure 4: Matching Pattern of the Model

Equation (19) defines the matching function. We also need two boundary conditions

$$z_{T,t}\left(z_t^*\right) = z_{\min}, z_{T,t}\left(z_{\max}\right) = z_t^* \tag{20}$$

The above two equations say that acquiring firm z_t^* will match with $z_{\min}$ and $z_{\max}$ will match with firm z_t^*.

In a unidimensional sorting model (as Becker, 1973), positive sorting arises if in the M&A technology function f has positive cross partial derivative, $f''_{z_T z} > 0$. Given f is a CES function, f satisfies this condition for any $\psi \leq 1$. In our model, acquiring firms have a trade-off between buying a small amount of capital from productive targets and buying a large amount of capital from unproductive targets.[15] Proposition 5 says that to obtain the positive sorting on acquiring firms productivity and target firms productivity, we need to have a stronger complementarity than Becker's model.

In addition, we can show that the decentralized equilibrium is also Pareto optimal.

Proposition 6 *The decentralized equilibrium is Pareto optimal.*

Proof. See appendix. ■

[15] Eeckout and Kircher (2012) studies this "quality vs quantity" tradeoff in a static environment.

4.5 Balanced Growth Path

The aggregate capital in this economy is defined as

$$K_t = \int k d\Gamma_t(k, z) \tag{21}$$

And we can also define the total output of the economy as

$$Y_t = \int_{z \geq z^*} [1 - \lambda s(z)] z k d\Gamma_t(k, z) + \int_{z < z^*} z k d\Gamma_t(k, z) \tag{22}$$

where $\lambda s(z)$ is the expected productivity loss of acquiring firm.

In the following parts, we focus on the balanced growth path (BGP) equilibrium and it is defined as:

Definition 7 *A Balanced growth path (BGP) equilibrium is a competitive equilibrium with a constant $g_K > 1$ such that (i) all value functions $J^A(z), J^T(z), J(z), P(z)$ and policy functions do not depend on time t; (ii) Y_t, K_t and C_t grow with same speed g_K.*

The following proposition shows there exists a BGP in the model.

Proposition 8 *The model has a BGP with constant growth rate g_K such that g_K is implicitly defined by*

$$\int_{z \geq z^*} \frac{m(z)}{1 - \frac{\omega}{g_K}\left(\lambda g^A(z) + (1-\lambda) g^I(z)\right)} dz + M(z^*) = \frac{g_K}{e\mu} \tag{23}$$

Aggregate output will be determined by

$$Y_t = Z K_t \tag{24}$$

Z is the aggregate TFP

$$Z = \int_{z \geq z^*} \frac{(1 - \lambda s(z)) z}{1 - \frac{\omega}{g_K}\left(\lambda g^A(z) + (1-\lambda) g^I(z)\right)} m(z) dz + + \int_{z^*}^{z_{\max}} z m(z) dz \tag{25}$$

Proof. See appendix. ■

First of all, we can see that if the firm's growth rate increases, the aggregate capital growth rate g_K increases as well. Thus we can see the positive link between M&A and the

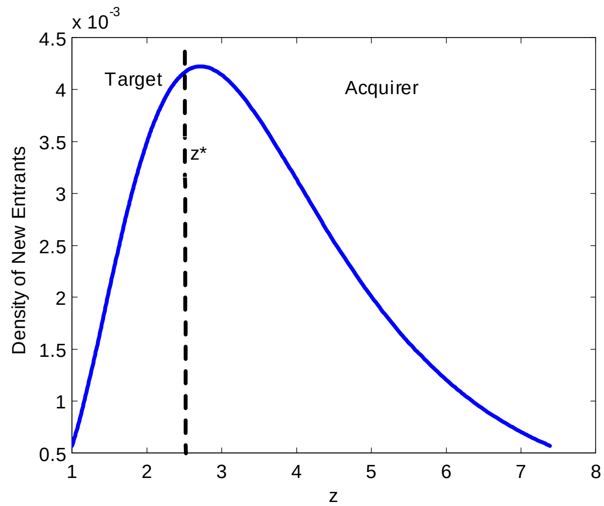


Figure 5: Distribution of Productivity: Entrants

aggregate growth rate. M&A can increase the growth rate of the firm hence increase the growth rate of the aggregate economy. Second, if the relative capital of new entrants $e\mu$ increases, g_K will increases as well.[16]

The aggregate TFP has two components. The first component is the acquirer's contribution to Z. $(1-\lambda s(z))\, z$ is the average productivity level of acquiring firms. $\frac{1}{1-\frac{\omega}{g_K}(\lambda g^A(z)+(1-\lambda)g^I(z))}$ is the acquiring firms' total capital share in the aggregate economy. Notice that if acquiring firms are more productive, they will have a higher $\left(\lambda g^A(z)+(1-\lambda)\, g^I(z)\right)$, hence they will have a higher market share in the economy. The second component is the target firms' contribution to Z.

On the BGP, the cutoff z^* will be a constant. Firms with productivity above z^* will always choose to invest. New entrants, if their productivity is below z^*, will always produce only one period and then sell all their capital (shown in figure 5). Hence acquiring firms will be more productive, larger and older than target firms in a BGP equilibrium. Firms above z^* will grow larger with growth rates $g^A(z)$ if they have access to acquisitions and $g^I(z)$ if they do not have access to acquisitions. Figure 6 shows the distribution of firm growth rates. The solid line represents the firm growth rate if the firm has access to acquisitions.

[16] On the balanced growth path, the number of firms is a constant but $e \neq 1-\omega$. The number of firms exiting from the market each period is $eM(z^*) + (1-\omega)\frac{e}{1-\omega}(1-M(z^*)) = e$. The first part is the firms that are acquired and the second part is the firms that are dead. $\frac{e}{1-\omega}(1-M(z^*))$ is the number of incumbents.

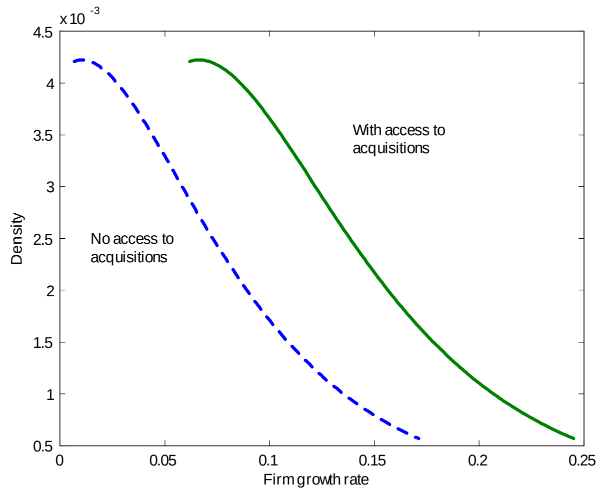


Figure 6: Distribution of Firm Growth Rate

The dashed line shows the firm growth rate if the firm does not have access to acquisitions. The difference between these two curves is the contribution of M&A to the firm growth rate.

On the BGP, productivity distribution will be fixed and only firm size will grow. Figure 7 draws the ln firm size distribution. On the BGP, the shape of the size distribution will be unchanged, but the distribution will shift to the right with a constant rate. The next propsition shows that the firm size distribution has a Pareto tail.

Proposition 9 *Define the average firm size as $\bar{K}_t$ and the relative size of firm j as $\frac{k_t(j)}{\bar{K}_t}$, then the distribution of the relative size conditional on productivity has a Pareto tail*

$$\lim_{x\to\infty} \frac{\Pr\left(\frac{k_t(j)}{\bar{K}_t} \geq x|z\right)}{x^{-\Theta(z)}} = constant \tag{26}$$

and $\Theta(z)$ satisfies

$$\omega\left[(1-\lambda)\, g^I(z)^{\Theta(z)} + \lambda g^A(z)^{\Theta(z)}\right] = g_K^{\Theta(z)} \tag{27}$$

and the unconditional distribution of relative firm has a Pareto tail with tail index $\Theta(z_{\max})$

$$\lim_{x\to\infty} \frac{\Pr\left(\frac{k_t(j)}{\bar{K}_t} \geq x\right)}{x^{-\Theta(z_{\max})}} = constant \tag{28}$$

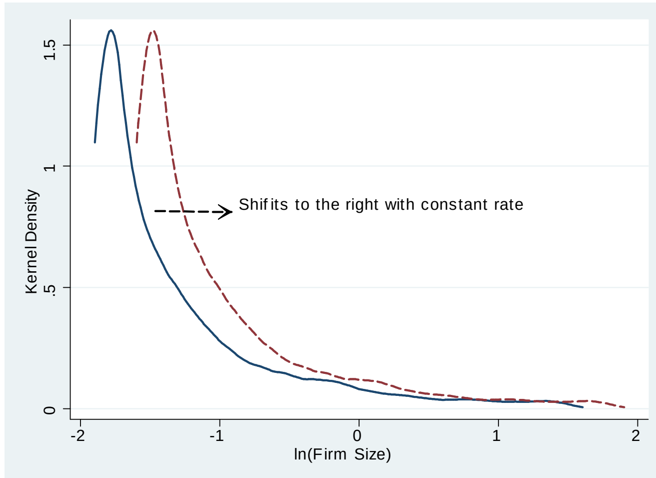


Figure 7: Distribution of Firm Size

Proof. See appendix. ■

The intuition of the proposition 9 is as follows: conditional on the productivity, firm growth rate does not depend on the size. Hence our model follows the Gibrat's law conditional on the productivity. It is well known that Gibrat's law will generate a size distribution with Pareto tail (Garbaix (2009)). Thus conditional on productivity, the firm size distribution has a Pareto tail. If pooling all firms together, the most productive firm will determine the tail of the size distribution.

Tonetti and Perla (2014) study a growth model in which unproductive firms can imitate productive firms. They start with a Pareto productivity distribution and get an equilibrium Pareto size distribution. However, in our model, productive firms try to raise the productivity of unproductive firms and the price is determined endogenously. In addition, starting from any productivity distribution, our model can generate a Pareto size distribution.

5 Empirical Evidence

In this section, we provide some empirical evidence of our model's implications. This section is organized as follows: we first calibrate the parameters of the model from the M&A data at the micro level and compare our model with M&A pattern. Then we get more evidence from information of new-startups. Finally, we provide some cross country evidence.

5.1 Evidence from M&A Pattern

5.1.1 Data

We use two data sets. The first one is the Compustat dataset. The second one is an M&A transaction data from the Thomson Reuters SDC Platinum database (SDC). SDC collects all M&A transactions in US that involve at least 5% of the ownership change of a company where the transaction is valued at $1 million or more (after 1992, all deals are covered) or where the value of the transaction was undisclosed. We download all US M&A transactions from 1978 to 2012. For most transactions, SDC contains a limited number of pre-transaction statistics on the merging parties, such as sales, employee counts and property, plant and equipment. In order to get more statistics, we merge the SDC data set with the Compustat data set. However, direct merging these two data sets is not possible since Compustat data only records most recent CUSIP codes while SDC data uses CUSIP codes at the time of M&A. Hence we first use historical CUSIP information in the CRSP data set and merge SDC data with CRSP data. Then we use CRSP identifier to link with Compustat data. There are 77901 transactions directly downloaded from the SDC data set. After matching CRSP translator, there are 6608 transactions in which we can find CRSP identifier (permno) for both acquirers and targets. After merging with Compustat data, 3255 transactions remain without any missing information on sales, employee counts or total assets.

5.1.2 Calibration

To calibrate the model, we assume consumer's utility is $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ with $\gamma = 3$. And we choose the depreciation rate $\delta = 0.1$. The probability of survival rate is chosen to be $\omega = 0.85$, the size of new entrant $\mu = 0.15$ (Thorburn (2000)) and the discount factor $\beta = 0.9$.

We assume the internal investment has a cost function as $\phi(\hat{\imath}) = \frac{v_i}{2}\hat{\imath}^2$. We choose v_i to match the M&A intensive margin: the share of M&A in total investment $\left(\frac{P(z_T)k_T}{P(z_T)k_T+\phi(i)}\right)$.

The productivity distribution of entrants $m(z)$ is a truncated log-normal distribution. We normalize the the mean of log productivity to be 1 and the standard deviation to match the firm growth rate dispersion. The $\log z_{\max}$ and $\log z_{\min}$ as two standard deviations away from the mean. q is calibrated to match the firm growth rate.

Table 2: Parameters

Parameters	Value	Moments
M&A Tech		
h	0.81	M&A/Output
θ	0.05	Sales dif
ϵ	0.35	Slope of M&A intensive margin
$\frac{1}{1-\psi}$	0.67	z_T/z
$1-\alpha$	0.55	Slope of z_T/z
Other Params		
λ	0.35	M&A extensive margin
v_i	54.3	M&A intensive margin
q	4.80	Firm growth rate
σ_z	0.5	Firm growth rate std.
ω	0.85	Thorburn (2000)
μ	0.15	Dunne et al. (1988)

Notes: This table reports the parameters used. M&A extensive margin = percentage of firms whose acquisitions>0; M&A intensive margin $=\frac{P(z_T)k_T}{P(z_T)k_T+\phi(i)}$.

The rest six parameters are related to the M&A technology: h, ψ,θ,α, ε and the probablity of accessing to M&A market λ. We calibrate them to jointly match the M&A share in total output, sales difference between acquiring and target firms, the productivity difference between target and acquiring firms $\frac{z_T}{z}$, the productivity matching function slope, extensive margin of the M&A and the slope of intensive margin. Extensive margin is the percentage of firms with acquisitions>0 in the Compustat database. The slope of intensive margin is the slope of regressing log M&A intensive margin on $\log(z)$. The parameters are shown in table 2.

Intuitively, M&A/output tells us the level of M&A cost. It helps us to calibrate h. The relative sales between targets and acquirers sheds light on the forgone cost sz. We use this moment to calibrate θ. Next, the slope of intensive margin implies the slope of price $P(z_T)$. It is helpful to calibrate ε.[17] Finally, $\frac{z_T}{z}$ and the slope of $\frac{z_T}{z}$ tell us how k_T can be transformed to k_M. We calibrate ψ and α to match these two moments

$\varepsilon=0.35$ indicates that in the M&A transaction, only 65% of the acquirers' productivity would be passed to newly merged firms. $1-\alpha=0.55$ means that there is a strong decreasing returns to scale on absorbing large target firms: when the relative size of the target increases by 1%, then the absorbing efficiency will decrease by 55%.

Table 3 reports the target moments of the data and the model. The model replicates the data moments reasonably good. We can see target firms are smaller and less productive

[17] In the appendix, we show that if $\psi=0$, $\ln P(z_T)$ has a slope $\frac{\varepsilon}{\alpha}$.

Table 3: Moments of the Data and Model

	Data	Model
Target sales/Acquirer sales	0.20	0.18
$\frac{z_T}{z}$	0.65	0.59
Slope of $\frac{z_T}{z}$	0.85	0.93
Extensive margin	0.30	0.30
Intensive margin	0.38	0.37
Slope of Intensive margin	0.14	0.21
M&A/Output	0.05	0.06
Firm growth rate	0.065	0.080
Firm growth rate std.	0.12	0.12

than acquiring firms,[18] which is consistent with the model prediction.

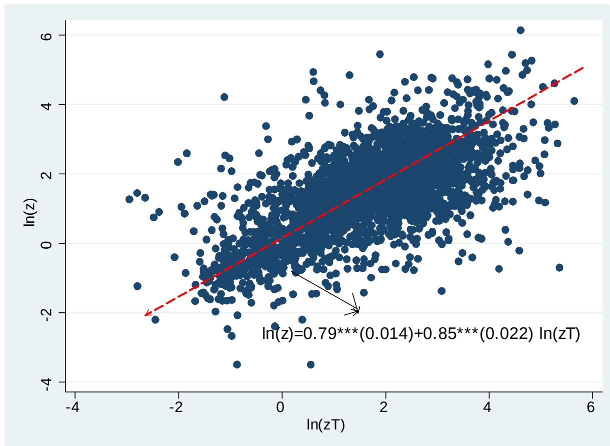
5.1.3 Positive Sorting Pattern in M&A

Our model predicts that there is a positive sorting pattern on between productivity of acquirers and targets. Figure 8 plots the sorting matching pattern of acquiring and target firms. The top graph plots sorting pattern of productivity, which is measured by log sales minus log assets. The horizontal line is the productivity of the acquiring firm and the vertical line is the productivity of the target firm. We can see that there is a strong positive assortative matching pattern on productivity: more productive acquirers tend to buy more productive targets. The linear fit function has a significant slope coefficient of 0.85 while the intercept is 0.79. The bottom graph plots the matching pattern of log productivity in the model. We plot log z on the x-axis and log z_T on the y-axis. There are two lines in the graph: the solid blue line is the matching function implied by the model. We can see that when log z is approximately 0.9, then the firm is indifferent between target and acquirer choice (the x-axis starts at 0.9 while y-axis ends at 0.9). The dashed red line is the linear fit function. It has a slope of 0.93 and an intercept of -1.05.

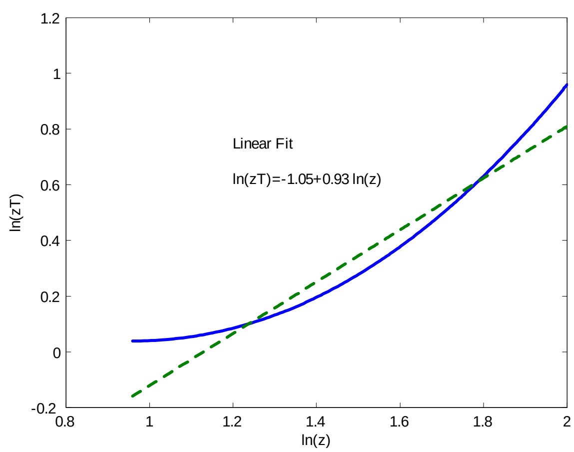
5.1.4 Targets are Young Firms

Our model predicts that the targets only survive one period. In the data, we find targets are younger than acquirers. We explore the firm age distribution in figure 9. The top two graphs plot the age distributions of the target firms and non-target firms in the data.

[18] David (2013) also documents this fact.



Data



Model

Figure 8: Productivity Sorting Pattern in M&A

Notes: This figure presents the log productivity matching patterns in the data and the model. Productivity in data is defined as ln(z)=ln(sales)-ln(assets). The dashed lines are the linear fits of the matching functions. *** denotes statistically significant at 1% level and standard errors are reported in brackets. Data source: SDC M&A database.

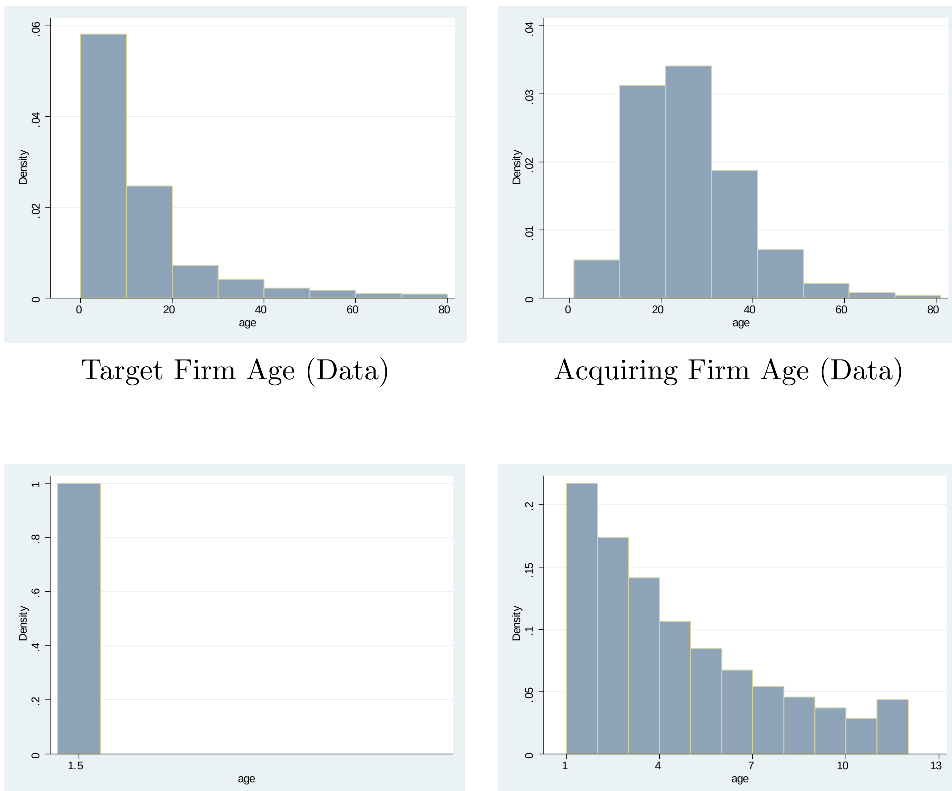
Most target firms' ages are between 0 to 10, while only 10% of acquiring firms' ages are less than 10. Thus target firms are much younger than acquiring firms. This pattern is consistent with the prediction of our model. The model predicts that the target firms will be acquired as soon as they enter the market. On the bottom two graphs in figure 9, we plot the age distributions of target firms and acquiring firms in the model. Target firms will only live one period and then they are acquired, while distribution of acquiring firms' age is a geometric distribution.

5.1.5 Growing through M&A or Internal Capital Accumulation?

The model distinguishes between investment through M&A and internal investment. First, we can directly look at the M&A intensive margin $\frac{P(z_T)\hat{k}_T}{\phi(\hat{\imath})+P(z_T)\hat{k}_T}$. From the Compustat, we can observe firms' capital expenditures (item 128) and acquisition expenditures (item 129). In the top graph of figure 10, we plot the log M&A intensive margin, which is the acquisition expenditures over capital expenditures plus acquisition value, against the firm's log productivity, which is measured by sales over capital. Each point on the figure is the average M&A intensive margin of firms at that productivity level. There is a significant negative correlation between z and intensive margin: more productive firms spend less money on M&A. The linear fit function has a significant slope of -0.14 and an intercept of -0.78.

In the model, the cost of growing through M&A, as defined by (12) is increasing in z. This is because on one hand, it is too costly for the high z firm to absorb the low z_T firm ($\hat{f}\left(\frac{z_T}{z}\right)$ is increasing on $\frac{z_T}{z}$), while on the other hand the forgone cost (sz) is also high for the productive firm. Hence the model predicts when z increases, M&A intensive margin ($\frac{P(z_T)\hat{k}_T}{\phi(\hat{\imath})+P(z_T)\hat{k}_T}$) decreases.

In the bottom graph of figure 10, we plot the M&A intensive margin in our model. The x-axis is the log productivity of acquiring firm while the y-axis is the log M&A intensive margin implied by the model. The blue solid line is the policy functions and the red dashed line is the linear fit function. We can see that our model also implies that more productive firms tend to rely less on M&A. In terms of slope magnitude, the linear fit function of the model has a slope of -0.21, which is slightly greater than the found in the data.



Target Firm Age (Data)

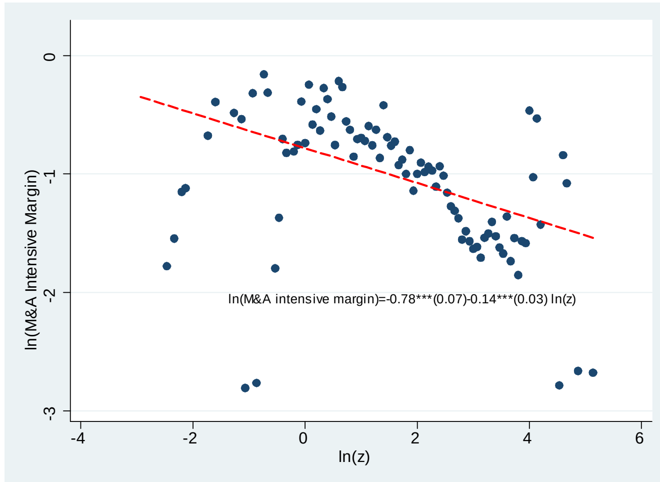
Acquiring Firm Age (Data)

Target Firm Age (Model)

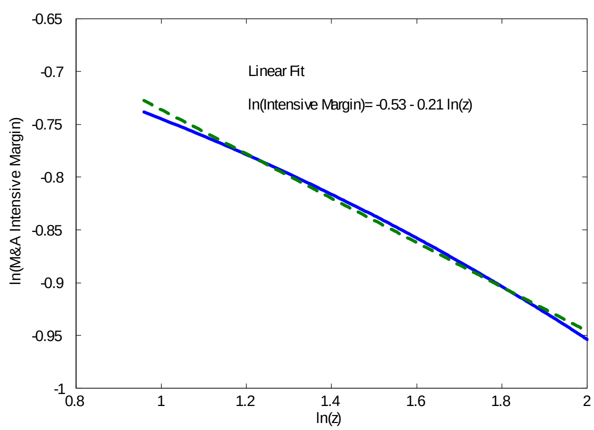
Acquiring Firm Age (Model)

Figure 9: Firm Age Distribution

Notes: This figure presents the age distributions of target and acquiring firms. Data source: SDC M&A database.



Data



Model

Figure 10: Intensive Margin of M&A and Productivity

Notes: This figure shows the log M&A intensive margin at different productivity level. Data comes from Compustat database. M&A intensive margin=acquisition expenditure(Compustat item 129)/(internal investment expenditure (Compustat item 128)+acquisition expenditure). Each point on the left graph is the average log M&A intensive margin across firms at a productivity level. The dashed lines are linear fits of log M&A intensive margin on log productivity. *** denotes statistically significant at 1% level. Standard errors are reported in brackets.

5.2 Evidence from New Start-ups

In this subsection, we study the model implication of new entrants. Our model predicts that for new start-ups, low productivity firms are acquired. Hence they should have lower return to investors. We get information of new start-ups from a Venture capital (VC) dataset provided by Thomson SDC VentureXpert database. VC finances new start-ups and then sells them to other firms (acquirers) or to the households (IPO). The VC dataset provides details on 23,000 portfolio companies of approximately 7,000 funds. For each company in the VC portfolio, we can observe information of each VC investment and the money received by VC when VC sells the company. Details of this data set are provided in the appendix.

There are two ways of exiting the portfolio companies for VC: selling those portfolio companies to other firms (acquisitions) or selling those companies to households (IPO). Standard finance theory predicts that these two exit strategies should provide the same return to the VC. However, from the venture capital data, we find acquired portfolio firms have a significantly lower return than IPO firms.[19]

In Figure 11, we plot the internal rate of return (IRR) density of IPO firms and acquired firms on the left graph.[20] The solid line is acquired firms' density function and the dashed line is IPO firms' density function. As the figure shows, IRR of IPO firms is significant higher than acquired firms' IRR. When we look at the numbers, the median IRR of IPO firms is about 130%, while median IRR of acquired firms is about 65%, only half of IPO firms.[21]

In our model, the new entrants are either acquired or not. Although our model does not explicitly model IPO process, we interpret those new entrants with high productivity which are not acquired as IPO firms. This is because the households directly own these firms. Hence we consider these firms are directly sold to households. While acquired entrants are different. Households do not directly own these firms after they are acquired. Households only hold stocks of acquiring firms. In the model, the IRR is defined as $q = \frac{J^T(z_T)}{IRR}$ for acquired firms, and $q = \frac{J(z)}{IRR}$ for IPO firms. We plot the density of IRR for these two groups on the left graph of Figure (11). The solid line shows the IRR of acquired firms. It

[19] Amit et al. (1998) finds a similar pattern.

[20] IRR is defined as rate of return such that NPV of investments equal 0.

[21] As robustness checks, we have checked whether the IRR difference between IPO firms and acquired firms disappears after controlling time effects, industry effects and broker fee. Our results are robust to all of these changes. Our data has lots of missing values. Susan Woodward pointed out that the IRR difference is greater if missing values are corrected.

ranges from 50% to 125%. The dashed line shows the IRR of IPO firms. It ranges from 125% to 320%. Comparing the average IRR of acquired and IPO firms, the first group is 106% while the second group is 195%.

5.3 Cross Country Evidence

The model has two predictions across countries: (1) M&A is positively correlated with growth rate; (2) If targets become relatively smaller, then growth rate is higher. The first prediction has been discussed before. When M&A becomes more efficient, expected firm value will increase. Hence from free entry condition, we can see the growth rate will also increase. The second prediction comes from when M&A becomes more efficient, the acquirer grows faster than the target. Hence the target will become relatively smaller.

Following Barro (1991), we do the following regression:

$$g_i = \beta_0 + \beta_1 \frac{M\&A}{GDP}_{1995} + \beta_2 GDP_{1995} + \beta_3 School_{1995} + \text{other controls} + error \tag{29}$$

g_i = average real GDP per capita growth rate from 1995 to 2005 of country i.

$\frac{\text{M\&A}}{\text{GDP}}_{1995}$ = initial M&A value in GDP in 1995.

GDP_{1995} = initial GDP per capita in 1995.

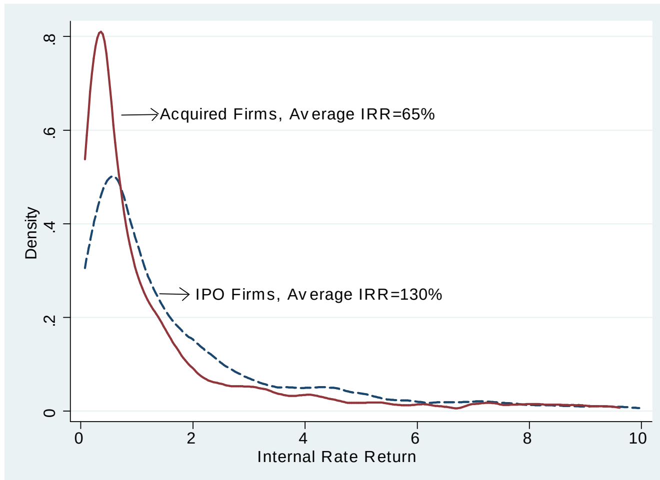
School_{1995} = initial human capital, measured by percentage of population who have primary (PRIM) or secondary degrees (SEC). This information is got from Barro-Lee database.[22]

Other controls include life expectancy, fertility rate and government consumption ratio. Table 4 shows the results. In the first column, we can see if initial M&A value increases by 1%, then the growth rate will increase by 0.6%. The effect is significant at 5% level. The second and third columns add new controls: stock market value in GDP and the total bank loan value in GDP.[23] Both of them are trying to control for the development of capital market in a country. We can see that after controlling these two variables, M&A is still positively correlated with growth. In the appendix, we also show that within US, the sector growth rate is also positively correlated with M&A.

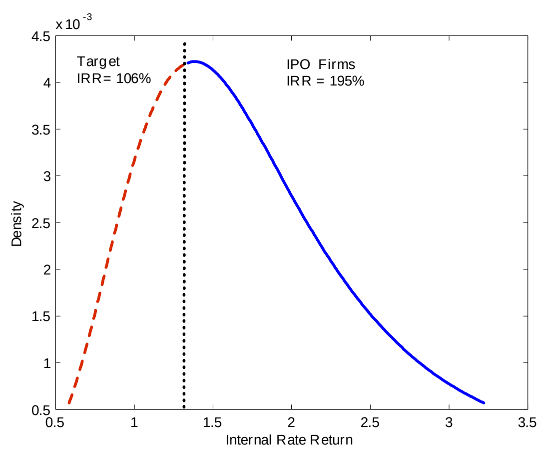
Figure 12 draws the target sales/acquirer sales and the growth rate. As predicted by the model, they are negatively correlated. And the relation is significant at 5% level.

[22] The website of the database is http://www.barrolee.com/data/dataexp.htm

[23] The data is obtained from World Bank financial sector database. See http://data.worldbank.org/indicator/FS.AST.DOMS.GD.ZS/countries



Data



Model

Figure 11: IRR Density of IPO Firms and Acquired Firms

Notes: This figure shows the distributions of Internal rate of return (IRR) in the data and the model. IRR is defined as the return to make NPV=0. Data source: SDC VentureXpert Database.

Table 4: M&A and Growth Rates across Countries

	(1)		(2)		(3)	
$\frac{M\&A}{GDP}_{1995}$	0.592**	(0.244)	0.600**	(0.250)	0.589**	(0.232)
GDP_{1995}	-0.003*	(0.002)	-0.002	(0.002)	-0.001	(0.002)
$PRIM_{1995}$	-0.011	(0.018)	-0.016	(0.017)	-0.009	(0.017)
SEC_{1995}	0.030**	(0.012)	0.034**	(0.015)	0.026*	(0.014)
Life expectancy	-0.001***	(0.000)	-0.001***	(0.000)	-0.001***	(0.000)
Fertility rate	-0.007***	(0.002)	-0.007***	(0.002)	-0.007***	(0.002)
Gov/GDP	-0.001**	(0.000)	-0.001***	(0.000)	-0.001***	(0.000)
$\frac{\text{Stock Mkt Value}}{\text{GDP}}$			-0.002	(0.003)		
$\frac{\text{Bank Loan}}{\text{GDP}}$					-0.001**	(0.000)
Constant	0.125***	(0.031)	0.130***	(0.034)	0.132***	(0.031)
N	75		63		74	
Adj R square	0.29		0.31		0.36	

Notes: This table reports the results of analyzing the M&A share and real GDP per capita growth rate across countries. The dependent variable is real GDP per capita growth rate. $\frac{\text{M\&A}}{\text{GDP}}_{1995}$ = initial M&A value in GDP in 1995. GDP_{1995} = initial GDP per capita in 1995. $PRIM_{1995}$ = percentage of population who have primary degrees. SEC_{1995} = percentage of population who have secondary degrees. Fertility rate = births per woman. Standard errors are reported in brackets. ***, ** and * denote statistically significant at the 1%, 5% and 10% levels, respectively.

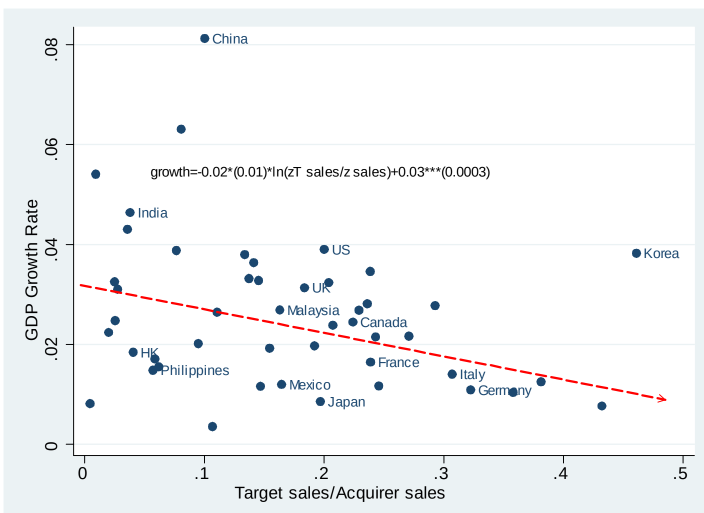


Figure 12: Relative sales and Growth Rate

Notes: This figure shows the relation between target sales/acquirer sales and growth rate across countries. Standard errors are reported in brackets. *** and ** denote statistically at 5% and 1% levels, respectively. Data source: SDC VentureXpert Database.

6 Growth Decomposition of US Economy

In this section, we explore a counterfactual experiment to understand how M&A can affect the growth rate. We shut down internal investment channel and M&A channel one by one. The results are shown in table 5. The first column is an economy in which firms can grow only through M&A. The second column is an economy where firms can grow only through internal capital accumulation. The third column is the benchmark model: firms can grow through both channels. We can see that when there is only M&A, the growth rate is about 2.08%, while when there is only internal capital accumulation, the growth rate is about 3.11%. Combining them together, the growth rate is about 3.96%. In other words, the M&A can account about 21% of the aggregate growth in our model.

It is interesting to compare our model with Perla and Tonetti (2014) and Lucas and Moll (2014). In their models, productivity is imitated on costly contact. The growth in their models is driven purely by the improvement in the productivity distribution: unproductive firms can increase their productivity by paying a contact cost. In our model, we consider M&A as a means of improving productivity. Productivity of unproductive firms can also be increased by paying an M&A cost. By choosing an appropriate M&A cost function, our model should be isomorphic with their models.

Greenwood et al. (1997) has stressed another important growth channel. They argue that the increase of internal investment can explain about 60% of GDP growth rate and productivity change can explain the remaining 40% of GDP growth rate.

We interpret the model with only internal investment as an exercise to evaluate the contribution of internal capital accumulation to growth. We find about 2/3 of the aggregate growth rate can be explained by internal investment, which is consistent with the found of Greenwood et al. (1997). We interpret the model with only M&A as an exercise to evaluate the importance of productivity increase. However the productivity increase is not driven by R&D, but it is resulted from improving unproductive firms' productivity. Our results suggest that the change of growth rate will be as high as 0.8% by shutting down M&A.

Besides the growth rate, the third row compares aggregate TFP in these two economies. Literature on capital reallocation has discussed how misallocation of resources can decrease the aggregate TFP, such as Klenow and Hsieh (2009), Midrigan and Xu (2014), David (2013). Our paper confirms this perspective. We can see that when shutting down the whole M&A process, TFP decreases by about 10%.

Table 5: Growth Contribution of M&A and Internal Capital Accumulation

	Only M&A	Only Internal Investment	Both
Growth Rate	2.08%	3.11%	3.96%
Firm growth rate	5.01%	6.61%	8.00%
TFP	5.85	4.70	5.21

Notes: This table shows the aggregate gains in three cases: firms can grow only through M&A, firms can grow only through internal investments and firms can growth through both channels.

Table 6: Growth Contribution of M&A, Transitory Productivity

	Only M&A	Only Internal Investment	Both
$\eta = 0.5$			
Growth Rate	0.21%	0.84%	0.93%
Firm Growth Rate	0.75%	1.58%	2.97%
TFP	4.14	4.00	4.11

Banerjee and Moll (2010) show that the effects of misallocation depend heavily on the persistence of productivity. For example, in our model, if productivity of firms is purely transitory, there will be no M&A at all. To get the sensitivity of our results, we assume each period with probability η, firms will redraw the productivity from distribution $m(z)$. Table 6 shows the growth decomposition when $\eta = 0.5$. First, the growth rate declines to 0.93%. Second, if shutting down M&A completely, the growth rate would decline only by 0.1%. However, it still accounts for over 10% of the aggregate growth rate. Hence, we argue even for very transitory productivity process, M&A still plays an important role to explain the growth rate.

7 Application: M&A Boom since 1990s

M&A becomes more and more important in the last a few decades. In figure 13, we plot total M&A transaction value in GDP from 1990 to 2005 (solid line). We can see that total M&A transaction value is about 1.5% of GDP in 1990 and then rises sharply from early 1990s. The peak is reached at 1998, with a value about 10%, which is more than 5 times the value in the 1990. From 2000, M&A transaction value decreases but is still significantly higher than the value in the 1990. The red dashed line plots the long-run trend of the M&A boom.[24] In this section, we focus on the long run trend of the boom.

[24] We use the HP filter with a smooth parameter 100 to get the long run trend.

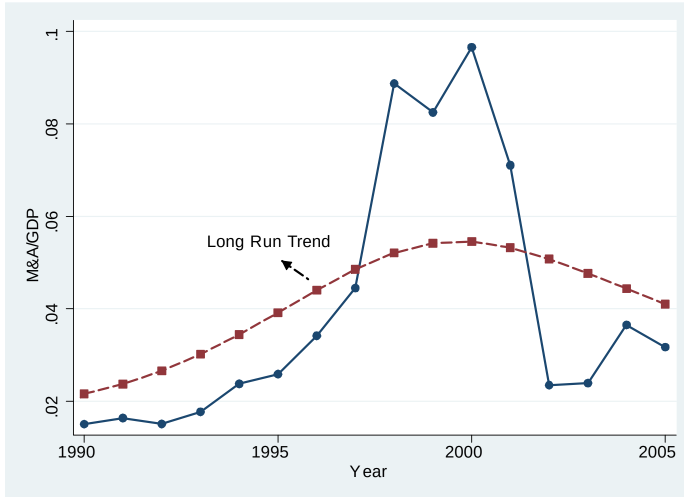


Figure 13: M&A Value Share in GDP (1990-2005)

Lots of previous research has sought to explain the M&A boom in the 1990s. Many see deregulation as the key driving force, such as Boone and Mulherin (2000) and Andrade et al. (2001). On the other hand, some people make the claim that the change of M&A technology is one important reason for the M&A boom in the 1990s. Specifically, the availability of IT technology makes M&A easier.[25] Hence this explanation suggests a decline of M&A cost. Table 7 provides more anecdotal evidence of potential reasons of M&A boom in 1990s of several industries.[26] We search news reports from Lexis–Nexis database, that analyze the merger activity at the time of the boom in an industry. We can see that most of these reports explain the M&A boom either through deregulation or through the technology improvement.

In this section, we would like to ask the questions: ecan the decline of M&A cost explain the M&A boom and what is the aggregate effect of the boom? To accomplish this goal, we calibrate our parameters in two subsamples: before and after the M&A boom. The big picture of our analysis is that we want to use some micro patterns in the M&A data to calibrate the parameters change in the M&A technology. Then we evaluate how much it can account the boom given the change of the M&A technologyr. Our calibration strategy is as follows:

[25] Source: Deloitte M&A consultant report (2005). http://www.deloitte.com/view/en_US/us/Services/consulting /221d1350a8efd110VgnVCM100000ba42f00aRCRD.htm

[26] This table follows Harford (2005).

Table 7: Reasons of M&A Boom

Industry	Date and Reason of MA Boom
Banking	Oct, 1996
	Deregulation and Information Technology (IT)
Communication	July, 1997
	Deregulation: Telecommunications Act in 1996
	IT technological changes
Computers	July, 1998
	Internet
Retail	Aug, 1996
	Strong growth and impact of internet
Wholesale	June, 1996
	Take advantage of new IT ability, grow by acquisition

Notes: This table shows the reasons of the M&A boom in different sectors. The reasons come from Lexis–Nexis searches of news reports analyzing the merger activities at the time of the boom. Source: Harford (2005, table 2).

(1) We fix some parameters same as table 2, including the consumer preference parameters γ and β, the probability of survival rate ω and the depreciation rate δ.

(2) We are interested at the transition paths, which are generally difficult to solve. To simplify the computation, we assume $\psi = 0$. In this case, $P_t(z_T)$ has a closed form solution, $P_t(z_T) = X_t z_T^{\frac{\varepsilon}{\alpha}}$.[27] Without solving the price function, we only need to solve for one number X_t. The functional form of price has a very intuitive explaination. When $\varepsilon = 0$, it means $f(z, z_T) = z$. Hence acquirers will replace the productivity of targets. The price should not depend on z_T and all targets have the same price X_t. On the other hand, when α is close to 0, it implies that quantity of capital from targets k_T does not matter so much. Firms do not trade off between the quality and quantity of targets but only focus on quality. Hence it will give a very steeper slope on the price.

(3) The parameters that we change are λ, q, v_i and σ_z and parameters that relate to the M&A functions: h, θ, α and ε. We assume there is a change of M&A technology in the year 1995 and it is expected in year 1990.[28] We separate the parameters into two groups: pre-change parameters and post-change parameters. We calibrate λ, h, θ, α and ε to jointly match productivity difference $\frac{z_T}{z}$, sales difference between acquiring and target firms, slope of $\frac{z_T}{z}$, extensive and intensive margin of the M&A in year 1990 and year 2005[29]. The details

[27] See equation (46) in the appendix for details.
[28] We choose this experiment since it matches the data best.
[29] Comparing to the benchmark calibration in table 2, we fix $\psi = 0$. Hence we do not match the slope of intensive margin.

Table 8: Parameters

	Pre-boom (1990)	After boom (2005)	Moments
M&A Tech			
h	0.61	0.67	Intensive margin
$1-\alpha$	0.40	0.38	Slope of $\frac{z_T}{z}$
ε	0.46	0.44	$\frac{z_T}{z}$
θ	0.09	0.01	Sales dif
Other Params			
λ	0.25	0.41	Extensive margin
v_i	42.1	41.0	M&A/output
σ_z	0.52	0.53	Firm growth rate std
q	4.2	4.9	Firm growth rate

of solving the model are provided in the appendix. The parameters are shown in table 8.

The first four parameters in table 8 shows parameters associated with M&A technology. We can see h increases, α increases, ε declines and θ declines. All of them suggest that the M&A technology becomes more efficient.

7.1 Can the M&A Technology Change Explain M&A Boom?

Figure 14 compares the prediction of M&A boom of the model with the data. The blue line with plus marker is the M&A/GDP we observe in the data. The green solid line is the prediction of the model. We can see that the data has a huge hump shape while the model can only generate a moderate hump. In terms of the magnitude, the M&A/GDP in the data will rise about 1.7% from 1990 to 2005 and the peak point is 5.7% in year 1998 which is about 4% larger than the value in 1990. The model predicts a 1.8% rise of M&A/GDP from 1990 to 2005 and the peak value is about 4.2%. That means the model can explain more than half of the M&A boom we observe in the data. An interesting point is that the model can generate a hump shape. It comes from the fact that more firms will sell in the transition dynamics than in the steady state. For example, we can consider two extreme case, on one hand, there is no M&A at all and at the other, only one firm will produce and all other firms will be acquired. In the first case, number of targets is zero while in the second case, number of targets is the number of new entrants. In the transition dynamics, all those incumbents whose productivity is below the $z_{\max}$ will gradually choose to sell. Hence we will observe a M&A boom.

Second, the dashed line in figure 14 is the prediction of the model if only parameters associated with M&A technology change. We interpret this line as the effect when only M&A

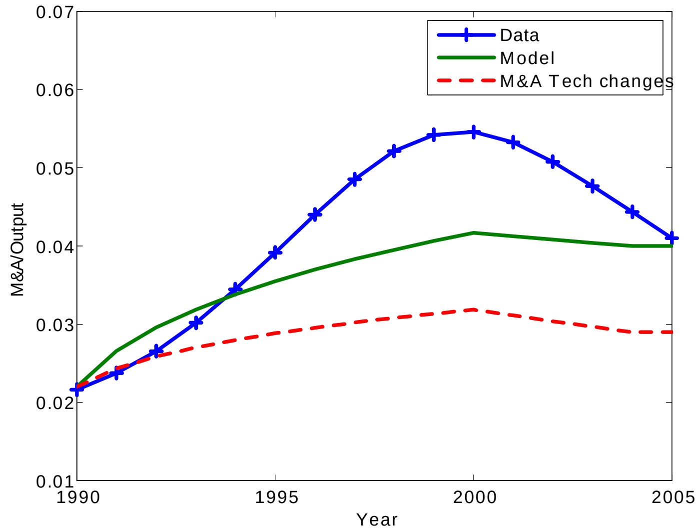


Figure 14: M&A/GDP

technology improves. We can see in this case the M&A/GDP will rise about 0.7% in two steady states, which accounts for 42% of the M&A/GDP change in the data (0.7%/1.7%). If we compare the M&A boom, it can account 18% change we observe in the data.

Figure 15 reports the transition dynamics of other 6 variables in the model. The first 5 graphs are the moments that are targeted: slope of matching function, extensive margin, intensive margin, relative sales and $\frac{z_T}{z}$. The last graph is the dynamic path of the growth rate g_K. First graph reports the dynamic path of relative sales. In the data (blue line), we can see target sales becomes relative smaller than acquirer sales (12% decline). The model generates a 10% decline. This is because the improvement of M&A technology will increase the growth rate of acquirers more than the average economy. Hence the size of the acquirer becomes relatively larger than targets. The dashed line draws the path when there is only M&A technology change. We can see the relative sales drop about 6.4%.

Second, from the dynamic path of the extensive margin, we observe a 13% rise. The model also predicts a 9.5% rise but it is driven by the change of λ. If we fix λ and only allow the M&A technology to change, we can see that the extensive margin in the model will decline (dashed line). It is because the improvement of M&A technology will push up the capital demand, as well as the price. More firms want to sell their capital. So the extensive margin of M&A will decrease.

Third, the intensive margin of the M&A in the data rises 3.5% and the model predicts a rise about 5.3%. However, if we only change M&A technology, the intensive margin will rise 7.4%. It suggest that the increase of λ will decrease the intensive margin. The reason is simple. If only λ increases, capital demand will increase. Hence the price will increase too. Acquirers, facing more expensive targets, will choose to invest more capital internally.

Fourth, in the data, the slope of the matching function does not show a clear pattern. The variation is huge. However, the model predicts a rise in the slope: 4.1% and the M&A technology can generate a 2.5% rise alone.

The next graph draws the path of $\frac{z_T}{z}$. We can see that in the data there is a declining trend: $\frac{z_T}{z}$ drops about 3.4% from 1990 to 2005. Our model predicts a declining trend too. It declines about 9%. The reason is as follows: the increase of capital demand will push z^* to increase to $z^{*\prime}$. Consider very productive firms ($z_{\max}$ for example), they will acquire more productive targets (shown in figure 16). However, less productive acquirers (think $z^{*\prime}$), they will acquire less productive firms after the change. Hence whether average $\frac{z_T}{z}$ drops or increases depends on which part dominates. The log normal assumption of the productivity distribution predicts that $\frac{z_T}{z}$ drops in the model.

Finally, we draw the path of growth rate. The data has a very volatile change of g_K but the model predicts the growth rate will increase by 0.2% because of this boom.

7.2 Change of Firm Size Distributions

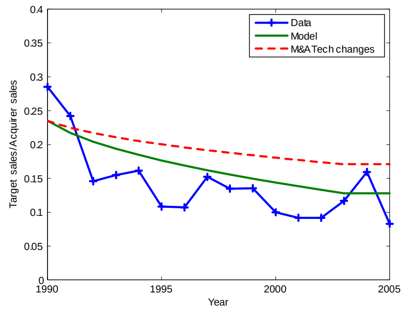
The model also sheds light on the distribution of firm size. In section 4, we have shown that the firm size distribution has a Pareto tail. In the extension model, this result still holds and the Pareto tail index is determined by

$$\omega \left[\left(1 - \hat{\lambda}\left(z_{\max}\right) \right) g^I \left(z_{\max}\right)^{\Theta} + \hat{\lambda}\left(z_{\max}\right) g^A \left(z_{\max}\right)^{\Theta} \right] = g_K^{\Theta} \tag{30}$$

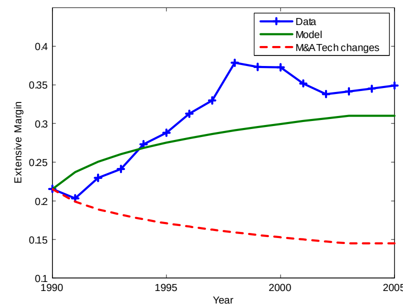
Hence when ω increases or the growth rate of firm $z_{\max}$ increases, Θ will decrease. Intuitively if firm's survival probability increases, productive firms will become larger. Thus the tail of the size distribution will be fatter. While if the growth rates of productive firms increase relative to g_K, the tail of the size distribution will become fatter as well.[30]

M&A can affect the distribution of firm size through changing the relative growth rate

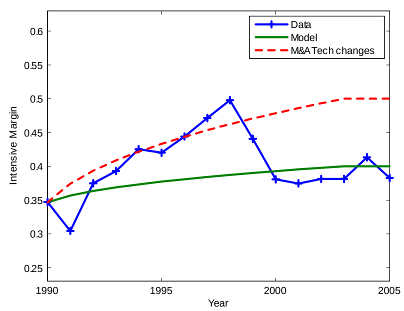
[30] Notice that $\omega \left[(1-\lambda) g^I \left(z_{\max}\right) + \lambda g^A \left(z_{\max}\right) \right] < g_K$. It implies $\Theta > 1$. Hence the size distribution does not satisfy Zipf's law.



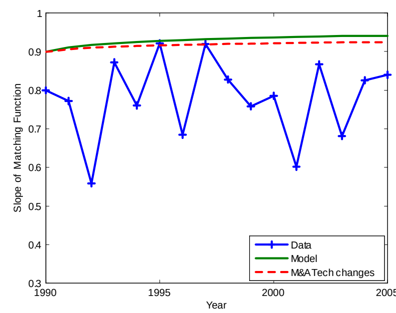
Target sales/Acquirer sales



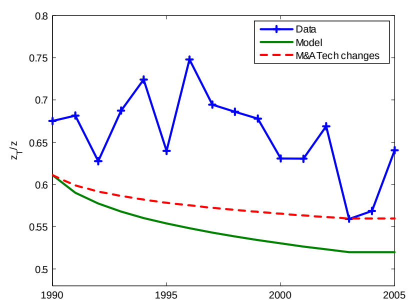
Extensive Margin



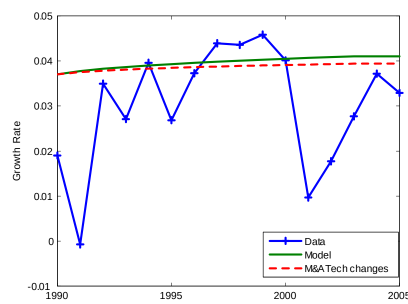
Intensive Margin



Slope of Matching Function



z_T/z



Growth Rate

Figure 15: Transition Dynamics of the Model and Data

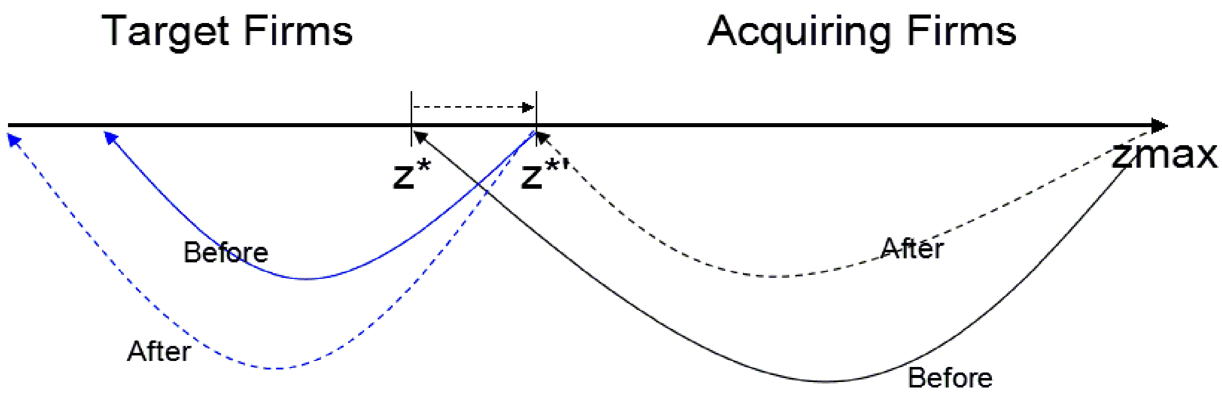


Figure 16: Matching Pattern before and after Boom

between firm $z_{\max}$ and aggregate economy. Think an extreme case: only $z_{\max}$ will be the acquirer. From the free entry condition, we can see the change of g_K is determined by the average of firm value $J(z)$. While the change of the growth rate of firm $z_{\max}$ depends on the firm value $J(z_{\max})$ and the M&A cost. When there is a decline of the M&A cost, the investment rate of firm $z_{\max}$ will increase due to the increase of $J(z_{\max})$ and the decline of M&A cost. Hence the change of the growth rate of firm $z_{\max}$ will be higher than g_K.

Figure 17 compares the firm size distribution of the model and the data. The top graph plot the log firm size distributions in 1990, 1995 and 2005 in the model. Firm we can see all distributions start at $\log(\mu) \approx -2$. This is where the firm enters. Then firm grows large. Comparing to these three distributions, we can see the tail becomes thicker.

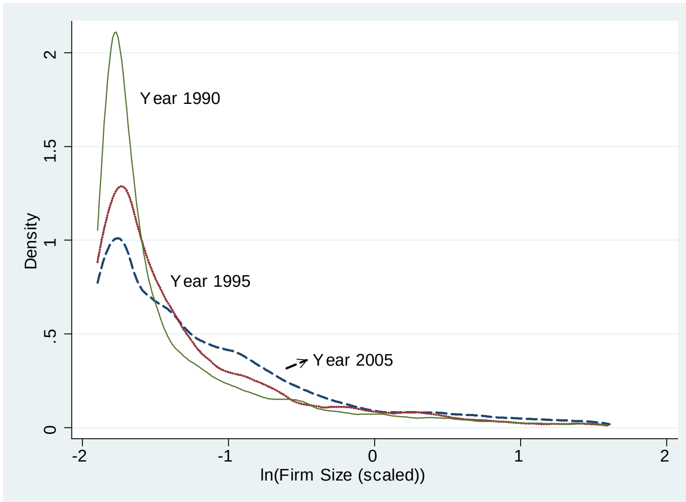
The bottom graph in figure 17 reports the fat tail index of firm size distribution in the data. The left hand graph is the fat tail index in the data. We order firms by relative sizes $\frac{k_{(1),t}}{\bar{K}_t} \geq ... \geq \frac{k_{(N),t}}{\bar{K}_t}$ year by year, stopping at a rank N, which is a cutoff still in the upper tail.[31] Then we estimate a "log-rank long size regression" as equation (31)

$$\ln(rank\ j \text{ at } t) = const - \hat{\Theta}_t \ln k_t(j) + noise \tag{31}$$

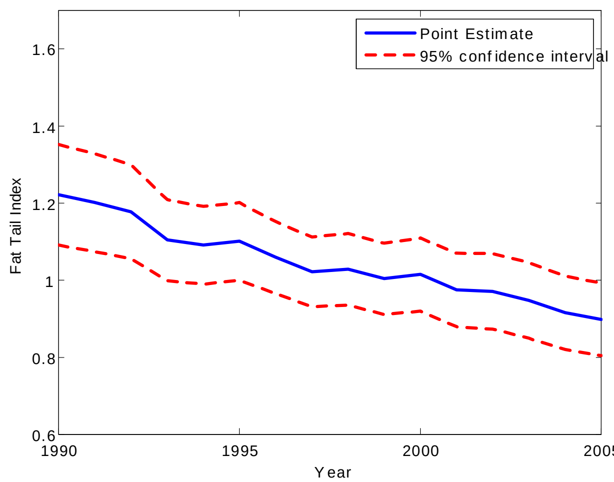
Equation (31) is estimated via OLS and we can get a sequence of $\hat{\Theta}_t$.[32] It is shown in

[31] However, there is not a consensus on how to pick the optimal cutoff. We choose the top 5% observations in the sample (see Gabaix (2009)).

[32] As noted by Gabaix (2009), the estimate has an asympototic standard error $\hat{\Theta}_t(N/2)^{-\frac{1}{2}}$ and the standard error returned in OLS is wrong.



Firm Size Distribution (Model)



Tail Index (data)

Figure 17: Firm Size Distribution in the Model and Data

Notes: This figure shows the model's transition path of Θ. We assume $\psi = 0$ (elasticity of substitution between z and z_T is 1). The top graph reports the estimates of the data. Solid line= point estimate; Dash line= 95% confidence interval. The bottom graph shows the predictions of the model. Dashed line=M&A technology change; Dashed dotted line= Deregulation; Solid line= Both changes occur.

the solid line. The dashed lines report 95% confidence intervals. In our sample, the tail index gradually declines from 1.2 to 0.9, and the decline is significant.[33] This is consistent with the model prediction.

8 Conclusion

In this paper, we study how M&A can affect the aggregate economy. In particular, we highlight the positive effects of M&A process on aggregate growth rate. Applying the model to the data, we argue that M&A is a quantitatively important driving force of aggregate growth, and one that has been neglected in previous academic research. Moreover, we assume the cost of M&A depends on the relative distance between acquiring and acquired firms. This assumption can help us to understand the relation between M&A pattern and growth across countries and some industry dynamics during M&A boom.

In our model, the M&A process is purely driven by the consideration of efficiency, while in reality M&A can increase the market power thereby harming some aspects of the market efficiency. Although we do not explicitly model this part in the paper, it is useful to take our paper as a benchmark. Nonetheless, our model may exaggerate the efficiency gain of M&A. To fully understand how M&A affects the aggregate economy, it would be interesting for the future research to introduce market power and strategic concern into the model.

When explaining the M&A boom, we introduce a productivity dependent anti-trust policy, which randomly blocks M&A. In the real world, the policy may not be random. To be more precise on the effect of deregulation, it is useful to get more information on how policy makers make decisions. We leave this as a topic to be explored in the future.

In this paper, we focus solely on US M&A. As cross border M&A is becoming more and more popular, it may be also interesting to study how M&A affect the cross country differences in an open economy.

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[33] It has been noticed that the inequality in the wealth distribution increases in the last a few decades as well (Benhabib et al. (2011)).

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9 Appendix

The appendix has four parts. In the first part, we provides more empirical evidence about the robustness of the positive correlation between M&A and growth rates. The second part provides data details of Venture capital dataset. The third part shows proofs of all the propositions. The last part discusses the details of solving the transition path.

9.1 M&A and Growth Rates across Sectors

Then we switch to the data across sectors in US. We do the following regressions

$$g_{i,t} = b_i + b_1 \frac{M\&A}{Sales}_{i,t-n} + \text{other controls} + error \tag{32}$$

$g_{i,t}$ = sales growth rate of sector i in year t. $\frac{\text{M\&A}}{\text{Sales}}_{i,t}$ = M&A value in total sales of sector i in year $t-n$. Other controls include time dummies. b_i is the fixed effect of sector i.

Table 9 shows that sector growth rate is positively correlated with M&A value. The dependent variable is 4-digit sector sales' growth rate. In the first three columns of table 9, we regress the sector growth rate on M&A value in 1-3 years ago. On average, if M&A value increases by 1%, the future growth rate will increase by 0.027% to 0.058%. The fourth column uses a dummy variable called "deregulation" to capture the M&A increase in a sector. Deregulation is 1 when the sector has an M&A related deregulation in that year based on Harford (2005, table 2). The result indicates that sector growth rate will increase by 0.028% after deregulation.

9.2 Venture Capital Data

The Venture capital (VC) data set is provided by Thomson SDC VentureXpert database. It provides details on portfolio companies, funds, firms, executives, (VC backed) IPOs and (VC) limited partners, which covers from 1967 to present, of approximately 7,000 funds (including private equity) into 23,000 portfolio companies. For each portfolio company, we can observe information of each investment from VC and the money received by VC

Table 9: M&A and Growth Rates across Sectors

	(1)	(2)	(3)	(4)
M&A/Sales(t-1)	0.0269***			
	(0.00650)			
M&A/Sales(t-2)		0.0584***		
		(0.00579)		
M&A/Sales(t-3)			0.0363***	
			(0.00694)	
Deregulation				0.0273*
				(0.0133)
Time Dummy	Yes	Yes	Yes	Yes
Fixed Effects	Yes	Yes	Yes	Yes
Obs.	18243	17777	17321	18243

Notes: This table reports the results of analyzing the M&A share and sector sales growth rate in US. The dependent variable is the sector's sales growth rate. M&A/Sales(t-n) denotes n periods lag. Deregulation=1 if sector/year has M&A related deregulation in Harford (2005, table 2) and 0 otherwise. *** and * denote statistically significant at the 1% and 10% levels, respectively. Standard errors are reported in brackets. Year dummies and sector fixed effects are controlled in all specifications. Data Source: Compustat.

when it exits the portfolio. When the portfolio company is acquired we can also observe the CUSIP number of both acquirer and target firm. We focus on projects that are either IPO or acquired. This gives 4323 IPO firms and 10222 acquired firms. We clean the data in the following procedures: (1) We drop those observations whose investment history is not consistent with number of rounds in the data. (2) We drop all observations that have positive funding investment after VC exits. (3) We drop all observations that have negative IPO or selling prices. (4) We drop all duplicated target CUSIP observations. After cleaning the data, we have 1651 IPO firms and 2652 acquired firms, covering from 1967 to 2012.

The summary statistics is reported in Table 10. The top panel reports acquired firm investment information. In those 2652 acquired firms, we can observe 27090 times investments. Hence each firm gets about 10.2 rounds investments from VC before it gets acquired. In each investment, portfolio firms will get $248,700 hence the total investment is $2,487,000 in 10 rounds. When the firm is acquired, VC usually can get 1.43 million dollars on average. The bottom panel reports IPO firms. They can get 9.5 rounds investment on average and in each round the investment is slightly lower than acquire firms group, only $143,800. However the first day value (computed using closing price) is about 4.5 million dollars.

Table 10: Summary Statistics of IPO and Acquired Firms

	Obs No.	Mean	Std.	Min	Max
			Acquired Firm		
Investment(thousands)	27090	24.87	348.46	1	35371.51
Investment round	27090	10.2	5.12	1	18
Transaction value(millions)	2652	1.43	8.83	0.0003	375.71
			IPO Firm		
Investment(thousands)	14298	14.38	203.07	1	35689.83
Investment round	14298	9.5	4.899034	1	17
First day value(millions)	1505	4.5	26.15	0.03	869.13

Notes: This table shows the summary statistics of SDC VentureXpert database. Investment. The top panel reports the acquired firms' information while the bottom panel reports the IPO firms' information. Investment = VC investments per round to the portfolio firms. Investment round = number of investment rounds made by VC. Transaction value = money that VC gets from selling acquired firms. First day value = value of VC stocks calculated at first IPO day's closed price.

9.3 Proof of All Propositions

9.3.1 Proof of Propositions 3-4

Proof: From equation (7) to equation (9), we guess all value functions are linear on k. Then we define $J_t^A(z) = \frac{V_t^A}{k}$, $J_t^T(z) = \frac{V_t^T}{k}$, $J_t^I(z) = \frac{V_t^I}{k}$. Substitute them into equation (7) to equation (9), we can verify this guess.

If $\pi_z > 0$ then assumption 3 implies $k^*(z) < \frac{1+r}{\omega} + \delta - 1$. We can see that mapping T maps a bounded function to a bounded function given $0 \leq \hat{k} \leq k^*(z)$. Then (i) T is monotone: if $J' > J$, we can see that $TJ' > TJ$. (ii) discounting property: $T(J + a) \leq TJ + \frac{\omega}{1+r}(1 - \delta + k^*)\, a$, $\frac{\omega}{1+r}(1 - \delta + k^*) < 1$. Hence $TJ = J$ has a unique fixed point. Since $\pi_z > 0$ and $\pi_{zz} > 0$ we can verify that T preserves monotonicity and convexity. Hence J is increasing and convex in z.

When $v \leq 1$, then we can see that if there is a gain in M&A, then the acquiring firm must be more productive than the target. It can be seen that if $v\left(s, \frac{k_T}{k}\right) f(z, z_T)\, k_T \geq z_T k_T$, we have $f(z, z_T) \geq z_T$. Given f is a CES function, we can see that $z \geq z_T$. Hence it must be the case that more productive firm acquire less productive firm.

9.3.2 Proof of Proposition 5

Proof: The idea of the proof is to verify whether in a positive sorting equilibrium, the second order condition holds. Define $\hat{f}\left(\frac{z_T}{z}\right) = \left[1-\varepsilon+\varepsilon\left(\frac{z_T}{z}\right)^{\psi}\right]^{\frac{1}{\psi}}$. From first order conditions, we have

$$s = \left[\frac{\theta}{\alpha}\frac{P\left(z_T\right)}{z}\left(\frac{\hat{k}_M}{h\hat{f}}\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\theta+\alpha}} \tag{33}$$

Then

$$\begin{aligned} sz + P\left(z_T\right)\left(\frac{\hat{k}_M}{\chi}\right)^{\frac{1}{\alpha}} &= \left(1+\frac{\theta}{\alpha}\right)P\left(z_T\right)\left(\frac{\hat{k}_M}{\chi}\right)^{\frac{1}{\alpha}} \\ &= \left(1+\frac{\theta}{\alpha}\right)\left(\frac{\alpha}{\theta}\right)^{\frac{\theta}{(\theta+\alpha)}} z^{\frac{\theta}{\theta+\alpha}} P\left(z_T\right)^{\frac{\alpha}{\theta+\alpha}}\left(\frac{\hat{k}_M}{h\hat{f}}\right)^{\frac{1}{\theta+\alpha}} \end{aligned}$$

Given $J(z)$, the choice of investments can be written as two separate problems

$$\max_{z_T,\hat{k}}\left[\frac{\omega}{1+r}J\left(z\right)\hat{k}_M - \left(1+\frac{\theta}{\alpha}\right)\left(\frac{\alpha}{\theta}\right)^{\frac{\theta}{(\theta+\alpha)}} z^{\frac{\theta}{\theta+\alpha}} P\left(z_T\right)^{\frac{\alpha}{\theta+\alpha}}\left(\frac{\hat{k}_M}{h\hat{f}}\right)^{\frac{1}{\theta+\alpha}}\right] \tag{34}$$

And

$$\max_{\hat{\imath}}\left[\frac{\omega}{1+r}J\left(z\right)\hat{\imath} - \phi\left(\hat{\imath}\right)\right] \tag{35}$$

The second one (35) is the optimal decision of internal investment and the first one (34) is the optimal decision problem of M&A. To discuss M&A pattern, we only need to focus on (34). We define $\hat{k}_T = \left(1+\frac{\theta}{\alpha}\right)\left(\frac{\alpha}{\theta}\right)^{\frac{\theta}{(\theta+\alpha)}} z^{\frac{\theta}{\theta+\alpha}}\left(\frac{\hat{k}_M}{h\hat{f}}\right)^{\frac{1}{\theta+\alpha}}$, then the problem can be written in a short way such that

$$\max_{z_T,M} F\left(z, z_T, \hat{k}_T\right) - w\left(z_T\right)\hat{k}_T$$

where $F\left(z, z_T, \hat{k}_T\right) = z + \beta\omega J\left(z\right)\hat{k}_M$, $w\left(z_T\right) = P\left(z_T\right)^{\frac{\alpha}{\theta+\alpha}}$. This function has a similar form as Eeckhout and Kircher (2012). The first order conditions are

$$F_{\hat{k}_T} - w\left(z_T\right) = 0 \tag{36}$$

$$F_{z_T} - w'\left(z_T\right)\hat{k}_T = 0 \tag{37}$$

And the second order condition requires that Hessian matrix to be negative definite. That is

$$H = \begin{bmatrix} F_{\hat{k}_T \hat{k}_T} & F_{\hat{k}_T z_T} - w' \\ F_{\hat{k}_T z_T} - w' & F_{z_T z_T} - w'' \hat{k}_T \end{bmatrix}$$

$F_{\hat{k}_T \hat{k}_T} < 0$ and

$$F_{\hat{k}_T \hat{k}_T} \left(F_{z_T z_T} - w'' \hat{k}_T \right) - \left(F_{\hat{k}_T z_T} - w' \right)^2 \geq 0 \tag{38}$$

Differentiate equations (36) and (37) with respect to z_T.

$$F_{\hat{k}_T z_T} - w'(z_T) = -F_{\hat{k}_T z} \frac{dz}{dz_T} - F_{\hat{k}_T \hat{k}_T} \frac{d\hat{k}_T}{dz_T} \tag{39}$$

$$F_{z_T z_T} - w'' \hat{k}_T = -F_{z_T z} \frac{dz}{dz_T} - \left(F_{z_T \hat{k}_T} - w' \right) \frac{d\hat{k}_T}{dz_T} \tag{40}$$

We substitute (39), (40) and (37) into condition (38), we get

$$\frac{dz}{dz_T} \left[F_{\hat{k}_T \hat{k}_T} F_{z z_T} - F_{\hat{k}_T z} F_{\hat{k}_T z_T} + F_{\hat{k}_T z} \frac{F_{Z_T}}{\hat{k}_T} \right] \geq 0 \tag{41}$$

Hence to have positive sorting we need

$$F_{\hat{k}_T \hat{k}_T} F_{z z_T} - F_{\hat{k}_T z} F_{M z_T} + F_{\hat{k}_T z} \frac{F_{Z_T}}{\hat{k}_T} \geq 0 \tag{42}$$

From the definition of $\hat{k}_T$, let us define

$$A = \frac{h}{\left(1 + \frac{\theta}{\alpha}\right)^{\theta + \alpha} \left(\frac{\alpha}{\theta}\right)^{\theta}} z^{-\theta}$$

then

$$\hat{k}_M = A \hat{f} \hat{k}_T^{\theta + \alpha}$$

Then

$$F_{\hat{k}_T \hat{k}_T} F_{z z_T} = \frac{\omega}{1 + r} J \frac{d^2 \hat{k}_M}{d\hat{k}_T^2} \left[\frac{\omega}{1 + r} J \frac{d^2 \hat{k}_M}{dz_T dz} + \frac{\omega}{1 + r} J(z) \frac{d\hat{k}_M}{dz_T} \right] \tag{43}$$

$$F_{\hat{k}_T z} F_{M z_T} = \frac{\omega}{1 + r} J \frac{d\hat{k}_M}{d\hat{k}_T dz_T} \left[\frac{\omega}{1 + r} J \frac{d^2 \hat{k}_M}{d\hat{k}_T dz} + \frac{\omega}{1 + r} J' \frac{d\hat{k}_M}{d\hat{k}_T} \right] \tag{44}$$

$$F_{\hat{k}_T z}\frac{F_{Z_T}}{\hat{k}_T} = \frac{\frac{\omega}{1+r}J}{\hat{k}_T}\frac{d\hat{k}_M}{dz_T}\left[\frac{\omega}{1+r}J\frac{d^2\hat{k}_M}{d\hat{k}_T dz} + \frac{\omega}{1+r}J'\frac{d\hat{k}_M}{d\hat{k}_T}\right] \tag{45}$$

After substitute the equations (43) to (44) into condition (42), we have

$$F_{\hat{k}_T\hat{k}_T}F_{zz_T} - F_{\hat{k}_T z}F_{Mz_T} + F_{\hat{k}_T z}\frac{F_{Z_T}}{\hat{k}_T} \propto (\theta+\alpha)(\theta+\alpha-1)\frac{A^2\hat{f}}{\hat{k}_T^2}\left[J\left(-\frac{\theta}{z}\frac{d\hat{f}}{dz_T} + \frac{d^2\hat{f}}{dzdz_T}\right) + J'\frac{d\hat{f}}{dz_T}\right]$$
$$-\frac{A^2}{\hat{k}_T^2}\frac{d\hat{f}}{dz_T}\left[\bar{J}\left(-\theta(\theta+\alpha)\frac{\hat{f}}{z} + (\theta+\alpha)\frac{d\hat{f}}{dz}\right) + J'(\theta+\alpha)\hat{f}\right][\theta+\alpha-1]$$

$$\begin{aligned}
F_{\hat{k}_T\hat{k}_T}F_{zz_T} - F_{\hat{k}_T z}F_{Mz_T} + F_{\hat{k}_T z}\frac{F_{Z_T}}{\hat{k}_T} &\propto (\theta+\alpha)\hat{f}\left[J\left(-\frac{\theta}{z}\frac{d\hat{f}}{dz_T} + \frac{d^2\hat{f}}{dzdz_T}\right) + J'\frac{d\hat{f}}{dz_T}\right] - \\
&\quad \frac{dQ}{dz_T}\left[J\left(-\theta(\theta+\alpha)\frac{\hat{f}}{z} + (\theta+\alpha)\frac{d\hat{f}}{dz}\right) + J'(\theta+\alpha)\hat{f}\right] \\
&= (\theta+\alpha)\hat{f}J\left(-\frac{\theta}{z}\frac{d\hat{f}}{dz_T} + \frac{d^2\hat{f}}{dzdz_T}\right) - J\left(-\theta(\theta+\alpha)\frac{\hat{f}}{z} + (\theta+\alpha)\frac{d\hat{f}}{dz}\right)\frac{d\hat{f}}{dz_T} \\
&\propto (\theta+\alpha)\hat{f}\left(-\frac{\theta}{z}\frac{d\hat{f}}{dz_T} + \frac{d^2\hat{f}}{dzdz_T}\right) - \left(-\theta(\theta+\alpha)\frac{\hat{f}}{z} + (\theta+\alpha)\frac{d\hat{f}}{dz}\right)\frac{d\hat{f}}{dz_T} \\
&= (\theta+\alpha)\hat{f}\frac{d^2\hat{f}}{dzdz_T} - (\theta+\alpha)\frac{d\hat{f}}{dz}\frac{d\hat{f}}{dz_T}
\end{aligned}$$

Hence $F_{\hat{k}_T\hat{k}_T}F_{zz_T} - F_{\hat{k}_T z}F_{Mz_T} + F_{\hat{k}_T z}\frac{F_{Z_T}}{\hat{k}_T} \geq 0 \Leftrightarrow \hat{f}\frac{d^2\hat{f}}{dzdz_T} \geq \frac{d\hat{f}}{dz}\frac{d\hat{f}}{dz_T}$. This condition is true if $\psi \leq 0$.

To get an intuition of this proposition, let us look at a special case when $\psi = 0$. In this case, f is a Cobb-Douglas function on z and z_T. We can show the following result.

Lemma 10 *If $\psi = 0$, price of the target firm is*

$$P_t(z_T) = X_t z_T^{\varepsilon/\alpha} \tag{46}$$

where X_t is a constant. The cost of getting 1 unit effective capital k_M for acquiring firm z from the target firm z_T is $\frac{X_t}{\left(hs_t^{\theta}\right)^{\frac{1}{\alpha}}}z^{\varepsilon/\alpha}$.

Proof. When $\psi = 0$, then the FOC of z_T is

$$\frac{P_t'(z_T)}{P_t(z_T)} = \frac{\varepsilon}{\alpha}\frac{1}{z_T} \tag{47}$$

Integrate we can get

$$P_t(z_T) = X_t z_T^{\frac{\varepsilon}{\alpha}} \tag{48}$$

Then we can verify the cost of getting 1 unit effective capital k_M from target firm z_T is $\frac{X_t}{\left(hs_t^{\theta}\right)^{\frac{1}{\alpha}}} z^{\varepsilon/\alpha}$. ■

Hence if f is a Cobb-Douglas function, the cost of getting 1 unit effective capital k_M from the target firm z_T is same for all z_T. Acquirers are indifferent between acquiring all target firms: whether purchasing lots of capital from unproductive target firm or small amount of capital from productive target firm does not matter. The Cobb-Douglas case is a boundary point. If we increase the complementarity between z and z_T, then intuitively acquirers are more likely to match with similar target firms.

9.3.3 Proof of Proposition 6

Proof: In this section, we explore the planner problem. Define total effective capital of firm z as $\tilde{K}_t(z)$. A social planner maximize the total output. He takes the distribution of $\tilde{K}_t(z)$ as state and optimally chooses the acquiring firm set A_t and target firm set T_t, the matching function $z_{T,t}: A_t \to T_t$, the time allocating to M&A $s_t(z)$, investment rates of firm $\hat{\imath}_t^A(z)$, $\hat{\imath}_t^I(z)$ and $\hat{k}_{M,t}(z)$. The social planner problem can be described as

$$W\left(\tilde{K}_t\right) = \max_{s_t(z),\hat{\imath}_t^A(z),\hat{k}_{M,t}(z),\hat{\imath}_t^I(z),z_{T,t}(z),A_t,T_t} \left\{U(C_t) + \beta W\left(\tilde{K}_{t+1}\right)\right\} \tag{49}$$

$$s.t.\ Y_t = C_t + \lambda \int \phi\left(\hat{\imath}_t^A\right) \tilde{K}_t(z)\, dz + (1-\lambda)\int \phi\left(\hat{\imath}_t^I\right) \bar{K}_t(z)\, dz + q e_{t+1}\mu \int \tilde{K}_t(z)\, dz \tag{50}$$

$$Y_t = \int_{z\in A_t} [1-\lambda s_t(z)]\, z\tilde{K}_t(z)\, dz + \int_{z\in T_t} z\tilde{K}_t(z)\, dz \tag{51}$$

$$\bar{K}_{t+1}(z) = \omega\left(1-\delta+\hat{k}_t(z)\right)\tilde{K}_t(z)\, I(z\in A_t) + e_{t+1}\mu\tilde{K}_t(z)\, m(z) \tag{52}$$

$$\lambda \int_{z\in A'} \left(\frac{\hat{k}_{M,t}(z)}{\chi\left(s_t(z), \frac{z_T}{z}\right)} \right)^{\frac{1}{\alpha}} \tilde{K}_t(z)\,dz = \int_{z\in T'} \tilde{K}_t(z)\,dz \quad \forall\ A' \subseteq A_t, T' \subseteq T_t \tag{53}$$

$$\hat{k}_t(z) = \lambda\left(\hat{\imath}_t^A(z) + \hat{k}_{t,M}(z)\right) + (1-\lambda)\,\hat{\imath}_t^I(z) \tag{54}$$

Equation (49) is the objective function of social planner, which is maximize the representative consumer's welfare. The first constraint (50) is the resource constraint: total output Y_t will be used as consumption C_t, the internal investment $\lambda \int \phi\left(\hat{\imath}_t^A\right) \tilde{K}_t(z)\,dz + (1-\lambda)\int \phi\left(\hat{\imath}_t^I\right) \tilde{K}_t(z)\,dz$, and new entrants' initial capital. Equation (51) is the definition of aggregate output, which is similar as decentralized market. Equation (52) is the capital evolution of this economy. Function I is an indicator function. Hence $\omega\hat{k}_t(z)\,\tilde{K}_t(z)\,I(z \in A_t)$ is the capital of the firm next period of acquiring firms that can survive. $e_{t+1}\mu\tilde{K}_t(z)\,m(z)$ is the capital of the new entrants next period. Equation (53) is the resource constraint of the M&A market. It has similar meaning in the decentralized market. Equation (54) defines the investment rate $\hat{k}_t(z)$. $\lambda\left(\hat{\imath}_t^A(z) + \hat{k}_{t,M}(z)\right)$ is the investment rate of acquiring firms who have access to M&A markets. $(1-\lambda)\,\hat{\imath}_t^I(z)$ is the investment rate of firms who do not have access to M&A markets.

From the proposition 1 of Eeckhout and Kircher (2012), we have the following lemma:

Lemma 11 *If $\psi \leq 0$, then the solution of social planner satisfies positive assortative matching (PAM) property.*

Proof. See proof of proposition 1 in Eeckhout and Kircher (2012). ■

Given that the social planner will choose a PAM equilibrium, then we can simplify the condition (53)

$$\lambda \int_{z} \left(\frac{\hat{k}_{M,t}(u)}{\chi\left(s_t(u), \frac{z_{T(u)}}{u}\right)} \right)^{\frac{1}{\alpha}} \tilde{K}_t(u)\,du = \int_{z_T(z)} \tilde{K}_t(u)\,du \tag{55}$$

Instead of solving the planner's Bellman equation directly, we follow the strategy of Lucas and Moll (2014) to use a much simpler equation for the marginal social value of type z firm's capital. This marginal value is defined more formally in Appendix A of Lucas and Moll (2014) but the idea follows that if we increase d unit of type z capital $\bar{K}(z) + d$, the increase of the aggregate output is

$$\tilde{j}(z,t) = \frac{\partial W\left(\tilde{K}_t\right)}{\partial \tilde{K}(z)}$$

We can define a Lagrangian problem of planner problem as

$$\begin{aligned} H\left(\tilde{K}_t\right) = & \max_{s_t(z),\hat{\imath}_t^A(z),\hat{k}_{M,t}(z),\hat{\imath}_t^I(z),z_{T,t}(z),A_t,T_t} \{U(C_t) + \beta W\left(\tilde{K}_{t+1}\right) + \\ & \int \Lambda_t(z) U'(C_t)\left[\int_{z_T(z)} \tilde{K}_t(u)\,du - \lambda \int_z \left(\frac{\hat{k}_{M,t}}{\chi\left(s(u), \frac{z_{T(u)}}{u}\right)}\right)^{\frac{1}{\alpha}} \tilde{K}_t(u)\,du\right] dz\} \end{aligned}$$

$$s.t.\ (50) - (52) \text{ and } (54)$$

where $\Lambda_t(z) U'(C_t)$ is the Lagrangian multiplier on resource constraint (55). Take derivative with respect to $\tilde{K}_t(z)$, we have if $z \in A_t$

$$\begin{aligned} \tilde{j}(z,t) \;=\; & \max_{s_t(z),\hat{\imath}_t^A(z),\hat{k}_{M,t}(z),\hat{\imath}_t^I(z)} \{U'(C_t)\left[(1-\lambda s_t(z))z - \lambda\phi\left(\hat{\imath}_t^A\right) - \right. \\ & \left.(1-\lambda)\phi\left(\hat{\imath}_t^I\right) - qe_{t+1}\mu\right] + \beta\omega\left(1-\delta+\hat{k}_t(z)\right)\tilde{j}(z,t+1) + \beta\omega e_{t+1}\mu\int \tilde{j}(u,t+1)\,m(u)\,du \\ & - \left(\frac{\hat{k}_{M,t}(z)}{\chi\left(s(z), \frac{z_{T(z)}}{z}\right)}\right)^{\frac{1}{\alpha}} \int_z \Lambda_t(z) U'(C_t)\,du\} \end{aligned}$$

Now we define $j(z,t)$ as

$$j(z,t)\,U'(C_t) = \tilde{j}(z,t) \tag{56}$$

Then we have if $z \in A_t$

$$j(z,t) = \max_{s_t(z),\hat{\imath}_t^A(z),\hat{k}_{M,t}(z),\hat{\imath}_t^I(z)} \left\{ \begin{array}{c} [1-\lambda s_t(z)]z - \lambda\phi\left(\hat{\imath}_t^A\right) - \\ (1-\lambda)\phi\left(\hat{\imath}_t^I\right) - qe_{t+1}\mu - \lambda\left(\frac{\hat{k}_{M,t}(z)}{\chi\left(s(z),\frac{z_{T(z)}}{z}\right)}\right)^{\frac{1}{\alpha}} \int_z \Lambda_t(z)\,du \\ +\beta\omega\frac{U'(C_{t+1})}{U'(C_t)}\left[\left(1-\delta+\hat{k}_t(z)\right)j(z,t+1) + e_{t+1}\mu\int j(u,t+1)\,m(u)\,du\right] \end{array} \right\} \tag{57}$$

Notice that the above choice of $s_t(z), \hat{\imath}_t^A(z), \hat{k}_{M,t}(z)$ and choice of $\hat{\imath}_t^I(z)$ can be separated in the above equation. Hence we can rewrite the above equation as two independent optimization problems

$$j^A(z,t) = \max_{s_t(z),\hat{\imath}_t^A(z),\hat{k}_{M,t}(z)} \left\{ \begin{array}{l} (1-s(z))z - q\hat{\imath}_t^A(z) - \phi\left(\hat{\imath}_t^A\right) - qe_{t+1}\mu - \left(\frac{\hat{k}_{M,t}(z)}{\chi\left(s(z),\frac{z_T(z)}{z}\right)}\right)^{\frac{1}{\alpha}} \int_z \Lambda_t(z)\,du \\ +\beta\omega\frac{U'(C_{t+1})}{U'(C_t)}\left[\left(1-\delta+\hat{k}_t(z)\right)j(z,t+1) + e_{t+1}\mu\int j(u,t+1)\,m(u)\,du\right] \end{array} \right\} \tag{58}$$

$$j^I(z,t) = \max_{\hat{\imath}_t^I(z)} \left\{ z - \phi\left(\hat{\imath}_t^I\right) - qe_{t+1}\mu + \beta\omega\frac{U'(C_{t+1})}{U'(C_t)} \left[\begin{array}{c} \left(1-\delta+\hat{k}_t(z)\right)j(z,t+1) + \\ e_{t+1}\mu\int j(u,t+1)\,m(u)\,du \end{array} \right] \right\} \tag{59}$$

$$j(z,t) = \lambda j^A(z,t) + (1-\lambda)\,j^I(z,t) \tag{60}$$

Similarly, we can get

$$j(z,t) = z - qe_{t+1}\mu + \beta\omega\frac{U'(C_{t+1})}{U'(C_t)}e_{t+1}\mu\int j(u,t+1)\,m(u)\,du + \int_z^{z_T(z_{\max})}\Lambda_t(u)\,du \quad \text{if } z\in T_t \tag{61}$$

If there is free entry condition such that

$$\beta\omega\frac{U'(C_{t+1})}{U'(C_t)}e_{t+1}\mu\int j(u,t+1)\,m(u)\,du = q$$

Hence equations (58) to (61) define the optimal decisions of the social planner. Then compare them with equations (11) to (15), they have the same forms. Firms will be the acquiring firm iff $\lambda j^A(z,t) + (1-\lambda)\,j^I(z,t) \geq z + \int_z^{z_T(z_{\max})}\Lambda_t(u)\,du$. Hence the social planner's solution will be the same with decentralized equilibrium.

9.3.4 Proof of Proposition 8

Proof: From proposition 4 and 5, we can see that if firms are in acquirer set A then they will quit the market only via exogenous death shocks: the new entrants whose productivity is z is $em(z)$. Then after $t-\tau$ periods, only $\omega^{t-\tau}$ fraction will survive. Hence at time t, the mass of firms with productivity z that enters at period τ is

$$n_{t,\tau}(z) = e\omega^{t-\tau}m(z) \quad \text{when } z \geq z^* \tag{62}$$

$$n_{t,\tau}(z) = \begin{cases} e \text{ if } \tau = t \\ 0 \text{ if } \tau < t \end{cases} \quad \text{when } z < z^* \tag{63}$$

Firm's growth rate is $g^A(z)$ when the firm can acquire target firms and $g^I(z)$ if it can not. If $z \geq z^*$, the aggregate capital of firms with productivity z that enters at period τ is

$$\sum_{j \in z} S_{t,\tau}(j) = \tilde{k}_\tau n_{t,\tau}(z) \sum_{n=0}^{t-\tau} \binom{t-\tau}{n} \lambda^n (1-\lambda)^{t-\tau-n} g^A(z)^n g^I(z)^{t-\tau-n} \tag{64}$$

$$= \tilde{k}_\tau n_{t,\tau}(z) \left[\lambda g^A(z) + (1-\lambda) g^I(z)\right]^{t-\tau} \tag{65}$$

The above equation says that the aggregate capital of firms in period t, whose productivity are z and ages are $t-\tau$, is equal to the initial capital of entrants $\bar{k}_\tau$ multiplied by the expected growth rate and the number of firms. Then we can simplify the aggregate capital in equation (18) as

$$K_t = e \int_{z \geq z^*} \sum_{\tau=0}^{t} \tilde{k}_\tau \omega^{t-\tau} \bar{g}(z)^{t-\tau} m(z)\, dz + eM(z^*)\, \tilde{k}_t \tag{66}$$

where $\bar{g}(z) = \lambda g^A(z) + (1-\lambda) g^I(z)$. Aggregate capital has two parts in (66). The first part is the capital of the acquiring firms. $S_{t,\tau}(z)$ is the acquirer z's total capital at time t. The second part is the capital of target firms that only live one period. Their size is $S_{t,t}(z) = \tilde{k}_t n_{t,t}(z)$ and they have a mass $n_{t,t}(z) = em(z)$. Guess K_t grow with constant rate g_K. Then

$$K_t = e \int_{z \geq z^*} \sum_{\tau=0}^{t} \mu K_t g_K^{\tau-t} \omega^{t-\tau} \bar{g}(z)^{t-\tau} m(z)\, dz + eM(z^*)\, \mu K_t \tag{67}$$

From consumer problem, we can see if $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$, then

$$\frac{1}{1+r_t} = \beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} = \frac{\beta}{g_K^{\gamma}}$$

When γ increases, we can see $\frac{1}{1+r_t}$ will decrease. The growth rate of the firm will decrease too. Given our parameters, we numerically verify

$$\frac{\omega}{g_K}\bar{g}(z) < 1,\ \forall\ z$$

Then (67) can be simplified to equation (23).

9.3.5 Proof of Proposition 9

Proof: Let us denote firm as j and its size as $k_t(j)$. We then have

$$\frac{k_t(j)}{\bar{K}_t} = g(j)\frac{k_{t-1}(j)}{\bar{K}_{t-1}} + \varepsilon \tag{68}$$

In equation (68),

$$g(j) = \begin{cases} \frac{g^A(j)}{g_K} & \text{with prob } \omega\lambda \\ \frac{g^I(j)}{g_K} & \text{with prob } \omega(1-\lambda) \\ 0 & \text{with prob } 1-\omega \end{cases}$$

ε denotes the capital of new entrant $\varepsilon = \mu$ if $g(j) = 0$. Otherwise $\varepsilon = 0$. Notice that $E(g(j)) = \omega\left(\lambda g^A(z) + (1-\lambda)g^I(z)\right) < 1$ from proposition 8. Then we have the following lemma.

Proof.

Lemma 12 *If $g^A(z) > 1$, then there exists $\Theta(z) > 0$ such that*

$$\omega\left(\lambda g^A(z)^{\Theta(z)} + (1-\lambda)g^I(z)^{\Theta(z)}\right) = g_K^{\Theta(z)} \tag{69}$$

and the conditional distribution of firm size satisfies

$$\lim_{x\to\infty}\frac{\Pr\left(k_t(z)/\bar{K}_t > x|z\right)}{x^{-\Theta(z)}} = c(z) \text{ for } z \text{ such that } g^A(z) > 1 \tag{70}$$

where $c(z)$ is a constant.

■

Proof. See Kesten (1973). ■

The above lemma says that conditional on firm productivity z, then firm's size distribution has a Pareto tail. Hence the distribution $\Pr\left(\frac{k_t(j)}{\bar{K}_t} > x\right)$ is a mixture of different

Pareto distributions.

Denote $\Theta_{\min} = \min\{\Theta(z)\}$, we have

$$\begin{aligned}
\frac{\Pr\left(k_t(j)/\bar{K}_t > x\right)}{x^{-\Theta_{\min}}} &= \int \frac{\Pr\left(k_t(j)/\bar{K}_t > x|z\right)}{x^{-\Theta_{\min}}} f(z)\,dz \qquad (71)\\
&= \int_{g^A(z)\leq 1} \frac{\Pr\left(k_t(j)/\bar{K}_t > x|z\right)}{x^{-\Theta_{\min}}} m(z)\,dz + \int_{g^A(z)>1} \frac{\Pr\left(k_t(j)/\bar{K}_t > x|z\right)}{x^{-\Theta_{\min}}} m(z)\,dz
\end{aligned}$$

In the first part, when $x \to \infty$, $\lim_{x\to\infty} \frac{\Pr(S_t(z)>x|z)}{x^{-\Theta_{\min}}} = 0$ since firm enters with size ε that has a boundary support while growth rate is less than 1 for these firms. Their size will shrink. Hence when x is large than the upper bound of ε support, $\Pr\left(\frac{k_t(j)}{\bar{K}_t} > x|z\right) = 0$. In the second part, if $z \in \arg\min\{\Theta(z)\}$, we have $\lim_{x\to\infty} \frac{\Pr\left(k_t(j)/\bar{K}_t>x|z\right)}{x^{-\Theta_{\min}}} = c(z)$ otherwise $\lim_{x\to\infty} \frac{\Pr(S_t(z)>x|z)}{x^{-\Theta_{\min}}} = 0$. Then we have

$$\lim_{x\to\infty} \frac{\Pr\left(k_t(j)/\bar{K}_t > x\right)}{x^{-\Theta_{\min}}} = \int_{z\in\arg\min\{\Theta(z)\},g(z)>1} c(z)\,m(z)\,dz \qquad (72)$$

Lemma 13 $\Theta(z)$ *is decreasing on* z. *Hence* $z_{\max} = \arg\min\{\Theta(z)\}$ *and* $\Theta_{\min} = \Theta(z_{\max})$

Proof. Take derivative in equation (69), we have

$$\frac{d\Theta}{dz} = -\frac{\lambda\Theta g^{A\Theta-1}\frac{dg^A}{dz} + (1-\lambda)\Theta g^{I\Theta-1}\frac{dg^I}{dz}}{\lambda g^A(z)^{\Theta(z)}\ln g^A + (1-\lambda) g^I(z)^{\Theta(z)}\ln g^I}$$

The numerator is greater than 0 since g^A and g^I are strictly increasing in z. Denote $F(\Theta) = \omega\lambda g^{A\Theta} + \omega(1-\lambda) g^{I\Theta} = 1$. The denominator is $\frac{dF}{d\Theta}$. Consider a small $\Delta > 0$, then we can see $F(\Theta+\Delta) = \omega\lambda\left(g^{A\Theta}\right)^{\frac{\Theta+\Delta}{\Theta}} + \omega(1-\lambda)\left(g^{I\Theta}\right)^{\frac{\Theta+\Delta}{\Theta}}$. $\frac{\Theta+\Delta}{\Theta} > 1$ Hence from Jensen inequality, we have

$$1 = F(\Theta)^{\frac{\Theta+\Delta}{\Theta}} < F(\Theta+\Delta)$$

Hence we have $\frac{dF}{d\Theta} > 0$. Thus $\frac{d\Theta}{dz} < 0$. ■

Then we can simplify equation (72) as

$$\lim_{x\to\infty} \frac{\Pr\left(k_t(j)/\bar{K}_t > x\right)}{x^{-\Theta_{\min}}} = c(z_{\max})\,m(z_{\max})$$

9.4 Solving the Transition Dynamics

In the transition path, when $\psi = 0$, we have

$$P_t\left(z_T\right) = X_t z_T^{\varepsilon/\alpha}$$

Firm size on the transition path now becomes to

$$S_{t,\tau}\left(z\right) = \sum_{j\in z} k_{t,\tau}\left(j\right) = \tilde{k}_\tau n_{t,\tau} \prod_{i=0}^{t-\tau-1} \left[\lambda g_{\tau+i}^A\left(z\right) + \left(1-\lambda\right) g_{\tau+i}^I\left(z\right)\right] \text{ if } z > z_{t-1}^*$$

The above equation says that at time t, firms who are going to survive from previous period are those z greater than z_{t-1}^*.

Market clearing condition is

$$\left(1-\delta\right)\int_{z_{\min}}^{z_t^*}\int_k k_t d\Gamma_t(z,k) = \int_{z_t^*}^{z_{\max}}\int_k \hat{k}_{T,t}\left(z\right) k_t d\Gamma_t(z,k) \tag{73}$$

We can simplify the equation as

$$\begin{aligned} & \left(1-\delta\right)\tilde{k}_t \int_{z_{\min}}^{z_{t-1}^*} m\left(z\right) dz + \left(1-\delta\right)\int_{z_{t-1}^*}^{z_t^*} S_t\left(z\right) dz \\ = & \int_{z_t^*}^{z_{\max}} \hat{k}_{T,t}\left(z\right)\sum_{\tau\le t} S_{t,\tau}\left(z\right) dz \end{aligned} \tag{74}$$

From equation (73) to (74), we use the condition that $\int_k k_t d\Gamma_t(z,k) = S_t\left(z\right)$. We also use the condition that those firms who are below z_{t-1}^* will be merged in the previous period. Hence only new entrants will sell the capital on the market in t. For firms between z_{t-1}^* and z_t^*, they are acquirers in $t-1$ but will sell the capital in period t. Hence both new entrants and incumbents will sell the capital. Let $x_{t,\tau}\left(z\right) = \prod_{i=0}^{t-\tau-1}\left[\lambda g_{\tau+i}^A\left(z\right) + \left(1-\lambda\right) g_{\tau+i}^I\left(z\right)\right]$.

After using the condition that $\tilde{k}_t = \mu e_t K_{t-1}$, we can simplify equation (74) to

$$\begin{aligned} & e_t K_{t-1}\int_{z_{\min}}^{z_{t-1}^*} m\left(z\right) dz + \int_{z_t^*}^{z_{\max}} m\left(z\right)\sum_{\tau\le t} e_\tau K_{\tau-1}\omega^{t-\tau} x_{t,\tau}\left(z\right) dz \\ = & \int_{z_t^*}^{z_{\max}} \hat{k}_{T,t}\left(z\right) m\left(z\right)\sum_{\tau\le t} e_\tau K_{\tau-1}\omega^{t-\tau} x_{t,\tau}\left(z\right) dz \end{aligned} \tag{75}$$

Let $K_t = K_\tau \prod_{s=0}^{t-\tau-1} g_{K,\tau+s}$, where $g_{K,t}$ is the growth rate in period t. We have

$$\begin{aligned} & e_t M\left(z_{t-1}^*\right) + \int_{z_{t-1}^*}^{z_t^*} m(z) \sum_{\tau \leq t} e_\tau \frac{\omega^{t-\tau} x_{t,\tau}(z)}{\prod_{s=0}^{t-\tau-1} g_{K,\tau-1+s}} dz \\ = & \int_{z_t^*}^{z_{\max}} \hat{k}_{T,t}(z) m(z) \sum_{\tau \leq t} e_\tau \frac{\omega^{t-\tau} x_{t,\tau}(z)}{\prod_{s=0}^{t-\tau-1} g_{K,\tau-1+s}} dz \end{aligned} \tag{76}$$

On the other hand, we can define the aggregate capital as

$$K_t = \sum_{\tau=0}^{t} e_\tau \tilde{k}_\tau \omega^{t-\tau} \int_{z \geq z_{t-1}^*} x_{t,\tau}(z) m(z) dz + e_t M\left(z_{t-1}^*\right) \tilde{k}_t$$

We can simplify it to

$$\frac{g_{K,t-1}}{\mu} = \sum_{\tau=0}^{t} e_\tau \frac{\omega^{t-\tau}}{\prod_{s=0}^{t-\tau-1} g_{K,\tau-1+s}} \int_{z \geq z_{t-1}^*} x_{t,\tau}(z) m(z) dz + e_t M\left(z_{t-1}^*\right) \tag{77}$$

The above equation defines aggregate growth rate $g_{K,t}$.

Finally, we have free entry condition each period:

$$q = \frac{1}{1+r_{t-1}} \int \max\left[\hat{\lambda}(z) J(z) + \left(1-\hat{\lambda}(z)\right) J^I(z), J^T(z)\right] m(z) dz \tag{78}$$

while

$$\frac{1}{1+r_t} = \frac{\beta}{g_{K,t}^{\gamma}}$$

To solve the problem, we follow the steps:

(1) Fix a large step T, solve two steady states before and after change.

(2) Guess a sequence of $\{X_t\}, \{g_{K,t}\}$.

(3) Given $\{X_t, g_{K,t}\}$, backward induct the value functions and policy functions.

(4) Recursively solve entry process e_t from equation (77).

(4) Check the market clearing condition (76) and free entry condition, as equation (78).